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Continuum random trees and branching processes with immigration

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Abstract

We study a genealogical model for continuous-state branching processes with immigration with a (sub)critical branching mechanism. This model allows the immigrants to be on the same line of descent. The corresponding family tree is an ordered rooted continuum random tree with a single infinite end defined by two continuous processes denoted by $(\overleftarrow{H}_t; t \geq 0)$ and $(\overrightarrow{H}_t; t \geq 0)$ that code the parts at resp. the left and the right hand side of the infinite line of descent of the tree. These processes are called the left and the right height processes. We define their local time processes via an approximation procedure and we prove that they enjoy a Ray–Knight property. We also discuss the important special case corresponding to the size-biased Galton–Watson tree in a continuous setting. In the last part of the paper we give a convergence result under general assumptions for rescaled discrete left and right contour processes of sequences of Galton–Watson trees with immigration. We also provide a strong invariance principle for a sequence of rescaled Galton–Watson processes with immigration that also holds in the supercritical case.

MSC: 60F17; 60G17; 60J80

Keywords: Continuous-state branching process; Immigration; Size-biased tree; Lévy tree; Continuum random tree; Height process; Contour process; Ray–Knight theorem; Limit theorem; Invariance principle; Galton–Watson branching process

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1. Introduction

1.1. The genealogy of Galton–Watson branching processes

Continuous-state branching processes with immigration (CSBPI for short) have been introduced by Kawasu and Watanabe in [19]. They are continuous analogues of Galton–Watson processes with immigration. In this paper we discuss a genealogical model for CSBPI's that can be described in a discrete setting as follows: let μ and ν be two probability measures on the set of non-negative integers denoted by $\mathbb{N}$. Recall that a Galton–Watson process $Z = (Z_n; n \geq 0)$ with *offspring distribution* μ and *immigration distribution* ν (a GWI(μ, ν)-process for short) is an $\mathbb{N}$-valued Markov chain whose transition probabilities are characterized by

$$\mathbf{E}\left[x^{Z_{n+1}} \mid Z_n\right] = g(x)^{Z_n} f(x), \quad x \in [0, 1], \tag{1}$$

where g (resp. f) stands for the generating function of μ (resp. ν).

The genealogical model we consider can be informally described as follows. Consider a population evolving at random roughly speaking according to a GWI(μ, ν)-process: The population can be decomposed in two kinds of individuals, namely the mutants and the non-mutants; there is exactly one mutant at a given generation; each individual gives birth to an independent number of children: the mutants in accordance with ν and the non-mutants in accordance with μ. We require that all the mutants are on the same infinite line of descent. Except in one part of the paper, we restrict our attention to a critical or subcritical offspring distribution μ:

$$\bar{\mu} := \sum_{k \geq 0} k\mu(k) \leq 1.$$

Then the resulting family tree is a tree with a single infinite end. We call such trees *sin-trees* following Aldous' terminology in [2] in Section 4. In order to code them by real-valued functions, all the discrete trees that we consider are ordered and rooted or equivalently are planar graphs (see Section 2.1 for more details). So we have to specify for each mutant how many of its children are on the left hand of the infinite line of descent and how many are on the right hand. We choose to dispatch them independently at random in accordance with a probability measure r on the set $\{(k, j) \in \mathbb{N}^* \times \mathbb{N}^* : 1 \leq j \leq k\}$. More precisely, with probability $r(k, j)$ a mutant has $k - 1$ non-mutant children and one mutant child who is the j-th child to be born; consequently there are $j - 1$ non-mutant children on the left hand of the infinite line of descent and $k - j$ on the right hand. The immigration distribution ν then is given by

$$\nu(k - 1) = \sum_{1 \leq j \leq k} r(k, j), \quad k \geq 1.$$

The probability measure r is called the *dispatching distribution* and the resulting random tree is called a (μ, r)-*Galton–Watson tree with immigration* (a GWI(μ, r)-tree for short). Indeed, if we denote by Z_n the number of non-mutants at generation n in the tree, then it is easy to see that $Z = (Z_n; n \geq 0)$ is GWI(μ, ν)-process.

Let us mention the special case $r(k, j) = \mu(k)/\bar{\mu}$, $1 \leq j \leq k$ that corresponds to the size-biased Galton–Watson tree with offspring distribution μ. This random tree arises naturally by conditioning (sub)critical GW-trees on non-extinction: see [1,2,20,21,28] and in the continuous case [26].

We shall code a GWI(μ, r)-tree τ by two real-valued functions in the following way: think of τ as a planar graph embedded in the clockwise oriented half-plane with unit edge length and consider a particle visiting continuously the edges of τ at speed one from the left to the right, going backward as less as possible; we denote by $\overleftarrow{C}_s(\tau)$ the distance from the root of the particle at time s and we call the resulting process $\overleftarrow{C}(\tau) := (\overleftarrow{C}_s(\tau); s \geq 0)$ the *left contour process* of τ. It is clear that the particle never reaches the right part of τ but $\overleftarrow{C}(\tau)$ completely codes the left part of τ. We denote by $\overrightarrow{C}(\tau)$ the process corresponding to a particle visiting τ from the right to the left so we can then reconstruct τ from $(\overleftarrow{C}(\tau), \overrightarrow{C}(\tau))$: see Section 2.2 for precise definitions and other codings of sin-trees.

1.2. Background on continuous-states branching processes with immigration

The main purpose of the paper is to provide a genealogical model for CSBPI's and to build a continuous family tree coded by two functions playing the role of $\overleftarrow{C}$ and $\overrightarrow{C}$. Before discussing it more specifically, let us recall from [19] that continuous-state branching processes with immigration are $[0, \infty]$-valued stochastically continuous Markov processes whose distribution is characterized by two functions on $[0, \infty)$: a *branching mechanism* ψ such that $(-\psi)$ is the Laplace exponent of a spectrally positive Lévy process denoted by $X = (X_t; t \geq 0)$ and an *immigration mechanism* φ that is the Laplace exponent of a subordinator denoted by $W = (W_t; t \geq 0)$:

$$\mathbf{E}\left[\mathrm{e}^{-\lambda X_t}\right] = \mathrm{e}^{t\psi(\lambda)}, \qquad \mathbf{E}\left[\mathrm{e}^{-\lambda W_t}\right] = \mathrm{e}^{-t\varphi(\lambda)}, \quad \lambda, t \geq 0.$$

More precisely, $Y^* = (Y^*_t; t \geq 0)$ is a (ψ, φ)-continuous-state branching process with immigration (a CSBPI(ψ, φ) for short) if its transition kernels are characterized by

$$\mathbf{E}\left[\mathrm{e}^{-\lambda Y^*_{a+b}} | Y^*_b\right] = \exp\left(-u(a,\lambda) Y^*_b - \int_0^a \mathrm{d}s\, \varphi(u(s,\lambda))\right), \quad a, b, \lambda \geq 0, \tag{2}$$

where $u(a, \lambda)$ is the unique nonnegative solution of the differential equation

$$\frac{\partial}{\partial a} u(a, \lambda) = -\psi(u(a, \lambda)) \quad \text{and} \quad u(0, \lambda) = \lambda, \quad a, \lambda \geq 0. \tag{3}$$

Note that this differential equation is equivalent to the integral equation

$$\int_{u(a,\lambda)}^{\lambda} \frac{\mathrm{d}u}{\psi(u)} = a. \tag{4}$$

Observe that ∞ is an absorbing state. In the paper, we will only consider *conservative* processes, that is processes such that a.s. $Y^*_t < \infty$, $t \geq 0$. This is equivalent to the analytical conditions

$$\int_{0+} \frac{\mathrm{d}u}{|\psi(u)|} = \infty \quad \text{and} \quad \psi(0) = \varphi(0) = 0. \tag{5}$$

Observe that if there is no immigration, that is if $\varphi = 0$, the process is simply a ψ-continuous-state branching process (a CSBP(ψ) for short). We shall denote CSBPI's and CSBP's in a generic way by resp. $Y^* = (Y^*_t; t \geq 0)$ and $Y = (Y_t; t \geq 0)$. We refer to [19] or [29] for a precise discussion of CSBPI's and to [7,15] for results on CSBP's.

Except in Section 4.1, we only consider CSBPI's with (sub)critical branching mechanism. This assumption is equivalent to the fact that X does not drift to $+\infty$. In that case ψ is of the form

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} \pi(\mathrm{d}r)(\mathrm{e}^{-\lambda r} - 1 + \lambda r), \quad \lambda \geq 0, \tag{6}$$

where $\alpha, \beta \geq 0$ and π is a σ-finite measure on $(0, \infty)$ such that $\int_{(0,\infty)} \pi(\mathrm{d}r)(r \wedge r^2) < \infty$. We also assume

$$\int_1^\infty \frac{\mathrm{d}u}{\psi(u)} < \infty, \tag{7}$$

which is equivalent to the a.s. extinction of CSBP(ψ) (see [15] for details). Note that (7) implies at least one of the two conditions: $\beta > 0$ or $\int_0^1 r\,\pi(\mathrm{d}r) = \infty$ that guarantee that X has infinite variation sample paths (see [5] for details).

We build the family tree corresponding to a CSBPI(ψ, φ) thanks to two continuous processes $(\overleftarrow{H}_t; t \geq 0)$ and $(\overrightarrow{H}_t; t \geq 0)$ called the *left and the right height processes* that are viewed as contour processes of the parts at the left and at the right hand of the infinite line of descent. More precisely, our construction relies on two auxilliary processes:

- The first one is the *height process* $H = (H_t; t \geq 0)$ introduced by Le Gall and Le Jan [25] coding the genealogy of ψ-continuous-state branching processes (see also [11] for related results). H is obtained as a functional $H(X)$ of the spectrally positive Lévy process X with exponent ψ. More precisely, for every $t \geq 0$, H_t "measures" in a local time sense the size of the set $\{s \leq t : X_{s-} = \inf_{[s,t]} X_r\}$ (see Section 3.1 for a more precise definition). Assumption (7) is equivalent for H to have a continuous modification. *From now on, we only consider this modification*.

  An important role is played by the excursion measure N of X above its minimum process. In the quadratic branching case $\psi(u) = c\,u^2$, X is a (scaled) Brownian motion, the height process H is a reflected Brownian motion and the "law" of H under N is the Ito measure of positive excursions of the linear Brownian motion: This is related to the fact that the contour process of Aldous' *Continuum Random Tree* is given by a normalized Brownian excursion (see [3,4]), or to the Brownian snake construction of superprocesses with a quadratic branching mechanism (see e.g. [24]). For a general ψ, limit theorems for the contour processes of discrete Galton–Watson trees given in [11], Chapter 2 and the Ray–Knight property for the local times of H proved in Theorem 1.4.1 [11] both strongly justify that the height process is the right object to code the genealogy of (sub)critical CSBP's and that H under the excursion measure N is the contour process of a continuum random tree that is called the Lévy tree: we refer to [12] for a precise definition of Lévy trees in term of random metric $\mathbb{R}$-trees space (see also [13] for related topics). All the results about height processes used in the paper are recalled in Section 3.1.
- The second process is a bivariate subordinator $(U, V) = ((U_t, V_t); t \geq 0)$, namely a $[0, \infty) \times [0, \infty)$-valued Lévy process started at 0. Its distribution is characterized by its Laplace exponent Φ:

  $$\mathbf{E}\left[\exp(-pU_t - qV_t)\right] = \exp(-t\,\Phi(p, q)).$$

  Φ is of the form

  $$\Phi(p, q) = dp + d'q + \int_{(0,\infty)^2} R(\mathrm{d}x\mathrm{d}y)(1 - \mathrm{e}^{-px-qy}),$$

where $d, d' \geq 0$ and R is a σ-finite measure on $(0, \infty)^2$ such that $\int R(\mathrm{d}x\mathrm{d}y)1 \wedge (x+y) < \infty$. The Lévy measure R plays the role of the discrete dispatching measure r. Roughly speaking, think of the population as being indexed by positive real numbers and let us make an informal analogy with the discrete model: If the height $t \in [0, \infty)$ in the family tree of the CSBPI corresponds to generation n in the discrete GWI-tree, then U_t (resp. V_t) corresponds to the sum of the numbers of immigrants at the left (resp. the right) hand of the infinite line of descent from generation 0 to generation n. Then, a jump of (U, V) occuring at time t corresponds to a total amount $U_t - U_{t-} + V_t - V_{t-}$ of immigrants arriving at height t in the family tree: $U_t - U_{t-}$ of them are put at the left hand of the infinite line of descent and they are the initial population of a CSBP(ψ); $V_t - V_{t-}$ of them are put at the right hand of the infinite line of descent and they are also the initial population of an independent CSBP(ψ). It implies that the real-valued subordinator $U + V$ has Laplace exponent φ and thus,

$$\varphi(p) = \Phi(p, p).$$

More precisely, we define $\overleftarrow{H}$ and $\overrightarrow{H}$ as follows. Let us first introduce the right-continuous inverses of U and V:

$$U_t^{-1} = \inf\{s \geq 0 : \ U_s > t\} \quad \text{and} \quad V_t^{-1} = \inf\{s \geq 0 : \ V_s > t\},$$

with the convention $\inf \emptyset = \infty$. Let H be the height process associated with the Lévy process X with Laplace exponent ψ and let (H', X') be an independent copy of (H, X). We set for any $t \geq 0$, $I_t = \inf_{[0,t]} X$ and $I'_t = \inf_{[0,t]} X'$. Then, we define $\overleftarrow{H}$ and $\overrightarrow{H}$ by

$$\overleftarrow{H}_t = H_t + U^{-1}_{-I_t} \quad \text{and} \quad \overrightarrow{H}_t = H'_t + V^{-1}_{-I'_t}, \quad t \geq 0. \tag{8}$$

The processes $\overleftarrow{H}$ and $\overrightarrow{H}$ are called respectively left and right height processes. Left and right height processes are continuous iff U^{-1} and V^{-1} are continuous, which happens iff U and V are not Poisson processes. This is equivalent to the analytical condition

$$dd' \neq 0 \quad \text{or} \quad R\left((0, \infty)^2\right) = \infty. \tag{9}$$

As for Lévy trees, it is possible to build a (ψ, Φ)-*immigration Lévy tree* via left and right height processes: To each $s \in \mathbb{R}$ corresponds a vertex in the continuum tree at height

$$J_s = \mathbf{1}_{(-\infty,0)}(s)\overleftarrow{H}_{-s} + \mathbf{1}_{[0,\infty)}(s)\overrightarrow{H}_s.$$

Suppose that $s \leq s'$. The common ancestor of vertices corresponding to s and s' is situated at height $m(s, s') = \inf\{J_u;\ u \in I(s, s')\}$, where $I(s, s')$ is taken as $[s, s']$ if $0 \notin [s, s']$ and as $\mathbb{R} \setminus [s, s']$ otherwise. Then, the distance separating the vertices corresponding to s and s' is given by

$$\mathbf{d}(s, s') = J_s + J_{s'} - 2m(s, s').$$

Check that $\mathbf{d}$ is a pseudo-metric on $\mathbb{R}$. We say that two real numbers s and s' are equivalent if they correpond to the same vertex, that is: $\mathbf{d}(s, s') = 0$. This equivalence relation is denoted by $s \sim s'$ and we formally define the (ψ, Φ)-*immigration Lévy tree* as the quotient set $\mathcal{T}^* = \mathbb{R}/\sim$ equipped with the metric $\mathbf{d}$ that makes it a random Polish space. Arguing as in [12], we can show that a.s. $(\mathcal{T}^*, \mathbf{d})$ is a real tree.

This genealogical model is clearly related to the model discussed in [22] by A. Lambert where all the population is on the left hand side of the infinite line of descent so that only one contour process is needed to encode the family tree of the CSBPI's. In Lambert's paper, this contour process is defined as a functional of a Markov process X^* generalizing the functional giving the height process introduced by Le Gall and Le Jan. This Markov process can be constructed either pathwise in terms of a spectrally positive Lévy process and an independent subordinator, or in distribution thanks to Itô's synthesis theorem. A. Lambert also defines in a weak sense the local time processes of the resulting contour process and states a generalized Ray–Knight theorem by proving they are distributed as a CSBPI.

1.3. Statements of the main results

In this paper, the model that we consider allows us to have population on both sides of the infinite line of descent which turns out to be a natural case to discuss for we can define continuous analogues of discrete size-biased trees. In particular we show in Theorem 1.3 stated below that, as in the discrete case, continuous analogues of size-biased trees are the limit of family trees of (sub)critical CSBP's conditioned on non-extinction. This special case strongly motivated the present work. In the more general case, we also provide a strong approximation for local time processes of right and left contour processes (Proposition 1.1), and we state the Ray–Knight property for them (Theorem 1.2). We prove functional weak convergence for rescaled contour processes of discrete GWI-trees (Theorem 1.5). We also state in Theorem 1.4 a general limit theorem for GWI processes (with possibly supercritical offspring distribution), in the same vein as Theorem 3.4 [17] of Grimvall.

More specifically, let us now give a detailed presentation of the main results of this paper. The first one defines local time processes of two contour processes by a strong approximation procedure.

Proposition 1.1. *Assume that (7) and (9) hold. Then, there exists two jointly measurable processes* $(\overleftarrow{L}^a_s;\ a, s \geq 0)$ *and* $(\overrightarrow{L}^a_s;\ a, s \geq 0)$ *such that:*

(i) *A.s. for any* $a > 0$, $(\overleftarrow{L}^a_s;\ s \geq 0)$ *and* $(\overrightarrow{L}^a_s;\ s \geq 0)$ *are continuous nondecreasing processes.*

(ii) *For all* $T > 0$ *the following limit holds in probability:*

$$\sup_{t\leq T}\left|\epsilon^{-1}\int_0^t \mathrm{d}s\mathbf{1}_{\{a<\overleftarrow{H}_s\leq a+\epsilon\}} - \overleftarrow{L}^a_t\right| \xrightarrow[\epsilon\to 0]{} 0.$$

The same limit holds for $\overrightarrow{H}$ *and* $\overrightarrow{L}^a$.

(iii) *A.s. for any continuous function* g *on* $[0, \infty)$ *with compact support and for any* $t \geq 0$,

$$\int_0^t \mathrm{d}s\, g(\overleftarrow{H}_s) = \int_0^\infty \mathrm{d}a\, \overleftarrow{L}^a_t g(a) \quad \textit{and} \quad \int_0^t \mathrm{d}s\, g(\overrightarrow{H}_s) = \int_0^\infty \mathrm{d}a\, \overrightarrow{L}^a_t g(a).$$

We next show that local time processes of $\overleftarrow{H}$ and $\overrightarrow{H}$ enjoy a "Ray–Knight" property: As U and V both drift to infinity, so do left and right height processes and it makes sense to define

$$\overleftarrow{L}^a_\infty = \lim_{s\to\infty} \overleftarrow{L}^a_s \quad \text{and} \quad \overrightarrow{L}^a_\infty = \lim_{s\to\infty} \overrightarrow{L}^a_s, \quad a \geq 0.$$

Then, the Ray–Knight theorem can be stated as follows.

Theorem 1.2. *Assume that (7) and (9) hold. Then, the process* $(\overleftarrow{L}^a_\infty + \overrightarrow{L}^a_\infty; a \geq 0)$ *is a CSBPI(ψ, φ) started at* 0 *with* $\varphi(\lambda) = \Phi(\lambda, \lambda)$, $\lambda \geq 0$.

Proposition 1.1 and Theorem 1.2 are proved in Section 3.2 while Section 3.3 is devoted to the study of the continuous analogue of size biased GW-trees, that corresponds to (ψ, Φ)-immigration Lévy trees where Φ is given by

$$\Phi(p, q) = \frac{\psi^*(p) - \psi^*(q)}{p - q}.$$

Here, we have set $\psi^*(\lambda) = \psi(\lambda) - \alpha\lambda$ and when $p = q$, the ratio $(\psi^*(p) - \psi^*(q))/(p - q)$ should be interpreted as $\psi'(p) - \alpha$. So $U + V$ is a subordinator with Laplace exponent $\varphi = \psi' - \alpha$ that is the immigration mechanism of the underlying CSBPI. Now, consider the height process H under its excursion measure N and denote by ζ the duration of the excursion. As a consequence of (7) we get $N(\sup_{s\in[0,\zeta]} H_s > a) \in (0, \infty)$ for any $a > 0$. Thus, we can define the probability measure $N_{(a)} = N(\cdot \mid \sup H > a)$ (see Section 3.1 for details). The main result proved in Section 3.3 can be stated as follows.

Theorem 1.3. *Assume that (7) holds. Then,*

$$\left(H_{t\wedge\zeta},\ H_{(\zeta-t)_+}; t \geq 0\right) \text{ under } N_{(a)} \xrightarrow[a\to\infty]{} \left(\overleftarrow{H}_t,\ \overrightarrow{H}_t; t \geq 0\right)$$

weakly in $C([0, \infty), \mathbb{R}^2)$.

(Here $(x)_+$ stands for the non-negative part $\max(0, x)$ of x.) The proof of this theorem relies on a lemma (Lemma 3.2) that is stated in Section 3.3 and that is an easy consequence of Lemma 3.4 in [12]. Let us mention that Lemma 3.2 is a generalization of Bismut's decomposition of the Brownian excursion.

Sections 4.1 and 4.2 are devoted to limit theorems for rescaled GWI processes and contours of GWI trees. The main result proved in Section 4.1 is a strong invariance principle for GWI processes. **In Section 4.1 and only in Section 4.1** we no longer restrict our attention to (sub)critical GWI-processes. More precisely, let $(\mu_p; p \geq 1)$ and $(\nu_p; p \geq 1)$ be **any** sequences of probability measures on $\mathbb{N}$ and let $x \in [0, \infty)$. We denote by $(Y_n^{*,p}; n \geq 0)$, $p \geq 1$ a sequence of GWI(μ_p, ν_p)-processes started at $Y_0^{*,p} = [px]$ and we denote by $(\gamma_p; p \geq 1)$ an increasing sequence of positive integers.

Theorem 1.4. *The three following assertions are equivalent:*

(i) *For any* $t \geq 0$ *the following convergence*

$$p^{-1} Y^{*,p}_{[\gamma_p t]} \xrightarrow[p\to\infty]{\text{(d)}} Z^*_t, \tag{10}$$

holds in distribution in $\mathbb{R}$*; Here* $(Z^*_t; t \geq 0)$ *stands for a non-constant and stochastically continuous process such that*

$$\forall t > 0, \quad \mathbf{P}(Z^*_t > 0) > 0 \quad \text{and} \quad \mathbf{P}(Z^*_t < \infty) = 1.$$

(ii) *We can find a non-constant spectrally positive Lévy process* $X = (X_t; t \geq 0)$ *with exponent* ψ *and a subordinator* $W = (W_t; t \geq 0)$ *with exponent* φ *such that (5) holds and such that the following convergences*

$$\mu_p\left(\frac{\cdot - 1}{p}\right)^{*p\gamma_p} \xrightarrow[p\to\infty]{\text{(d)}} \mathbf{P}(X_1 \in \cdot) \quad \text{and} \quad \nu_p\left(\frac{\cdot}{p}\right)^{*\gamma_p} \xrightarrow[p\to\infty]{\text{(d)}} \mathbf{P}(W_1 \in \cdot) \tag{11}$$

hold in distribution in $\mathbb{R}$ *(here* $*$ *denotes the convolution product of measures).*

(iii) *There exists a non-constant and conservative CSBPI*(ψ, φ) *denoted by* $Y^* = (Y^*_t; t \geq 0)$, *started at* $Y^*_0 = x$ *and such that*

$$\left(p^{-1} Y^{*,p}_{[\gamma_p t]}; t \geq 0\right) \xrightarrow[p\to\infty]{\text{(d)}} Y^* \tag{12}$$

weakly in the cadlag functions space $\mathbb{D}([0,\infty), \mathbb{R})$ *endowed with the Skorohod topology.*

In regard of Theorem 3.4 [17] due to Grimvall that concerns limits of GW-processes without immigration, the latter limit theorem is very natural. However it turns out to be new. To prove (ii) $\Longrightarrow$ (iii) we adapt an argument contained in the proof of Theorem 3.4 [17]; our main contribution is the proof of (i) $\Longrightarrow$ (ii).

In Section 4.2 we prove a limit theorem for the genealogy of a sequence of (sub)critical GWI-processes: let $(\mu_p; p \geq 1)$ be a sequence of offspring distributions such that $\bar{\mu}_p = \sum_{k\geq 0} k\mu_p(k) \leq 1$ and denote by $g^{(p)}$ the corresponding generating functions. Define recursively $g^{(p)}_n$ by $g^{(p)}_n = g^{(p)}_{n-1} \circ g^{(p)}$ with $g^{(p)}_0 = \mathrm{Id}$. Let $(r_p; p \geq 1)$ be a sequence of dispatching distributions and let τ_p be a GWI(μ_p, r_p)-tree. For any $n \geq 0$, we also denote by $Y^{*,p}_n$ the number of non-mutants at generation n in τ_p. Recall that $(\gamma_p; p \geq 1)$ stands for an increasing sequence of positive integers. We suppose that

$$\mu_p\left(\frac{\cdot - 1}{p}\right)^{*p\gamma_p} \xrightarrow[p\to\infty]{\text{(d)}} \mathbf{P}(X_1 \in \cdot) \quad \text{and} \quad r_p\left(\frac{\cdot}{p}, \frac{\cdot}{p}\right)^{*\gamma_p} \xrightarrow[p\to\infty]{\text{(d)}} \mathbf{P}(U_1 \in \cdot; V_1 \in \cdot) \tag{13}$$

weakly in resp. $\mathbb{R}$ and $\mathbb{R}^2$. Here X stands for a spectrally positive Lévy process whose exponent ψ satisfies (6) and (7), and (U, V) stands for a bivariate subordinator whose exponent Φ satisfies (9). We also make the additional assumption

$$\liminf_{p\to\infty} g^{(p)}_{[\delta\gamma_p]}(0)^p > 0 \tag{14}$$

which implies that extinction times of $GW(\mu_p)$-processes converge in a distribution in the γ_p^{-1} time-scale. (14) turns out to be a necessary condition in order to have a strong convergence of contour processes of rescaled Galton–Watson trees: see [11] Chapter 2 p. 54 for a precise discussion of this point.

Denote by $\overleftarrow{H}$ and $\overrightarrow{H}$ left and right height processes associated with a (ψ, Φ)-immigration Lévy tree as defined by (8) and denote by $\overleftarrow{L}$ and $\overrightarrow{L}$ their corresponding local times. The main result proved in Section 4.2 is the following.

Theorem 1.5. *Assume that (13) and (14) hold. For any* $p \geq 0$, *let* τ_p *be a* $GW(\mu_p)$*-tree. Then,*

$$\left(\left(\gamma_p^{-1}\overleftarrow{C}_{2p\gamma_p t}(\tau_p), \gamma_p^{-1}\overrightarrow{C}_{2p\gamma_p t}(\tau_p)\right)_{t\geq 0}; \left(p^{-1}Y^{*,p}_{[p\gamma_p a]}\right)_{a\geq 0}\right)$$
$$\xrightarrow[p\to\infty]{\text{(d)}} \left(\left(\overleftarrow{H}_t, \overrightarrow{H}_t\right)_{t\geq 0}; \left(\overleftarrow{L}^a_\infty + \overrightarrow{L}^a_\infty\right)_{a\geq 0}\right)$$

in distribution in $\mathbb{D}([0,\infty), \mathbb{R}^2) \times \mathbb{D}([0,\infty), \mathbb{R})$.

This result relies on combinatorial formulas stated in Section 2.2, on Theorem 1.4 and also on Theorem 2.3.1 [11] that guarantees a similar convergence for rescaled contour processes of sequences of Galton–Watson trees without immigration.

This paper is organized as follows: In Section 2.1, we set definitions and notations concerning discrete trees. In Section 2.2, we discuss various codings of sin-trees that are used in the proof of Theorem 1.5. In Section 3.1 we recall important properties of the height process that are needed to prove Proposition 1.1, Theorems 1.2 and 1.3 in Section 3.2; Section 3.3. Sections 4.1 and 4.2 are devoted to proofs of Theorems 1.4 and 1.5.

2. Sin-trees and sin-forests

2.1. Definitions and examples

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the set of the nonnegative integers, set $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ and denote the set of finite words written with positive integers by $\mathbb{U} := \{\varnothing\} \cup \bigcup_{n\geq 1} (\mathbb{N}^*)^n$. Let $u \in \mathbb{U}$ be the word $u_1 \dots u_n$, $u_i \in \mathbb{N}^*$. We denote the length of u by $|u|$: $|u| = n$. Let $v = v_1 \dots v_m \in \mathbb{U}$. Then the word uv stands for the concatenation of u and v: $uv = u_1 \dots u_n v_1 \dots v_m$. Observe that $\mathbb{U}$ is totally ordered by the *lexicographical order* denoted by $\leq$. A rooted ordered tree t is a subset of $\mathbb{U}$ satisfying the following conditions

(i) $\varnothing \in t$ and $\varnothing$ is called the *root* of t.
(ii) If $v \in t$ and if $v = uj$ for $j \in \mathbb{N}^*$, then, $u \in t$.
(iii) For every $u \in t$, there exists $k_u(t) \geq 0$ such that $uj \in t$ for every $1 \leq j \leq k_u(t)$.

We denote by $\mathbb{T}$ the set of ordered rooted trees. We define on $\mathbb{U}$ the *genealogical* order $\preccurlyeq$ by

$$\forall u, v \in \mathbb{U}, \quad u \preccurlyeq v \Longleftrightarrow \exists w \in \mathbb{U} : v = uw.$$

If $u \preccurlyeq v$, we say that u is an ancestor of v. If u is distinct from the root, it has an unique predecessor with respect to $\preccurlyeq$ who is called its parent and who is denoted by $\overleftarrow{u}$. We define the youngest common ancestor of u and v by the $\preccurlyeq$-maximal element $w \in \mathbb{U}$ such that $w \preccurlyeq u$ and $w \preccurlyeq v$ and we denote it by $u \wedge v$. We also define the distance between u and v by $\mathbf{d}(u, v) = |u| + |v| - 2|u \wedge v|$ and we use the notation $[\![u, v]\!]$ for the shortest path between u and v. Let $t \in \mathbb{T}$ and $u \in t$. We define the tree t shifted at u by $\theta_u(t) = \{v \in \mathbb{U} : uv \in t\}$ and we denote by $[t]_u$ the tree t cut at the node u: $[t]_u := \{u\} \cup \{v \in t : v \wedge u \neq u\}$. Observe that $[t]_u \in \mathbb{T}$. For any $u_1, \dots, u_k \in t$ we also set $[t]_{u_1,\dots,u_k} := [t]_{u_1} \cap \dots \cap [t]_{u_k}$ and

$$[t]_n = [t]_{\{u \in t:\, |u|=n\}} = \{u \in t : |u| \leq n\}, \quad n \geq 0.$$

Let us denote by $\mathcal{G}$ the σ-field on $\mathbb{T}$ generated by the sets $\{t \in \mathbb{T} : \ u \in t\}$, $u \in \mathbb{U}$. All random objects introduced in this paper are defined on an underlying probability space denoted by $(\Omega, \mathcal{F}, \mathbf{P})$. A random tree is then a $\mathcal{F}$-$\mathcal{G}$ measurable mapping $\tau : \Omega \longrightarrow \mathbb{T}$. We say that a sequence of random trees $(\tau_k;\ k \geq 0)$ converges in distribution to a random tree τ iff

$$\forall n \geq 0,\ \forall t \in \mathbb{T}, \quad \mathbf{P}\left([\tau_k]_n = t\right) \xrightarrow[k\to\infty]{} \mathbf{P}\left([\tau]_n = t\right)$$

and we denote it by $\tau_k \xrightarrow[k\to\infty]{\text{distr}} \tau$.

Let μ be a probability distribution on $\mathbb{N}$. We call *Galton–Watson tree* with offspring distribution μ (a GW(μ)-tree for short) any $\mathcal{F} - \mathcal{G}$ measurable random variable τ whose distribution is characterized by the following conditions:

(i) $\mathbf{P}(k_\varnothing(\tau) = i) = \mu(i),\ i \geq 0$.
(ii) For every $i \geq 1$ such that $\mu(i) \neq 0$, the shifted trees $\theta_1(\tau), \dots, \theta_i(\tau)$ under $\mathbf{P}(\cdot \mid k_\varnothing(\tau) = i)$ are independent copies of τ under $\mathbf{P}$.

Remark 2.1. Let $u_1, \ldots, u_k \in \mathbb{U}$ such that $u_i \wedge u_j \notin \{u_1, \ldots, u_k\}$, $1 \le i, j \le k$, and let τ be a GW(μ)-tree. Then, as a consequence of the definition of GW-trees, conditional on the event $\{u_1, \ldots, u_k \in \tau\}$, $\theta_{u_1}(\tau), \ldots, \theta_{u_k}(\tau)$ are i.i.d. GW(μ)-trees independent of $[\tau]_{u_1,\ldots,u_k}$.

We often consider a forest (i.e. a sequence of trees) instead of a single tree. More precisely, we define the forest f associated with the sequence of trees $(t_l\,; l \ge 1)$ by the set

$$f = \{(-1, \varnothing)\} \cup \bigcup_{l \ge 1} \{(l, u),\ u \in t_l\}$$

and we denote by $\mathbb{F}$ the set of forests. Vertex $(-1, \varnothing)$ is viewed as a fictive root situated at generation -1. Let $u' = (l, u) \in f$ with $l \ge 1$; the height of u' is defined by $|u'| := |u|$ and its ancestor is defined by $(l, \varnothing)$. For convenience, we denote it by $\varnothing_l := (l, \varnothing)$. As already specified, all ancestors $\varnothing_1, \varnothing_2, \ldots$ are descendants of $(-1, \varnothing)$ and are situated at generation 0. Most notations concerning trees extend to forests: The lexicographical order $\le$ is defined on f by taking first the individuals of t_1, next those of $t_2 \ldots$ etc and leaving $(-1, \varnothing)$ unordered. The genealogical order $\preccurlyeq$ on f is defined tree by tree in an obvious way. Let $v' \in f$. The youngest common ancestor of u' and v' is then defined as the $\preccurlyeq$-maximal element of w' such that $w' \preccurlyeq u'$ and $w' \preccurlyeq v'$ and we keep denoting it by $u' \wedge v'$. The number of children of u' is $k_{u'}(f) := k_u(t_l)$ and the forest f shifted at u' is defined as the tree $\theta_{u'}(f) := \theta_u(t_l)$. We also define $[f]_{u'}$ as the forest $\{u'\} \cup \{v' \in f : v' \wedge u' \ne u'\}$ and we extend in an obvious way notations $[f]_{u'_1,\ldots,u'_k}$ and $[f]_n$. For convenience of notation, we often identify f with the sequence $(t_l\,; l \ge 1)$. When $(t_l\,; l \ge 1) = (t_1, \ldots, t_k, \varnothing, \varnothing, \ldots)$, we say that f is a finite forest with k elements and we write $f = (t_1, \ldots, t_k)$.

We formally define the set of trees with a single infinite line of descent (called *sin-trees* for short) by

$$\mathbb{T}_{\text{sin}} = \{t \in \mathbb{T} : \forall n \ge 0,\ \#\{v \in t : |v| = n \text{ and } \#\theta_v(t) = \infty\} = 1\}.$$

Let $t \in \mathbb{T}_{\text{sin}}$. For any $n \ge 0$, we denote by $u_n^*(t)$ the unique individual u on the infinite line of descent (i.e. such that $\#\theta_u(t) = \infty$) situated at height n. Observe that $u_0^*(t) = \varnothing$. We use the notation $\ell_\infty(t) = \{u_n^*(t);\ \ n \ge 0\}$ for the infinite line of descent of t and we denote by $(l_n(t)\,; n \ge 1)$ the sequence of positive integers such that $u_n^*(t)$ is the word $l_1(t) \ldots l_n(t) \in \mathbb{U}$. We also introduce the set of *sin-forests* $\mathbb{F}_{\text{sin}}$ that is defined as the set of forests $f = (t_l\,; l \ge 1)$ such that all the trees t_l are finite except one sin-tree t_{l_0}. We extend to sin-forests notations u_n^*, and l_n by setting $l_n(f) = l_n(t_{l_0})$, $u_n^*(f) = (\,l_0, u_n^*(t_{l_0})\,)$ and $u_0^*(f) = \varnothing_{l_0}$.

We now precisely define Galton–Watson trees with immigration introduced in Section 1: Recall that a GWI tree is characterized by

- its *offspring distribution* μ on $\mathbb{N}$ that we suppose critical or subcritical: $\bar{\mu} = \sum_{k \ge 0} k\mu(k) \le 1$;
- its *dispatching distribution* r defined on the set $\{(k, j) \in \mathbb{N}^* \times \mathbb{N}^* : 1 \le j \le k\}$ that prescribes the distribution of the number of immigrants and their positions with respect to the infinite line of descent.

More precisely, τ is a GWI(μ, r)-tree if it satisfies the two following conditions:

(i) The sequence $S = ((k_{u_n^*(\tau)}(\tau), l_{n+1}(\tau));\, n \ge 0)$ is i.i.d. with distribution r.
(ii) Conditional on S, the trees $\{\theta_{u_n^*(\tau)i}\,(\tau),\quad n \in \mathbb{N},\ 1 \le i \le k_{u_n^*(\tau)}(\tau) \quad \text{with} \quad i \ne l_{n+1}(\tau)\}$ are mutually independent GW(μ)-trees.

We define a GWI(μ, r)-forest with $l \ge 1$ elements by the forest $\varphi = (\tau, \tau_1, \ldots, \tau_{l-1})$ where the τ_i's are i.i.d. GW(μ)-trees independent of the GWI(μ, r)-tree τ. It will be sometimes convenient to insert τ at random in the sequence $(\tau_1, \dots, \tau_{l-1})$ but unless otherwise specified we choose to put the random sin-tree first in a random sin-forest.

Example 2.1. *The size-biased GW-tree.* Recall from Section 1 that a GW(μ) size-biased tree is a GWI(μ, r)-tree with $r(k, j) = \mu(k)/\bar{\mu}$, $1 \leq j \leq k$. The term "size-biased" can be justified by the following elementary result. Let φ be a random forest corresponding to a sequence of l independent GW(μ)-trees and let $\varphi_\flat$ be a GWI(μ, r)-forest with l elements where r is taken as above and where the position of the unique random sin-tree in $\varphi_\flat$ is picked uniformly at random among l possible choices. Check that for any nonnegative measurable functional G on $\mathbb{F} \times \mathbb{U}$:

$$\mathbf{E}\left[\sum_{u\in\varphi} G\left([\varphi]_u, u\right)\right] = \sum_{n\geq 0} l\,\bar{\mu}^n \mathbf{E}\left[G\left([\varphi_\flat]_{u_n^*(\varphi_\flat)}, u_n^*(\varphi_\flat)\right)\right] \tag{15}$$

and in particular

$$\frac{\mathrm{d}\mathbf{P}([\varphi_\flat]_n \in \cdot)}{\mathrm{d}\mathbf{P}([\varphi]_n \in \cdot)} = \frac{Z_n(\varphi)}{l\bar{\mu}^n},$$

where $Z_n(\varphi) = \#\{u \in \varphi : |u| = n\}$, $n \geq 0$.

Example 2.2. *A two-types GW-tree.* Let ρ be a probability measure on $\mathbb{N} \times \mathbb{N}$. Consider a population process with two types (say type 1 and type 2) whose branching mechanism is described as follows: all the individuals in the tree have the same offspring distribution; namely, one individual has k children of type 1 and l children of type 2 with probability $\rho(k, l)$. We order the children putting first those with type 1 and next those with type 2. Assume that we start with one ancestor with type 1. If we ignore the types, the resulting family tree is a GW(μ)-tree where μ is given by

$$\mu(n) = \sum_{k+l=n} \rho(k, l).$$

We assume that μ is (sub)critical. For any $n \geq 1$, denote by A_n the event of a line of descent from generation n to the ancestor that only contains individuals with type 1. Then we can prove easily

$$\tau \text{ under } \mathbf{P}(\cdot \mid A_n) \xrightarrow[n\to\infty]{\text{distr}} \tau_\infty,$$

where τ_∞ stands for a GWI(μ, r)-tree where r is given by

$$r(k, l) = \frac{1}{m}\sum_{j=l}^{k} \rho(j, k-j) \quad \text{with } m = \sum_{k\geq 0} k\rho(k, \mathbb{N}).$$

Example 2.3. *Ascending particle on a GW-tree.* Let $(\pi_n;\ n \geq 0)$ be a sequence of probability measures on $\mathbb{N}$ such that $\pi_n(\{1, \dots n\}) = 1$. Let τ be a critical or subcritical GW(μ)-tree. Consider a particle climbing τ at random in the following way: it starts at the root $\varnothing$; suppose it is at vertex $u \in \tau$ at time n, then there are two cases: if $k_u(\tau) > 0$, then at time $n + 1$ the particle goes to $v = uj$ with probability $\pi_{k_u(\tau)}(j)$; if $k_u(\tau) = 0$, then the particle stays at u at time $n + 1$. The particle is thus stopped at a final position denoted by U. We can show that $[\tau]_n$ conditional on $\{|U| \geq n\}$ is distributed as $[\tau_\infty]_n$ where τ_∞ is a GWI(μ, r)-tree with

$$r(k, l) = \frac{\mu(k)}{1-\mu(0)}\pi_k(l), \quad 1 \leq l \leq k.$$

Consequently,

$$\tau \text{ under } P(\cdot \mid |U| \geq n) \xrightarrow[n\to\infty]{\text{distr}} \tau_\infty.$$

2.2. Codings of sin-trees

Let us first recall how to code a finite tree $t \in \mathbb{T}$. Let $u_0 = \varnothing < u_1 < \cdots < u_{\#t-1}$ be the vertices of t listed in lexicographical order. We define the *height process* of t by $H_n(t) = |u_n|$, $0 \leq n < \#t$. $H(t)$ clearly characterizes the tree t.

We also need to code t in a third way by a path $D(t) = (D_n(t); 0 \leq n \leq \#t)$ that is defined by $D_{n+1}(t) = D_n(t) + k_{u_n}(t) - 1$ and $D_0(t) = 0$. $D(t)$ is sometimes called the Lukaciewicz path associated with t. It is clear that we can reconstruct t from $D(t)$. Observe that the jumps of $D(t)$ are not smaller than -1. Moreover $D_n(t) \geq 0$ for any $0 \leq n < \#t$ and $D_{\#t}(t) = -1$. We recall from [25] without proof the following formula that allows to write the height process as a functional of $D(t)$:

$$H_n(t) = \#\left\{0 \leq j < n : D_j(t) = \inf_{j \leq k \leq n} D_k(t)\right\}, \quad 0 \leq n < \#t. \tag{16}$$

Remark 2.2. If τ is a critical or subcritical GW(μ)-tree, then it is clear from our definition that $D(\tau)$ is a random walk started at 0 that is stopped at -1 and whose jump distribution is given by $\rho(k) = \mu(k+1)$, $k \geq -1$. Thus (16) allows to write $H(\tau)$ as a functional of a random walk.

The previous definition of D and of the height process can be easily extended to a forest $f = (t_l\,; l \geq 1)$ of finite trees as follows: Since all trees t_l are finite, it is possible to list all the vertices of f but $(-1, \varnothing)$ in lexicographical order: $u_0 = \varnothing_1 < u_1 < \cdots$ by visiting first t_1, then $t_2 \ldots$ etc. We then simply define the height process of f by $H_n(f) = |u_n|$ and $D(f)$ by $D_{n+1}(f) = D_n(f) + k_{u_n}(f) - 1$ with $D_0(f) = 0$. Set $n_p = \#t_1 + \cdots + \#t_p$ and $n_0 = 0$ and observe that

$$H_{n_p+k}(f) = H_k(t_{p+1}) \quad \text{and} \quad D_{n_p+k}(f) = D_k(t_{p+1}) - p, \quad 0 \leq k < \#t_{p+1},\ p \geq 0.$$

We thus see that the height process of f is the concatenation of the height processes of trees composing f. Moreover the n-th visited vertex u_n is in t_p iff $p = 1 - \inf_{0\leq k\leq n} D_k(f)$. Then, it is easy to check that (16) remains true for every $n \geq 0$ when $H(t)$ and $D(t)$ are replaced by resp. $H(f)$ and $D(f)$.

Let us now explain how to code sin-trees. Let $t \in \mathbb{T}_{\text{sin}}$. A particle visiting t in lexicographical order never reaches the part of t at the right hand of the infinite line of descent. So we need two height processes or equivalently two contour processes to code t. More precisely, the left part of t is the set $\{u \in t : \exists v \in \ell_\infty(t) \text{ s.t. } u \leq v\}$. This set can be listed in a lexicographically increasing sequence of vertices denoted by $\varnothing = u_0 < u_1 < \cdots$ etc. We simply define the *left height process* of t by $\overleftarrow{H}_n(t) = |u_n|$, $n \geq 0$. $\overleftarrow{H}(t)$ completely codes the left part of t. To code the right part we consider the "mirror image" $t^\bullet$ of t. More precisely, let $v \in t$ be the word $c_1 c_2 \ldots c_n$. For any $j \leq n$, denote by $v_j := c_1 \ldots c_j$ the j-th ancestor of v with $v_0 = \varnothing$. Set $c_j^\bullet = k_{v_{j-1}}(t) - c_j + 1$ and $v^\bullet = c_1^\bullet \ldots c_n^\bullet$. We then define $t^\bullet$ as $\{v^\bullet,\ v \in t\}$ and we define the *right height process* of t as $\overrightarrow{H}(t) := \overleftarrow{H}(t^\bullet)$.

We next give another way to code a sin tree by two processes called left contour and right contour processes of the sin-tree t, that are denoted by resp. $\overleftarrow{C}(t)$ and $\overrightarrow{C}(t)$. Informally speaking, $\overleftarrow{C}(t)$ is the distance-from-the-root process of a particle starting at the root and moving clockwise on t viewed as a unit edge length graph embedded in the oriented half plane. We define $\overrightarrow{C}(t)$ as the contour process corresponding to the anti-clockwise journey. So we can also write $\overrightarrow{C}(t) = \overleftarrow{C}(t^\bullet)$. More precisely, $\overleftarrow{C}(t)$ (resp. $\overrightarrow{C}(t)$) can be recovered from $\overleftarrow{H}(t)$ (resp. $\overrightarrow{H}(t)$) through the following transform: Set $b_n = 2n - \overleftarrow{H}_n(t)$ for $n \geq 0$. Then observe that

$$C_s(t) = \begin{cases} \overleftarrow{H}_n(t) - s + b_n & \text{if } s \in [b_n, b_{n+1} - 1), \\ s - b_{n+1} + \overleftarrow{H}_{n+1}(t) & \text{if } s \in [b_{n+1} - 1, b_{n+1}]. \end{cases} \tag{17}$$

The contour process is close to the height process in the following sense: Define a mapping $q : \mathbb{R}_+ \longrightarrow \mathbb{Z}_+$ by setting $q(s) = n$ iff $s \in [b_n, b_{n+1})$. Check for every integer $m \geq 1$ that

$$\begin{aligned} \sup_{s\in[0,m]} |\overleftarrow{C}_s(t) - \overleftarrow{H}_{q(s)}(t)| &\leq \sup_{s\in[0,b_m]} |\overleftarrow{C}_s(t) - \overleftarrow{H}_{q(s)}(t)| \\ &\leq 1 + \sup_{n\leq m} |\overleftarrow{H}_{n+1}(t) - \overleftarrow{H}_n(t)|. \end{aligned} \tag{18}$$

Similarly, it follows from the definition of b_n that

$$\sup_{s\in[0,m]} \left|q(s) - \frac{s}{2}\right| \leq \sup_{s\in[0,b_m]} \left|q(s) - \frac{s}{2}\right| \leq \frac{1}{2} \sup_{n\leq m} \overleftarrow{H}_n(t) + 1. \tag{19}$$

We now give a decomposition of $\overleftarrow{H}(t)$ and $\overrightarrow{H}(t)$ along $\ell_\infty(t)$ that is well suited to GWI-trees and that is used in Section 3.2: Recall that $(u_n; n \geq 0)$ stands for the sequence of vertices of the left part of t listed in lexicographical order. Let us consider the set $\{u^*_{n-1}(t)i; 1 \leq i < l_n(t); n \geq 1\}$ of individuals at the left hand of $\ell_\infty(t)$ having a brother on $\ell_\infty(t)$. To avoid trivialities, we assume that this set is not empty and we denote by $v_1 < v_2 < \cdots$ etc. the (possibly finite) sequence of its elements listed in the lexicographical order.

The forest $f(t) = (\theta_{v_1}(t), \theta_{v_2}(t), \ldots)$ is then composed of the bushes rooted at the left hand of $\ell_\infty(t)$ listed in the lexicographical order of their roots. Define $L_n(t) := (l_1(t) - 1) + \cdots + (l_n(t) - 1)$ for any $n \geq 1$ and $L_0(t) = 0$; then, consider the p-th individual of $f(t)$ with respect to the lexicographical order on $f(t)$; it is easy to check that this individual is in a bush rooted in t at height

$$\alpha(p) = \inf\left\{k \geq 0 : \; L_k(t) \geq 1 - \inf_{j\leq p} D_j(f(t))\right\}.$$

Thus the corresponding individual in t is $u_{\mathbf{n}(p)}$ where $\mathbf{n}(p)$ is given by

$$\mathbf{n}(p) = p + \alpha(p) \tag{20}$$

(note that the first individual of $f(t)$ is labelled by 0). Conversely, let us consider u_n that is the n-th individual of the left part of t with respect to the lexicographical order on t. Set $\mathbf{p}(n) = \#\{k < n : \; u_k \notin \ell_\infty(t)\}$ that is the number of individuals coming before u_n and not belonging to $\ell_\infty(t)$. Then

$$\mathbf{p}(n) = \inf\{p \geq 0 : \mathbf{n}(p) \geq n\} \tag{21}$$

and the desired decomposition follows:

$$\overleftarrow{H}_n(t) = n - \mathbf{p}(n) + H_{\mathbf{p}(n)}(f(t)). \tag{22}$$

Since $n - \mathbf{p}(n) = \#\{0 \leq k < n : u_k \in \ell_\infty(t)\}$, we also get

$$\alpha(\mathbf{p}(n) - 1) \leq n - \mathbf{p}(n) \leq \alpha(\mathbf{p}(n)). \tag{23}$$

Observe that if $u_n \notin \ell_\infty(t)$, then we actually have $n - \mathbf{p}(n) = \alpha(\mathbf{p}(n))$. Proofs of these identities follow from simple counting arguments and they are left to the reader. Similar formulas hold for $\overrightarrow{H}(t)$ taking $t^\bullet$ instead of t in (20)–(23).

Remark 2.3. The latter decomposition is particularly useful when we consider a GWI(μ, r)-tree τ: In this case $(f(\tau), f(\tau^\bullet))$ is independent of $(L(\tau), L(\tau^\bullet))$, $f(\tau)$ and $f(\tau^\bullet)$ are mutually independent and $f(\tau)$ (resp. $f(\tau^\bullet)$) is a forest of i.i.d. GW(μ)-trees if for a $k \geq 2$ we have $r(k, 2) + \ldots + r(k, k) \neq 0$ (resp. $r(k, k-1) + \ldots + r(k, 1) \neq 0$), it is otherwise an empty forest. Moreover, the process $(L(\tau), L(\tau^\bullet))$ is a $\mathbb{N} \times \mathbb{N}$-valued random walk whose jump distribution is given by

$$\mathbf{P}\left(L_{n+1}(\tau) - L_n(\tau) = m; L_{n+1}(\tau^\bullet) - L_n(\tau^\bullet) = m'\right) = r(m + m' + 1, m + 1).$$

3. Continuum random sin-trees

3.1. The continuous time height process

In this section we recall from [25] the definition of the analogue in continuous time of the discrete height process defined in Section 2.2. We also recall from [11] several related results used in the next sections.

To define the continuous-time height process, we use an analogue of (16) where the role of the random walk is played by a spectrally positive Lévy process $X = (X_t;\ t \geq 0)$. The (sub)criticality of μ corresponds to the fact that X does not drift to $+\infty$. We also assume that X has a path of infinite variation (in the finite variation case, the height process is basically a discrete process and so is the underlying tree: see [25,27] for a discussion with applications to queuing processes). As already mentionned in the introduction, this happens if the exponent ψ of X satisfies conditions (6) and (7). By analogy with (16), the height process $H = (H_t;\ t \geq 0)$ associated with X is defined in such a way that for every $t \geq 0 H_t$ measures the size of the set:

$$\left\{ s \in [0, t] : X_{s-} = \inf_{s \leq r \leq t} X_r \right\}. \tag{24}$$

To make this precise, we use a time-reversal argument: For any $t > 0$, we define the Lévy process reversed at time t by

$$\widehat{X}^t_s = X_t - X_{(t-s)-}, \quad 0 \leq s \leq t$$

(with the convention $X_{0-} = 0$). Then, $\widehat{X}^t$ is distributed as X up to time t. Let us set for any $s \geq 0$,

$$S_s = \sup_{r \leq s} X_r \quad \text{and} \quad \widehat{S}^t_s = \sup_{r \leq s} \widehat{X}^t_r.$$

Then, the set (24) is the image of

$$\{s \in [0,t] : \widehat{S}^t_s = \widehat{X}^t_s\}$$

under the time reversal operation $s \to t-s$. Recall that under our assumptions $S-X$ is a strong Markov process for which 0 is a regular value. So, we can consider its local time process at 0 that is denoted by $L(X)$. We define the *height process* by

$$H_t = L_t(\widehat{X}^t), \quad t \geq 0. \tag{25}$$

To complete the definition, we still need to specify the normalization of the local time: let us introduce the right-continuous inverse of $L(X)$:

$$L^{-1}_t = \inf\{s \geq 0 : L_s(X) > t\}$$

(with the convention that $\inf \emptyset = \infty$). Define K_t by $X_{L^{-1}_t}$ if $t < L_\infty(X)$ and by ∞ otherwise. A classical result of fluctuation theory (see [5,6]) asserts that $(K_t;\ t \geq 0)$ is a subordinator whose Laplace exponent is given by

$$\mathbf{E}[\exp(-\lambda K_t)] = \exp(-ct\psi(\lambda)/\lambda), \quad t, \lambda \geq 0.$$

Here, c is a positive constant that only depends on the normalization of $L(X)$. We fix the normalization so that $c = 1$. When $\beta > 0$, standard results on subordinators imply for any $t \geq 0$,

$$H_t = \frac{1}{\beta}\mathrm{m}\left(\{\widehat{S}^t_s; 0 \leq s \leq t\}\right),$$

where m stands for the Lebesgue measure on the real line. In particular when X is a Brownian motion, we see that H is distributed as a reflected Brownian motion.

Let us briefly recall the "Ray–Knight theorem" for H (Theorem 1.4.2 [25] and Theorem 1.4.1 [11]), that can be viewed as a generalization of famous results about linear Brownian motion. For any $a, t \geq 0$, we introduce the local time L^a_t of H at time t and at level a that can be defined via the following approximation

$$\lim_{\epsilon \to 0} \mathbf{E}\left[\sup_{0 \leq s \leq t}\left|\frac{1}{\epsilon}\int_0^s \mathrm{d}r\,\mathbf{1}_{\{a < H_r \leq a+\epsilon\}} - L^a_s\right|\right] = 0 \tag{26}$$

(see Proposition 1.3.3 [11] for details). Next, set for any $r \geq 0$: $T_r = \inf\{s \geq 0 : X_s = -r\}$ and $Y_a = L^a_{T_r}$, $a \geq 0$. Then, Theorem 1.4.1 [11] asserts that $(Y_a; a \geq 0)$ is a CSBP(ψ) started at r.

Although the height process is in general not Markovian, we can still develop an excursion theory of H away from 0: Recall notation $I_t = \inf_{s \leq t} X_s$. Observe that for any $t \geq 0$, H_t only depends on the values taken by $X - I$ on the excursion interval that straddles t. Under our assumptions, $X - I$ is a strong Markov process for which 0 is a regular value so that $-I$ can be chosen as the local time of $X - I$ at level 0. We denote by N the corresponding excursion measure. Let (g_i, d_i), $i \in \mathcal{I}$ be the excursion intervals of $X - I$ above 0. We can check that $\mathbf{P}$-a.s.

$$\bigcup_{i \in \mathcal{I}}(g_i, d_i) = \{s \geq 0 : X_s - I_s > 0\} = \{s \geq 0 : H_s > 0\}.$$

Denote by $h_i(s) = H_{g_i+s}$, $0 \leq s \leq \zeta_i = d_i - g_i$, $i \in \mathcal{I}$ the excursions away from 0. Then, each H_i can be written as a functional of the corresponding excursion of $X - I$ away from 0 on

(g_i, d_i). Consequently, the point measure

$$\mathcal{M}(\mathrm{d}r\mathrm{d}\omega) = \sum_{i\in\mathcal{I}} \delta_{(-I_{g_i}, h_i)}(\mathrm{d}r\mathrm{d}\omega) \tag{27}$$

is a Poisson point measure with intensity $dr\, N(\mathrm{d}\omega)$. Note that in the Brownian case, N is the Ito excursion measure of positive excursions of the reflected linear Brownian motion.

From now on until the end of this section we argue under N. Let ζ denote the duration of the excursion. Local time processes of the height process $(L^a_s; 0 \le s \le \zeta)$, $a \ge 0$ can be defined under N through the same approximation as before, namely

$$\lim_{\epsilon\to 0} \sup_{a\ge 0} N\left(\mathbf{1}_V \sup_{0\le s\le t\wedge\zeta} \left| \frac{1}{\epsilon}\int_0^s \mathrm{d}r \mathbf{1}_{\{a<H_r\le a+\epsilon\}} - L^a_s \right|\right) = 0, \quad t\ge 0, \tag{28}$$

where V is any measurable subset of excursions such that $N(V) < \infty$. The above mentioned Ray–Knight Theorem for H implies

$$N\left(1-\exp(-\lambda L^a_\zeta)\right) = u(a,\lambda), \quad a,\lambda \ge 0, \tag{29}$$

where we recall that u is defined by (4). Set $v(a) = \lim_{\lambda\to\infty} u(a,\lambda)$ to be a positive and finite quantity by (7) satisfying $a = \int_{v(a)}^\infty \mathrm{d}u/\psi(u)$. By a simple argument discussed in Corollary 1.4.2 [11], we get

$$N\left(L^a_\zeta > 0\right) = N\left(\sup_{s\le\zeta} H_s > a\right) = v(a). \tag{30}$$

Let $a > 0$ and set $N_{(a)} = N(\,\cdot \mid \sup H > a)$ that is a well-defined probability measure. The Lévy tree coded by H under $N_{(a)}$ enjoys a branching property that can be stated as follows: set

$$\widetilde{\tau}^a_t = \inf\left\{ s\ge 0 : \int_0^s \mathrm{d}r \mathbf{1}_{\{H_r\le a\}} > t\right\}.$$

We denote by $\mathcal{H}_a$ the σ-field generated by $(X_{\widetilde{\tau}^a_t},\ t\ge 0)$ and by the class of the N-negligible sets of $\mathcal{F}$. We introduce the excursion intervals of H above a:

$$\bigcup_{i\in\mathcal{I}(a)} (g_i, d_i) = \{s\ge 0 : H_s > a\}$$

and we recall from Proposition 1.3.1 [11] the following result.

Proposition 3.1 (*Proposition 1.3.1 [11]*). *The process $(L^a_s,\ s\ge 0)$ is measurable with respect to $\mathcal{H}_a$. Then, under $N_{(a)}$ and conditional on $\mathcal{H}_a$ the point measure*

$$\mathcal{M}_a(\mathrm{d}l\mathrm{d}\omega) = \sum_{i\in\mathcal{I}(a)} \delta_{(L^a_{g_j}, H_{(g_i+\cdot)\wedge d_i}-a)}(\mathrm{d}l\,\mathrm{d}\omega) \tag{31}$$

is independent of $\mathcal{H}_a$. Moreover it is a Poisson point measure with intensity

$$\mathbf{1}_{[0,L^a_\zeta]}(l)\,\mathrm{d}l\ N(\mathrm{d}\omega).$$

Remark 3.1. Proposition 1.3.1 [11] is actually stated under $\mathbf{P}$ for the so-called *exploration process* $(\rho_t; t\ge 0)$ that is a Markov process taking its values in the space of the finite measures of $[0, \infty)$ and that is related to the height process in the following way: **P**-a.s. for any $t \geq 0$, the topological support of ρ_t is the compact interval $[0, H_t]$. Thus it easy to deduce from Proposition 1.3.1 [11] a statement for the height process under **P** and our statement follows from the fact that $N_{(a)}$ is the distribution under **P** of the first excursion of H away from 0 that reaches level a.

3.2. *Proof of Theorem 1.2*

Recall from Section 1 the definition of the left and right height processes $\overleftarrow{H}$ and $\overrightarrow{H}$ of a (ψ, Φ)-immigration Lévy tree. We first prove Proposition 1.1.

Proof of Proposition 1.1. We only need to consider $\overleftarrow{H}$. Recall the notation (g_i, d_i), $i \in \mathcal{I}$ for excursion intervals of H away from 0. Set for any $a \geq 0$ and any $i \in \mathcal{I}$

$$\zeta_i = d_i - g_i, \qquad h_i = H_{g_i \wedge (\cdot + d_i)} \quad \text{and} \quad a_i = (a - U^{-1}_{-I_{g_i}})_+,$$

where for any $x \in \mathbb{R}$ we have set $(x)_+ = x \vee 0$. Recall notation $T_r = \inf\{s \geq 0 : X_s = -r\}$, $r \geq 0$. Define for any $s \geq 0$,

$$\overleftarrow{L}^a_s = (-I_s - U_{a-})_+ \wedge \Delta U_a + \sum_{i \in \mathcal{I}} L^{a_i}_{s \wedge d_i} - L^{a_i}_{s \wedge g_i}.$$

It is clear from the definition that $(s, a) \to \overleftarrow{L}^a_s$ is jointly measurable. Since the mappings

$$s \longrightarrow (-I_s - U_{a-})_+ \wedge \Delta U_a \quad \text{and} \quad s \to L^{a_i}_{s \wedge d_i} - L^{a_i}_{s \wedge g_i}$$

are non-decreasing and continuous, then $s \to \overleftarrow{L}^a_s$ is a non-decreasing mapping and it is continuous on every open interval (g_i, d_i), $i \in \mathcal{I}$ and also on $[T_{U_{a-}}, \infty)$. Let $s \in [0, T_{U_{a-}})$. Suppose that s does not belong to an excursion interval of H away from 0, that is $H_s = 0$. Then, $\overleftarrow{H}_s = U^{-1}_{-I_s} < a$ and it easy to check that the continuity of H implies the existence of a non-empty open interval centered around s on which $\overleftarrow{L}^a$ is a constant mapping. These observations imply (i).

Point (iii) follows from point (ii) by standard arguments. It remains to prove (ii): By (26), we see that for any $i \in \mathcal{I}$, $(L^a_{(s+g_i)\wedge d_i} - L^a_{g_i}; a, s \geq 0)$ only depends on excursion h_i. So it makes sense to denote it by $(L^a_s(h_i); a, s \geq 0)$. Since $\mathbf{P}(U_{a-} = U_a) = 1$ and by the definition of $\overleftarrow{H}$ we a.s. get for any $T \geq 0$,

$$\sup_{0\leq t\leq T} \left| \epsilon^{-1} \int_0^t \mathrm{d}s \mathbf{1}_{\{a < \overleftarrow{H}_s \leq a+\epsilon\}} - \overleftarrow{L}^a_t \right| \leq \sum_{i \in \mathcal{I}} \mathbf{1}_{[0,U_a]}(-I_{g_i})$$
$$\times \sup_{t \in [0, T\wedge\zeta_i]} \left| \epsilon^{-1} \int_0^t \mathrm{d}s \mathbf{1}_{\{a_i < h_i(s) \leq a_i+\epsilon\}} - L^{a_i}_t(h_i) \right|.$$

By conditionning on U, we get a.s.

$$\mathbf{E}\left[\sup_{0\leq t\leq T} \left| \epsilon^{-1} \int_0^t \mathrm{d}s \mathbf{1}_{\{a < \overleftarrow{H}_s \leq a+\epsilon\}} - \overleftarrow{L}^a_t \right| \,\middle|\, U \right] \leq \int_0^{U_a} \mathrm{d}x \, \mathrm{n}_\epsilon(a - U^{-1}_x), \tag{32}$$

where we have set for any $y \geq 0$,

$$\mathrm{n}_\epsilon(y) = N\left(\mathbf{1}_{V(y)} \sup_{0\leq s\leq T\wedge\zeta} \left| \frac{1}{\epsilon} \int_0^s \mathrm{d}r \mathbf{1}_{\{y < H_r \leq y+\epsilon\}} - L^y_s \right| \right),$$

with $V(y) = \{\sup H > y\}$. By (30), $N(V(y)) = v(y) < \infty$ so (28) applies and we get $\mathrm{n}_\epsilon(y) \to 0$ when ϵ goes to 0, for any fixed $y \geq 0$. Moreover,

$$\mathrm{n}_\epsilon(y) \leq N(L_\zeta^y) + N\left(\epsilon^{-1}\int_0^\zeta \mathrm{d}s\mathbf{1}_{\{y<H_s\leq y+\epsilon\}}\right) = N(L_\zeta^y) + \epsilon^{-1}\int_y^{y+\epsilon} \mathrm{d}a N(L_\zeta^a)$$

by (28) once again. Then, use (4) and (29) to get

$$N(L_\zeta^a) = \frac{\partial}{\partial\lambda}N\left(1-\mathrm{e}^{-\lambda L_\zeta^a}\right)_{|\lambda=0} = \frac{\partial}{\partial\lambda}u(a,\lambda)_{|\lambda=0} = \mathrm{e}^{-\alpha a} \leq 1.$$

Thus, $\mathrm{n}_\epsilon(y) \leq 2$ and $\int_0^{U_a}\mathrm{d}x\mathrm{n}_\epsilon(a - U_x^{-1})$ tends a.s. to 0 when ϵ goes to 0 by dominated convergence. (ii) follows from (32) by an easy argument. ■

Proof of Theorem 1.2. Since a.s. (U, V) has no fixed discontinuity, we can write for any $a \geq 0$ a.s.

$$\overleftarrow{L}_\infty^a + \overrightarrow{L}_\infty^a = \sum_{i\in\mathcal{I}} L_{\zeta_i}^{a_i}(h_i) + \sum_{j\in\mathcal{I}'} L_{\zeta_j'}^{a_j'}(h_j'), \tag{33}$$

with an obvious notation for h_j', a_j' and ζ_j', $j \in \mathcal{I}'$. Fix $0 \leq b_1 < \cdots < b_n$. Deduce from (33)

$$\mathbf{E}\left[\exp\left(-\lambda_1(\overleftarrow{L}_\infty^{b_1} + \overrightarrow{L}_\infty^{b_1}) - \cdots - \lambda_n(\overleftarrow{L}_\infty^{b_n} + \overrightarrow{L}_\infty^{b_n})\right)\right] \tag{34}$$

$$= \mathbf{E}\left[\exp\left(-\int_0^{U_{b_n}}\mathrm{d}x\,\omega(U_x^{-1}) - \int_0^{V_{b_n}}\mathrm{d}x\,\omega(V_x^{-1})\right)\right], \tag{35}$$

where we have set for any $s > 0$:

$$\omega(s) = N\left(1-\exp\left(-\lambda_1 L_\zeta^{(b_1-s)_+} - \cdots - \lambda_n L_\zeta^{(b_n-s)_+}\right)\right).$$

Let $(Y_a;\ a \geq 0)$ denote a CSBP(ψ) started at $Y_0 = 1$. The Ray–Knight property of the local times of H then implies

$$\omega(s) = -\log\mathbf{E}\left[\exp\left(-\lambda_1 Y_{(b_1-s)_+} - \cdots - \lambda_n Y_{(b_n-s)_+}\right)\right]. \tag{36}$$

Now use the Lévy–Ito decomposition of (U, V) to get a.s.

$$\int_0^{U_{b_n}}\mathrm{d}x\omega(U_x^{-1}) + \int_0^{V_{b_n}}\mathrm{d}x\omega(V_x^{-1})$$
$$= (d+d')\int_0^{b_n}\omega(s)\mathrm{d}s + \sum_{0\leq s\leq b_n}(\Delta U_s + \Delta V_s)\omega(s).$$

Recall that $\varphi(\lambda) = \Phi(\lambda,\lambda)$, $\lambda \geq 0$ and deduce from the previous identity:

$$\mathbf{E}\left[\exp\left(-\int_0^{U_{b_n}}\mathrm{d}x\,\omega(U_x^{-1}) - \int_0^{V_{b_n}}\mathrm{d}x\,\omega(V_x^{-1})\right)\right] = \exp\left(-\int_0^{b_n}\varphi(\omega(s))\mathrm{d}s\right).$$

Denote by $(Y_a^*;\ a \geq 0)$ a CSBPI(ψ,φ) started at $Y_0^* = 0$. An elementary computation (left to the reader) shows that

$$\mathbf{E}\left[\exp\left(-\lambda_1 Y_{b_1}^* - \cdots - \lambda_n Y_{b_n}^*\right)\right] = \exp\left(-\int_0^{b_n}\varphi(\omega(s))\mathrm{d}s\right),$$

which completes the proof of Theorem 1.2 by (34) and (36). ■

Remark 3.2. Let us explain how Theorem 1.2 extends to a CSBPI(ψ,φ) started at an arbitrary state $r \geq 0$. Set $\overleftarrow{H}^r_t = H_t + U^{-1}_{(-I_t - r)_+}$. Thus, $\overleftarrow{H}^0 = \overleftarrow{H}$. Observe that $\overleftarrow{H}^r$ coincides with H up to time $T_r = \inf\{s \geq 0 : \ X_s = -r\}$. Then, use the Markov property at time T_r to show that $\overleftarrow{H}^r_{T_r + \cdot}$ is independent of $\overleftarrow{H}^r_{\cdot \wedge T_r}$ and distributed as $\overleftarrow{H}^0$. Proposition 1.1 and (26) make possible to define a local time process for $\overleftarrow{H}^r$ denoted by $(\overleftarrow{L}^{r,a}_s ; \ a, s \geq 0)$ that satisfies properties (i), (ii) and (iii) of Proposition 1.1. Moreover, the previous observations imply that

$$(\overleftarrow{L}^{r,a}_{T_r} ; \ a \geq 0) \quad \text{and} \quad (\overleftarrow{L}^{r,a}_\infty - \overleftarrow{L}^{r,a}_{T_r} ; \ a \geq 0)$$

are two independent processes: the first one is distributed as a CSBP(ψ) started at r and the the second one is a CSBPI(ψ, φ) started at 0. Then deduce from (4) and (3) that the sum of these two processes is distributed as a CSBPI(ψ, φ) started at r.

3.3. *Proof of Theorem 1.3*

In this section we discuss the ψ-size-biased Lévy tree case, namely

$$\Phi(p, q) = \frac{\psi^*(p) - \psi^*(q)}{p - q}$$

where we have set $\psi^*(\lambda) = \psi(\lambda) - \alpha$. Let us introduce the last time under level a for the left and the right height processes:

$$\overleftarrow{\sigma}_a = \sup\{s \geq 0 : \overleftarrow{H}_s \leq a\} \quad \text{and} \quad \overrightarrow{\sigma}_a = \sup\{s \geq 0 : \overrightarrow{H}_s \leq a\}.$$

One important argument in the proof of Theorem 1.3 is the following lemma.

Lemma 3.2. *Assume that (7) holds. Then, for any positive measurable function F and G,*

$$N\left(\int_0^\zeta \mathrm{d}s F\left(H_{\cdot\wedge s}\right) G\left(H_{(\zeta-\cdot)\wedge(\zeta-s)}\right)\right) = \int_0^\infty \mathrm{d}a \mathrm{e}^{-\alpha a} \mathbf{E}\left[F(\overleftarrow{H}_{\cdot\wedge\overleftarrow{\sigma}_a}) G(\overrightarrow{H}_{\cdot\wedge\overrightarrow{\sigma}_a})\right],$$

and for any $a > 0$,

$$N\left(\int_0^\zeta \mathrm{d}L^a_s F\left(H_{\cdot\wedge s}\right) G\left(H_{(\zeta-\cdot)\wedge(\zeta-s)}\right)\right) = \mathrm{e}^{-\alpha a} \mathbf{E}\left[F(\overleftarrow{H}_{\cdot\wedge\overleftarrow{\sigma}_a}) G(\overrightarrow{H}_{\cdot\wedge\overrightarrow{\sigma}_a})\right].$$

Proof. The second point of the lemma is an easy consequence of the first one and of (28). Thus, we only have to prove the first point. To that end, we introduce M_f the space of all finite measures on $[0, \infty)$. If $\mu \in M_f$, we denote by $H(\mu) \in [0, \infty]$ the supremum of the (topological) support of μ. We also introduce a "killing operator" on measures defined as follows. For every $x \geq 0$, $k_x \mu$ is the element of M_f such that $k_x \mu([0, t]) = \mu([0, t]) \wedge (\mu([0, \infty)) - x)_+$ for every $t \geq 0$. Let M_f^* stand for the set of all measures $\mu \in M_f$ such that $H(\mu) < \infty$ and the topological support of μ is $[0, H(\mu)]$. If $\mu \in M_f^*$, we denote by Q_μ the law under $\mathbf{P}$ of the process H^μ defined by

$$H^\mu_t = H(k_{-I_t}\mu) + H_t, \quad \text{if } t \leq T_{\langle \mu, 1\rangle},$$
$$H^\mu_t = 0, \quad \text{if } t > T_{\langle \mu, 1\rangle},$$

where $T_{\langle \mu, 1\rangle} = \inf\{t \geq 0 : X_t = -\langle \mu, 1\rangle\}$. Our assumption $\mu \in M_f^*$ guarantees that H^μ has continuous sample paths, and we can therefore view Q_μ as a probability measure on the space

$C_+([0,\infty))$ of nonnegative continuous functions on $[0,\infty)$. For every $a \geq 0$, we let $\mathbb{M}_a$ be the probability measure on $(M_f^*)^2$ that is the distribution of $(\mathbf{1}_{[0,a]}(t)\,\mathrm{d}U_t, \mathbf{1}_{[0,a]}(t)\,\mathrm{d}V_t)$.

The main argument of the proof is the key-Lemma 3.4 [11] that asserts that for any nonnegative measurables functions F and G on $C_+([0,\infty))$,

$$N\left(\int_0^\zeta \mathrm{d}s\, F\left(H_{(s-\cdot)_+}\right) G\left(H_{(s+\cdot)\wedge\zeta}\right)\right) = \int_0^\infty \mathrm{d}a\, \mathrm{e}^{-\alpha a} \int \mathbb{M}_a(\mathrm{d}\mu \mathrm{d}\nu) \int Q_\mu(\mathrm{d}h) Q_\nu(\mathrm{d}h') F(h) G(h'). \tag{37}$$

We use (37) to complete the proof as follows: First observe that (37) implies that the height process is reversible under N, namely

$$(H_s; 0 \leq s \leq \zeta) \overset{\text{(law)}}{=} (H_{\zeta-s}; 0 \leq s \leq \zeta) \text{ under } N.$$

Then fix $r \geq 0$. By reversing one-by-one the excursions of H away from 0 on $[0, T_r]$, we get

$$(r + I_{(T_r-\cdot)_+},\ H_{(T_r-\cdot)_+}) \overset{\text{(law)}}{=} (-I_{\cdot\wedge T_r},\ H_{\cdot\wedge T_r}). \tag{38}$$

Next, fix $a > 0$ and set $\mu = \mathbf{1}_{[0,a]}(t)\, dU_t$. Note that for any x,

$$H(k_x\mu) = a - U_x^{-1}. \tag{39}$$

Deduce from (39), (38) and the definition of the left height process that

$$\overleftarrow{H}_{\cdot\wedge\overleftarrow{\sigma}_a} \overset{\text{(law)}}{=} H^\mu_{(T_{\langle\mu,1\rangle}-\cdot)_+}.$$

A similar identity holds for the right height process and the lemma follows from (37). ■

Remark 3.3. This lemma can be viewed as the continuous counterpart of identity (15).

Remark 3.4. In the Brownian case $\psi(\lambda) = \lambda^2/2$, left and right height processes are two independent three-dimensional Bessel processes and the lemma is a well-known identity due to Bismut [8] (See [9] for a generalization to spectrally Lévy processes and [10] to general Lévy processes).

Proof of Theorem 1.3. Let $b > 0$. For any ω in $\mathbb{D}([0,\infty),\mathbb{R})$ we introduce $\tau_b(\omega) = \inf\{s \geq 0 : \omega(s) > b\}$ (with the usual convention $\inf\emptyset = \infty$). To simplify notations we set

$$\widehat{H} = H_{(\zeta-\cdot)_+},\ \tau_b = \tau_b(H) \quad \text{and} \quad \widehat{\tau}_b = \tau_b(\widehat{H}).$$

We only have to prove the following convergence for any bounded measurable function F,

$$\lim_{a\to\infty} N_{(a)}\left(F(H_{\cdot\wedge\tau_b}, \widehat{H}_{\cdot\wedge\widehat{\tau}_b})\right) = \mathbf{E}\left[F(\overleftarrow{H}_{\cdot\wedge\overleftarrow{\tau}_b}, \overrightarrow{H}_{\cdot\wedge\overrightarrow{\tau}_b})\right], \quad b > 0, \tag{40}$$

(with an evident notation for $\overleftarrow{\tau}_b$ and $\overrightarrow{\tau}_b$) since it implies for any $t > 0$

$$\lim_{b\to\infty}\lim_{a\to\infty} N_{(a)}(\tau_b, \widehat{\tau}_b \leq t) = \lim_{b\to\infty} \mathbf{P}\left(\overleftarrow{\tau}_b, \overrightarrow{\tau}_b \leq t\right) = 0.$$

Let us prove (40): deduce from (28) that N-a.e. the topological support of $\mathrm{d}L^b_\cdot$ is included in $\subset [\tau_b, \zeta - \widehat{\tau}_b]$. Thus, by Lemma 3.2

$$N\left(L^b_\zeta F(H_{\cdot\wedge\tau_b}, \widehat{H}_{\cdot\wedge\widehat{\tau}_b})\right) = \mathrm{e}^{-\alpha b}\mathbf{E}\left[F(\overleftarrow{H}_{\cdot\wedge\overleftarrow{\tau}_b}, \overrightarrow{H}_{\cdot\wedge\overrightarrow{\tau}_b})\right]. \tag{41}$$

Recall from Section 3.1 the notation (g_j, d_j), $j \in \mathcal{I}(b)$ for the excursion intervals of H above level b. For any $a > b$, we set

$$Z_b^a = \# \left\{ j \in \mathcal{I}(b) : \sup_{s \in (g_j, d_j)} H_s > a \right\}$$

that is the number of excursions above level b reaching level $a - b$. Deduce from Proposition 3.1 that conditional on $\mathcal{H}_b$ under $N_{(b)}$, the random variable Z_b^a is independent of $\mathcal{H}_b$ and distributed as a Poisson random variable with parameter $L_\zeta^b N(\sup H > a - b) = L_\zeta^b v(a - b)$. Then use (41) and the obvious inclusion $\{\sup H > a\} \subset \{\sup H > b\}$ to get

$$N_{(a)} \left(Z_b^a F(H_{\cdot \wedge \tau_b}, \widehat{H}_{\cdot \wedge \widehat{\tau}_b}) \right) = \frac{v(a-b)}{v(a)} \mathrm{e}^{-\alpha b} \mathbf{E} \left[F(\overleftarrow{H}_{\cdot \wedge \overleftarrow{\tau}_b}, \overrightarrow{H}_{\cdot \wedge \overrightarrow{\tau}_b}) \right]. \tag{42}$$

Let C be a bounding constant for F. Then,

$$\left| N_{(a)} \left(Z_b^a F(H_{\cdot \wedge \tau_b}, \widehat{H}_{\cdot \wedge \widehat{\tau}_b}) \right) - N_{(a)} \left(F(H_{\cdot \wedge \tau_b}, \widehat{H}_{\cdot \wedge \widehat{\tau}_b}) \right) \right| \leq C N_{(a)} \left(Z_b^a ;\ Z_b^a \geq 2 \right). \tag{43}$$

Now, observe that

$$\begin{aligned} N_{(a)} \left(Z_b^a ;\ Z_b^a \geq 2 \right) &= \frac{v(a-b)}{v(a)} N \left(L_\zeta^b (1 - \exp(-L_\zeta^b v(a-b))) \right) \\ &= \frac{v(a-b)}{v(a)} \left(\frac{\partial}{\partial \lambda} u(b, 0) - \frac{\partial}{\partial \lambda} u(b, v(a-b)) \right). \end{aligned}$$

Since $\lim_{a \to \infty} v(a - b) = 0$ and $v(a) = u(b, v(a - b))$, we get

$$\lim_{a \to \infty} v(a-b)/v(a) = \left(\frac{\partial}{\partial \lambda} u(b, 0) \right)^{-1} = \mathrm{e}^{\alpha b}$$

and

$$\lim_{a \to \infty} \frac{\partial}{\partial \lambda} u(b, v(a-b)) = \frac{\partial}{\partial \lambda} u(b, 0).$$

Thus, $\lim_{a \to \infty} N_{(a)} \left(Z_b^a ;\ Z_b^a \geq 2 \right) = 0$ and (40) follows from the latter limits combined with (41)–(43). ■

4. Limit theorems

4.1. Proof of Theorem 1.4

Recall the notations of Section 1: Let $(\mu_p ; p \geq 1)$ and $(\nu_p ; p \geq 1)$ be *any* sequences of probability measures on $\mathbb{N}$. In particular, we no longer assume that the μ_p's are (sub)critical. Let $(\gamma_p ; p \geq 1)$ be an increasing sequence of positive integers. Denote by $g^{(p)}$ and $f^{(p)}$ the generating functions of resp. μ_p and ν_p. Let $x \in [0, \infty)$. Recall that for any $p \geq 1$, $(Y_n^{*,p} ; n \geq 0)$ stands for a GWI(μ_p, ν_p)-process started at $Y_0^{*,p} = [px]$. We also need to introduce for any $p \geq 1$ a random walk $(W_n^p ; n \geq 0)$ independent of the $Y^{*,p}$'s, started at 0 and whose jumps distribution is ν_p. We denote by $(Y_n^p ; n \geq 0)$, $p \geq 1$ a sequence of GW(μ_p)-processes started at $Y_0^p = p$.

One important ingredient of the proof is Theorem 3.4 [17] due to Grimvall that is the exact analogue of Theorem 1.4 without immigration. For convenience of notation we re-state it as a lemma.

Lemma 4.1 (*Theorem 3.4 [17]*). *The three following assertions are equivalent*

(a) *For any $t \geq 0$,*

$$p^{-1} Y^p_{[\gamma_p t]} \xrightarrow[p\to\infty]{(\mathrm{d})} Z_t \tag{44}$$

where the process $(Z_t; t \geq 0)$ is a stochastically continuous process such that

$$\forall t > 0, \quad \mathbf{P}(Z_t > 0) > 0 \quad \textit{and} \quad \mathbf{P}(Z_t < \infty) = 1.$$

(b) *There exists a spectrally positive Lévy process $X = (X_t; t \geq 0)$ with exponent ψ satisfying (5) such that the following convergence*

$$\mu_p \left(\frac{\cdot - 1}{p} \right)^{*p\gamma_p} \xrightarrow[p\to\infty]{(\mathrm{d})} \mathbf{P}(X_1 \in \cdot) \tag{45}$$

holds weakly in $\mathbb{R}$.

(c) *There exists a conservative stochastically continuous CSBP(ψ) denoted by $Y = (Y_t; t \geq 0)$ started at $Y_0 = 1$ such that the convergence*

$$\left(p^{-1} Y^p_{[\gamma_p t]}; t \geq 0 \right) \xrightarrow[p\to\infty]{(\mathrm{d})} Y \tag{46}$$

holds weakly in $\mathbb{D}([0,\infty), \mathbb{R})$.

Remark 4.1. Theorem 3.4 [17] is stated with a different scaling: we refer to the proof Theorem 2.1.1 [11] to derive Lemma 4.1 from Theorem 3.4 [17].

Since obviously Theorem 1.4(iii) $\Longrightarrow$ Theorem 1.4(i), we only have to prove (i) $\Longrightarrow$ (ii) and (ii) $\Longrightarrow$ (iii).

Proof of (i) $\Longrightarrow$ (ii)**.** For any $t, \lambda \in [0,\infty)$ and any $p \geq 1$, we set

$$u_p(t,\lambda) = -p \log \left(g^{(p)}_{[\gamma_p t]}(\mathrm{e}^{-\lambda/p}) \right), \qquad \varphi_p(\lambda) = -\gamma_p \log \left(f^{(p)}(\mathrm{e}^{-\lambda/p}) \right)$$

$$b_p(t,\lambda) = \int_0^{[\gamma_p t]/\gamma_p} \mathrm{d}s \; \varphi_p(u_p(s,\lambda)), \qquad d_p(t,\lambda) = -\log \mathbf{E}\left[\mathrm{e}^{-\lambda p^{-1} Y^{*,p}_{[\gamma_p t]}} \right]$$

and $d(t,\lambda) = -\log \mathbf{E}[\exp(-\lambda Z^*_t)]$. First deduce from (1)

$$\frac{[px]}{p} u_p(t,\lambda) + b_p(t,\lambda) = d_p(t,\lambda). \tag{47}$$

The convergence of Theorem 1.4(i) combined with Dini's theorem implies the following assertions:

- For any $\lambda \geq 0$, $d(0,\lambda) = x\lambda$ and $d(\cdot,\lambda)$ is continuous on $[0,\infty)$.
- For any $t, \lambda > 0$, $d(t,\lambda) \in (0,\infty)$ and $\lim_{\lambda\to 0} d(t,\lambda) = 0$.
- For any $t \geq 0$, $d_p(t,\cdot) \underset{p\to\infty}{\longrightarrow} d(t,\cdot)$ uniformly on every compact subsets of $[0,\infty)$.

Let T be any denumerable dense subset of $(0,\infty)$ and let E be any infinite subset of $\mathbb{N}$. By use of Helly's selection theorem combined with Cantor's diagonal procedure, we can find an increasing sequence $(p_k; k \geq 1)$ of elements of E, a set of measures $(\mathrm{m}_t, t \in T)$ on $[0,\infty)$ and a measure n on $[0, \infty)$ such that $\mathrm{m}_t([0, \infty)) \le 1$, $t \in T$, $\mathrm{n}([0, \infty)) \le 1$ and such that for all $t \in T$,

$$\forall r \in [0, \infty) \text{ s.t. } \mathrm{m}_t(\{r\}) = 0, \quad \lim_{k \to \infty} \mathbf{P}(p_k^{-1} Y^{p_k}_{[\gamma_{p_k} t]} \le r) = \mathrm{m}_t([0, r]), \tag{48}$$

$$\forall r \in [0, \infty) \text{ s.t. } \mathrm{n}(\{r\}) = 0, \quad \lim_{k \to \infty} \mathbf{P}(p_k^{-1} W^{p_k}_{\gamma_{p_k}} \le r) = \mathrm{n}([0, r]). \tag{49}$$

Define for any $\lambda \ge 0$ and any $t \in T$,

$$u(t, \lambda) = -\log \int_{[0,\infty)} \mathrm{e}^{-\lambda y} \mathrm{m}_t(\mathrm{d}y) \quad \text{and} \quad \varphi(\lambda) = -\log \int_{[0,\infty)} \mathrm{e}^{-\lambda y} \mathrm{n}(\mathrm{d}y),$$

with the convention $-\log(0) = \infty$ so that $m_t = 0$ iff $u(t, \lambda) = \infty$ for a certain $\lambda \ge 0$. By Dini's theorem and standard monotonicity arguments, we deduce from (48) and (49) that for any $t \in T$ the following convergences hold

$$u_{p_k}(t, \cdot) \underset{k \to \infty}{\longrightarrow} u(t, \cdot) \quad \text{and} \quad \varphi_{p_k} \underset{k \to \infty}{\longrightarrow} \varphi \tag{50}$$

as $[0, \infty]$-valued functions uniformly on every compact subset of the open interval $(0, \infty)$.

We consider two cases: $x \ne 0$ and $x = 0$ and **we first suppose** $x \ne 0$. By (47), we get

$$\frac{[px]}{p} u_p(t, \lambda) \le d_p(t, \lambda).$$

Then we pass to the limit along $(p_k; k \ge 1)$ to show that $u(t, \lambda) \le x^{-1} d(t, \lambda) < \infty$, $t \in T$, $\lambda > 0$. Thus $\mathrm{m}_t \ne 0$ for any $t \in T$ and it makes sense to define the function b on $T \cup \{0\} \times [0, \infty)$ by

$$b(t, \lambda) = d(t, \lambda) - x u(t, \lambda) \quad \text{if } t \in T, \lambda \ge 0$$

and $b(0, \lambda) = 0$, $\lambda \ge 0$. Deduce from (50) that for any $t \in T$

$$b_{p_k}(t, \cdot) \underset{k \to \infty}{\longrightarrow} b(t, \cdot), \tag{51}$$

uniformly on every compact subset of the open interval $(0, \infty)$. We first prove the following claim.

Claim 1. *There exists* $t_0 \in T$ *such that* $\mathrm{m}_{t_0} \ne \delta_0$

Proof of Claim 1. Suppose that $u(t, \lambda) = 0$, for any $t \in T$ and any $\lambda > 0$. Denote by $q_p \in [0, 1]$ the smallest solution in $[0, 1]$ of $g^{(p)}(z) = z$. Observe that $t \to g^{(p)}_{[\gamma_p t]}(z)$ is non-decreasing for $0 \le z \le q_p$ and non-increasing for $q_p \le z \le 1$. Thus, for any $p \ge 1$ and any $\lambda \ge 0$, $u_p(\cdot, \lambda)$ is monotone. Since T is dense, then a standard monotonicity argument implies that

$$u_{p_k}(t, \lambda) \underset{k \to \infty}{\longrightarrow} 0, \quad t \ge 0, \ \lambda > 0. \tag{52}$$

We take $u(\cdot, \lambda) = 0$, $\lambda \ge 0$, then we also get

$$b(t, \lambda) = d(t, \lambda), \quad t \ge 0, \ \lambda > 0 \tag{53}$$

and since the $b_{p_k}(\cdot, \lambda)$'s are non-decreasing and $d(\cdot, \lambda)$ is continuous, (51) holds for any $t \ge 0$.

Now, set $s_p = \gamma_p^{-1}([\gamma_p(t + s)] - [\gamma_p t])$ and use the Markov property for $Y^{*,p}$ at time $[\gamma_p t]$ to get

$$d_p(s + t, \lambda) = d_p(t, u_p(s_p, \lambda)) + b_p(s_p, \lambda). \tag{54}$$

Since $s_{p_k} \to s$, since the $b_{p_k}(\cdot, \lambda)$'s and the $u_{p_k}(\cdot, \lambda)$'s are monotone and since their limits are continuous, we get

$$u_{p_k}(s_{p_k}, \lambda) \xrightarrow[k\to\infty]{} 0 \quad \text{and} \quad b_{p_k}(s_{p_k}, \lambda) \xrightarrow[k\to\infty]{} d(s, \lambda), \qquad t \geq 0, \ \lambda > 0.$$

Use this to pass to the limit in (54) to get

$$d(s+t, \lambda) = d(s, 0) + b(t, \lambda) = d(t, \lambda), \qquad t, s \geq 0, \ \lambda > 0.$$

It then implies that $d(s, \lambda) = x\lambda$, $s \geq 0$ and the process Z^* has to be a constant process which contradicts the assumptions of Theorem 1.4(i). ■

Claim 2. $\mathrm{n} \neq 0$.

Proof of Claim 2. Recall from the proof of Claim 1 that for any $p \geq 1$ and any $\lambda \geq 0$, $u_p(\cdot, \lambda)$ is monotone. Since the $f^{(p)}$'s are non-decreasing, we get for any $p \geq 1$ and any $t \geq 0$,

$$\frac{[t\gamma_p]}{\gamma_p} \varphi_p \left(\lambda \wedge u_p(t, \lambda)\right) \leq b_p(t, \lambda) \leq d_p(t, \lambda) \tag{55}$$

(use (47)) for the right member). Let $t_0 \in T$ satisfying Claim 1. Then, for any $\lambda > 0$, $u(t_0, \lambda) \in (0, \infty)$. So it makes sense to pass to the limit in (55) along $(p_k; k \geq 1)$ with $t = t_0$. We obtain

$$t_0 \, \varphi\left(\lambda \wedge u(t_0, \lambda)\right) \leq d(t_0, \lambda) < \infty, \tag{56}$$

which implies the claim. ■

Claim 3. *For any $\lambda > 0$, $b(\cdot, \lambda)$ extends to a non-decreasing continuous function on* $[0, \infty)$.

Provided that Claim 3 holds, Dini's theorem combined with a monotonicity argument implies the following convergence :

$$b_{p_k}(t, \cdot) \xrightarrow[k\to\infty]{} b(t, \cdot) \qquad t \in [0, \infty) \tag{57}$$

holds uniformly on every compact subset of the open interval $(0, \infty)$.

Proof of Claim 3. Fix $p \geq 1$ and $\lambda > 0$ and recall that $u_p(\cdot, \lambda)$ is monotone and that φ_p is non-decreasing. Thus, we get for any $0 \leq s < t$,

$$\begin{aligned} 0 \leq b_p(t, \lambda) - b_p(s, \lambda) &\leq \frac{[\gamma_p t] - [\gamma_p s]}{\gamma_p} \varphi_p(u_p(t, \lambda) \vee u_p(s, \lambda)) \\ &\leq \frac{[\gamma_p t] - [\gamma_p s]}{\gamma_p} \varphi_p \left(\frac{p}{[px]} (d_p(t, \lambda) \vee d_p(s, \lambda)) \right), \end{aligned}$$

by (47). Next, by (51) and Claim 2

$$0 \leq b(t, \lambda) - b(s, \lambda) \leq (t - s)\, \varphi\left(x^{-1}(d(t, \lambda) \vee d(s, \lambda))\right), \qquad \lambda > 0, \ s, t \in T, \tag{58}$$

which completes the proof of the claim. ■

Thus, it makes sense to extend the definition of u on $[0, \infty) \times [0, \infty)$ by setting

$$u(t, \lambda) := x^{-1}\left(d(t, \lambda) - b(t, \lambda)\right) \qquad t \geq 0, \ \lambda > 0 \tag{59}$$

and $u(t, 0) = 0$, $t \geq 0$. Observe that $b(t, \lambda) \leq d(t, \lambda)$ and $u(t, \lambda) \leq x^{-1} d(t, \lambda)$, for any $t \geq 0$ and any $\lambda > 0$. If λ goes to 0, then $u(t, 0+) = b(t, 0+) = 0$ for any $t \geq 0$. It implies in particular that for any $t \in T$ m_t is a probability distribution. This result combined with (57) also implies that the convergence

$$u_{p_k}(t, \cdot) \underset{k \to \infty}{\longrightarrow} u(t, \cdot), \quad t \in [0, \infty) \tag{60}$$

holds uniformly on every compact subset of the closed interval $[0, \infty)$. Now, set for any $t, y \in [0, \infty)$,

$$Y_t(k, y) = p_k^{-1} Y^{p_k}_{[\gamma_{p_k} t]} \quad \text{with } Y_0^{p_k} = [p_k y].$$

As a consequence of (60) and of the continuity theorem for Laplace exponents (see [14] p. 431), there exists a family of probability measures $(P_t(y, \mathrm{d}z); t, y \geq 0)$ on $[0, \infty)$ such that the distribution of $Y_t(k, y)$ converges weakly to $P_t(y, \mathrm{d}z)$. In particular, $P_t(1, \mathrm{d}z) = \mathrm{m}_t(\mathrm{d}z)$, $t \in T$. Since $(t, y) \to \int P_t(y, \mathrm{d}z) \exp(-\lambda z) = \exp(-y u(t, \lambda))$ is continuous for any $\lambda \geq 0$, the mapping $(t, y) \to P_t(y, B)$ is measurable for any Borel set $B \subset [0, \infty)$. Moreover, the Markov property and the branching property for the Y^{p_k}'s imply that for any $t, s, y, y' \geq 0$:

$$\int P_t(y, \mathrm{d}y') P_s(y', \mathrm{d}z) = P_{t+s}(y, \mathrm{d}z) \quad \text{and}$$
$$P_t(y, \mathrm{d}z) * P_t(y', \mathrm{d}z) = P_t(y + y', \mathrm{d}z). \tag{61}$$

By Theorem 4 [30] of Silverstein (see also the correspondence between spectrally positive Lévy processes and CSBPs in Theorems 1 and 2 [23]) there exists a spectrally positive Lévy process X with exponent ψ satisfying (5) such that $u(t, \lambda)$ is the unique nonnegative solution of the differential equation (3). Then the $(P_t(y, \mathrm{d}z); t, y \geq 0)$ are the transition kernels of a non-zero conservative CSBP(ψ). The Lévy–Khintchine formula implies that ψ is of the form

$$\psi(\lambda) = \alpha_0 \lambda + \beta \lambda^2 \int_{(0,\infty)} \pi(\mathrm{d}r) \left(\mathrm{e}^{-\lambda r} - 1 + \lambda r \mathbf{1}_{\{r<1\}} \right) \tag{62}$$

with $\alpha_0 \in \mathbb{R}$, $\beta \geq 0$ and $\int \pi(\mathrm{d}r) 1 \wedge r^2 < \infty$.

Concerning the immigration exponent, use (56) to get $\varphi(0+) = d(t, 0+) = 0$. Thus, n is a true probability distribution that has to be infinitely divisible on $[0, \infty)$ since it is obtained as a weak limit of marginals of rescaled random walks. φ is therefore the Laplace exponent of a conservative subordinator denoted by W. It has to be of the form

$$\varphi(\lambda) = \kappa \lambda + \int_{(0,\infty)} \rho(\mathrm{d}r) \left(1 - \mathrm{e}^{-\lambda r} \right) \tag{63}$$

with $\kappa \geq 0$ and $\int \rho(\mathrm{d}r) 1 \wedge r < \infty$. Deduce from (47), (50) and (60) that

$$d(t, \lambda) = x u(t, \lambda) + \int_0^t \mathrm{d}s \, \varphi(u(s, \lambda)), \quad \lambda, t \geq 0. \tag{64}$$

We now need to show **uniqueness** for limiting functions u, ψ and φ: let $\widetilde{u}$, $\widetilde{\psi}$ and $\widetilde{\varphi}$ be obtained by repeating the previous procedure from another denumerable dense subset $\widetilde{T} \subset [0, \infty)$ and another subsequence $(\widetilde{p}_k; k \geq 1)$. Thus, we must have

$$d(t, \lambda) = x u(t, \lambda) + \int_0^t \mathrm{d}s \, \varphi(u(s, \lambda)) = x \widetilde{u}(t, \lambda) + \int_0^t \mathrm{d}s \, \widetilde{\varphi}(\widetilde{u}(s, \lambda)), \quad t, \lambda \geq 0.$$

Differentiate twice the latter equation at $t = 0$ to get

$$-x\psi + \varphi = -x\widetilde{\psi} + \widetilde{\varphi} \quad \text{and} \quad (x\psi' - \varphi')\psi = (x\widetilde{\psi}' - \widetilde{\varphi}')\widetilde{\psi}.$$

Differentiate the first expression and deduce from the second one the following equation:

$$(x\psi' - \varphi')(\psi - \widetilde{\psi}) = 0. \tag{65}$$

Suppose that $x\psi' = \varphi'$ on a non-empty open interval (a, b). Differentiate twice this expression to get

$$\int_{(0,\infty)} \rho(\mathrm{d}r) r^3 \mathrm{e}^{-\lambda r} = -x \int_{(0,\infty)} \pi(\mathrm{d}r) r^3 \mathrm{e}^{-\lambda r}, \quad \lambda \geq 0,$$

by (62) and (63). Thus, $\rho = \pi = 0$, $\beta = 0$ and $x\alpha_0 = \kappa$, which imply that Z^* is constant: This contradicts the assumption of Theorem 1.4(i). Accordingly, $x\psi'$ and φ' must differ at a point and by continuity $x\psi' \neq \varphi'$ on a non-empty open interval. Thus by (65), we get $\psi = \widetilde{\psi}$ and $\widetilde{\varphi} = \varphi$, which implies the desired uniqueness. ∎

Thus, we have shown in the $x \neq 0$ case that there exists a non-constant CSBP(ψ): $Y = (Y_t; t \geq 0)$ started at $Y_0 = 1$ and a subordinator W with Laplace exponent φ both satisfying (5) such that for any denumerable dense subset $T \subset (0, \infty)$ and any infinite subset $E \subset \mathbb{N}$, we can find an increasing sequence $(p_k; k \geq 1)$ of elements of E (depending on T) that satisfies for any $t \in T$ and any $\lambda \geq 0$:

$$\lim_{k\to\infty} \mathbf{E}\left[\mathrm{e}^{-\lambda p_k^{-1} Y^{p_k}_{[\gamma_{p_k} t]}}\right] = \mathbf{E}\left[\mathrm{e}^{-\lambda Y_t}\right] \quad \text{and} \quad \lim_{k\to\infty} \mathbf{E}\left[\mathrm{e}^{-\lambda p_k^{-1} W^{p_k}_{\gamma_{p_k}}}\right] = \mathbf{E}\left[\mathrm{e}^{-\lambda W_1}\right].$$

It easy to prove that these limits imply for any $t \geq 0$ that the following convergences

$$p^{-1} Y^p_{[\gamma_p t]} \xrightarrow[p\to\infty]{(\mathrm{d})} Y_t \quad \text{and} \quad p^{-1} W^p_{\gamma_p} \xrightarrow[p\to\infty]{(\mathrm{d})} W_1$$

hold in distribution in $\mathbb{R}$. Theorem 1.4(ii) follows from Lemma 4.1 (a) $\Longrightarrow$ (b). ∎

It remains to consider the $x = 0$ case: Let $t, s \in [0, \infty)$; for any $p \geq 1$, we may and will choose s_p such that $|s_p - s| \leq 1/\gamma_p$ and $[\gamma_p(s_p + t)] = [\gamma_p s] + [\gamma_p t]$. Use the Markov property for $Y^{*,p}$ at $[\gamma_p s]$ to get

$$d_p(s_p + t, \lambda) = d_p(s, u_p(t, \lambda)) + b_p(t, \lambda). \tag{66}$$

Since $b_p = d_p$, we get for any $t, s, \lambda, K \geq 0$, any $t \in T$ and any $p \geq 1$:

$$d_p(s_p + t, \lambda) - d_p(t, \lambda) \geq d_p(s, K \wedge u_p(t, \lambda)). \tag{67}$$

Observe next that since $b_p = d_p$, the $d_p(\cdot, \lambda)$'s and $d(\cdot, \lambda)$ are non-decreasing. Since $d(\cdot, \lambda)$ is continuous, Dini's Theorem implies that $d_p(s_p + t, \lambda)$ tends to $d(s + t, \lambda)$ when p goes to ∞. Choose t in T and pass to the limit in (67) along the subsequence $(p_k, k \geq 0)$ to get

$$d(s + t, \lambda) - d(t, \lambda) \geq d(s, K \wedge u(t, \lambda)), \quad t \in T,\ s \geq 0,\ \lambda > 0.$$

If $u(t, \lambda) = \infty$, then let λ go to 0 in the previous inequality to get $d(s, K) = 0$, for every $s, K \geq 0$, which contradicts the assumption of Theorem 1.4(i). Thus, $u(t, \lambda) \neq \infty$ and consequently $\mathrm{m}_t \neq 0$, for any $t \in T$. Then, use (50) to pass to the limit in (66) and to get

$$d(s + t, \lambda) = d(s, u(t, \lambda)) + d(t, \lambda), \quad \lambda > 0,\ s \geq 0,\ t \in T. \tag{68}$$

Fix $s_0 > 0$. Since $\mathbf{P}(Z_{s_0} > 0) > 0$, $d(s_0, \cdot)$ has to be a continuous increasing mapping. Then, it admits a continuous increasing inverse denoted by Δ: $\Delta(d(s_0, \lambda)) = \lambda$, $\lambda \geq 0$. We extend the definition of u on $[0, \infty) \times [0, \infty)$ by setting

$$u(t, \lambda) := \Delta\left(d(s_0 + t, \lambda) - d(t, \lambda)\right).$$

Then observe that $u(t, 0+) = 0$ for any $t \in T$. Thus, $\mathrm{m}_t([0, \infty)) = 1$, $t \in T$. Now recall that for any fixed $\lambda \geq 0$ and any $p \geq 1$, $u_p(\cdot, \lambda)$ is monotone. It implies (60) by a standard monotonicity argument. Now, use (56) to deduce $\varphi(\lambda) < \infty$, $\lambda > 0$ and $\varphi(0+) = 0$, which both imply that n is a probability measure on $[0, \infty)$. Then, use similar arguments to those used in the $x \neq 0$ case to complete the proof of Theorem 1.4(ii). ■

Proof of Theorem 1.4 (ii)$\Longrightarrow$(iii). The weak convergence of finite dimensional marginals is a straightforward consequence of Lemma 4.1 $(a) \Longrightarrow (b)$ combined with a simple computation based on (1). So, it remains to prove tightness. To that end, we adapt an argument of Grimvall [17]: Fix $a \geq 0$ and denote by $(Y_n^{*,p}(a); n \geq 0)$ a GWI(μ_p, ν_p)-process started at $Y_0^{*,p}(a) = [pa]$. Denote by $\mathbf{Q}_a^{(p)}(\cdot)$ the distribution of $p^{-1}(Y_1^{*,p}(a) - [ap])$. Theorem 2.2' [16] (see also Lemma 3.6 [17]) asserts that the sequence of the distributions of the processes $p^{-1}Y^{*,p}_{[\gamma_p \cdot]}$, $p \geq 1$ is a tight sequence in $\mathbb{D}([0, \infty), \mathbb{R})$ if the two following conditions are satisfied:

(d) For any $t \geq 0$, $\lim_{M \to \infty} \limsup_{p \to \infty} \mathbf{P}\left(\sup_{0 \leq s \leq t} p^{-1} Y^{*,p}_{[\gamma_p s]} > M\right) = 0$.

(e) For every compact set $C \subset [0, \infty)$,

$$\{(\mathbf{Q}_a^{(p)})^{*\gamma_p},\ a \in C,\ p \geq 1\}$$

is a tight family of probability measures on $\mathbb{R}$.

Proof of (e). Observe that

$$\left(\mathbf{Q}_a^{(p)}\right)^{*\gamma_p} = \mu_p\left(\frac{\cdot - 1}{p}\right)^{*[ap]\gamma_p} * \nu_p\left(\frac{\cdot}{p}\right)^{*\gamma_p}.$$

Thus, (e) easily follows from Theorem 1.4(ii). ■

Proof of (d). Fix $t > 0$. Let K be any positive real number. Observe that for any $p \geq 1$, any $\lambda, y > 0$ and any $s \in [0, t]$, we have

$$\begin{aligned}
\mathbf{P}\left(p^{-1} Y^{*,p}_{[\gamma_p s]}(y) \leq K\right) &\leq \exp\left(K - d_p(s, \lambda)\right) \\
&\leq \exp\left(K - \frac{[py]}{p} u_p(s, \lambda)\right) \\
&\leq \exp\left(K - \frac{[py]}{p} \lambda \wedge u_p(t, \lambda)\right)
\end{aligned}$$

(use (47) and the monotonicity of the $u_p(\cdot, \lambda)$'s). Now, since Theorem 1.4(ii) implies Lemma 4.1(c), we get $\inf\{\lambda \wedge u_p(t, \lambda),\ \ p \geq 1\} > 0$. Thus, it proves that for any $K > 0$, there exists $M(K) > 0$ such that

$$\mathbf{P}\left(p^{-1} Y^{*,p}_{[\gamma_p s]}(y) > K\right) > 1/2, \quad y \geq M(K),\ s \in [0, t],\ p \geq 1. \tag{69}$$

Now use the Markov property and (69) to get

$$\mathbf{P}\left(p^{-1}Y^{*,p}_{[\gamma_p t]}(x) > K\right) \geq \frac{1}{2}\mathbf{P}\left(\sup_{0\leq s\leq t} p^{-1}Y^{*,p}_{[\gamma_p s]}(x) > M(K)\right), \quad p \geq 1$$

and (d) follows from the one-dimensional marginals convergence. ■

4.2. *Proof of Theorem 1.5*

We now consider a sequence of *(sub)critical* offspring distributions $(\mu_p; p \geq 1)$. Recall notations of Section 1 and consider a sequence $(\tau_p; p \geq 1)$ of GWI(μ_p, r_p)-trees where μ_p and r_p satisfy (13) and (14). For any $p \geq 1$, denote by $H^p = (H^p_k; k \geq 0)$ and by $D^p = (D^p_k; k \geq 0)$ resp. the height process and the random walk associated with the forest $f(\tau_p)$ containing the "left bushes" of τ_p. We also denote by $H^{\bullet,p}$ and by $D^{\bullet,p}$ the processes corresponding to the forest $f(\tau^\bullet_p)$ that contains the "right bushes" of τ_p. Recall from Section 2.2 the notations $(L_k(\tau_p); k \geq 0)$ and $(L_k(\tau^\bullet_p); k \geq 0)$, $p \geq 1$. We denote by Σ the function such that

$$\overleftarrow{H}^p = \Sigma(H^p, D^p, L(\tau_p)) \quad \text{and} \quad \overrightarrow{H}^p = \Sigma(H^{\bullet,p}, D^{\bullet,p}, L(\tau^\bullet_p)), \tag{70}$$

that is specified by (20)–(23).

We use Corollary 2.5.1 [11] asserting that (13) and (14) imply the joint convergence

$$\left(p^{-1}D^p_{[p\gamma_p t]}, \gamma_p^{-1}H^p_{[p\gamma_p t]}; t \geq 0\right) \xrightarrow[p\to\infty]{\text{(d)}} (X_t, H_t;\ t \geq 0) \tag{71}$$

holds in distribution in $\mathbb{D}([0,\infty), \mathbb{R}^2)$. We also get an analogous convergence for $H^{\bullet,p}$ and $D^{\bullet,p}$ since they have the same distribution.

Next, we use Remark 2.3 and a standard argument to deduce from the right limit of (13) that the following convergence

$$\left(p^{-1}L_{[\gamma_p t]}(\tau_p),\ p^{-1}L_{[\gamma_p t]}(\tau^\bullet_p); t \geq 0\right) \xrightarrow[p\to\infty]{\text{(d)}} (U_t, V_t; t \geq 0), \tag{72}$$

holds in distribution in $\mathbb{D}([0,\infty), \mathbb{R}^2)$. Thus, by (71), (72) and Skorohod's representation theorem, we may assume that the following convergences

$$\lim_{p\to\infty}\left(p^{-1}D^p_{[p\gamma_p t]}, \gamma_p^{-1}H^p_{[p\gamma_p t]}\right)_{t\geq 0} = (X, H),$$
$$\lim_{p\to\infty}\left(p^{-1}D^{\bullet,p}_{[p\gamma_p t]}, \gamma_p^{-1}H^{\bullet,p}_{[p\gamma_p t]}\right)_{t\geq 0} = (X', H')$$

and

$$\lim_{p\to\infty}\left(p^{-1}L_{[\gamma_p t]}(\tau_p),\ p^{-1}L_{[\gamma_p t]}(\tau^\bullet_p)\right)_{t\geq 0} = (U, V)$$

hold a.s. in $\mathbb{D}([0,\infty), \mathbb{R}^2)$, where (X, H), (X', H') and (U, V) are independent processes and where (X, H) and (X', H') have the same distribution. For convenience of notation, we keep denoting in the same way random processes involved in the latter almost sure convergences, so we may also assume that (70) and (8) hold. We first prove

$$\left(\gamma_p^{-1}\overleftarrow{H}^p_{[p\gamma_p t]};\ t \geq 0\right) \xrightarrow[p\to\infty]{} \left(\overleftarrow{H}_t; t \geq 0\right) \tag{73}$$

a.s. in $\mathbb{D}([0,\infty),\mathbb{R})$. To that end, let us introduce for any $\omega \in \mathbb{D}([0,\infty),\mathbb{R})$, the right-continuous inverse of ω

$$S_x(\omega) = \inf\{s \geq 0 : \omega(s) > x\}, \quad x \in \mathbb{R}$$

(with the convention $\inf \emptyset = \infty$). Set $\mathcal{V}(\omega) = \{x \in \mathbb{R} : S_{x-}(\omega) < S_x(\omega)\}$. It is easy to check that $\omega \to S_x(\omega)$ is continuous in $\mathbb{D}([0,\infty),\mathbb{R})$ at any ω such that $x \notin \mathcal{V}(\omega)$ (see Proposition 2.11, Chapter VI [18]). By (9), the process $x \to S_x(U) = U_x^{-1}$ has a.s. continuous sample paths. Then it implies a.s.

$$\lim_{p\to\infty} S_x\left(p^{-1}L_{[\gamma_p \cdot]}(\tau_p)\right) = U_x^{-1}, \quad x \in \mathbb{Q}_+.$$

Since U^{-1} is a continuous increasing process, standard arguments imply

$$\left(S_x(p^{-1}L_{[\gamma_p\cdot]}(\tau_p)); x \geq 0\right) \xrightarrow[p\to\infty]{} \left(U_x^{-1}; x \geq 0\right) \tag{74}$$

a.s. in $\mathbb{D}([0,\infty),\mathbb{R})$ (see Theorem 2.15, Chapter VI [18]). Let us set

$$\alpha^p(n) := \inf\left\{k \geq 0 :\ L_k(\tau_p) \geq 1 - \inf_{j\leq n} D_j^p\right\}, \quad p, n \geq 1.$$

Since $t \to I_t = \inf_{s\in[0,t]} X_s$ is a continuous process, the following convergence

$$\left(\inf_{j\leq[p\gamma_p t]} p^{-1}D_j^p; t \geq 0\right) \xrightarrow[p\to\infty]{\text{(d)}} (I_t; t \geq 0)$$

a.s. holds uniformly on every compact subsets of $[0,\infty)$. Thus, by (74)

$$\left(\gamma_p^{-1}\alpha^p([p\gamma_p t]); t \geq 0\right) \xrightarrow[p\to\infty]{\text{(d)}} (U_{-I_t}^{-1}; t \geq 0) \tag{75}$$

a.s. uniformly on every compact subsets of $[0,\infty)$. Next, we set

$$\pi^p(n) := \inf\{k \geq 0 : k + \alpha^p(k) \geq n\}, \quad p, n \geq 1.$$

Then, by (22) and (23), we get

$$\overleftarrow{H}_n^p = n - \pi^p(n) + H_{\pi(n)}^p \quad \text{and} \quad \alpha^p(\pi^p(n) - 1) \leq n - \pi^p(n) \leq \alpha^p(\pi^p(n)). \tag{76}$$

We easily deduce from (75) that $(p\gamma_p)^{-1}\pi^p([p\gamma_p\cdot])$ a.s. converges to the identity map uniformly on every compact subset of $[0,\infty)$. Then, (73) follows from (76) and (75). Use similar arguments to prove the corresponding convergence for $\overrightarrow{H}^p$. Thus, we have proved that the following convergence

$$\left(\gamma_p^{-1}\overleftarrow{H}_{[p\gamma_p t]}^p, \gamma_p^{-1}\overrightarrow{H}_{[p\gamma_p t]}^p; t \geq 0\right) \xrightarrow[p\to\infty]{\text{(d)}} (\overleftarrow{H}_t, \overrightarrow{H}_t;\ t \geq 0) \tag{77}$$

holds in distribution in $\mathbb{D}([0,\infty),\mathbb{R}^2)$. The joint convergence of the corresponding contour processes is a consequence of (18) and (19): denote by q_p the function associated with τ_p as defined in Section 2.2; set $y_p(s) = (p\gamma_p)^{-1}q_p(p\gamma_p s)$; by (18) we get for every $T > 0$,

$$\sup_{s\leq T}\left|\frac{1}{\gamma_p}\overrightarrow{C}_{p\gamma_p s}(\tau_p) - \frac{1}{\gamma_p}\overrightarrow{H}_{p\gamma_p y_p(s)}^p\right| \leq \frac{1}{\gamma_p} + \frac{1}{\gamma_p}\sup_{n\leq Tp\gamma_p}|\overrightarrow{H}_{n+1}^p - \overrightarrow{H}_n^p| \xrightarrow[p\to\infty]{} 0 \tag{78}$$

in probability by (77). On the other hand, we get from (19)

$$\sup_{s\le T}\left|y_p(s)-\frac{s}{2}\right|\le\frac{1}{2p\gamma_p}\sup_{k\le Tp\gamma_p}\overrightarrow{H}^p_k+\frac{1}{p\gamma_p}\xrightarrow[p\to\infty]{}0 \tag{79}$$

in probability by (77). Using similar arguments, we prove analogue convergences in probability for the right contour processes. Then,

$$\left(\gamma_p^{-1}\overleftarrow{C}_{2p\gamma_p t}(\tau_p),\gamma_p^{-1}\overrightarrow{C}_{2p\gamma_p t}(\tau_p);t\ge 0\right)\xrightarrow[p\to\infty]{(\mathrm{d})}(\overleftarrow{H}_t,\overrightarrow{H}_t;\ t\ge 0). \tag{80}$$

The convergence of the sequence $p^{-1}Y^{*,p}_{[p\gamma_p\cdot]}$ is a consequence of Theorem 1.4: we easily see that (13) implies (11) by taking $W_1=U_1+V_1$ and thus,

$$\nu_p(k-1)=\sum_{1\le j\le k}r_p(k,j),\quad k\ge 1.$$

To get the desired joint convergence of the contour processes with the GWI process, argue exactly as in the proof of Corollary 2.5.1 [11] p. 63–64. ■

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References

[1] D. Aldous, J. Pitman, Tree-valued Markov chains derived from Galton–Watson processes, Ann. Inst. H. Poincaré 34 (1998) 637–686.
[2] D.J. Aldous, Asymptotic fringe distributions for general families of random trees, Ann. Appl. Probab. 1 (2) (1991) 228–266.
[3] D.J. Aldous, The continuum random tree I, Ann. Probab. 19 (1991) 1–28.
[4] D.J. Aldous, The continuum random tree III, Ann. Probab. 21 (1993) 248–289.
[5] J. Bertoin, Lévy Processes, Cambridge Univ. Press, 1996.
[6] N.H. Bingham, Fluctuation theory in continuous time, Adv. Appl. Probab. 7 (1975) 705–766.
[7] N.H. Bingham, Continuous branching processes and spectral positivity, Stochastic Process. Appl. 4 (1976) 217–242.
[8] J.M. Bismut, Last exit decomposition and regularity at the boundary of transition probabilities, Zeitschrift für Wahr. 69 (1985) 65–98.
[9] L. Chaumont, Conditionings and paths decompositions for Lévy processes, Stochastic Process. Appl. 64 (1) (1996) 39–54.
[10] T. Duquesne, Path decompositions for real Lévy processes, Ann. I. H. Poincaré PR 39 (2) (2003) 339–370.
[11] T. Duquesne, J-F. Le Gall, Random trees, Lévy processes and spatial branching processes, Astérisque (281) (2002).
[12] T. Duquesne, J.-F. Le Gall, Probabilistic and fractal aspects of Lévy trees, Probab. Theory Related Fields 131 (4) (2005) 553–603.
[13] S.N. Evans, J. Pitman, A. Winter, Rayleigh processes, real trees, and root growth with re-grafting, Probab. Theory Related Fields (2005) (in press).
[14] W. Feller, An Introduction to Probability Theory and Its Applications, second ed., vol. II, Wiley, New York, 1971.
[15] D.R. Grey, Asymptotic behaviour of continuous-time continuous state-space branching processes, J. Appl. Probab. 11 (1974) 669–677.
[16] A. Grimvall, On the transition from a Markov chain to a continuous time process, J. Stoch. Processes 1 (1973) 335–368.
[17] A. Grimvall, On the convergence of a sequence of branching processes, Ann. Probab. 2 (1974) 1027–1045.
[18] J. Jacod, A.N. Shiryaev, Limit Theorems for Stochastic Processes, Springer, Berlin, 1987.

[19] K. Kawazu, S. Watanabe, Branching processes with immigration and related limit theorems, Probab. Theo. Appl. 16 (1) (1971) 36–53.
[20] H. Kesten, Subdiffusive behavior of random walk on a random cluster, Ann. Inst. H. Poincaré Probab. Statist. 22 (1987) 425–487.
[21] A. Lambert, Quasi-stationary distributions and the continuous-state branching process conditioned to be never extinct, Electron. J. Probab. 12 (2007).
[22] A. Lambert, The genealogy of continuous-state branching processes with immigration, Probab. Theory Related Fields 122 (1) (2002) 42–70.
[23] J. Lamperti, Continuous-state branching processes, Bull. Amer. Math. Soc. 73 (1967) 382–386.
[24] J.-F. Le Gall, Brownian excursions, trees and measure-valued branching processes, Ann. Probab. 19 (1991) 1399–1439.
[25] J.-F. Le Gall, Y. Le Jan, Branching processes in Lévy processes: the exploration process, Ann. Probab. 26 (1) (1998) 213–252.
[26] Zheng-Hu Li, Asymptotic behaviour of continuous time and state branching processes, J. Austral. Math. Soc. Ser. A 68 (1) (2000) 68–84.
[27] V. Limic, A LIFO queue in heavy traffic, prepublication, 1999.
[28] R. Lyons, R. Pemantle, Y. Peres, Conceptual proof of *LlogL* criteria for mean behavior of branching processes, Ann. Probab. 23 (1995) 1125–1138.
[29] M.A. Pinsky, Limit theorems for continuous state branching processes with immigration, Bull. Am. Math. Soc. 78 (1972) 242–244.
[30] M.L. Silverstein, A new approach to local times, J. Math. Mech. 17 (1968) 1023–1054.