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Note on nonstability of the linear functional equation of higher order

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ABSTRACT

We provide a complete solution of the problem of Hyers–Ulam stability for a large class of higher order linear functional equations in single variable, with constant coefficients. We obtain this by showing that such an equation is nonstable in the case where at least one of the roots of the characteristic equation is of module 1. Our results are related to the notions of shadowing (in dynamical systems and computer science) and controlled chaos. They also correspond to some earlier results on approximate solutions of functional equations in single variable.

1. Introduction

Throughout this paper, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ stand, as usual, for the sets of positive integers, nonnegative integers, integers, reals and complex numbers, respectively. In what follows $m \in \mathbb{N}$, X is a nontrivial normed space over a field $K \in \{\mathbb{R}, \mathbb{C}\}$, $a_0, \dots, a_{m-1} \in K$, A is a nonempty set, $F : A \to X$, $f : A \to A$, and f^j denotes the j-th iterate of f for $j \in \mathbb{N}_0$.

In this paper, we investigate the problem of Hyers–Ulam stability of the linear functional equation of the form

$$\varphi(f^m(x)) = \sum_{j=0}^{m-1} a_j \varphi(f^j(x)) + F(x), \tag{1}$$

with the unknown function $\varphi : A \to X$. It is one of the most important functional equations in single variable and many results have been given (see [1,2] and the references therein) on continuity, convexity, differentiability and analyticity of solutions for it. One of the simplest examples of Eq. (1), with $A \in \{\mathbb{Z}, \mathbb{N}_0\}$, is the linear recurrence (or difference equation)

$$y_{n+m} = \sum_{j=0}^{m-1} a_j y_{n+j} + b_n, \quad \forall n \in A \tag{2}$$

for sequences $(y_n)_{n\in A}$ in X, where $(b_n)_{n\in A}$ is a fixed sequence in X; clearly (1) becomes (2) with $f(n) = n + 1$, $y_n := \varphi(n) = \varphi(f^n(0))$ and $b_n := F(n)$. The problem of stability of (2) corresponds to the notions of shadowing (in dynamical systems and computer science) and controlled chaos (see, e.g., [3–6]). Our investigation is also connected to the results in [7–11], concerning the existence of approximate solutions of functional equations in single variable (for more information on such functional equations, see e.g., [12,1,2]).

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Stability of some particular forms of (1) has been studied, e.g., in [13–17]; in [18, Theorem 2] it has been proved that, in the case where X is a Banach space, f is bijective and the characteristic equation

$$r^m - \sum_{j=0}^{m-1} a_j r^j = 0 \tag{3}$$

has no roots of module 1, the equation is Hyers–Ulam stable, or more precisely that, for every $\delta \in \mathbb{R}$ and $\gamma : A \to X$ with

$$\sup_{x\in A} \left\| \gamma(f^m(x)) - \sum_{j=0}^{m-1} a_j \gamma(f^j(x)) - F(x) \right\| \le \delta, \tag{4}$$

there is a solution $\varphi : A \to X$ of (1) with

$$\sup_{x\in A} \|\gamma(x) - \varphi(x)\| \le \frac{\delta}{|1-|r_1||\cdot\dots\cdot|1-|r_m||},$$

where $r_1, \dots, r_m$ denote the complex roots of (3) (for more details on this kind of stability and some examples of very recent results, see e.g., [19,13,20]). Moreover, it is known that the equation can be, in some cases, nonstable if the characteristic equation has a root of module 1 (see [21] and [18, Example 1]); however it is not known if this is always the case. In this paper, we show a result that solves the problem of Hyers–Ulam stability of (1) completely for injective f with at least one non-periodic point.

Let $S \subset A$ and $\mathcal{D} \subset X^A$ be nonempty. In what follows, we say that functional equation (1) is nonstable on the set S, in the class of functions $\mathcal{D}$, provided there is a function $\gamma \in \mathcal{D}$ such that

$$\sup_{x\in S} \left\| \gamma(f^m(x)) - \sum_{j=0}^{m-1} a_j \gamma(f^j(x)) - F(x) \right\| < \infty, \tag{5}$$

and there does not exist any solution $\varphi \in \mathcal{D}$ of (1) with $\sup_{x\in S} \|\gamma(x) - \varphi(x)\| < \infty$; if $S = A$, then, for simplicity, we omit the part 'on the set S'.

It makes sense to introduce the class $\mathcal{D}$ in the definition of nonstability (and in analogous possible suitable definitions of stability) for the functional equations in a single variable, because the existence, uniqueness and behaviour of their solutions strictly depends on the regularity, both of the given functions and the solutions considered (see, e.g., [2, 0.0B]).

2. Auxiliary lemmas

From now on, $r_1, \dots, r_m$ denote the complex roots of (3); if $m > 1$, then $b_0, \dots, b_{m-2}$ stand for the unique complex numbers with

$$z^m - \sum_{j=0}^{m-1} a_j z^j = (z - r_1)\left(z^{m-1} - \sum_{j=0}^{m-2} b_j z^j\right), \quad \forall z \in \mathbb{C}.$$

Remark 1. Clearly, $a_{m-1} = r_1 + b_{m-2}$, $a_0 = -r_1 b_0$ and, in the case $m > 3$, $a_j = -r_1 b_j + b_{j-1}$ for $j = 1, \dots, m-2$. Observe yet that, if $r_1, a_0, \dots, a_{m-1} \in \mathbb{R}$, then $b_0, \dots, b_{m-2} \in \mathbb{R}$.

We start with a lemma which will be our main tool for investigation of stability of Eq. (1).

Lemma 1. *Let $r_1 \in K$, $m > 1$, $T_i \subset A$ be nonempty for $i = 1, 2$, $\psi_0, \psi : A \to X$,*

$$\sup_{x\in T_1} \|\psi_0(f(x)) - r_1\psi_0(x) - F(x)\| =: \delta < \infty, \tag{6}$$

and

$$\sup_{x\in T_i} \left\| \psi(f^{m-1}(x)) - \sum_{j=0}^{m-2} b_j \psi(f^j(x)) - \psi_0(x) \right\| =: \delta_i < \infty, \quad i = 1, 2. \tag{7}$$

Then the following three conclusions are valid.

(i) *If $T_1 \cap f^{-1}(T_1) \neq \emptyset$, then*

$$\sup_{x\in T_1\cap f^{-1}(T_1)} \left\| \psi(f^m(x)) - \sum_{j=0}^{m-1} a_j \psi(f^j(x)) - F(x) \right\| \le \delta + (1 + |r_1|)\delta_1. \tag{8}$$

(ii) *If ψ_0 is unbounded on a nonempty set $D \subset T_1 \cup T_2$, then ψ is unbounded on the set*

$$D_0 := \bigcup_{i=0}^{m-1} f^i(D).$$

(iii) *The existence of a solution* $\varphi : A \to X$ *of Eq.* (1) *with*

$$\sup_{x\in T_0} \|\psi(x) - \varphi(x)\| < \infty,$$

where $T_0 := \bigcup_{i=0}^{m-1} f^i(T_2)$, *implies the existence of a solution* $\hat{\eta} : A \to X$ *of the functional equation*

$$\hat{\eta}(f(x)) = r_1\hat{\eta}(x) + F(x) \tag{9}$$

with

$$\sup_{x\in T_2} \|\psi_0(x) - \hat{\eta}(x)\| < \infty. \tag{10}$$

Proof. It is easily seen that, by (6), (7) and Remark 1, for each $x \in T_1 \cap f^{-1}(T_1)$

$$\begin{aligned}\left\|\psi(f^m(x)) - \sum_{j=0}^{m-1} a_j\psi(f^j(x)) - F(x)\right\| &\le \left\|\psi(f^{m-1}(f(x))) - \sum_{j=0}^{m-2} b_j\psi(f^j(f(x))) - \psi_0(f(x))\right\| \\ &\quad + |r_1|\left\|\psi(f^{m-1}(x)) - \sum_{j=0}^{m-2} b_j\psi(f^j(x)) - \psi_0(x)\right\| + \|\psi_0(f(x)) - r_1\psi_0(x) - F(x)\| \\ &\le (1+|r_1|)\delta_1 + \delta.\end{aligned}$$

Clearly, if ψ_0 is unbounded on a set $D \subset T_1 \cup T_2$, then ψ is unbounded on D_0 in view of (7).

Suppose that there is a solution $\varphi : A \to X$ of (1) with $\sup_{x\in T_0} \|\psi(x) - \varphi(x)\| =: M < \infty$. Define $\hat{\eta} : A \to X$ by

$$\hat{\eta}(x) := \varphi(f^{m-1}(x)) - \sum_{j=0}^{m-2} b_j\varphi(f^j(x)). \tag{11}$$

Then (see Remark 1) $\hat{\eta}$ is a solution to (9) and from (7) it follows that, for each $x \in T_2$,

$$\begin{aligned}&\|\psi_0(x) - \hat{\eta}(x)\| \\ &\quad \le \delta_2 + \|\psi(f^{m-1}(x)) - \varphi(f^{m-1}(x))\| + \sum_{j=0}^{m-2} |b_j|\|\psi(f^j(x)) - \varphi(f^j(x))\| \le \delta_2 + \left(1 + \sum_{j=0}^{m-2}|b_j|\right)M < \infty. \quad \square\end{aligned}$$

In what follows, for each $x \in A$, we write $C_f^*(x) := \{y \in A : f^n(y) = f^k(x) \text{ with some } k, n \in \mathbb{N}\}$, $C_f^+(x) := \{f^n(x) : n \in \mathbb{N}\}$ and $C_f^-(x) := \{y \in A : f^n(y) = x \text{ with some } n \in \mathbb{N}\}$; we say that $C_f^*(x)$ ($C_f^+(x)$, $C_f^-(x)$, respectively) is the orbit (positive orbit, negative orbit, resp.) of x under f. As usual, if $n \in \mathbb{N}$ and $D \subset A$, then $f^{-n}(D) := \{y \in A : f^n(y) \in D\}$ and, in the case where f is injective, $x_0 \in A$ and $f^{-n}(\{x_0\}) \neq \emptyset$, we simply denote by $f^{-n}(x_0)$ the unique element of the set $f^{-n}(\{x_0\})$.

The following hypothesis will be useful in the sequel.

($\mathcal{H}$) $A_f^* \neq \emptyset$ is a set of non-periodic points of f in A (i.e., $f^n(x^*) \neq x^*$ for $x^* \in A_f^*$, $n \in \mathbb{N}$) such that $C_f^*(x^*) \cap C_f^*(y^*) = \emptyset$ for every $x^*, y^* \in A_f^*$ with $x^* \neq y^*$.

We need the following two auxiliary lemmas. The proof of the first one is an easy induction.

Lemma 2. *Assume that* $r_1 \in K$ *and* $\varphi_0 : A \to X$ *is a solution of Eq.* (9). *Then*

$$\varphi_0(f^n(x)) = r_1^n\varphi_0(x) + \sum_{k=1}^{n} r_1^{n-k}F(f^{k-1}(x)), \quad \forall n \in \mathbb{N},\ x \in A,$$

and, in the case where f *is injective and* $r_1 \neq 0$,

$$\varphi_0(f^{-n}(x)) = r_1^{-n}\varphi_0(x) - \sum_{k=1}^{n} r_1^{k-n-1}F(f^{-k}(x)), \quad \forall n \in \mathbb{N},\ x \in f^n(A).$$

Lemma 3. *Suppose that* ($\mathcal{H}$) *is valid,* f *is injective,* $S^+, S^- \subset A_f^*$, $C_f^-(y^*)$ *is infinite for each* $y^* \in S^-$, $r_1 \in K$, $|r_1| = 1$, $\xi : (S^+ \cup S^-) \to X$, *and* $\eta : A \to X$ *is a solution of Eq.* (9). *Then, for any* $\delta > 0$, *there is a function* $\psi_0 : A \to X$, *unbounded on* $C_f^+(x^*)$ *and* $C_f^-(y^*)$ *for* $x^* \in S^+$, $y^* \in S^-$, *such that*

$$\sup_{x\in A}\|\psi_0(f(x)) - r_1\psi_0(x) - F(x)\| \le \delta, \tag{12}$$

$$\psi_0(f(x)) = r_1\psi_0(x) + F(x), \quad \forall x \in (S^- \setminus S^+) \cup \bigcup_{w^*\in S^+\setminus S^-} C_f^-(w^*) \cup \bigcup_{z^*\in S^-\setminus S^+} C_f^+(z^*), \tag{13}$$

$$\psi_0(x) = \eta(x), \quad \forall x \in A \setminus \bigcup_{x^*\in S^+\cup S^-} C_f^*(x^*), \tag{14}$$

$$\psi_0(z^*) = \xi(z^*), \quad \forall z^* \in S^+ \cup S^-, \tag{15}$$

and, for every solution $\hat{\varphi} : A \to X$ of Eq. (9) and every $x^* \in S^+, y^* \in S^-$,

$$\sup_{x\in C_f^+(x^*)} \|\psi_0(x) - \hat{\varphi}(x)\| = \infty, \qquad \sup_{x\in C_f^-(y^*)} \|\psi_0(x) - \hat{\varphi}(x)\| = \infty. \tag{16}$$

Proof. Take $\delta > 0$ and $u \in X$ with $0 < \|u\| \le 1$. Let functions $\mu^+ : S^+ \to \{-1, 1\}$ and $\mu^- : S^- \to \{-1, 1\}$ be given by

$$\mu^+(x^*) := \begin{cases} 1, & \sup_{n\in\mathbb{N}} \left\| \sum_{k=1}^{n} r_1^{n-k} F(f^{k-1}(x^*)) + n r_1^n \delta u \right\| = \infty; \\ -1, & \text{otherwise,} \end{cases}$$

$$\mu^-(y^*) := \begin{cases} 1, & \sup_{n\in\mathbb{N}} \left\| \sum_{k=1}^{n} r_1^{k-n-1} F(f^{-k}(y^*)) + n r_1^{-n} \delta u \right\| = \infty; \\ -1, & \text{otherwise,} \end{cases}$$

for every $x^* \in S^+,\ y^* \in S^-$. Define $\psi_0 : A \to X$ by (14), (15), and

$$\psi_0(f^n(x^*)) := r_1\psi_0(f^{n-1}(x^*)) + F(f^{n-1}(x^*)) + \mu^+(x^*) r_1^n \delta u, \quad \forall x^* \in S^+,\ n \in \mathbb{N},$$
$$\psi_0(f^{-n}(y^*)) := r_1^{-1}\Big(\psi_0(f^{-n+1}(y^*)) - F(f^{-n}(y^*)) - \mu^-(y^*) r_1^{-n+1}\delta u\Big), \quad \forall y^* \in S^-,\ n \in \mathbb{N},$$
$$\psi_0(f^n(z^*)) := r_1\psi_0(f^{n-1}(z^*)) + F(f^{n-1}(z^*)), \quad \forall z^* \in S^- \setminus S^+,\ n \in \mathbb{N},$$
$$\psi_0(f^{-n}(w^*)) := r_1^{-1}\Big(\psi_0(f^{-n+1}(w^*)) - F(f^{-n}(w^*))\Big), \quad \forall w^* \in S^+ \setminus S^-,\ n \in \mathbb{N}, f^{-n}(\{w^*\}) \ne \emptyset.$$

Since the sets

$$S_1 := A \setminus \bigcup_{x^*\in S^+\cup S^-} C_f^*(x^*), \qquad S_2 := S^+ \cup S^-, \qquad S_3 := \bigcup_{x^*\in S^+} C_f^+(x^*),$$
$$S_4 := \bigcup_{y^*\in S^-} C_f^-(y^*), \qquad S_5 := \bigcup_{z^*\in S^-\setminus S^+} C_f^+(z^*), \qquad S_6 := \bigcup_{w^*\in S^+\setminus S^-} C_f^-(w^*)$$

are pairwise disjoint and

$$A = \bigcup_{i=1}^{6} S_i,$$

that definition is correct. We prove that ψ_0 satisfies (12) and (13).

In view of (14), for each $x \in S_1$ we have

$$\psi_0(f(x)) - r_1\psi_0(x) - F(x) = 0,$$

because η is a solution to (9). Further, according to the definition of ψ_0, we get the same for $x \in (S^- \setminus S^+) \cup S_5$ and $x \in S_6$, which proves (13). Finally, if $x \in S^+ \cup S_3$, then $x = f^{n-1}(x^*)$ for some $x^* \in S^+$ and $n \in \mathbb{N}$, and the definition of ψ_0 yields

$$\|\psi_0(f(x)) - r_1\psi_0(x) - F(x)\| = \|\psi_0(f^n(x^*)) - r_1\psi_0(f^{n-1}(x^*)) - F(f^{n-1}(x^*))\| = \|\mu^+(x^*) r_1^n \delta u\| \le \delta;$$

if $x \in S_4$, then $x = f^{-n}(y^*)$ for some $y^* \in S^-$ and $n \in \mathbb{N}$, and then

$$\|\psi_0(f(x)) - r_1\psi_0(x) - F(x)\| = \|\psi_0(f^{-n+1}(y^*)) - r_1\psi_0(f^{-n}(y^*)) - F(f^{-n}(y^*))\| = \|\mu^-(y^*) r_1^{-n+1}\delta u\| \le \delta.$$

This completes the proof of (12).

Now we show by induction that, for each $n \in \mathbb{N}$,

$$\psi_0(f^n(x^*)) = r_1^n\psi_0(x^*) + \sum_{k=1}^{n} r_1^{n-k} F(f^{k-1}(x^*)) + \mu^+(x^*) n r_1^n \delta u, \quad \forall x^* \in S^+. \tag{17}$$

The case $n = 1$ follows directly from the definition of ψ_0 (also with $n = 1$). So, take a positive integer n and suppose that (17) holds. Then, according to the definition of ψ_0 and the inductive hypothesis,

$$\begin{aligned}\psi_0(f^{n+1}(x^*)) &= r_1\psi_0(f^n(x^*)) + F(f^n(x^*)) + \mu^+(x^*)r_1^{n+1}\delta u\\ &= r_1\left(r_1^n\psi_0(x^*) + \sum_{k=1}^{n} r_1^{n-k}F(f^{k-1}(x^*)) + \mu^+(x^*)nr_1^n\delta u\right) + F(f^n(x^*)) + \mu^+(x^*)r_1^{n+1}\delta u\\ &= r_1^{n+1}\psi_0(x^*) + \sum_{k=1}^{n+1} r_1^{n+1-k}F(f^{k-1}(x^*)) + \mu^+(x^*)(n+1)r_1^{n+1}\delta u.\end{aligned}$$

In a similar way we can prove that, for each $n \in \mathbb{N}$,

$$\psi_0(f^{-n}(y^*)) = r_1^{-n}\psi_0(y^*) - \sum_{k=1}^{n} r_1^{k-n-1}F(f^{-k}(y^*)) - \mu^-(y^*)nr_1^{-n}\delta u, \quad \forall y^* \in S^-. \tag{18}$$

Let $\hat{\varphi} : A \to X$ be an arbitrary solution of Eq. (9). Then, by Lemma 2 and (17) and (18), for every $x^* \in S^+, y^* \in S^-$ we have

$$\psi_0(f^n(x^*)) - \hat{\varphi}(f^n(x^*)) = r_1^n(\psi_0(x^*) - \hat{\varphi}(x^*)) + n\mu^+(x^*)r_1^n\delta u, \quad \forall n \in \mathbb{N},$$

and

$$\psi_0(f^{-n}(y^*)) - \hat{\varphi}(f^{-n}(y^*)) = r_1^{-n}(\psi_0(y^*) - \hat{\varphi}(y^*)) - n\mu^-(y^*)r_1^{-n}\delta u, \quad \forall n \in \mathbb{N}.$$

Consequently, for every $x^* \in S^+,\ y^* \in S^-$,

$$\sup_{x\in C_f^+(x^*)} \|\psi_0(x) - \hat{\varphi}(x)\| = \infty, \qquad \sup_{x\in C_f^-(y^*)} \|\psi_0(x) - \hat{\varphi}(x)\| = \infty.$$

Finally, take $x^* \in S^+$ ($y^* \in S^-$, respectively). Observe that, on account of (17) and (18), for each $n \in \mathbb{N}$, in the case $\mu^+(x^*) = 1$ ($\mu^-(y^*) = 1$, resp.),

$$\|\psi_0(f^n(x^*))\| \geq \left\|\sum_{k=1}^{n} r_1^{n-k}F(f^{k-1}(x^*)) + nr_1^n\delta u\right\| - \|\psi_0(x^*)\|$$

($\|\psi_0(f^{-n}(y^*))\| \geq \left\|\sum_{k=1}^{n} r_1^{k-n-1}F(f^{-k}(y^*)) + nr_1^{-n}\delta u\right\| - \|\psi_0(y^*)\|$, resp.) and, in the case $\mu^+(x^*) = -1$ ($\mu^-(y^*) = -1$, resp.),

$$\|\psi_0(f^n(x^*))\| \geq 2n\delta\|u\| - \left\|\sum_{k=1}^{n} r_1^{n-k}F(f^{k-1}(x^*)) + nr_1^n\delta u\right\| - \|\psi_0(x^*)\|$$

($\|\psi_0(f^{-n}(y^*))\| \geq 2n\delta\|u\| - \left\|\sum_{k=1}^{n} r_1^{k-n-1}F(f^{-k}(y^*)) + nr_1^{-n}\delta u\right\| - \|\psi_0(y^*)\|$, resp.). Consequently, in either case, ψ_0 is unbounded on $C_f^+(x^*)$ and $C_f^-(y^*)$ for every $x^* \in S^+$ and $y^* \in S^-$. $\square$

3. The main results

Remark 2. Assume $r_1 \neq 0$. Then the set $M := \{n \in \{0, \dots, m-1\} : a_n \neq 0\}$ is not empty. Write $\rho := \min M < m$. Then (cf. Remark 1), in the case $\rho < m-1$, we have $b_\rho \neq 0$ and $b_j = 0$ for $j < \rho$; in the case $\rho = m-1$, $b_j = 0$ for $j = 0, \dots, m-2$ and $a_{m-1} = r_1$.

Now we are in a position to prove the following theorem, which shows that (under suitable assumptions) Eq. (1) is nonstable on the set S of all points of a collection of arbitrarily chosen infinite positive and negative orbits of f in A, i.e., that there exist functions $\psi : A \to X$ satisfying inequality (19) (with some $\delta > 0$) and such that $\sup_{x\in S} \|\psi(x) - \varphi_0(x)\| = \infty$ for each solution $\varphi_0 : A \to X$ of (1); from (22) it results that the class of such functions $\psi : A \to X$ is not small. Moreover, on the set $A \setminus S$ such functions ψ can be 'quite close' to a solution of Eq. (1), and even to a given solution of the equation.

In the next theorem, given two sets $S^+, S^- \subset A_f^*$, we use the following denotations:

$$S_f := (S^- \setminus S^+) \cup \bigcup_{w^*\in S^+\setminus S^-} C_f^-(w^*) \cup \bigcup_{z^*\in S^-\setminus S^+} C_f^+(z^*),$$

$$A_f := A \setminus \bigcup_{x^*\in S^+\cup S^-} C_f^*(x^*), \qquad S_\rho^* := \bigcup_{i=\rho}^{m-1} f^i(S^+ \cup S^-).$$

Theorem 1. *Let* $(\mathcal{H})$ *be valid,* $|r_{j_0}| = 1$ *for some* $j_0 \in \{1, \dots, m\}$, f *be injective,* $S^+, S^- \subset A_f^*$, $C_f^-(y^*)$ *be infinite for each* $y^* \in S^-$, *and* $\xi^* : S_\rho^* \to X$. *Suppose that* $\varphi : A \to X$ *is a solution of Eq. (1). Then, for each* $\delta > 0$, *there exists a function* $\psi : A \to X$ *such that*

$$\sup_{x\in A} \left\| \psi(f^m(x)) - \sum_{j=0}^{m-1} a_j \psi(f^j(x)) - F(x) \right\| \le \delta, \tag{19}$$

$$\psi(f^m(x)) = \sum_{j=0}^{m-1} a_j \psi(f^j(x)) + F(x), \quad \forall x \in S_f, \tag{20}$$

$$\psi(x) = \varphi(x), \quad \forall x \in A_f, \tag{21}$$

$$\psi(y) = \xi^*(y), \quad \forall y \in S_\rho^*, \tag{22}$$

and

$$\sup_{x\in C_f^+(x^*)} \|\psi(x) - \varphi_0(x)\| = \infty, \qquad \sup_{x\in C_f^-(y^*)} \|\psi(x) - \varphi_0(x)\| = \infty \tag{23}$$

for each solution $\varphi_0 : A \to X$ *of Eq. (1) and every* $x^* \in S^+$, $y^* \in S^-$.

Moreover, if $r_{j_0} \in K$, *then* ψ *can be chosen such that it is unbounded on* $C_f^+(x^*)$ *and* $C_f^-(y^*)$ *for every* $x^* \in S^+$, $y^* \in S^-$.

Proof. The case $m = 1$ follows from Lemma 3, because then $a_0 = r_1$. So, let $m > 1$. Clearly, without loss of generality, we may assume that $j_0 = 1$.

Take $\delta > 0$. First consider the situation where $r_1 \in K$. Write

$$\xi(x^*) := \xi^*(f^{m-1}(x^*)) - \sum_{j=\rho}^{m-2} b_j \xi^*(f^j(x^*)), \quad \forall x^* \in S^+ \cup S^-. \tag{24}$$

Since $\hat{\eta} : A \to X$, defined by (11), is a solution of Eq. (9), from Lemma 3 it follows that there exists a function $\psi_0 : A \to X$ such that, for every solution $\hat{\varphi} : A \to X$ of (9) and $x^* \in S^+$, $y^* \in S^-$, conditions (12), (15) and (16) hold and ψ_0 is unbounded on $C_f^+(x^*)$ and $C_f^-(y^*)$.

Define $\psi : A \to X$ by (21), (22),

$$\psi(f^{\rho-n}(w^*)) := a_\rho^{-1}\left[\psi(f^{m-n}(w^*)) - \sum_{j=\rho+1}^{m-1} a_j \psi(f^{j-n}(w^*)) - F(f^{-n}(w^*))\right],$$
$$\forall w^* \in S^+ \setminus S^-, \ n \in \mathbb{N}, f^{\rho-n}(\{w^*\}) \ne \emptyset,$$

$$\psi(f^{m+n-1}(z^*)) := \sum_{j=\rho}^{m-1} a_j \psi(f^{j+n-1}(z^*)) + F(f^{n-1}(z^*)), \quad \forall z^* \in S^- \setminus S^+, \ n \in \mathbb{N},$$

$$\psi(f^{m-1+n}(x^*)) := \sum_{j=\rho}^{m-2} b_j \psi(f^{j+n}(x^*)) + \psi_0(f^n(x^*)), \quad \forall x^* \in S^+, \ n \in \mathbb{N}, \tag{25}$$

and, with $b_{m-1} := -1$,

$$\psi(f^{\rho-n}(y^*)) := \begin{cases} -b_\rho^{-1}\left[\displaystyle\sum_{j=\rho+1}^{m-1} b_j \psi(f^{j-n}(y^*)) + \psi_0(f^{-n}(y^*))\right], & \text{if } \rho < m-1; \\ \psi_0(f^{-n}(y^*)), & \text{if } \rho = m-1, \end{cases} \quad \forall y^* \in S^-, \ n \in \mathbb{N}. \tag{26}$$

Since the sets

$A_1 := \{f^{m-1+n}(x^*) : x^* \in S^+, \ n \in \mathbb{N}\}$,
$A_2 := \{f^{\rho-n}(w^*) : w^* \in S^+ \setminus S^-, \ n \in \mathbb{N}, \ f^{\rho-n}(\{w^*\}) \ne \emptyset\}$,
$A_3 := \{f^{m+n-1}(z^*) : z^* \in S^- \setminus S^+, \ n \in \mathbb{N}\}$, $\quad A_4 := \{f^{\rho-n}(y^*) : y^* \in S^-, \ n \in \mathbb{N}\}$,

A_f and S_ρ^* are disjoint and $A = A_f \cup S_\rho^* \cup A_1 \cup A_2 \cup A_3 \cup A_4$, the definition of ψ is correct. Further, from (25) and (26) we deduce at once that

$$\psi(f^{m-1}(x)) = \sum_{j=0}^{m-2} b_j \psi(f^j(x)) + \psi_0(x) \tag{27}$$

for every $x^* \in S^+$ and $x \in C_f^+(x^*)$ ($y^* \in S^-$ and $x \in C_f^-(y^*)$, respectively).

Next, we prove (20). Take $x \in S_f$. If $x = f^{n-1}(z^*)$ with some $z^* \in S^- \setminus S^+$ and $n \in \mathbb{N}$, then the definition of ψ implies that

$$\psi(f^m(x)) = \psi(f^{m+n-1}(z^*)) = \sum_{j=\rho}^{m-1} a_j \psi(f^{j+n-1}(z^*)) + F(f^{n-1}(z^*)) = \sum_{j=\rho}^{m-1} a_j \psi(f^j(x)) + F(x).$$

If $x = f^{-n}(w^*)$ with some $w^* \in S^+ \setminus S^-$ and $n \in \mathbb{N}$, then the definition of ψ yields

$$\begin{aligned}\psi(f^m(x)) &= \psi(f^{m-n}(w^*)) = a_\rho \psi(f^{\rho-n}(w^*)) + \sum_{j=\rho+1}^{m-1} a_j \psi(f^{j-n}(w^*)) + F(f^{-n}(w^*)) \\ &= \sum_{j=\rho+1}^{m-1} a_j \psi(f^j(x)) + a_\rho \psi(f^\rho(x)) + F(x) = \sum_{j=0}^{m-1} a_j \psi(f^j(x)) + F(x),\end{aligned}$$

which completes the proof of (20).

Observe that, in view of (15), (22), (24), and Remark 2, equality (27) is also valid for $x \in S^+ \cup S^-$, which means that condition (7) holds with $\delta_1 = \delta_2 = 0$ and $T_1 = T_2 \in \{C_f^-(y^*) \cup \{y^*\}, C_f^+(x^*) \cup \{x^*\}\}$ for every $x^* \in S^+$, $y^* \in S^-$. Hence, on account of Lemma 1, for every $x^* \in S^+$, $x \in C_f^+(x^*) \cup \{x^*\}$ ($y^* \in S^-$, $x \in C_f^-(y^*)$, resp.) we have

$$\left\| \psi(f^m(x)) - \sum_{j=0}^{m-1} a_j \psi(f^{m-1}(x)) - F(x) \right\| \le \delta.$$

This, (20) and (21) yield (19). Moreover, from Lemma 1 (with $D = T_1 = T_2$) we deduce that, for every $x^* \in S^+$ ($y^* \in S^-$, resp.), ψ is unbounded on $C_f^+(x^*)$ ($C_f^-(y^*)$, resp.) and

$$\sup_{z \in C_f^+(x^*)} \|\psi(z) - \varphi_0(z)\| = \infty$$

($\sup_{z \in C_f^-(y^*)} \|\psi(z) - \varphi_0(z)\| = \infty$, resp.) for each solution $\varphi_0 : A \to X$ of Eq. (1).

To complete the proof consider the case $K = \mathbb{R}$. Then, X^2, endowed with the linear structure and the Taylor norm $\|\cdot\|_T$ defined by:

$$(x, y) + (z, w) := (x + z, y + w), \qquad (\alpha + i\beta)(x, y) := (\alpha x - \beta y, \beta x + \alpha y),$$
$$\|(x, y)\|_T := \sup_{0 \le \theta \le 2\pi} \|(\cos\theta)x + (\sin\theta)y\|$$

for $x, y, z, w \in X$, $\alpha, \beta \in \mathbb{R}$, is a complex normed space (see e.g. [22, p. 39], [23] or [24, 1.9.6, p. 66]). It is easily seen that

$$\max\{\|x\|, \|y\|\} \le \|(x, y)\|_T \le \|x\| + \|y\|, \quad \forall (x, y) \in X^2. \tag{28}$$

Write $\overline{F}(x) := (F(x), F(x))$, $\overline{\varphi}(x) := (\varphi(x), \varphi(x))$ for $x \in A$ and $\overline{\xi^*}(x) := (\xi^*(x), \xi^*(x))$ for $x \in S_\rho^*$. From the previous part of the proof, it results that there is $\overline{\psi} : A \to X^2$ with

$$\sup_{x \in A} \left\| \overline{\psi}(f^m(x)) - \sum_{j=0}^{m-1} a_j \overline{\psi}(f^j(x)) - \overline{F}(x) \right\|_T \le \delta, \tag{29}$$

$$\overline{\psi}(f^m(x)) = \sum_{j=0}^{m-1} a_j \overline{\psi}(f^j(x)) + \overline{F}(x), \quad \forall x \in S_f, \tag{30}$$

$$\overline{\psi}(x) = \overline{\varphi}(x), \quad \forall x \in A_f, \tag{31}$$

$$\overline{\psi}(y) = \overline{\xi^*}(y), \quad \forall y \in S_\rho^*, \tag{32}$$

and

$$\sup_{x \in C_f^+(x^*)} \|\overline{\psi}(x) - \overline{\varphi}(x)\|_T = \infty, \qquad \sup_{x \in C_f^-(y^*)} \|\overline{\psi}(x) - \overline{\varphi}(x)\|_T = \infty$$

for every $x^* \in S^+$, $y^* \in S^-$ and every solution $\overline{\varphi} : A \to X^2$ of the equation

$$\overline{\varphi}(f^m(x)) = \sum_{j=0}^{m-1} a_j \overline{\varphi}(f^j(x)) + \overline{F}(x). \tag{33}$$

Define $p_1, p_2 : X^2 \to X$ by: $p_i(x_1, x_2) := x_i$ for $x_1, x_2 \in X$, $i = 1, 2$. Take $x^* \in S^+$ and suppose there are solutions $\varphi_1, \varphi_2 : A \to X$ of (1) with

$$\sup_{x\in C_f^+(x^*)} \|p_1(\overline{\psi}(x)) - \varphi_1(x)\| < \infty, \qquad \sup_{x\in C_f^+(x^*)} \|p_2(\overline{\psi}(x)) - \varphi_2(x)\| < \infty.$$

Write $\overline{\varphi}(x) := (\varphi_1(x), \varphi_2(x))$ for $x \in A$. Then $\overline{\varphi}$ is a solution of (33) and, by (28),

$$\begin{aligned}\sup_{x\in C_f^+(x^*)} \|\overline{\psi}(x) - \overline{\varphi}(x)\|_T &\le \sup_{x\in C_f^+(x^*)} \left\{\|p_1(\overline{\psi}(x)) - p_1(\overline{\varphi}(x))\| + \|p_2(\overline{\psi}(x)) - p_2(\overline{\varphi}(x))\|\right\}\\ &\le \sup_{x\in C_f^+(x^*)} \|p_1(\overline{\psi}(x)) - \varphi_1(x)\| + \sup_{x\in C_f^+(x^*)} \|p_2(\overline{\psi}(x)) - \varphi_2(x)\| < \infty.\end{aligned}$$

This is a contradiction. Repeating similar reasoning for $y^* \in S^-$ we deduce that, for every $x^* \in S^+,\ y^* \in S^-$ there are $j^+(x^*),\ j^-(y^*) \in \{1, 2\}$ such that

$$\sup_{x\in C_f^+(x^*)} \|p_{j^+(x^*)}(\overline{\psi}(x)) - \varphi_0(x)\| = \infty, \qquad \sup_{x\in C_f^-(y^*)} \|p_{j^-(y^*)}(\overline{\psi}(x)) - \varphi_0(x)\| = \infty \tag{34}$$

for each solution $\varphi_0 : A \to X$ of (1). Now, it is enough to take

$$\psi(x) := p_1(\overline{\psi}(x)), \quad \forall x \in A_f, \tag{35}$$

$$\psi(x) := p_{j^+(x^*)}(\overline{\psi}(x)), \quad \forall x \in C_f^*(x^*),\ x^* \in (S^+ \setminus S^-), \tag{36}$$

$$\psi(x) := p_{j^-(y^*)}(\overline{\psi}(x)), \quad \forall x \in C_f^*(y^*),\ y^* \in (S^- \setminus S^+), \tag{37}$$

and, for every $x^* \in S^+ \cap S^-$,

$$\psi(f^k(x^*)) := p_{j^+(x^*)}(\overline{\psi}(f^k(x^*))), \qquad \psi(f^j(x^*)) := p_{j^-(x^*)}(\overline{\psi}(f^j(x^*))), \quad \forall k, j \in \mathbb{Z},\ k \ge m, j < m. \tag{38}$$

Since (32) yields

$$p_1(\overline{\psi}(x)) = \xi^*(x) = p_2(\overline{\psi}(x)), \quad \forall x \in S_\rho^*, \tag{39}$$

it is easily seen that, in view of (31) and (34), conditions (21)–(23) are valid for each solution $\varphi_0 : A \to X$ of Eq. (1) and every $x^* \in S^+, y^* \in S^-$.

It remains to prove (19) and (20). To this end, in view of (28)–(30), it is enough to show that, for each $x \in A$, there is $i \in \{1, 2\}$ with

$$\psi(f^m(x)) - \sum_{j=\rho}^{m-1} a_j\psi(f^j(x)) - F(x) = p_i\left(\overline{\psi}(f^m(x)) - \sum_{j=\rho}^{m-1} a_j\overline{\psi}(f^j(x)) - \overline{F}(x)\right). \tag{40}$$

Note that, according to (35), this is the case for $x \in A_f$ with $i = 1$. Further, by (36) (by (37), respectively), for each $x^* \in (S^+ \setminus S^-)$ ($y^* \in (S^- \setminus S^+)$, resp.), (40) is true for $x \in C_f^*(x^*)$ ($x \in C_f^*(y^*)$, resp.) with $i = j^+(x^*)$ (with $i = j^-(y^*)$, resp.). Finally take $x^* \in S^+ \cap S^-$, $k \in \mathbb{Z}$ and write $x := f^k(x^*)$. Clearly, $f^j(x) = f^{j+k}(x^*)$ for $j \in \mathbb{Z}$. If $k \ge 0$, then $j + k \ge \rho$ for $j \ge \rho$ and consequently, on account of (38) and (39), (40) holds with $i = j^+(x^*)$. If $k < 0$, then (38) and (39) yield (40) with $i = j^-(x^*)$, because $j + k < m$ for $j \le m$. □

Remark 3. The information that function ψ in Theorem 1 can be unbounded seems to be interesting especially when F is bounded, because in such a case every bounded function $\gamma : A \to X$ satisfies (5).

Remark 4. It follows from Theorem 1 that, under suitable assumptions, Eq. (1) is nonstable in X^A. One of those assumptions is that (1) has a solution $\varphi \in X^A$, which is always the case when

$$A = \bigcup_{x^*\in A_f^*} C_f^*(x^*);$$

it is just enough to choose arbitrarily $\varphi(f^i(x^*))$ for $i = \rho, \ldots, m-1, x^* \in A_f^*$ and write, analogously as in the proof of Theorem 1,

$$\varphi(f^{m+n-1}(x^*)) := \sum_{j=\rho}^{m-1} a_j\varphi(f^{j+n-1}(x^*)) + F(f^{n-1}(x^*)), \quad \forall n \in \mathbb{N},$$

$$\psi(f^{\rho-n}(x^*)) := a_\rho^{-1}\left[\psi(f^{m-n}(x^*)) - \sum_{j=\rho+1}^{m-1} a_j\psi(f^{j-n}(x^*)) - F(f^{-n}(x^*))\right], \quad \forall n \in \mathbb{N},\ f^{\rho-n}(\{x^*\}) \ne \emptyset.$$

Next, observe that if (1) does not have any solution in X^A and F is bounded, then (1) is nonstable in X^A (see Remark 3).

The following corollary and two very simple theorems show that nonstability of (1) in X^A also can be obtained under an assumption somewhat weaker than injectivity of f (Corollary 1) or even without any such assumption (Theorems 2 and 3). This means that injectivity of f is not necessary to get some nonstability results for (1), though it plays a crucial role in the proof of Theorem 1.

Corollary 1. *Assume that $|r_j| = 1$ for some $j \in \{1, \dots, m\}$, Eq. (1) has a solution $\varphi \in X^A$, $(\mathcal{H})$ is valid, and f is injective on the set $C_f^*(x^*)$ (i.e., $f(x) \neq f(y)$ for $x, y \in C_f^*(x^*)$ with $x \neq y$) for some $x^* \in A_f^*$. Then Eq. (1) is nonstable in X^A.*

Proof. According to Theorem 1 (with $A := C_f^*(x^*)$, $S^+ := \{x^*\}$ and $S^- := \emptyset$) there is a function $\psi : C_f^*(x^*) \to X$ with

$$\sup_{x \in C_f^*(x^*)} \left\| \psi(f^m(x)) - \sum_{j=0}^{m-1} a_j \psi(f^j(x)) - F(x) \right\| =: \delta < \infty$$

and such that $\sup_{x \in C_f^+(x^*)} \|\psi(x) - \varphi_0(x)\| = \infty$ for each solution $\varphi_0 : C_f^*(x^*) \to X$ of Eq. (1). Write $\psi(x) := \varphi(x)$ for $x \in A \setminus C_f^*(x^*)$. Since φ is a solution of (1) and $f(A \setminus C_f^*(x^*)) \cap C_f^*(x^*) = \emptyset$, we have

$$\sup_{x \in A} \left\| \psi(f^m(x)) - \sum_{j=0}^{m-1} a_j \psi(f^j(x)) - F(x) \right\| = \sup_{x \in C_f^*(x^*)} \left\| \psi(f^m(x)) - \sum_{j=0}^{m-1} a_j \psi(f^j(x)) - F(x) \right\| = \delta.$$

However, for each solution $\varphi_0 : A \to X$ of (1),

$$\sup_{x \in A} \|\psi(x) - \varphi_0(x)\| \geq \sup_{x \in C_f^+(x^*)} \|\psi(x) - \varphi_0(x)\| = \infty. \quad \square$$

Theorem 2. *Let $m > 1$, $S \subset A$ be nonempty, $f(S) \subset S$, $\sup_{x \in S} \|F(x)\| < \infty$,*

$$\lim_{n \to \infty} \left\| \sum_{k=0}^{n} F(f^k(x_0)) \right\| = \infty, \tag{41}$$

holds for some $x_0 \in S$, and

$$\sum_{j=0}^{m-1} a_j = 1. \tag{42}$$

Then Eq. (1) is nonstable on S in X^A.

Proof. It follows from (42) that $r_j = 1$ for some $j \in \{1, \dots, m\}$. Without loss of generality, we may assume that $j = 1$. Since $\sup_{x \in S} \|F(x)\| < \infty$, the function $\psi_0 : A \to X$, $\psi_0(x) \equiv 0$ satisfies (6) with $T_1 := S$. Further, it is easily seen that, by (41) and Lemma 2, every solution $\hat{\eta} : A \to X$ of (9) is unbounded on S $\left(\text{because } \hat{\eta}(f^n(x_0)) = \hat{\eta}(x_0) + \sum_{k=0}^{n-1} F(f^k(x_0)) \text{ for } n \in \mathbb{N}\right)$, whence $\sup_{x \in S} \|\hat{\eta}(x) - \psi_0(x)\| = \infty$. Finally, observe that $\psi : A \to X$, $\psi(x) \equiv 0$ fulfils (7) with $T_1 = T_2 = S$ and $\delta_1 = \delta_2 = 0$. Consequently, Lemma 1 completes the proof. $\square$

Theorem 3. *Let $m > 1$, $S \subset A$ be nonempty, $f(S) \subset S$, and $r_1 \in K$. Assume that there is a function $\psi_0 : A \to X$ such that (6) holds with $T_1 := S$ and Eq. (9) has no solution $\hat{\eta} : A \to X$ that satisfies condition (10) with $T_2 := S$. Further, suppose that the functional equation*

$$\gamma(f^{m-1}(x)) = \sum_{j=0}^{m-2} b_j \gamma(f^j(x)) + \psi_0(x) \tag{43}$$

is nonstable on S in the class of functions X^A or has a solution in that class. Then Eq. (1) is nonstable on S in the class of functions X^A.

Proof. According to the assumptions on Eq. (43), there is a function $\psi : A \to X$ satisfying (7) with $T_1 = T_2 = S$ and some real $\delta_1 = \delta_2 \geq 0$. Hence Lemma 1 completes the proof. $\square$

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