The Cauchy problem for hyperbolic systems with Hölder continuous coefficients with respect to time

Yasuo Yuzawa

Center for Tsukuba Advanced Research Alliance, University of Tsukuba, Tennoudai 1-1-1, Tsukuba city, Ibaraki 315-8577, Japan

Received 17 August 2004; revised 13 December 2004
Available online 5 February 2005

Abstract

We discuss the local existence and uniqueness of solutions of certain nonstrictly hyperbolic systems, with Hölder continuous coefficients with respect to time variable. We reduce the nonstrictly hyperbolic systems to the parabolic ones, then we shall prove them by use of Tanabe–Sobolevski’s method.

© 2005 Elsevier Inc. All rights reserved.

Keywords: Weakly hyperbolic; Nonstrictly hyperbolic; Gevrey well-posedness; Hölder coefficients

1. Introduction

We consider the following Cauchy problem:

\[
\begin{align*}
\partial_t u(t, x) - \sum_{j=1}^d A_j(t) \partial_j u(t, x) &= f(t, x), \\
u(0, x) &= u_0(x),
\end{align*}
\]

\[(t, x) \in [0, T] \times \mathbb{R}^d, \quad x \in \mathbb{R}^d,
\]

where each \( A_j \) is \( N \times N \) matrix function, \( f \), \( u \) and \( u_0 \) are \( N \) component vector functions and \( \partial_t = \partial / \partial t \), \( \partial_j = \partial / \partial x_j \).

E-mail address: yuzawa@wslab.risk.tsukuba.ac.jp

0022-0396/$ - see front matter © 2005 Elsevier Inc. All rights reserved.
We assume that this system has weak hyperbolicity, that is,

\[(A.I) \text{ All eigenvalues of } \sum_{j=1}^{d} A_j(t) \xi_j \text{ are real valued in } [0, T] \times \mathbb{R}^d \setminus \{0\} \text{ and their multiplicity does not exceed } \nu.\]

Many papers are devoted to the study of wellposedness in the Gevrey classes for the Cauchy problem (1.1). When all \( A_j \) are smooth enough with respect to \( t \), then this property has proved for the order \( 1 \leq s < 1 + 1/(\nu - 1) \) by Bronshtein [1] in the higher-order scalar case and by Kajitani [4] in the system case, respectively. Moreover, they have showed it in the case that the coefficient also depend on \( x \). When each \( A_j \) has only \( \mu \)-Hölder continuity in \( t \) for some \( 0 < \mu \leq 1 \), the Cauchy problem is also wellposed in the Gevrey classes but the Gevrey order must be lower than the smooth case. The first result in the Hölder continuous case was derived by Colombini et al. [2]. They proved that the Cauchy problem to the second-order equation \( u_{tt} = a(t)u_{xx} \) was Gevrey wellposed for the order \( 1 \leq s < 1 + \mu/2 \) and, it is important, this order is optimal. This work has been continued in various directions. Nishitani [7] extended to the second-order equations with coefficients also depending on \( x \), and then Ohya and Tarama [8] extended that the higher-order scalar equation was Gevrey wellposed for \( 1 \leq s < 1 + \mu/\nu \). The system case was investigated by Kajitani [5], and he showed that the weakly hyperbolic systems were wellposed in the Gevrey classes for \( 1 \leq s < 1 + \mu/(\nu + 1) \), and in [6] he also derived the energy inequality for \( 1 \leq s < 1 + \min(\mu/(\nu + 1), (2 - \mu)/(2\nu - 1)) \) and applied to the nonlinear Cauchy problem.

Recently, D’Ancona et al. [3] proved the Gevrey wellposedness for \( 1 \leq s < 1 + \mu/\nu \) to \( 3 \times 3 \) weakly hyperbolic systems with coefficients depending on \( t \). To prove it, they derived the energy estimates to the approximate symbols. In this paper, we shall extend their result to any \( N \times N \) systems by using the other approach, semi-group method called Tanabe–Sobolevski method (cf. [6,9,10]) and consequently obtain the energy estimates.

To state our results we shall introduce Gevrey classes and these properties.

**Definition 1.1.** Let \( s \geq 1 \), then we denote by \( \gamma^{(s)}(\mathbb{R}^d) \) the set of all functions satisfying the following condition: for any compact subset \( K \) of \( \mathbb{R}^d \), there exist constants \( C_K > 0 \) and \( A_K > 0 \) such that

\[ |\partial_x^\alpha u(x)| \leq C_K A_K^{[2]|x|!s} \]

for any \( x \in K \) and \( \alpha \in \mathbb{N}^d \) and we define \( \gamma^{(s)}_0(\mathbb{R}^d) = \gamma^{(s)}(\mathbb{R}^d) \cap C_0(\mathbb{R}^d) \).

**Definition 1.2.** Let \( k \) be an integer and \( 0 < \mu \leq 1 \). For a topological space \( Y \), we denote by \( C^{k,\mu}([0, T]; Y) \) the set of functions \( u(t, y) \) which are \( k \) times differentiable in \( Y \) with respect to \( t \) and \( (\partial/\partial t)^k u(t, y) \) are \( \mu \)-Hölder continuous in \( Y \). More precisely, there exists a constant \( C \) such that

\[ \|\partial_t^k u(t, \cdot) - \partial_t^k u(t', \cdot)\|_Y \leq C |t - t'|^\mu \]

for any \( t, t' \in [0, T] \).
Definition 1.3. For $\rho \geq 0$, $s > 1$, and $l \in \mathbb{R}$, we define

$$H_{A(\rho)}^l(\mathbb{R}^d) = \{ u \in L^2_x(\mathbb{R}^d) ; \langle \bar{\zeta} \rangle^l e^{A(\rho)\bar{\zeta}} \hat{u}(\bar{\zeta}) \in L^2_\bar{\zeta}(\mathbb{R}^d) \},$$

where $\langle \bar{\zeta} \rangle = \sqrt{1 + |\bar{\zeta}|^2}$, $A(\rho) = A(\rho, \bar{\zeta}; s) = \rho(\bar{\zeta})^{1/s}$ and $\hat{u}(\bar{\zeta})$ stands for a Fourier transform of $u(x)$:

$$\hat{u}(\bar{\zeta}) = \int_{\mathbb{R}^d} e^{-ix \cdot \bar{\zeta}} u(x) \, dx,$$

and for $\rho < 0$ we define $H_{A(\rho)}^l(\mathbb{R}^d)$ as the dual space of $H_{-A(\rho)}^{-l}(\mathbb{R}^d)$.

When $\rho = 0$, $H_{A(0)}^l = H^l_0$ are usual Sobolev spaces and we write them as $H^l$ in brief. When $\rho > 0$, $H_{A(\rho)}^l$ are Hilbert spaces with inner products

$$(u, v)_{H_{A(\rho)}^l} = \langle \langle \bar{\zeta} \rangle^l e^{\rho(\bar{\zeta})^{1/s}} u, \langle \bar{\zeta} \rangle^l e^{\rho(\bar{\zeta})^{1/s}} v \rangle_{L^2}$$

and we define the norm of $H_{A(\rho)}^l$ by $\|u\|_{H_{A(\rho)}^l} = \|\langle \bar{\zeta} \rangle^l e^{\rho(\bar{\zeta})^{1/s}} \hat{u}(\bar{\zeta})\|_{L^2}$.

Definition 1.4. We define $e^{A(\rho)} = e^{\rho(D)^{1/s}}$ a pseudodifferential operator of order infinity such as

$$e^{A(\rho)}u(x) = e^{\rho(D)^{1/s}}u(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \bar{\zeta} + A(\rho, \bar{\zeta})} \hat{u}(\bar{\zeta}) \, d\bar{\zeta}$$

for $u(x)$ in $H_{A(\rho)}^l$.

Definition 1.5. Let $\rho(t)$ be positive definite in $[0, T]$ and $X$ a function space. Then, we denote by $e^{-A(\rho(t))}C^{k, \mu}([0, T]; X)$ the class of functions $f(t, x)$ for which to every $t \in [0, T]$,

$$e^{A(\rho(t))}f(t, x) = e^{\rho(t)(D)^{1/s}}f(t, x) \in C^{k, \mu}([0, T]; X).$$

We note the relations between $\gamma^{(s)}_0(\mathbb{R}^d)$ and $H_{A(\rho)}^l(\mathbb{R}^d)$.

Proposition 1.1 (cf. Kajitani [4]). Lemma 1.2 For any $u(x)$ in $\gamma^{(s)}_0(\mathbb{R}^d)$ and $l \in \mathbb{R}$, there exists a constant $\rho_u > 0$ such that $u(x)$ in $H_{A(\rho_u)}^l(\mathbb{R}^d)$.

Conversely, if $u(x)$ is of a compact support and belongs to $H_{A(\rho)}^l(\mathbb{R}^d)$ for some $\rho > 0$, then $u$ also belongs to $\gamma^{(s)}_0(\mathbb{R}^d)$. 
Now, we shall state the main theorems.

**Theorem 1.1.** Let \( 1 \leq s < 1 + \mu/p \) and \( \sigma = (v - 1)(1 - 1/s) \). Assume that (A.I) and the following condition (A.II):

(A.II) each \( A_j(t) \) belongs to \( C^0,\mu([0, T]) \) for \( j = 1, \ldots, d \). Then for every \( u_0(x) \) in \( H^l_{A(T)} \) and every \( f(t, x) \) in \( e^{-A(T-t)}C([0, T]; H^l) \), there is a pseudodifferential operator \( W(t, D; \tau) \) such that

\[
u(t, x) = W(t, D; 0)u_0(x) - i \int_0^t W(t, D; r)f(r, x)\,dr\]

is a unique solution of the Cauchy problem (1.1). Moreover \( u(t, x) \) is in \( e^{-A(T-t)}C([0, T]; H^l) \cap e^{-A(T-t)}C^1([0, T]; H^{l-1}) \) and satisfies

\[
\|u(t, \cdot)\|_{H^{l-\sigma}_{A(T-t)}} \leq C \left( \|u_0\|_{H^l_{A(T)}} + \int_0^t \|f(r, \cdot)\|_{H^{l}_{A(T-r)}}\,dr \right) \tag{1.2}
\]

for any \( l \in \mathbb{R} \) and \( 0 < t \leq T \).

Considering the property of the finite propagation of the solution for the weakly hyperbolic system and Proposition 1.1, the following theorem is concluded by Theorem 1.1.

**Theorem 1.2.** Assume that (A.I) and (A.II). If \( 1 \leq s < 1 + \mu/p \), then for any \( f(t, x) \) in \( C([0, T]; \gamma^{(s)}(\mathbb{R}^d)) \) and \( u_0(x) \) in \( \gamma^{(s)}(\mathbb{R}^d) \), there is a solution \( u(t, x) \) in \( C^1([0, T]; \gamma^{(s)}(\mathbb{R}^d)) \) of Eq. (1.1).

2. The proof of theorem

In this section, we assume that \( 1 \leq s < 1 + \mu/p \) and \( \kappa = 1/s \) without permission. We reintroduce Eq. (1.1):

\[
\begin{cases}
\partial_t u(t, x) - \sum_{j=1}^d A_j(t)\partial_j u(t, x) = f(t, x), & (t, x) \in [0, T] \times \mathbb{R}^d, \\
u(0, x) = u_0(x), & x \in \mathbb{R}^d.
\end{cases} \tag{1.1}
\]

Let \( v(t, x) \) be \( e^{A(T-t)}u(t, x) = e^{(T-t)(D)^\kappa}u(t, x) \), then we can reduce the problem (1.1) to

\[
\begin{cases}
D_v(t, x) - i(D)^\kappa v(t, x) - \sum_{j=1}^d A(t)D_j v(t, x) = g(t, x), \\
v(0, x) = v_0(x),
\end{cases} \tag{2.1}
\]
where $v_0(x) = e^{A(t)}u_0(x)$, $g(t, x) = -ie^{A(T-t)}f(t, x)$, $D_t = -i\hat{c}_t$, $D_j = -i\hat{c}_j$, $D = (D_1, D_2, \ldots, D_d)$. We shall solve problem (2.1) by constructing a semi-group for a generator

$$a(t) = a(t, D) = i\langle D \rangle I + \sum_{j=1}^{d} A_j(t)D_j. \quad (2.2)$$

Let $\tau$ be fixed in $[0, T]$. We define

$$V_0(t, \tau; \xi) = e^{i\langle D \rangle(t-\tau)} = \sum_{n=0}^{\infty} \frac{(i(t-\tau))^n}{n!} \left( i\langle \xi \rangle I + \sum_{j=1}^{d} A_j(\tau)\xi_j \right)^n. \quad (2.3)$$

Since assumption (A.1), it is well defined and easy to see that $V_0$ satisfies

$$\begin{align*}
\begin{cases}
D_t V_0(t, \tau; \xi) = a(\tau, \xi)V_0(t, \tau; \xi), & t > \tau, \\
V_0(\tau, \tau; \xi) = I.
\end{cases}
\end{align*} \quad (2.4)$$

We note that $V_0(t, \tau; \xi)$ can be expressed

$$V_0(t, \tau; \xi) = \frac{1}{2\pi i} \int_{\gamma} e^{i\lambda(t-\tau)}(\lambda I - a(\tau, \xi))^{-1} d\lambda, \quad (2.5)$$

where $\gamma$ is a simple closed curve which includes in the whole of the eigen values of $a(\tau, \xi)$.

Next, we shall construct a semi-group $V(t, \tau)$ for a generator $a(t)$;

$$\begin{align*}
\begin{cases}
D_t V(t, \tau) = a(t)V(t, \tau), & t > \tau, \\
V(\tau, \tau) = I.
\end{cases}
\end{align*} \quad (2.6)$$

In order to construct $V(t, \tau)$, we shall use Tanabe–Sobolevski’s method. (cf. [6,9]) We denote $V(t, \tau; \xi)$, the symbol of $V(t, \tau)$, by

$$V(t, \tau; \xi) = V_0(t, \tau; \xi) + \int_{\tau}^{t} V_0(t, r; \xi)\Phi(r, \tau; \xi) dr. \quad (2.7)$$

If $V(t, \tau; \xi)$ satisfies (2.6), $\Phi(t, \tau; \xi)$ must satisfy the following equation:

$$\Phi(t, \tau; \xi) = R(t, \tau; \xi) + \int_{\tau}^{t} R(t, r; \xi)\Phi(r, \tau; \xi) dr. \quad (2.8)$$
where \( R(t, \tau; \zeta) = (a(t, \zeta) - a(\tau, \zeta))V_0(t, \tau; \zeta) \). Conversely, if \( \Phi(t, \tau; \zeta) \) satisfies (2.8), then \( V(t, \tau; \zeta) \) satisfies (2.6). We shall construct a solution \( \Phi(t, \tau; \zeta) \) of Eq. (2.8) as follows:

\[
\Phi(t, \tau; \zeta) = \sum_{j=0}^{\infty} \Phi_j(t, \tau; \zeta),
\]

where

\[
\begin{align*}
\Phi_0(t, \tau; \zeta) &= R(t, \tau; \zeta), \\
\Phi_j(t, \tau; \zeta) &= \int_{\tau}^t R(t, r; \zeta) \Phi_{j-1}(r, \tau; \zeta) \, dr \quad (j \geq 1).
\end{align*}
\]

**Proposition 2.1.** (i) There exists a constant \( C_0 > 0 \) such that

\[
|V_0(t, \tau; \zeta)| \leq C_0 (\zeta_1)^\sigma e^{-\frac{1}{2} (t-\tau) (\zeta_1)^\kappa},
\]

for any \( \zeta \in \mathbb{R}^d \), where \( \sigma = (\nu - 1)(1 - \kappa) \) and \( C_0 \) is independent of \( \tau \).

(ii) There exists a constant \( C_1 > 0 \) such that

\[
|\Phi_j(t, \tau; \zeta)| \leq C_1^{j+1} j!^{-1} ((t-\tau) (\zeta_1)^\kappa) j (\zeta_1)^{(1-\kappa)\nu + \kappa (1-\mu) - (\kappa (\nu+\mu)-\nu) j} e^{-\frac{1}{2} (t-\tau) (\zeta_1)^\kappa},
\]

for any \( \zeta \in \mathbb{R}^d \) and \( j \geq 0 \), where \( C_1 \) is independent of \( \tau \).

(iii)

\[
\left| \int_{\tau}^t V_0(t, r; \zeta) \Phi_j(r, \tau; \zeta) \, dr \right| \leq C_0 C_1^{j+1} (j+1)!^{-1} ((t-\tau) (\zeta_1)^\kappa) j+1 \times (\zeta_1)^{(\sigma - (\kappa (\nu+\mu)-\nu)) - (\kappa (\nu+\mu)-\nu) j} e^{-\frac{1}{2} (t-\tau) (\zeta_1)^\kappa}
\]

for any \( \zeta \in \mathbb{R}^d \) and \( j \geq 0 \).

To prove this proposition, we prepare the following lemma.

**Lemma 2.1.** Let \( \Omega \) be a domain in \( \mathbb{C} \). Assume that \( f(z) \) is a regular function near \( \Omega \), \( \gamma \) is a closed curve on \( \Omega \) and \( \{\lambda_1, \lambda_2, \ldots, \lambda_d\} \) is a subset in the interior of \( \gamma \). Then,

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f(\lambda)}{\prod_{j=1}^{d} (\lambda - \lambda_j)} \, d\lambda
\]

\[
= \int_0^1 \cdots \int_0^1 \theta_2 \theta_3 \theta_4 \cdots \theta_{d-1} f^{(d-1)}(q(\lambda_1, \lambda_2, \ldots, \lambda_d; \theta)) \, d\theta_1 d\theta_2 \cdots d\theta_{d-1},
\]

where \( q(\lambda_1, \lambda_2, \ldots, \lambda_d; \theta) = \theta_1 \theta_2 \cdots \theta_{d-1} \lambda_1 + (1 - \theta_1) \theta_2 \cdots \theta_{d-1} \lambda_2 + (1 - \theta_2) \theta_3 \cdots \theta_{d-1} \lambda_3 + \cdots + (1 - \theta_{d-2}) \theta_{d-1} \lambda_d + (1 - \theta_{d-1}) \lambda_d. \)
Proof. We shall prove by mathematical induction. When \( d = 1 \), this is Cauchy’s integral theorem. We assume that this is valid for \( d \leq m - 1 \). Suppose that \( \lambda_j \neq \lambda_{j'} \) for any \( j \neq j' \). We note that

\[
\sum_{l=1}^{m} \frac{1}{\prod_{j \neq l} (\lambda_l - \lambda_j)} = 0
\]

and

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f(\lambda)}{\prod_{j=1}^{m-1} (\lambda - \lambda_j)} d\lambda = \sum_{l=1}^{m-1} \frac{f(\lambda_l)}{\prod_{j \neq l} (\lambda_l - \lambda_j)}.
\]

then, by use of the assumption of the induction,

(l.h.s. for \( d = m \))

\[
= \sum_{l=1}^{m} \frac{f(\lambda_l)}{\prod_{j \neq l} (\lambda_l - \lambda_j)} = \sum_{l=1}^{m} \frac{f(\lambda_l)}{\prod_{j, j \neq l} (\lambda_l - \lambda_j)} - \sum_{l=1}^{m} \frac{f(\lambda_m)}{\prod_{j \neq l} (\lambda_l - \lambda_j)}
\]

\[
= \sum_{l=1}^{m-1} \frac{\lambda_l - \lambda_m}{\prod_{j \neq l, m} (\lambda_l - \lambda_j)} \int_{0}^{1} f'(\theta_{m-1} \lambda_l + (1 - \theta_{m-1})\lambda_m) d\theta_{m-1}
\]

\[
= \sum_{l=1}^{m-1} \frac{1}{\prod_{j \neq l, m} (\lambda_l - \lambda_j)} \int_{0}^{1} f'(\theta_{m-1} \lambda_l + (1 - \theta_{m-1})\lambda_m) d\theta_{m-1}
\]

\[
= \int_{0}^{1} \cdots \int_{0}^{1} \theta_{2} \theta_{3} \cdots \theta_{m-3} \left( \frac{1}{m-2} \right) \left. \frac{d^{m-2}}{d\lambda_{m-2}} \right|_{\lambda_{l=q(1,2,\ldots,m-1);0}}^{} d\theta_{1} \cdots d\theta_{m-1}
\]

\[
= (r.h.s. \text{ for } d = m).
\]

From the continuity of both sides with respect to a parameter of \( \{\lambda_{1}, \lambda_{2}, \ldots, \lambda_{d}\} \), it is valid for any \( \{\lambda_{j}\} \). Thus the proof of Lemma 2.1 is finished. \( \square \)

**Proof of Proposition 2.1.** (i) Let \( \tau \) be fixed in \((0, T)\) and \( \xi^0 \) in \( S^{d-1} \). We can number the eigen values of \( \sum_{j=1}^{d} A_j(\tau)\xi^0_j \) as

\[
\lambda_{1,1}(\tau, \xi^0) = \lambda_{1,2}(\tau, \xi^0) = \cdots = \lambda_{1,i_1}(\tau, \xi^0) < \lambda_{2,1}(\tau, \xi^0) = \cdots = \lambda_{2,i_2}(\tau, \xi^0) < \cdots < \lambda_{p,1}(\tau, \xi^0) = \cdots = \lambda_{p,i_p}(\tau, \xi^0),
\]
where $1 \leq p \leq d$. We note that $\max\{i_l; 1 \leq l \leq p\} \leq v$. By the continuity and homogeneity of $\lambda_{j,k}(\tau, \xi)$ with respect to $\xi$, there exist constants $C^0 = C(\xi^0) > 0$ and $\delta^0 = \delta(\xi^0) > 0$ such that

$$|\lambda_{k,a}(\tau, \xi) - \lambda_{l,b}(\tau, \xi)| \leq 2C^0(\xi^0)$$

for any $\alpha, \beta$ and $k \neq l$, \hspace{1cm} (2.14)

$$|\lambda_{k,a}(\tau, \xi) - \lambda_{k,b}(\tau, \xi)| \leq C^0(\xi^0)$$

for any $\alpha$ and $\beta$ \hspace{1cm} (2.15)

for any $|\xi|/|\xi| - \xi^0| < \delta^0$. Now,

$$\det(\lambda I - a(\tau, \xi)) = \prod_{1 \leq j \leq p, 1 \leq k \leq i_j}(\lambda - i(\xi)^k - \lambda_{j,k}(\tau, \xi)).$$

Then,

$$V_0(t, \tau; \xi) = \frac{1}{2\pi i} \int_{\gamma_1} e^{i\lambda (t-\tau)} \frac{co(\lambda I - i(\xi)^k I - a(\tau, \xi))}{\prod(\lambda - i(\xi)^k - \lambda_{j,k}(\tau, \xi))} d\lambda$$

$$= \frac{1}{2\pi i} \left( \int_{\gamma_1} + \int_{\gamma_2} + \cdots + \int_{\gamma_p} \right) d\lambda,$$

where each interior of $\gamma_h$ includes only $\{\lambda_{h,1}, \lambda_{h,2}, \ldots, \lambda_{h,i_h}\}$ for $1 \leq h \leq p$. It is important that by virtue of (2.14) and (2.15), we can take $\{\gamma_h\}_{h=1,...,p}$ independently of $\xi$ if $|\xi|/|\xi| - \xi^0| < \delta^0$. Put

$$F_h(z) = e^{iz(t-\tau)} \frac{co(z I - i(\xi)^k I - a(\tau, \xi))}{\prod_{j \neq h, 1 \leq l \leq i_j}(z - i(\xi)^k - \lambda_{j,l}(\tau, \xi))},$$

then, from Lemma 2.1,

$$I_h := \frac{1}{2\pi i} \int_{\gamma_h} e^{i\lambda (t-\tau)} \frac{co(\lambda I - i(\xi)^k I - a(\tau, \xi))}{\prod(\lambda - i(\xi)^k - \lambda_{j,l}(\tau, \xi))} d\lambda$$

$$= \int_0^1 \cdots \int_0^1 \theta_{i_h-2}^{i_h-1} F_h(q(i(\xi)^k + \lambda_{h,1}, i(\xi)^k + \lambda_{h,2}, \ldots, i(\xi)^k + \lambda_{h,i_h}; \theta)) d\theta.$$

From the Leibniz's formula and

$$\left| \frac{d^{i_h}}{d\lambda^{i_h}} \right|_{\lambda=q} e^{i\lambda(t-\tau)} = (t - \tau)^{i_h} e^{-i\lambda(t-\tau)} \right|_{\lambda=q} \leq C(t - \tau)^{i_h} e^{-(\xi)^k(t-\tau)},$$
\[
\frac{d^2}{d\lambda^2} \bigg|_{\lambda=q} (C^0(\lambda I - i(\xi)K I - a(\tau, \xi))) \leq C(\xi)^{d-1-l_x}, \\
\frac{d^3}{d\lambda^3} \bigg|_{\lambda=q} \left( \prod_{j \neq h, 1 \leq l \leq i_j} (z - i(\xi)^k - \lambda_j, l(\tau, \xi)) \right) \leq C(\xi)^{-(d-i_h+l_3)},
\]

where \( q = q(i(\xi)^0 + \lambda_{h,1}, i(\xi)^0 + \lambda_{h,2}, \ldots, i(\xi)^0 + \lambda_{h,i_h}; 0) \), then

\[
|I_h| \leq C \sum_{l_1+l_2+l_3 = i_h-1} \frac{(i_h - 1)!}{l_1! l_2! l_3!} (t - \tau)^l(\xi)^{(d-1-l_x)-(d-i_h+l_3)} e^{-\xi^k(t-\tau)}
\]

\[
\leq C \sum_{l_1+l_2+l_3 = i_h-1} (t - \tau)^l(\xi)^{(i_h-1)-(l_2+l_3)} e^{-\xi^k(t-\tau)}
\]

\[
\leq C \sum_{l_1+l_2+l_3 = i_h-1} (t - \tau)^l(\xi)^l e^{-\xi^k(t-\tau)} \leq C(1 + ((t - \tau)(\xi)^{i_h-1}) e^{-\xi^k(t-\tau)}
\]

for any \( 1 \leq h \leq p \). Thus we have

\[
|V_0(t, \tau; \xi)| \leq C(1 + ((t - \tau)(\xi)^{i-1}) e^{-\xi^k(t-\tau)}
\]

\[
\leq C(1 + (t - \tau)(\xi)^{y-1}) e^{-\xi^k(t-\tau)}
\]

\[
\leq C(1 + (t - \tau)(\xi)^{y-1})(1-k)(y-1) e^{-\xi^k(t-\tau)}
\]

\[
\leq C(\xi)^{\sigma} e^{-\xi^k(t-\tau)}
\]

for any \( |\xi|/|\xi| - \xi^0| < \delta^0 \), where \( \sigma = (y-1)(1-k) \). By virtue of Heine–Borel’s covering theorem, there exists a constant \( C_0 > 0 \) such that

\[
|V_0(t, \tau; \xi)| \leq C_0(\xi)^{\sigma} e^{-\xi^k(t-\tau)}
\]

for \( \xi \in S^{d-1} \). Since the homogeneity of \( \lambda_{h,j}(\tau, \xi) \) with respect to \( \xi \), we obtain (i).

(ii) We shall prove (ii) by use of mathematical induction. When \( j = 0 \), from (2.11) there is a constant \( C' > 0 \) such that

\[
|\Phi_0(t, \tau; \xi)| = |(a(t, \xi) - a(\tau, \xi))V_0(t, \tau; \xi)| \leq C'(t - \tau)^{\mu}(\xi)C_0(\xi)^{\sigma} e^{-\xi^k(t-\tau)}
\]

\[
\leq C_0C'(t - \tau)^{\mu}(\xi)^{\sigma+1-k\mu} e^{-\xi^k(t-\tau)}
\]

\[
\leq 4C_0C'(\xi)^{(1-k)y + k(1-\mu)} e^{-\xi^k(t-\tau)}
\]

so \( \Phi_0 \) satisfies (2.12). Suppose that \( \Phi_j \) satisfies (2.12), then

\[
|\Phi_{j+1}(t, \tau; \xi)| = \left| \int_{\tau}^{t'} R(t, r; \xi) \Phi_j(r, \tau; \xi) \, dr \right|
\]
We note that for symbol $V(t, \tau)$, we can observe that $\|V(t, \tau)\|_{H^l} \leq C_l \|w\|_{H^{l+\sigma}}$ for any $l \in \mathbb{R}$ and $w \in H^\infty$. From Proposition 2.1, we can observe that $\Phi(t, \tau)$ is well defined by (2.9), so $V(t, \tau)$ can be also defined by (2.7). Precisely,

**Proposition 2.2.** (i) There exists a constant $C_2 > 0$ such that

$$\|V_0(t, \tau)w\|_{H^l} \leq C_2 \|w\|_{H^{l+\sigma}} \tag{2.16}$$

$$\|\Phi(t, \tau)w\|_{H^l} \leq C_2 \|w\|_{H^{l+(1-\mu)+(1-\kappa)\tau}} \tag{2.17}$$

for any $l \in \mathbb{R}$ and $w \in H^\infty$, where $V_0(t, \tau)$ and $\Phi(t, \tau)$ are pseudodifferential operators with symbols $V_0(t, \tau; \zeta)$ and $\Phi(t, \tau; \zeta)$, respectively, $\sigma = (1 - 1)(1 - \kappa)$ and $C_2$ is independent of $\tau$.

(ii) There exists a constant $C_3 > 0$ such that

$$\|V(t, \tau)w\|_{H^l} \leq C_3 \|w\|_{H^{l+\sigma}} \tag{2.18}$$

for any $l \in \mathbb{R}$ and $w \in H^\infty$, where $V(t, \tau)$ is a pseudodifferential operator with a symbol $V(t, \tau; \zeta)$ and $C_3$ is independent of $\tau$.

**Proof.** (i): From (2.11),

$$\|V_0(t, \tau)w\|_{L^2} = \|V_0(t, \tau; \zeta)\hat{w}\|_{L^2} \leq C_0 \|\langle \zeta \rangle^\sigma e^{-\frac{1}{2}(t-\tau)\langle \zeta \rangle^\kappa} \hat{w}\|_{L^2} \leq C \|w\|_{H^\sigma}.$$

We note that for $j > 0$

$$\sup_{x \geq 0} x^j e^{-x} = \frac{x^j}{\sum_{n=0}^{\infty} x^n n! - 1} < j!,$$

then from (2.12),

$$|\Phi_j(t, \tau; \zeta)| \leq C_1^{j+1} j!^{-1} ((t - \tau)\langle \zeta \rangle^\kappa)^j e^{-\frac{1}{2}(t-\tau)\langle \zeta \rangle^\kappa} \langle \zeta \rangle^{1-v}\kappa(1-\mu)-(\kappa(v+\mu)-v)j \leq C_1^{j+1} 4j^j \langle \zeta \rangle^{(1-\kappa)v}\kappa(1-\mu)-(\kappa(v+\mu)-v)j.$$
Since $\kappa = 1/s > v/(\mu + v)$, we have

$$|\phi_j(t, \tau; \zeta)| \leq \frac{C_1}{2^j} (\zeta)^{\kappa(1-\mu)+(1-\kappa)v}$$

for $\langle \zeta \rangle \geq (8C_1)^{1/(\kappa(v+\mu)-v)}$ and $j \geq 0$. Therefore there is a constant $C_2 > 0$, independently of $\tau$, such that

$$\|\phi(t, \tau) w\|_{L^2} = \left\| \sum_{j=0}^{\infty} \phi_j(t, \tau; \zeta) \hat{w} \right\|_{L^2} \leq C_2 \|w\|_{H^{\kappa(1-\mu)+(1-\kappa)v}}.$$

(ii) Similarly to the proof of (i),

$$\left| \int_{\tau}^{t} V_0(t, r; \zeta) \phi_j(r, \tau; \zeta) \, dr \right| \leq C_0 (4C_1)^{j+1} \langle \zeta \rangle^{\sigma-(\kappa(v+\mu)-v)-(\kappa(v+\mu)-v)j},$$

then we have

$$\left| \int_{\tau}^{t} V_0(t, r; \zeta) \phi_j(r, \tau; \zeta) \, dr \right| \leq \frac{4C_0C_1}{2^j} \langle \zeta \rangle^{\sigma-(\kappa(v+\mu)-v)}$$

for $\langle \zeta \rangle \geq (8C_1)^{1/(\kappa(v+\mu)-v)}$ and $j \geq 0$. Therefore, there is a constant $C'_2 > 0$ such that

$$\left\| \int_{\tau}^{t} V_0(t, r) \phi(r, \tau) w \, dr \right\|_{L^2} \leq C'_2 \|w\|_{H^\sigma}.$$  \hspace{1cm} (2.19)

From (2.7), (2.16) and (2.19), we have (2.18). □

Put

$$v(t) = V(t, 0) v_0 + \int_{0}^{t} V(t, r) g(r) \, dr$$

then, it is obvious that $v(t)$ is a solution of Eq. (2.1), moreover, there is a constant $C > 0$ such that

$$\|v(t, \cdot)\|_{L^2} \leq C \left( \|v_0\|_{H^\sigma} + \int_{0}^{t} \|g(r, \cdot)\|_{H^\sigma} \, dr \right),$$  \hspace{1cm} (2.20)

where $\sigma = (v-1)(1-\kappa)$. Since $u(t, x) = e^{-A(T-t)} v(t, x)$, we have Theorem 1.1 provided that

$$W(t, D; \tau) = e^{-A(T-t)} V(t, \tau) e^{A(T-\tau)}.$$
Acknowledgments

The author would like to express his deep thanks to Professor Kajitani for valuable suggestions that greatly improved the presentation.

References