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Asymptotic splitting in the three-dimensional problem of elasticity for non-homogeneous piezoelectric plates

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ABSTRACT

A novel asymptotic approach to the theory of non-homogeneous anisotropic plates is suggested. For the problem of linear static deformations we consider solutions, which are slowly varying in the plane of the plate in comparison to the thickness direction. A small parameter is introduced in the general equations of the theory of elasticity. According to the procedure of asymptotic splitting, the principal terms of the series expansion of the solution are determined from the conditions of solvability for the minor terms. Three-dimensional conditions of compatibility make the analysis more efficient and straightforward. We obtain the system of equations of classical Kirchhoff's plate theory, including the balance equations, compatibility conditions, elastic relations and kinematic relations between the displacements and strain measures. Subsequent analysis of the edge layer near the contour of the plate is required in order to satisfy the remaining boundary conditions of the three-dimensional problem. Matching of the asymptotic expansions of the solution in the edge layer and inside the domain provides four classical plate boundary conditions. Additional effects, like electromechanical coupling for piezoelectric plates, can easily be incorporated into the model due to the modular structure of the analysis. The results of the paper constitute a sound basis to the equations of the theory of classical plates with piezoelectric effects, and provide a trustworthy algorithm for computation of the stressed state in the three-dimensional problem. Numerical and analytical studies of a sample electromechanical problem demonstrate the asymptotic nature of the present theory.

1. Introduction

Theories of thin plates, existing in the literature, generally fall into one of the four major categories. Historically, the first one is the "method of hypotheses", or engineering approach, which came to its logical conclusion in the classical treatise by Timoshenko and Woinowsky-Krieger (1959) (for a modern variant of the analysis of composite piezoelectric plates see, e.g. Reddy (2004)). This theory, which appears nearly in each course on strength of materials all over the world, is based on standard assumptions concerning the distribution of mechanical entities over the thickness of the plate, negligibility of certain values, etc. Balance and kinematic relations for the plate are obtained using some of the equations of the theory of elasticity. Although the resulting two-dimensional formulation is quite reasonable, and all assumptions have theoretical and practical argumentation, some more fundamental and thorough research in the field of reduction of the three-dimensional problem to a two-dimensional one is needed.

The following theoretical step is the application of a variational method. Based on the weak formulation of the three-dimensional problem, in which an appropriate variational principle replaces the field equations, this approach uses the famous Galerkin–Ritz method with an approximation of the unknowns over the thickness. The variational principle produces two-dimensional equations and natural boundary conditions for the coefficients in the approximations, which can be interpreted in terms of the theory of plates. The application of this method to piezoelectric plates can be found e.g. in Reddy (1999, 2004), Fiedler et al. (2010). Krommer and Irschik (2000), Krommer (2002, 2004) studied the consistency conditions concerning approximations of electrical and mechanical entities in the models of shear-deformable plates; a classical plate without shear with non-local effects due to electric coupling is the subject of the research by Krommer (2003).

Being very useful in engineering applications, the traditional variational method cannot reveal the particularity of thin bodies: the resulting theory strongly depends on the chosen set of approximating functions. Advantageous approximations can only be selected

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based on the previous experience, and the correspondence to the true solution of the original three-dimensional problem is estimated based on numerical modeling. A different approach is the variational-asymptotic method (VAM), proposed by Berdichevsky (1983, 2009): stationary points of a functional with a small parameter are determined asymptotically, when the small parameter tends to zero. In its application to plates this results in the convergence of the plate and three-dimensional solutions as the thickness of the plate is decreasing. Recently this method has been applied for the case of composite plates with significantly different material properties of the layers (Berdichevsky, 2010), and to piezoelectric plates (Liao and Yu, 2009). Clarity and attractivity of the idea of the method are balanced by the absence of a formal mathematical proof of its validity (Berdichevsky, 1983) along with a certain dependence of the results on the decisions, made during the analysis. An analogy can be drawn to the competitive numerical methods of finite elements and finite differences: although the first method, based on a weak formulation, is very powerful and robust in applications, some particular situations require the fine work of using the method of finite differences, which is based on field equations.

Apart stands the direct approach to plates as material surfaces with a certain set of degrees of freedom of particles. Being free from logical contradictions, this powerful method easily deals with such complicated problems as nonlinear deformations of curved shells (Berdichevsky, 1983, 2009; Eliseev, 2006; Eliseev and Vetyukov, 2010). However, practical applications of this method require a combination with the full three-dimensional analysis, which is needed prior to the solution of the two-dimensional problem in order to provide the form of the strain energy function (stiffnesses), and allows to find the stressed state of the actual solid body for the computed force factors of the reduced model. This three-dimensional analysis, which is of crucial importance for the successful application of the combined approach, does not necessarily need to include the effects of geometric nonlinearity or initial curvature of the shell, i.e. the present problem of linear plate deformation is sufficient to be used in this procedure. The same is true for the challenging problem of geometrically nonlinear modeling of piezoelectric shells (Vetyukov and Krommer, 2010a).

The literature concerning the solution of three-dimensional equations with the help of various series expansions is voluminous. We can mention e.g. an interesting work by Batista (2010), where an iterative algorithm for successive correction of the solution of the homogeneous three-dimensional problem leads to an infinite series expansion with respect to the thickness coordinate. In the present study we focus on the more traditional approach, in which the three-dimensional solution is sought as a series expansion with respect to a small parameter, which is related to the thickness of the plate. Thus, two-dimensional equations for the leading order terms of the series expansion for a homogeneous plate were obtained by Goldenveizer (1961), Maugin and Attou (1990) used his asymptotic method with a change of unknowns for the analysis of coupled electromechanical equations; Carvalho et al. (2009) presented a more mathematical approach. An advanced study of piezoelectric plates with material non-homogeneity in all three dimensions is reported by Kalamkarov and Kolpakov (2001); the usage of powerful mathematical techniques, typical for the analysis of periodic structures, makes the latter work quite complicated. An example of an asymptotic analysis, in which the layers of the structural material and piezoelectric material in the plate are considered explicitly with piecewise solutions and conditions on the interfaces between the layers, is presented by Mauritson (2009).

Relevant to the present study are the works by Wang and Tarn (1994), Tarn (1997). In the equations for the components of stresses and displacements, written for a particular material structure, a small parameter is introduced by means of non-dimensional variables. The analysis proceeds by giving particular orders of smallness to different components of the stress tensor a priori. The conditions of solvability for the minor terms of the series expansion of the solution with respect to the small parameter take part in the formulation of the two-dimensional equations. Cheng et al. (2000), Cheng and Batra (2000), Reddy and Cheng (2001) presented further extension of the method, suitable for numerical analysis of piezoelectric plates.

The present study uses the procedure of asymptotic splitting (Eliseev, 2003, 2006), which has already been systematically applied to the theories of thin rods with a non-homogeneous cross-section (Yeliseyev and Orlov, 1999), of thin-walled rods of open profile (Eliseev, 2003; Vetyukov, 2010), and of homogeneous thin plates (Eliseev, 2003). In comparison to the above mentioned works featuring formal asymptotic expansions, the advantages are the following:

- According to the logics of the procedure of asymptotic splitting, the principal terms of the expansion are determined from the conditions of solvability for the minor terms, which makes the procedure formal and straightforward as soon as we have the original three-dimensional problem with a small parameter formulated; no assignment of orders of smallness to different components of stresses and displacements ("scaling") needs to be done a priory, which is intrinsic to some earlier works (e.g. Maugin and Attou, 1990; Wang and Tarn, 1994; Cheng et al., 2000 etc.).
- Instead of the complete recurrent system of relations between successive terms in the series expansion of the solution, we consider only those equalities, which appear to be relevant for the analysis.
- The invariant form of the relations helps to keep the analysis compact.
- The procedure has a clear modular structure: we study the stresses, strains, displacements and electrical effects almost independently, each of these stages provides a corresponding part of the complete theory. The formulation is completed by the constitutive relations of the three-dimensional model.
- Independence of different stages of the procedure is provided by the three-dimensional conditions of compatibility, which play a key role in the analysis. An alternative attempt to proceed with the analysis of the field of displacements without this intermediate step would increase the complexity of the analysis, as the asymptotic magnitude of the field of displacements is two orders higher than that of the strains. This thesis is justified by the experience of application of the procedure of asymptotic splitting for other continua (see, e.g. Yeliseyev and Orlov, 1999; Vetyukov, 2010). The conditions of compatibility provide important constraints on the strain measures of the plate theory. Moreover, the analysis of the edge layer and the matching procedure are strongly based on the conditions of compatibility.
- Material parameters and possible non-homogeneity of the cross-section come into play only at the final stage of the procedure, when the constitutive relations are derived. Variation of the material properties over the thickness does not affect the linear distribution of the in-plane part of the strain tensor, which follows from compatibility conditions. Moreover, possible jumps in the coefficients do not require special treatment: the consistency of the procedure is preserved even for multilayer plates.

An especially strong point of the present study is the analysis of the edge layer. Matching two asymptotic expansions, derived for the inner part of the plate and near its contour, we arrive at four classical boundary conditions. The slightly simplified version of the analysis for the case of simple material properties and straight boundaries, presented in the paper, has principle differences from the analogous attempts, existing in the literature (Goldenveizer, 1961, 1969, 1994; Dauge and Gruais, 1998; Lin, 2004; Tarn and Huang, 2002): the three-dimensional conditions of compatibility allow us to avoid the notion of displacements in the formulation of static boundary conditions, and the usage of the mathematically strict Prandtl's method of matching of asymptotic expansions increases the reliability of the results.

The theoretical basis of the paper is particularly important for modeling and development of smart structures, which feature piezoelectric sensors and actuators, see e.g. Reddy (1999), Nader (2008). To this end the present analysis is accomplished by accounting for the piezoelectric effect in the material of the structure, which results in a plate theory with additional field variables, namely voltage and free charge on the surface. The closed system of equations includes additional relations for those variables, i.e. the equations of the electric circuit between the electrodes on the opposite sides of the plate. Application of the results of the analysis in comparison with a numerical three-dimensional solution is demonstrated on a sample problem, which experimentally proves the asymptotic nature of the presented theory.

2. The three-dimensional elastic problem

A plate is a three-dimensional body with the position vector of a point

$$\boldsymbol{r} = \boldsymbol{x} + z\boldsymbol{k}, \quad -\frac{h}{2} \leqslant z \leqslant \frac{h}{2}, \quad \boldsymbol{x} \in \Omega. \tag{1}$$

Here, the out-of-plane unit vector is denoted as $\boldsymbol{k}$, the corresponding Cartesian coordinate is z, and $\boldsymbol{x}$ is the plane part of the position vector; h is the thickness and Ω is the domain in the plane of the plate.

The general system of equations of elasticity includes the balance equation and the boundary conditions for the stress tensor $\boldsymbol{\tau}_3$ (the index '3' will distinguish three-dimensional entities from their two-dimensional counterparts):

$$\nabla_3 \cdot \boldsymbol{\tau}_3 + \boldsymbol{f} = 0, \quad \boldsymbol{k} \cdot \boldsymbol{\tau}_3|_{z=\pm\frac{h}{2}} = 0; \tag{2}$$

∇_3 is Hamilton's operator. A formulation with free surfaces at $z = \pm\frac{h}{2}$ does not reduce the generality because of the arbitrariness of the volumetric force $\boldsymbol{f}$. The side surface conditions at the boundary of the domain $\partial\Omega$ will be formulated and analyzed in Section 6.

The field of displacements $\boldsymbol{u}_3$ and the field of strains $\boldsymbol{\varepsilon}_3$ are kinematically related:

$$\boldsymbol{\varepsilon}_3 = \nabla_3 \boldsymbol{u}_3^S; \tag{3}$$

$(\ldots)^S$ defines the symmetric part of a tensor.

The problem is closed with the relation of elasticity:

$$\boldsymbol{\tau}_3 = {}^4\mathbf{C} \cdot\cdot \boldsymbol{\varepsilon}_3, \quad {}^4\mathbf{C} = {}^4\mathbf{C}(z); \tag{4}$$

the fourth-rank tensor of material elastic properties can vary over the thickness of the plate.

The condition of compatibility means that for a given field of $\boldsymbol{\varepsilon}_3$ one can find such a field of displacements $\boldsymbol{u}_3$ that (3) holds; hence,

$$\nabla_3 \times (\nabla_3 \times \boldsymbol{\varepsilon}_3)^T = 0. \tag{5}$$

Another mathematically equivalent form of this condition can be advantageous in some cases:

$$\Delta_3 \boldsymbol{\varepsilon}_3 + \nabla_3\nabla_3 \ \mathrm{tr}\ \boldsymbol{\varepsilon}_3 = 2(\nabla_3\nabla_3 \cdot \boldsymbol{\varepsilon}_3)^S. \tag{6}$$

Due to the structure of our problem, it appears to be convenient to separate the in-plane and out-of-plane parts of vectors and tensors. The in-plane part will be denoted with an index $\perp$:

$$\mathbf{I}_\perp = \mathbf{I} - \boldsymbol{kk}, \quad \boldsymbol{f}_\perp = \mathbf{I}_\perp \cdot \boldsymbol{f}, \quad \boldsymbol{\tau}_{3\perp} \equiv \boldsymbol{\tau}_\perp = \mathbf{I}_\perp \cdot \boldsymbol{\tau}_3 \cdot \mathbf{I}_\perp, \quad \boldsymbol{r}_\perp = \boldsymbol{x}. \tag{7}$$

Then, we introduce

$$\begin{aligned}
\boldsymbol{\varepsilon}_3 &= \varepsilon_z\boldsymbol{kk} + \boldsymbol{\gamma k} + \boldsymbol{k\gamma} + \boldsymbol{\varepsilon}_\perp, \\
\boldsymbol{\tau}_3 &= \sigma_z\boldsymbol{kk} + \boldsymbol{sk} + \boldsymbol{ks} + \boldsymbol{\tau}_\perp, \\
\boldsymbol{u}_3 &= u_z\boldsymbol{k} + \boldsymbol{u}_\perp,
\end{aligned} \tag{8}$$

in which $\boldsymbol{\gamma}$ and $\boldsymbol{s}$ are out-of-plane shear strain and stress vectors.

3. Asymptotic splitting in the equations of balance

3.1. Starting point for the asymptotic analysis

In order to indicate the thinness of the plate, we introduce a formal small parameter λ in the expression of the position vector of a point of the plate: instead of (1) we write

$$\boldsymbol{r} = \lambda^{-1}\boldsymbol{x} + z\boldsymbol{k}. \tag{9}$$

Now the magnitudes of z and $\boldsymbol{x}$ in (9) have formally the same order. The corresponding form of Hamilton's operator will be

$$\nabla_3 = \lambda\nabla + \boldsymbol{k}\partial_z. \tag{10}$$

Here ∇ is the differential operator with respect to the in-plane position vector $\boldsymbol{x}$.

Another possible argumentation to (10) is that we seek for solutions, which vary in the plane much slower than over the thickness (i.e. z is a "fast" variable), and therefore the derivatives with respect to the in-plane coordinates $\boldsymbol{x}$ acquire a corresponding order of smallness.

We seek for the unknown field of stresses in the form of a power series in the small parameter:

$$\boldsymbol{\tau}_3 = \lambda^{-2} \overset{0}{\boldsymbol{\tau}} + \lambda^{-1} \overset{1}{\boldsymbol{\tau}} + \ldots \tag{11}$$

Our goal is to find those terms in the solution, which dominate as the plate is getting thinner and $\lambda \to 0$. It means that we are interested rather in the convergence of the solution to the principal terms than in the convergence of the series itself. Only the term $\overset{0}{\boldsymbol{\tau}}$ is of real interest to us, and the role of the rest of the series is to determine the principal term from the conditions of solvability for the minor terms. The simple advantage of the dimensionless formal small parameter in comparison to e.g. ratio between the thickness and the span of the plate is that it can be simply set equal to *1* after the analysis is finished and the terms of interest are determined. The deeper idea is that this formalism allows us to consider not just a thin plate, but rather a special class of "plate" solutions with a particular asymptotic behavior.

The leading power λ^{-2} in (11) is known already from the asymptotic analysis of a homogeneous plate (Eliseev, 2003), otherwise it would have been necessary to estimate it with the help of trial-and-error.

The formal mathematical proof of the approach is yet incomplete in the sense that we do not show the existence of the solution in the form (11). However, already published results indicate its validity; for an additional discussion see Yeliseyev and Orlov (1999), Vetyukov (2010).

Now we can write the balance equations with the small parameter. Using (10) in (2), we obtain

$$\begin{aligned}
&\lambda\nabla \cdot \boldsymbol{\tau}_\perp + \partial_z\boldsymbol{s} + \boldsymbol{f}_\perp = 0, \\
&\lambda\nabla \cdot \boldsymbol{s} + \partial_z\sigma_z + f_z = 0, \\
&\sigma_z|_{z=\pm\frac{h}{2}} = 0, \quad \boldsymbol{s}|_{z=\pm\frac{h}{2}} = 0.
\end{aligned} \tag{12}$$

Substituting the series expansion (11) in (12), we begin the asymptotic procedure.

3.2. First step: λ^{-2}

First we balance the principal terms in the resulting equations and boundary conditions, which have the order of smallness λ^{-2}. The results are:

$$\begin{aligned}&\partial_z \overset{0}{\boldsymbol{s}} = 0, \quad \overset{0}{\boldsymbol{s}}\Big|_{z=\pm\frac{h}{2}} = 0 \Rightarrow \overset{0}{\boldsymbol{s}} = 0,\\ &\partial_z \overset{0}{\sigma}_z = 0, \quad \overset{0}{\sigma}_z\Big|_{z=\pm\frac{h}{2}} = 0 \Rightarrow \overset{0}{\sigma}_z = 0.\end{aligned} \tag{13}$$

The most important in-plane part of the stress tensor $\overset{0}{\boldsymbol{\tau}}_\perp$ is yet to be determined as a final result of the procedure.

3.3. Second step: λ^{-1}

Now we proceed to the next term, taking (13) into account. We immediately conclude that $\overset{1}{\sigma}_z = 0$, and the shear stress is coupled with the principal term in the stress tensor:

$$\nabla \cdot \overset{0}{\boldsymbol{\tau}}_\perp + \partial_z \overset{1}{\boldsymbol{s}} = 0, \quad \overset{1}{\boldsymbol{s}}\Big|_{z=\pm\frac{h}{2}} = 0 \Rightarrow \overset{1}{\boldsymbol{s}}\Big|_{z=\zeta} = -\nabla \cdot \int_{-\frac{h}{2}}^{\zeta} \overset{0}{\boldsymbol{\tau}}_\perp dz. \tag{14}$$

The boundary condition at the upper surface $z = \frac{h}{2}$ leads to the in-plane balance equation in terms of plate theory:

$$\nabla \cdot \mathbf{T} = 0, \quad \mathbf{T} = h\left\langle \overset{0}{\boldsymbol{\tau}}_\perp \right\rangle; \quad \langle \ldots \rangle \equiv \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}}, \ldots dz. \tag{15}$$

The standard notation for the mean value over the thickness is introduced in (15) and used in the following. The equation of balance of the in-plane force factor $\mathbf{T} = \mathbf{T}(\boldsymbol{x})$ appears as a result of the condition of solvability for the shear stress at the second step of the asymptotic procedure. It would have been possible to account for the in-plane external forces in this balance equation: assuming $\boldsymbol{f} = f_z\boldsymbol{k} + \lambda^{-1}\boldsymbol{f}_\perp$, we would arrive at a non-homogeneous balance equation in the plane.

In order to obtain the second classical equation of balance of moments, the third step of the asymptotic procedure is needed.

3.4. Third step: λ^0

Collecting terms of the order λ^0 in the second equality of (12), we arrive at

$$\begin{aligned}&\nabla \cdot \overset{1}{\boldsymbol{s}} + \partial_z \overset{2}{\sigma}_z + f_z = 0, \quad \overset{2}{\sigma}_z\Big|_{z=\pm\frac{h}{2}} = 0 \Rightarrow \nabla \cdot \boldsymbol{Q} + q = 0,\\ &q \equiv h\langle f_z \rangle, \quad \boldsymbol{Q} \equiv h\left\langle \overset{1}{\boldsymbol{s}} \right\rangle.\end{aligned} \tag{16}$$

The form of the vector of transversal forces $\boldsymbol{Q}$ and the corresponding balance equation are in correspondence with the classical plate theory. We proceed by using (14):

$$\boldsymbol{Q} = -\nabla \cdot \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{h}{2}}^{\zeta} \overset{0}{\boldsymbol{\tau}}_\perp dz d\zeta = -\nabla \cdot \mathbf{M}. \tag{17}$$

The formally introduced expression of the tensor of internal moment $\mathbf{M} = \mathbf{M}(\boldsymbol{x})$ can be transformed to a more classical form with the help of integration by parts:

$$\begin{aligned}&\mathbf{M} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \mathbf{e} d\zeta, \quad \mathbf{e} = \int_{-\frac{h}{2}}^{\zeta} \overset{0}{\boldsymbol{\tau}}_\perp dz \Rightarrow \mathbf{M} = \zeta \mathbf{e}\Big|_{\zeta=-\frac{h}{2}}^{\frac{h}{2}} - \int_{-\frac{h}{2}}^{\frac{h}{2}} \zeta \partial_\zeta \mathbf{e} d\zeta\\ &\Rightarrow \mathbf{M} = -h\left\langle \left(z - \frac{h}{2}\right) \overset{0}{\boldsymbol{\tau}}_\perp \right\rangle.\end{aligned} \tag{18}$$

The factor $z - \frac{h}{2}$ could be replaced by $z - z_0$ with arbitrary "reference" level z_0, from which the moments are counted: this results in additional terms of the kind $\left(z_0 - \frac{h}{2}\right)\mathbf{T}$ in the moment $\mathbf{M}$, which do not affect $\boldsymbol{Q}$ due to the balance Eq. (15). Classically (e.g. for isotropic plates) one uses $z_0 = 0$; hence,

$$\mathbf{M} = -h\left\langle z \overset{0}{\boldsymbol{\tau}}_\perp \right\rangle. \tag{19}$$

Eliminating the transversal force, we write the second classical equation of balance:

$$\nabla \cdot \nabla \cdot \mathbf{M} - q = 0. \tag{20}$$

4. Asymptotic analysis for the strains

4.1. Principal terms

The field of strains $\boldsymbol{\varepsilon}_3$ (8) must be compatible. We use the condition (6) and the differential operator in the form (10). Balancing the in-plane and out-of-plane components of the resulting equation, we arrive at

$$\begin{aligned}&\partial_z^2 \boldsymbol{\varepsilon}_\perp - 2\lambda \partial_z \nabla \boldsymbol{\gamma}^S + \lambda^2 \nabla\nabla \varepsilon_z = 0,\\ &\partial_z(\nabla \text{ tr } \boldsymbol{\varepsilon}_\perp - \nabla \cdot \boldsymbol{\varepsilon}_\perp) = \lambda(\nabla\nabla \cdot \boldsymbol{\gamma} - \Delta \boldsymbol{\gamma}),\\ &\Delta \text{ tr } \boldsymbol{\varepsilon}_\perp - \nabla \cdot \nabla \cdot \boldsymbol{\varepsilon}_\perp = 0.\end{aligned} \tag{21}$$

Due to the linearity of the relation between strains and stresses, the field of strains must have the same asymptotic behavior with respect to the formal small parameter:

$$\boldsymbol{\varepsilon}_3 = \lambda^{-2} \overset{0}{\boldsymbol{\varepsilon}} + \lambda^{-1} \overset{1}{\boldsymbol{\varepsilon}} + \ldots \tag{22}$$

From the first relation in (21) we immediately conclude, that the plane part of the principal term of the strain tensor is linearly distributed over the thickness:

$$\overset{0}{\boldsymbol{\varepsilon}}_\perp = -\boldsymbol{\kappa} z + \boldsymbol{\varepsilon}. \tag{23}$$

The coefficients $\boldsymbol{\kappa}(\boldsymbol{x})$ and $\boldsymbol{\varepsilon}(\boldsymbol{x})$ are functions of the in-plane coordinates. In the following they will be related to the corresponding classical plate strain measures, namely to the curvature and to the reference surface strain.

The second and the third equation in (21) lead to additional conditions for the newly introduced plate strain measures:

$$\nabla \cdot \boldsymbol{\kappa} = \nabla \text{ tr } \boldsymbol{\kappa}, \quad \nabla \cdot \nabla \cdot \boldsymbol{\varepsilon} = \Delta \text{ tr } \boldsymbol{\varepsilon}. \tag{24}$$

The first relation here is the condition of compatibility for the bending strain measure, which is in correspondence with the more general compatibility condition for curved shells (Eliseev, 2006; Vetyukov, 2010). The second equality is a well-known condition of compatibility for the plane problem of elasticity.

The out-of-plane components of the principal term of the strain tensor, which are coupled with the minor terms in the series expansion (22), will be determined in the following section from the results of the first step of the procedure for stresses.

4.2. Elastic relations

Deriving elastic relations in terms of the theory of plates can be done in a straightforward manner, which is very much similar to the existing approximation techniques. At first we need the linear relation between the in-plane part of the stress tensor $\overset{0}{\boldsymbol{\tau}}$ and the plate strain measures $\boldsymbol{\kappa}, \boldsymbol{\varepsilon}$. To this end we consider the elastic relation (4), written for the principal terms, together with the conditions (13). Solving this linear system we obtain the out-of-plane components of the strain tensor and the in-plane stresses:

$$\overset{0}{\boldsymbol{s}} = 0, \quad \overset{0}{\sigma}_z = 0, \quad \overset{0}{\boldsymbol{\tau}} = {}^4\mathbf{C} \cdot\cdot \overset{0}{\boldsymbol{\varepsilon}} \Rightarrow \overset{0}{\varepsilon}_z, \quad \overset{0}{\gamma}, \quad \overset{0}{\boldsymbol{\tau}}_\perp = {}^4\mathbf{C}_* \cdot\cdot \overset{0}{\boldsymbol{\varepsilon}}_\perp$$
$$= {}^4\mathbf{C}_* \cdot\cdot (-\boldsymbol{\kappa} z + \boldsymbol{\varepsilon}). \tag{25}$$

It should be noted, that the modified tensor of elastic properties ${}^4\mathbf{C}_*$ cannot be identified with the plane part of the original one: ${}^4\mathbf{C}_* \neq {}^4\mathbf{C}_\perp$; this is similar to the difference between the plane strain and plane stress problems.

Integrating over the thickness according to (15) and (19), we arrive at the linear relations

$$\begin{aligned} &\mathbf{T} = {}^4\mathbf{A} \cdot\cdot \boldsymbol{\varepsilon} + {}^4\mathbf{B} \cdot\cdot \boldsymbol{\kappa}, \quad \mathbf{M} = {}^4\mathbf{B} \cdot\cdot \boldsymbol{\varepsilon} + {}^4\mathbf{D} \cdot\cdot \boldsymbol{\kappa}; \\ &{}^4\mathbf{A} = h\langle {}^4\mathbf{C}_* \rangle, \quad {}^4\mathbf{B} = -h\langle {}^4\mathbf{C}_* z \rangle, \quad {}^4\mathbf{D} = h\langle {}^4\mathbf{C}_* z^2 \rangle. \end{aligned} \tag{26}$$

These particular values of the plate stiffnesses correspond to the reference layer coordinate $z_0 = 0$, and the same integrals are known to result from the approach with approximations, see e.g. Krommer (2003).

To better clarify the relation (25), we explicitly compute the tensor ${}^4\mathbf{C}_*$ for an orthotropic material (which is isotropic in the plane of the plate). The constitutive relation has the following structure:

$$\begin{aligned} &\boldsymbol{\tau}_3 = {}^4\mathbf{C} \cdot\cdot \boldsymbol{\varepsilon}_3, \quad \boldsymbol{\tau}_\perp = C_1 \ \mathrm{tr}\ \boldsymbol{\varepsilon}_\perp \mathbf{I}_\perp + C_2 \boldsymbol{\varepsilon}_\perp + C_3 \varepsilon_z \mathbf{I}_\perp, \\ &\boldsymbol{s} = C_4 \boldsymbol{\gamma}, \quad \sigma_z = C_5 \varepsilon_z + C_3 \ \mathrm{tr}\ \boldsymbol{\varepsilon}_\perp; \end{aligned} \tag{27}$$

the elastic moduli C_i are functions of z; for a multilayer plate they are piecewise constants, see sample problem in Section 8. From (25) it then follows

$$\overset{0}{\boldsymbol{\gamma}} = 0, \quad \overset{0}{\varepsilon}_z = -\frac{C_3}{C_5} \ \mathrm{tr}\ \overset{0}{\boldsymbol{\varepsilon}}_\perp \Rightarrow \overset{0}{\boldsymbol{\tau}}_\perp = \left(C_1 - \frac{C_3^2}{C_5} \right) \mathrm{tr}\ \overset{0}{\boldsymbol{\varepsilon}}_\perp \mathbf{I}_\perp + C_2 \overset{0}{\boldsymbol{\varepsilon}}_\perp. \tag{28}$$

The particular difference between ${}^4\mathbf{C}_*$ and ${}^4\mathbf{C}_\perp$ can be seen here: the plane part of the tensor ${}^4\mathbf{C}_\perp$ does not include the moduli C_3 and C_5.

The elastic relations (26) complete the system of equations of statics for a plate, the other two groups of relations being the balance Eqs. (15), (20) and the compatibility conditions (24). The picture is however yet incomplete as long as the field of displacements and the boundary conditions in terms of the plate theory are not analyzed.

5. Asymptotic analysis for the displacements

We study the asymptotic behavior of the displacement vector $\boldsymbol{u}_3$ in the three-dimensional problem. The consequences of the kinematic relation (3) for the given asymptotic expansion of the strain tensor (22), (23) will be analyzed.

We rewrite (3) in components with the small parameter:

$$\lambda \nabla \boldsymbol{u}_\perp^S = \boldsymbol{\varepsilon}_\perp, \quad \partial_z u_z = \varepsilon_z, \quad \partial_z \boldsymbol{u}_\perp + \lambda \nabla u_z = 2\boldsymbol{\gamma}. \tag{29}$$

According to the previous experience with homogeneous plates (Eliseev, 2003), we seek for the displacements in the form of a power series with the principal term of the order λ^{-4}:

$$\boldsymbol{u}_3 = \lambda^{-4} \overset{0}{\boldsymbol{u}} + \lambda^{-3} \overset{1}{\boldsymbol{u}} + \ldots \tag{30}$$

Substituting (22), (23) and (30) in (29), we begin the asymptotic procedure.

5.1. First step: λ^{-4}

Balancing the principal terms, we obtain

$$\partial_z \overset{0}{\boldsymbol{u}} = 0 \Rightarrow \overset{0}{\boldsymbol{u}} = \overset{0}{\boldsymbol{u}}(\boldsymbol{x}). \tag{31}$$

The leading term in the series expansion (30) is a function of the in-plane coordinates.

5.2. Second step: λ^{-3}

At this stage the following equations appear:

$$\nabla \overset{0}{\boldsymbol{u}}{}_\perp^S = 0, \quad \partial_z \overset{1}{u}_z = 0, \quad \partial_z \overset{1}{\boldsymbol{u}}_\perp + \nabla \overset{0}{u}_z = 0. \tag{32}$$

The first equality here means that the plane part of the principal term $\overset{0}{\boldsymbol{u}}_\perp$ represents a rigid body motion of the plate and, therefore, can be omitted in the following. Together with the third equality this results in

$$\overset{0}{\boldsymbol{u}}_\perp = 0, \quad \overset{1}{\boldsymbol{u}}_\perp = -z \nabla w(\boldsymbol{x}) + \boldsymbol{u}(\boldsymbol{x}); \quad w \equiv \overset{0}{u}_z. \tag{33}$$

The in-plane displacement of the plate model $\boldsymbol{u}(\boldsymbol{x})$ is asymptotically smaller than the lateral displacement w.

5.3. Third step: λ^{-2}

The last step is the most important one as it produces actual kinematic relations for the plate model. Balancing coefficients at λ^{-2}, we arrive at

$$\nabla \overset{1}{\boldsymbol{u}}{}_\perp^S = -\boldsymbol{\kappa} z + \boldsymbol{\varepsilon} \Rightarrow -z \nabla\nabla w + \nabla \boldsymbol{u}^S = -\boldsymbol{\kappa} z + \boldsymbol{\varepsilon}. \tag{34}$$

This naturally leads to the expected kinematic relations

$$\boldsymbol{\varepsilon} = \nabla \boldsymbol{u}^S, \quad \boldsymbol{\kappa} = \nabla\nabla w. \tag{35}$$

Only the first of the kinematic relations (29) was applied at the third step. The other two also have a non-zero right-hand side, and it is the role of the lower order terms starting with $\overset{2}{\boldsymbol{u}}$ to fulfill these conditions. For the consistency of the theory it is important that all three-dimensional equations will be satisfied, but the exact look of the lower order correction terms is not of interest from the asymptotic point of view.

6. Asymptotic analysis of the edge layer

6.1. Introduction

The above series expansions for $\boldsymbol{\tau}_3$ and $\boldsymbol{\varepsilon}_3$ are the so-called outer expansions. They are valid only far from the contour of the plate and cannot fit the boundary conditions at $\partial\Omega$. It is therefore necessary to consider another expansion of the solution of the three-dimensional problem, which is valid in a thin domain close to the contour of the plate and which allows satisfying the boundary conditions. In the literature on solid mechanics this small domain is traditionally called edge layer (Nayfeh, 1973; Dauge and Gruais, 1998), and the expansion inside this layer is called inner expansion.

Both expansions represent the same solution of the three-dimensional problem, but in different regions of the plate. Therefore the inner expansion must turn into the outer one as we move away from the contour. A special procedure, which is called matching, is intended to provide such equivalence of expansions (Nayfeh, 1973). Matching allows to completely determine the inner expansion and, what is more important, to provide boundary conditions for the outer one.

In the following we demonstrate the method of matched expansions for the case of a straight edge of the plate. Static boundary conditions for the outer expansion are obtained, which can directly be interpreted in terms of the plate theory.

6.2. Basic equations

In Cartesian coordinates x, y, z the domain of the plate is described by the inequalities $-\frac{h}{2} \leqslant z \leqslant \frac{h}{2}$, $y < 0$. The edge of the plate $\partial\Omega$ is the plane $y = 0$, and the outer normal to the edge surface is

$\boldsymbol{n} = \boldsymbol{j}$ ($\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are the unit vectors in the directions of Cartesian coordinates).

The three-dimensional solution is assumed to be changing rapidly over the thickness of the plate and in the direction of the normal $\boldsymbol{n}$ inside the domain. There are two "fast" variables in the edge layer, and the position vector differs from (9):

$$\boldsymbol{r} = \lambda^{-1} x \boldsymbol{i} + \boldsymbol{r}_{\dashv}, \quad \boldsymbol{r}_{\dashv} = y\boldsymbol{j} + z\boldsymbol{k}; \tag{36}$$

the index $\dashv$ denotes the part in the plane yz. Corresponding Hamilton's operator in the edge layer will be

$$\nabla_3 = \lambda \boldsymbol{i}\partial_x + \nabla_{\dashv}, \quad \nabla_{\dashv} = \boldsymbol{j}\partial_y + \boldsymbol{k}\partial_z. \tag{37}$$

The derivatives along the edge acquire an order of smallness, as x is a "slow" variable in the edge layer.

Relations (2) and (6) remain valid and we again use expansions (11) and (22). The static boundary conditions on the edge of the plate are the following:

$$\boldsymbol{j} \cdot \boldsymbol{\tau}_3|_{y=0} = \boldsymbol{N}(x,z), \quad \boldsymbol{N} = \lambda^{-2}(N_x \boldsymbol{i} + N_y \boldsymbol{j}) + \lambda^{-1} N_z \boldsymbol{k}. \tag{38}$$

The powers of λ are chosen such that all the components of the external traction force $\boldsymbol{N}$ influence the principal term of the expansion.

We introduce the following components:

$$\begin{aligned} \boldsymbol{\tau}_3 &= \boldsymbol{\tau}_{\dashv} + \boldsymbol{i}\boldsymbol{s} + \boldsymbol{s}\boldsymbol{i} + \sigma_x \boldsymbol{i}\boldsymbol{i}, \\ \boldsymbol{\varepsilon}_3 &= \boldsymbol{\varepsilon}_{\dashv} + \boldsymbol{i}\boldsymbol{\gamma} + \boldsymbol{\gamma}\boldsymbol{i} + \varepsilon_x \boldsymbol{i}\boldsymbol{i}; \end{aligned} \tag{39}$$

the shear components $\boldsymbol{s}$ and γ have here a meaning, different from (8). It allows to write (2) and (6) in the following way:

$$\begin{aligned} &\nabla_{\dashv} \cdot \boldsymbol{\tau}_{\dashv} + \lambda \partial_x \boldsymbol{s} + \boldsymbol{f}_{\dashv} = 0, \\ &\nabla_{\dashv} \cdot \boldsymbol{s} + \lambda \partial_x \sigma_x + f_x = 0; \end{aligned} \tag{40}$$

$$\begin{aligned} &\lambda^2 \partial_x^2 \boldsymbol{\varepsilon}_{\dashv} - 2\lambda \partial_x \nabla_{\dashv} \boldsymbol{\gamma}^S + \nabla_{\dashv}\nabla_{\dashv} \varepsilon_x = 0, \\ &\lambda \partial_x (\nabla_{\dashv} \cdot \boldsymbol{\varepsilon}_{\dashv} - \nabla_{\dashv} \operatorname{tr} \boldsymbol{\varepsilon}_{\dashv}) = \Delta_{\dashv} \boldsymbol{\gamma} - \nabla_{\dashv}\nabla_{\dashv} \cdot \boldsymbol{\gamma}, \\ &\Delta_{\dashv} \operatorname{tr} \boldsymbol{\varepsilon}_{\dashv} - \nabla_{\dashv} \cdot \nabla_{\dashv} \cdot \boldsymbol{\varepsilon}_{\dashv} = 0. \end{aligned} \tag{41}$$

For the sake of clarity let us now restrict the consideration to a plate with isotropic homogenous material properties. We write the material constitutive law in the following form:

$$\boldsymbol{\tau}_3 = 2\mu\Big(\boldsymbol{\varepsilon}_3 + \frac{\nu}{1-2\nu} \operatorname{tr} \boldsymbol{\varepsilon}_3 \mathbf{I}\Big), \tag{42}$$

μ and ν being the shear modulus and Poisson's ratio respectively. For the components (39) this material law can be written in the following form:

$$\begin{aligned} \boldsymbol{\tau}_{\dashv} &= 2\mu\Big(\boldsymbol{\varepsilon}_{\dashv} + \frac{\nu}{1-2\nu}(\operatorname{tr} \boldsymbol{\varepsilon}_{\dashv} + \varepsilon_x)\mathbf{I}_{\dashv}\Big), \\ \boldsymbol{s} &= 2\mu\gamma, \\ \sigma_x &= 2\mu\Big(\varepsilon_x + \frac{\nu}{1-2\nu}(\operatorname{tr} \boldsymbol{\varepsilon}_{\dashv} + \varepsilon_x)\Big). \end{aligned} \tag{43}$$

The complete system of equations consists of (40), (41), (43) together with the boundary conditions on the upper and lower surfaces (2) and on the edge (38). Using the asymptotic expansions (11) and (22), we obtain a set of problems for each power of the small parameter.

6.3. First step

The equations of the first step (for $\overset{0}{\boldsymbol{\tau}}_3$ and $\overset{0}{\boldsymbol{\varepsilon}}_3$) can be splitted into two separate problems, which has been many times mentioned in the literature (see, e.g. Goldenveizer, 1961). The first one is the antiplane strain problem:

$$\nabla_{\dashv} \cdot \overset{0}{\boldsymbol{s}} = 0, \quad \Delta_{\dashv} \overset{0}{\boldsymbol{\gamma}} = \nabla_{\dashv}\nabla_{\dashv} \cdot \overset{0}{\boldsymbol{\gamma}}. \tag{44}$$

The second one is a plane strain problem:

$$\nabla_{\dashv} \cdot \overset{0}{\boldsymbol{\tau}}_{\dashv} = 0, \quad \Delta_{\dashv} \operatorname{tr} \overset{0}{\boldsymbol{\varepsilon}}_{\dashv} = \nabla_{\dashv} \cdot \nabla_{\dashv} \cdot \overset{0}{\boldsymbol{\varepsilon}}_{\dashv}. \tag{45}$$

Additionally we have an equation for $\overset{0}{\varepsilon}_x$:

$$\nabla_{\dashv}\nabla_{\dashv} \overset{0}{\varepsilon}_x = 0, \tag{46}$$

which means that

$$\overset{0}{\varepsilon}_x = \boldsymbol{a}(x) \cdot \boldsymbol{r}_{\dashv} + b(x); \tag{47}$$

the functions $\boldsymbol{a}$ and b are yet unknown.

Hooke's law (43) completes each of the two problems (44) and (45). The problems are nevertheless uncoupled and can be solved independently in the considered case of the material isotropy.

The general solution of (44) is

$$\overset{0}{\boldsymbol{s}} = \nabla_{\dashv}\psi \times \boldsymbol{i}, \quad \Delta_{\dashv}\psi = 2\mu C(x), \tag{48}$$

the function C cannot be specified yet. We seek ψ in the following form:

$$\begin{aligned} \psi &= \psi_*(z) + \psi_0(y,z), \\ \psi_* &= \Big(z + \frac{h}{2}\Big)\Big(\mu C\Big(z - \frac{h}{2}\Big) + \langle N_x \rangle\Big), \\ \psi_0 &= \sum_k C_k(x)\psi_k(z)e^{\alpha_k y}, \end{aligned} \tag{49}$$

ψ_k and α_k being the eigenfunctions and the eigenvalues of the boundary problem

$$\psi'' + \alpha\psi = 0, \quad \psi|_{\pm\frac{h}{2}} = 0. \tag{50}$$

The solution (48), (49) satisfies homogenous boundary conditions on the surfaces $z = \pm\frac{h}{2}$, and the boundary condition on the edge takes the form:

$$\psi_0|_{y=0,z=\zeta} = \int_{-\frac{h}{2}}^{\zeta} N_x dz - \psi_*. \tag{51}$$

This condition can be used to determine C_k in (49).

The solution of (45) is written with the Airy stress function Φ:

$$\overset{0}{\boldsymbol{\tau}}_{\dashv} = \boldsymbol{i} \times \nabla_{\dashv}\nabla_{\dashv}\Phi \times \boldsymbol{i}, \quad \Phi = \Phi_*(z) + \Phi_0(y,z), \tag{52}$$

where

$$\begin{aligned} \Phi_* &= -\Big(z + \frac{h}{2}\Big)^2 \Big(\frac{2}{h^2}\langle zN_y \rangle \Big(z - \frac{h}{2}\Big) + \frac{1}{h}\Big\langle \Big(\frac{h}{2} - z\Big)N_y \Big\rangle\Big), \\ \Phi_0 &= \sum_k D_k(x)\Phi_k(z)e^{\omega_k y}. \end{aligned} \tag{53}$$

Here Φ_k and ω_k are the eigenfunctions and the eigenvalues of the following fourth order boundary problem:

$$\Phi^{IV} + 2\omega^2\Phi'' + \omega^4\Phi = 0, \quad \Phi'|_{\pm\frac{h}{2}} = \Phi|_{\pm\frac{h}{2}} = 0. \tag{54}$$

The solution (52), (54) satisfies boundary conditions on the surfaces $z = \pm\frac{h}{2}$. Boundary conditions on the edge $y = 0$ serve to determine D_k.

Finally, the principal term of σ_x can be found from Hooke's law (43):

$$\overset{0}{\sigma}_x = -\nu\Delta_{\dashv}\Phi - 2\mu(1+\nu)\overset{0}{\varepsilon}_x. \tag{55}$$

This is the end of the first step. The formulas for the principal terms of the inner expansion still contain unknown functions of x: C, $\boldsymbol{a}$, b. They should be found during the matching procedure.

6.4. Matching procedure for the first step

The matching procedure establishes a correspondence between the inner and the outer expansions. In this case it is sufficient to apply Prandtl's method of matching (Nayfeh, 1973). It requires that the limit of the outer expansion for $y \to 0$ is equal to the limit of the inner one for $y \to -\infty$:

$$\lim_{y\to-\infty} \overset{0}{\boldsymbol{\tau}}_{(i)} = \lim_{y\to 0} \overset{0}{\boldsymbol{\tau}}_{(o)}, \tag{56}$$

in which $\overset{0}{\boldsymbol{\tau}}_{(i)}$ is the term of the inner expansion, determined by formulas (48), (52), (55), and $\overset{0}{\boldsymbol{\tau}}_{(o)}$ is the term of the outer expansion. Evaluating the limits in (56) and balancing the terms with the same powers of z, we obtain the following boundary conditions for the components of the tensors $\boldsymbol{\kappa}$ and $\boldsymbol{\varepsilon}$ from the outer expansion (23):

$$\begin{aligned}
&\frac{2\mu}{1-\nu}(\kappa_y + \nu\kappa_x)\Big|_{y=0} = -\frac{12}{h^2}\langle zN_y\rangle,\\
&\frac{2\mu}{1-\nu}(\varepsilon_y + \nu\varepsilon_x)\Big|_{y=0} = \langle N_y\rangle,\\
&2\mu\varepsilon_{xy}|_{y=0} = \langle N_x\rangle.
\end{aligned} \tag{57}$$

Moreover, the second and the third conditions in (57) can be combined into one vectorial equality:

$$\boldsymbol{j}\cdot\left\langle \overset{0}{\boldsymbol{\tau}}_{\perp(o)}\right\rangle\Big|_{y=0} = \langle N_x\boldsymbol{i} + N_y\boldsymbol{j}\rangle, \tag{58}$$

which is a well-known boundary condition for the plane stress problem. And, finally, the first condition in (57) can be written as

$$\boldsymbol{j}\cdot\left\langle z\overset{0}{\boldsymbol{\tau}}_{\perp(o)}\right\rangle\cdot\boldsymbol{j}|_{y=0} = \langle zN_y\rangle. \tag{59}$$

This is Kirchhoff's plate boundary condition for the bending moment.

The remaining equations in (56) allow us to find all unknown parameters of the inner expansion: C, $\boldsymbol{a}$ and b. Thus, for example,

$$C(x) = -\kappa_{xy}|_{y=0}. \tag{60}$$

6.5. Second step

It is known, that the classical Kirchhoff's theory of plates features four scalar boundary conditions. We already have three. The second step should be considered in order to determine the last boundary condition for the outer expansion.

The equations of the second step are similar to the problems (44) and (45). Here, we only need the plane stress problem:

$$\nabla_\dashv\cdot\overset{1}{\boldsymbol{\tau}}_\dashv + \partial_x\overset{0}{\boldsymbol{s}} = 0, \quad \Delta_\dashv\,\mathrm{tr}\,\overset{1}{\boldsymbol{\varepsilon}}_\dashv = \nabla_\dashv\cdot\nabla_\dashv\cdot\overset{1}{\boldsymbol{\varepsilon}}_\dashv, \tag{61}$$

and the equation for $\overset{1}{\varepsilon}_x$:

$$\nabla_\dashv\nabla_\dashv\overset{1}{\varepsilon}_x = 2\partial_x\nabla_\dashv\overset{0}{\gamma}{}^S. \tag{62}$$

These equations should be supplemented with Hooke's law (43).

The solution of (61) can be decomposed into two terms:

$$\begin{aligned}
&\overset{1}{\boldsymbol{\tau}}_\dashv = \boldsymbol{\tau}_* + \boldsymbol{\tau}_0,\\
&\boldsymbol{\tau}_* = \partial_x\psi_0(\boldsymbol{kj}+\boldsymbol{jk}) - \partial_x\partial_z\left(\psi_* y + 2\int\psi_0 dy\right)\boldsymbol{jj},\\
&\boldsymbol{\tau}_0 = \boldsymbol{i}\times\nabla_\dashv\nabla_\dashv\Phi\times\boldsymbol{i}.
\end{aligned} \tag{63}$$

Here $\boldsymbol{\tau}_*$ is a particular solution of the non-homogeneous first equation in (61), which also satisfies the boundary conditions on the surfaces $z = \pm\frac{h}{2}$; $\boldsymbol{\tau}_0$ is the general solution of the homogeneous equation. The Airy stress function Φ can be represented as

$$\Phi = \Phi_1(z) + \varphi(z)y + \Phi_0(y,z). \tag{64}$$

Here Φ_1 is a particular solution of the problem

$$\Phi^{IV}(z) = -2(\partial_z\partial_y\partial_x\psi_0)|_{y=0}, \quad \Phi|_{z=\pm\frac{h}{2}} = \Phi'|_{z=\pm\frac{h}{2}} = 0. \tag{65}$$

The function φ has the form

$$\begin{aligned}
&\varphi = \frac{\varphi_0}{2}\left(1+\frac{2z}{h}\right)^2(h-z),\\
&\varphi_0 = \langle N_z\rangle + \langle z\partial_z N_x\rangle - 2\mu\frac{h^3}{12}C'(x),
\end{aligned} \tag{66}$$

where $C(x)$ is given by (60). And, finally,

$$\Phi_0 = \sum_k E_k(x)\Phi_k(z)e^{\omega_k y}, \tag{67}$$

where Φ_k and ω_k are the eigenfunctions and the eigenvalues of the problem (54). The unknown functions $E_k(x)$ can be found from the boundary conditions at the edge $y = 0$.

6.6. Matching procedure for the second step

Here, we again need to distinguish the entities of the inner and outer expansions by the indices (i) and (o).

At this step the components of $\overset{1}{\boldsymbol{\tau}}_{\dashv(i)}$ should be matched to the corresponding terms in the outer expansion. Those are the components $\overset{1}{\sigma}_{y(o)}$, $\overset{1}{\sigma}_{z(o)}$ and $\overset{1}{\tau}_{yz(o)}$ of $\overset{1}{\boldsymbol{\tau}}_{(o)}$. As the principal term $\overset{0}{\sigma}_{y(o)}$ already took part in the matching procedure for the first step in Section 6.4, further analysis of the correction term $\overset{1}{\sigma}_{y(o)}$ is not needed. Matching the other two components gives one new boundary condition for the outer expansion:

$$\frac{2\mu}{1-\nu}\partial_y((2-\nu)\kappa_x + \kappa_y)\Big|_{y=0} = -\frac{h^2}{12}(\langle N_z\rangle + \langle z\partial_x N_x\rangle). \tag{68}$$

Analogously to (59), this condition can be written in the form:

$$\left(\nabla\cdot\left\langle z\overset{0}{\boldsymbol{\tau}}_{\perp(o)}\right\rangle\cdot\boldsymbol{j} + \partial_x\left(\boldsymbol{i}\cdot\left\langle z\overset{0}{\boldsymbol{\tau}}_{\perp(o)}\right\rangle\cdot\boldsymbol{j}\right)\right)\Big|_{y=0} = \langle N_z\rangle + \langle z\partial_x N_x\rangle. \tag{69}$$

This is Kirchhoff's plate boundary condition for the combination of the transversal force (first term) and the derivative of the twisting moment with respect to x.

6.7. Boundary conditions: summary

The derived boundary conditions 58, 59 and 69 (four scalar relations) can be written in terms of the force factors, introduced for the plate in Sections 3.3 and 3.4:

$$\begin{aligned}
&\boldsymbol{j}\cdot\mathbf{T} = \boldsymbol{p}_\perp,\\
&-\boldsymbol{j}\cdot\mathbf{M}\cdot\boldsymbol{j} = -\boldsymbol{i}\cdot\boldsymbol{m},\\
&\boldsymbol{j}\cdot\mathbf{Q} - \partial_x(\boldsymbol{i}\cdot\mathbf{M}\cdot\boldsymbol{j}) = p_z + \partial_x\boldsymbol{j}\cdot\boldsymbol{m};\\
&\boldsymbol{p}\equiv h\langle\boldsymbol{N}\rangle, \quad \boldsymbol{m}\equiv h\langle z\boldsymbol{k}\times\boldsymbol{N}\rangle.
\end{aligned} \tag{70}$$

The classical plate boundary conditions in an invariant form can e.g. be deduced from the results of application of the direct approach to a curved shell (see, Eliseev and Vetyukov, 2010):

$$\begin{aligned}
&\boldsymbol{n}\cdot(\mathbf{T} + \mathbf{Q}\boldsymbol{k}) - \partial_l(\boldsymbol{n}\cdot\mathbf{M}\cdot\boldsymbol{l} - \boldsymbol{n}\cdot\boldsymbol{m})\boldsymbol{k} = \boldsymbol{p},\\
&\boldsymbol{n}\cdot\mathbf{M}\cdot\boldsymbol{n} = -\boldsymbol{l}\cdot\boldsymbol{m}.
\end{aligned} \tag{71}$$

Here $\boldsymbol{n}$ is again the outer normal to the edge, $\boldsymbol{l}$ is the unit vector along the edge ($\boldsymbol{n}\times\boldsymbol{l} = \boldsymbol{k}$), and ∂_l is the derivative along the edge. The conditions (70) and (71) are equivalent in the present case, when $\boldsymbol{n} = \boldsymbol{j}$, $\boldsymbol{l} = -\boldsymbol{i}$ and $\partial_l = -\partial_x$. In (71) we take into account that vector $\boldsymbol{m}$ in Eliseev and Vetyukov (2010) erroneously had a different sign.

It should be noted, that the case of kinematic clamping of the boundary of the plate is simple and does not require an additional asymptotic analysis. From (33) the following logical relation follows immediately:

$$\boldsymbol{u}_3|_{\boldsymbol{x}\in\partial\Omega} = 0 \Rightarrow \boldsymbol{u} = 0, \quad w = 0, \quad \boldsymbol{n}\cdot\nabla w \equiv \partial_n w = 0. \tag{72}$$

7. Piezoelectric effect: coupled problem

7.1. Basic equations

When the material of the cross-section exhibits piezoelectric effects, the coupled system of electromechanical equations needs to be considered (see, e.g. Nowacki, 1979; Nader, 2008). The equations of Section 2 must be supplemented with the equation for the electric displacement vector $\boldsymbol{D}$:

$$\nabla_3 \cdot \boldsymbol{D} = 0. \tag{73}$$

Instead of the single constitutive relation (4) we deal with

$$\begin{aligned} \boldsymbol{\tau}_3 &= {}^4\mathbf{C} \cdot \cdot \boldsymbol{\varepsilon}_3 - \boldsymbol{E} \cdot {}^3\mathbf{e}, \\ \boldsymbol{D} &= {}^3\mathbf{e} \cdot \cdot \boldsymbol{\varepsilon}_3 + \boldsymbol{\epsilon} \cdot \boldsymbol{E}. \end{aligned} \tag{74}$$

Here $\boldsymbol{\epsilon}$ is the tensor of dielectric constants; the third-rank tensor of piezoelectric constants ${}^3\mathbf{e}$ has non-zero components only in the piezoelectric material. The electric field vector $\boldsymbol{E}$ is related to the gradient of the electric potential φ:

$$\boldsymbol{E} = -\nabla_3 \varphi. \tag{75}$$

We consider the practically important case of a plate with two electrodes at the upper and lower surfaces; the voltage between the electrodes is the potential difference:

$$v = \varphi\big|_{z=-\frac{h}{2}}^{\frac{h}{2}}. \tag{76}$$

The consideration of an electroded piezoelectric layer, bonded to a non-conducting substrate plate, requires the same procedure with some minor modifications.

The voltage can either be prescribed (actuation), or should be determined in the course of the solution of the problem (sensing). The complementary boundary condition is written in terms of the free charge density per unit surface area σ:

$$\boldsymbol{n} \cdot \boldsymbol{D} = -\sigma. \tag{77}$$

7.2. Asymptotic analysis

As the asymptotic orders of the mechanical entities with respect to the external loading are already determined, the form of the equations in Section 7.1 naturally implies that both the fields of electric displacement $\boldsymbol{D}$ and potential φ have their series expansions starting from λ^{-2}:

$$\boldsymbol{D} = \lambda^{-2} \overset{0}{\boldsymbol{D}} + \lambda^{-1} \overset{1}{\boldsymbol{D}} + \cdots, \quad \varphi = \lambda^{-2} \overset{0}{\varphi} + \lambda^{-1} \overset{1}{\varphi} + \cdots. \tag{78}$$

It will be sufficient to consider only the principal terms of the expansions (78). From (73) and (10) it follows that

$$\partial_z \overset{0}{D}_z = 0 \Rightarrow \overset{0}{D}_z = \text{const} = D_{z0}. \tag{79}$$

Writing the second constitutive relation (74) for the principal terms and projecting on the vertical direction, we arrive at

$$D_{z0} = \boldsymbol{k} \cdot {}^3\mathbf{e} \cdot \cdot \overset{0}{\boldsymbol{\varepsilon}}_3 - \boldsymbol{k} \cdot \boldsymbol{\epsilon} \cdot \boldsymbol{k} \partial_z \overset{0}{\varphi}. \tag{80}$$

The relative asymptotic correlation between the externally applied mechanical and electric factors can be drawn such that both influence the principal terms in the solution. The electric voltage $\lambda^{-2} v$ between the two electrodes on the upper and the lower surfaces of the plate and the total charge $\lambda^{-2}\Sigma$ on the upper electrode need to be two orders of smallness larger than the volumetric mechanical force f_z in order to have comparable effects on the behavior of the structure,

$$h\left\langle \partial_z \overset{0}{\varphi} \right\rangle = \overset{0}{\varphi}\Big|_{z=-\frac{h}{2}}^{\frac{h}{2}} = v, \quad \int_{\Omega_e} D_{z0} = -\Sigma. \tag{81}$$

Here Ω_e is the two-dimensional domain, occupied by the electrodes.

Now we proceed to the principal terms in the first of the constitutive relations (74); with (80), we write:

$$\begin{aligned} \overset{0}{\boldsymbol{\tau}} &= {}^4\mathbf{C} \cdot \cdot \overset{0}{\boldsymbol{\varepsilon}}_3 + \boldsymbol{k} \cdot {}^3\mathbf{e} \partial_z \overset{0}{\varphi} \\ &= \left({}^4\mathbf{C} + \frac{\boldsymbol{k} \cdot {}^3\mathbf{e} \boldsymbol{k} \cdot {}^3\mathbf{e}}{\boldsymbol{k} \cdot \boldsymbol{\epsilon} \cdot \boldsymbol{k}} \right) \cdot \cdot \overset{0}{\boldsymbol{\varepsilon}}_3 - \frac{\boldsymbol{k} \cdot {}^3\mathbf{e}}{\boldsymbol{k} \cdot \boldsymbol{\epsilon} \cdot \boldsymbol{k}} D_{z0}. \end{aligned} \tag{82}$$

The next logical step is to apply the procedure, discussed in Section 4.2, in order to determine the out-of-plane strain components. As $\boldsymbol{k} \cdot \overset{0}{\boldsymbol{\tau}} = 0$, from (82) we find $\overset{0}{\varepsilon}_z$ and $\overset{0}{\boldsymbol{\gamma}}$ as linear functions of $\overset{0}{\boldsymbol{\varepsilon}}_\perp = -\boldsymbol{\kappa} z + \boldsymbol{\varepsilon}$, and D_{z0}. We substitute them into the relation for the voltage, which follows by integration of (80):

$$h^{-1} v = \left\langle \frac{\boldsymbol{k} \cdot {}^3\mathbf{e} \cdot \cdot \overset{0}{\boldsymbol{\varepsilon}}_3}{\boldsymbol{k} \cdot \boldsymbol{\epsilon} \cdot \boldsymbol{k}} \right\rangle - D_{z0} \left\langle \frac{1}{\boldsymbol{k} \cdot \boldsymbol{\epsilon} \cdot \boldsymbol{k}} \right\rangle. \tag{83}$$

This results in

$$D_{z0} = -\sigma = \mathbf{p} \cdot \cdot \boldsymbol{\varepsilon} + \mathbf{m} \cdot \cdot \boldsymbol{\kappa} + c v. \tag{84}$$

Capacity of the plate c depends on its mechanical properties. Integrating further the in-plane part of (82) over the thickness, we finally obtain the internal plate force factors $\mathbf{T}$, $\mathbf{M}$ and the distributed surface charge σ as linear functions of $\boldsymbol{\kappa}$, $\boldsymbol{\varepsilon}$ and v:

$$\mathbf{T} = {}^4\mathbf{A} \cdot \cdot \boldsymbol{\varepsilon} + {}^4\mathbf{B} \cdot \cdot \boldsymbol{\kappa} + \tilde{\mathbf{p}} v, \quad \mathbf{M} = {}^4\mathbf{B} \cdot \cdot \boldsymbol{\varepsilon} + {}^4\mathbf{D} \cdot \cdot \boldsymbol{\kappa} + \tilde{\mathbf{m}} v. \tag{85}$$

Fourth-rank tensors ${}^4\mathbf{A}$, ${}^4\mathbf{B}$ and ${}^4\mathbf{D}$ determine the elastic properties of the plate at constant voltage. Practical computation of coefficients $\tilde{\mathbf{p}}$, $\tilde{\mathbf{m}}$ shows their equivalence to $\mathbf{p}$, $\mathbf{m}$ from (84) at least for the case of orthotropic materials. General proof or denial of this fact, which is important for the usage of these constitutive relations in the framework of a direct approach (Vetyukov and Krommer, 2010a) with the function of enthalpy per unit area of the plate, is yet to be found. From a purely mechanical point of view, the voltage can be considered as a source of eigenstrains in the plate, analogous to the temperature effects; see Vinson (1993) for this well-known analogy.

7.3. Solution of the coupled plate problem: workflow

Two degrees of freedom exist for each pair of electrodes on the opposite sides of the plates: total charge Σ or voltage v, see (81). Depending on the impedance of the electric circuit, which connects the electrodes, a relation between Σ and v complements the structural equations. The two typical cases are the following:

- *Actuation*. The voltage v is prescribed, which results in the generalized forces, and, consequently, in the deformation of the structure. A proper distribution of those generalized forces can be used to assign to the plate a desired deformation (see, Huber et al. (2009) for an application to a frame structure).
- *Sensing*. The measured voltage v is interpreted by an observer in terms of structural entities: displacements, amplitudes, etc. Although unknown, the voltage is the same for the whole pair of opposing electrodes. The system of equations is completed by the condition for the total charge: $\Sigma = 0$, as the external electric circuit is open. As in the case of actuation a proper distribution of the sensing can be used to measure arbitrary kinematic entities of interest (see, Krommer and Irschik (2007) for the three-dimensional elastic case; see, Krommer and Vetyukov (2009) and Vetyukov and Krommer (2010b) for the application to the geometrically nonlinear behavior of rod and shell structures).

8. Sample problem

8.1. Problem formulation

We consider axisymmetric deformation of a circular plate, which consists of two layers: the upper one is made of the piezoelectric material PZT-5A, and the lower one is aluminium. Radius of the plate is $a = 1$ m, thickness of each layer is $h/2$. Total thickness h will be varied in the simulations.

In Fig. 1 the problem is presented in cylindrical coordinates r, θ, z; axis z is the axis of revolution, and local basis $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$, $\boldsymbol{k} \equiv \boldsymbol{e}_z$, directed along the coordinate lines, will be used in the following. The center of the plate $r = 0$ is kinematically fixed, and the outer edge $r = a$ is free from kinematical constraints. Both mechanical and electric loading are considered in the model: either the voltage v between the upper and lower electrodes on the piezoelectric layer is prescribed, or the distributed moment $\boldsymbol{m} = -\boldsymbol{e}_\theta m_a$ is applied at $r = a$. In the latter case the electric circuit is open, and voltage v is an additional unknown, which needs to be determined from the condition that the total charge on the upper electrode remains zero.

8.2. Analytical solution

An analytical solution of the problem within the framework of the plate formulation at hand is easy to obtain. All axisymmetric planar vector and tensor fields have two components:

$$\mathbf{M} = M_r(r)\boldsymbol{e}_r\boldsymbol{e}_r + M_\theta(r)\boldsymbol{e}_\theta\boldsymbol{e}_\theta, \quad \boldsymbol{Q} = Q_r(r)\boldsymbol{e}_r + Q_\theta(r)\boldsymbol{e}_\theta, \ldots \tag{86}$$

We begin with the balance Eqs. (15)–(17), which in the absence of mechanical loading in the domain read:

$$T_r' + \frac{1}{r}(T_r - T_\theta) = 0, \quad Q_r' = 0, \quad Q_r = M_r' + \frac{1}{r}(M_r - M_\theta). \tag{87}$$

The boundary conditions follow from (71) with $\boldsymbol{n} = \boldsymbol{e}_r$ and $\boldsymbol{l} = \boldsymbol{e}_\theta$:

$$T_r(a) = 0, \quad Q_r(a) = 0, \quad M_r(a) = m_a. \tag{88}$$

We conclude that $Q_r = 0$ and proceed to the kinematic relations (35) with the radial displacement u and the transversal displacement w:

$$\begin{aligned} &\boldsymbol{u} = u(r)\boldsymbol{e}_r + w(r)\boldsymbol{k}, \\ &\varepsilon_r = u', \quad \varepsilon_\theta = \frac{u}{r}, \quad \kappa_r = w', \quad \kappa_\theta = \frac{w'}{r}. \end{aligned} \tag{89}$$

As the piezoelectric material is orthotropic, the constitutive relations (84) and (85) read:

$$\begin{aligned} &\mathbf{T} = A_1 \operatorname{tr} \boldsymbol{\varepsilon}\mathbf{I}_\perp + A_2\boldsymbol{\varepsilon} + B_1 \operatorname{tr} \boldsymbol{\kappa}\mathbf{I}_\perp + B_2\boldsymbol{\kappa} + vp\mathbf{I}_\perp, \\ &\mathbf{M} = D_1 \operatorname{tr} \boldsymbol{\kappa}\mathbf{I}_\perp + D_2\boldsymbol{\kappa} + B_1 \operatorname{tr} \boldsymbol{\varepsilon}\mathbf{I}_\perp + B_2\boldsymbol{\varepsilon} + vm\mathbf{I}_\perp, \\ &-\sigma = p \operatorname{tr} \boldsymbol{\varepsilon} + m \operatorname{tr} \boldsymbol{\kappa} + cv. \end{aligned} \tag{90}$$

The coefficients in (90) are related to the material parameters of the structure and of the piezoelectric material; for the explicit formulas and numerical values see, Appendix A.

From (87)–(90) follows a system of equations for two components of displacements, which can be written in the following form:

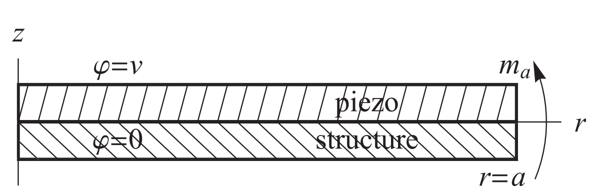

Fig. 1. Sample problem: axisymmetric deformation of a circular plate.

$$\begin{aligned} &(A_1 + A_2)g_u + (B_1 + B_2)g_w = 0, \\ &(B_1 + B_2)g_u + (D_1 + D_2)g_w = 0, \\ &g_u \equiv r^2u' + ru' - u, \quad g_w \equiv r^2w''' + rw' - w'. \end{aligned} \tag{91}$$

As $(A_1 + A_2)(D_1 + D_2) - (B_1 + B_2)^2 \neq 0$, we conclude that $g_u = 0$ and $g_w = 0$. Because $\boldsymbol{u}(0) = 0$, we can write

$$u = \varepsilon_0 r, \quad w = \frac{1}{2}\kappa_0 r^2, \quad \boldsymbol{\varepsilon} = \varepsilon_0\mathbf{I}_\perp, \quad \boldsymbol{\kappa} = \kappa_0\mathbf{I}_\perp. \tag{92}$$

The constant strains follow from the boundary conditions (88):

$$\begin{aligned} &(2A_1 + A_2)\varepsilon_0 + (2B_1 + B_2)\kappa_0 + pv = 0, \\ &(2B_1 + B_2)\varepsilon_0 + (2D_1 + D_2)\kappa_0 + mv = m_a. \end{aligned} \tag{93}$$

The charge σ appears to be independent from r; in the case of open circuit additional condition applies:

$$2m\kappa_0 + 2p\varepsilon_0 + cv = 0. \tag{94}$$

From the system (93) we obtain the maximal transversal displacement for the case of pure electric loading, when $m_a = 0$:

$$w_v = \frac{a^2 v((2A_1 + A_2)m - (2B_1 + B_2)p)}{2h^2((2B_1 + B_2)^2 - (2A_1 + A_2)(2D_1 + D_2))}. \tag{95}$$

In the non-local problem with the open circuit both the maximal transversal displacement w_m and the voltage v_m depend on m_a; explicit solution of the system (93) and (94) is too lengthy to be included in the present text.

It should be noted, that the problem of finding the voltage v_m at a given mechanical loading could be solved with the help of the compatibility conditions (24), which simplifies the analysis when the field of displacements is not of interest.

8.3. Numerical analysis

The problem was solved numerically in a three-dimensional formulation. Using ABAQUS software, we created an axisymmetric finite element model with the parameters, used in the analytical study; the thickness of the plate was varied in the range from 0.001 m to 0.016 m. A regular mesh, consisting of square elements with quadratic geometric order (CAX8RE or CAX8R depending on the layer), featured 10 elements over the thickness of the plate. Attempts of further mesh refinement evidenced that the obtained solutions are practically converged. For the thinnest plate the mesh included 10^5 elements, so that the obtained accuracy is reachable only in the axisymmetric test case.

We started with the problem of actuation: $m_a = 0$, $v = 1$ V. The analytical solution (95) for this case gives

$$w_v h^2 = -2.4617 \cdot 10^{-10} m^3. \tag{96}$$

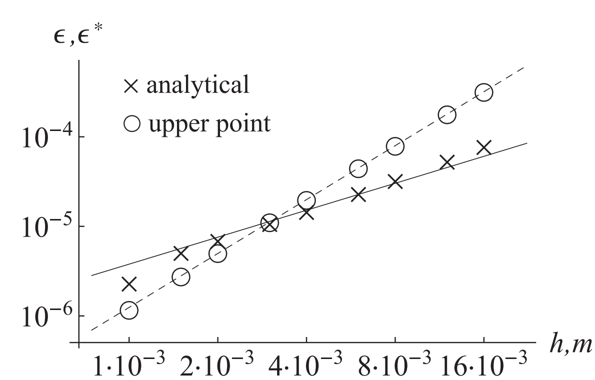

Fig. 2. Relative differences of the numerical and analytical solutions at different thickness values in comparison with the local effects in the cross-section.

Relative differences of the numerical solutions from this reference value are shown in Fig. 2 in a logarithmic scale. With crosses we presented the values $\epsilon = |w_{num} - w_v|/w_{num}$, where the deflections of the mid-point $w_{num} \equiv u_z|_{r=a,z=0}$ were taken from numerical results. The analysis is incomplete without the accounting for the local deformations in the cross-section. Circles in Fig. 2 answer to the relative difference of the vertical deflection between the upper and the middle points of the cross-section: $\epsilon^* = |w_{num} - w^*_{num}|/w_{num}$, $w^*_{num} \equiv u_z|_{r=a,z=h/2}$.

In order to estimate the asymptotic orders we approximated the numerical results by analytical curves: $\epsilon = \epsilon_0 h$ and $\epsilon^* = \epsilon^*_0 h^2$, the constants were obtained by a least squares fitting. The two corresponding straight lines match the points in Fig. 2. We can conclude that the relative error ϵ has the first order of smallness. Although the local effect ϵ^* is one order of smallness higher, as it is indeed predicted by the second equality in (32), in the present problem it starts playing a role already at $h \approx 0.003$ m; this shows a good example of mutual relation between terms of different orders in a formal asymptotic expansion.

For the plate with the thickness $h = 0.004$ m we analyzed three-dimensional fields: in-plane and out-of-plane strains $\varepsilon_r, \varepsilon_z$, out-of-plane electric field $E_z = -\partial_z \varphi$ and in-plane stress σ_r. Except for the edge layer, the fields should not depend on r. Distributions over the thickness coordinate z, restored by the analytical results (for an additional discussion of the procedure see Section 9), are compared with the nodal values of the finite element solution in Fig. 3. As it is foreseen by the asymptotic study, the in-plane strains are linearly distributed over the thickness; the relative difference between the end values $\varepsilon_r|_{z=\pm h/2}$ and the analytic ones $\varepsilon_0 \mp \kappa_0 h/2$ was $\epsilon_\varepsilon \approx 4.6 \cdot 10^{-5}$. In restoring the three-dimensional fields for piezoelectric layers it is important to account for quadratic terms in the distribution of electric potential φ: assuming that in the piezoelectric layer $\partial_z \varphi = \text{const} = 2v/h$, one would end up with an error of almost 50% in the value of stress at the upper surface $\sigma_r|_{z=h/2}$.

For the same value of the thickness we solved the non-local problem with the mechanical moment applied. The analytical solution gives $w_m = 8.17556 \cdot 10^{-4}$ m, $v_m = 5.5989$ V. In the numerical analysis the moment m_a was formed by a linearly varying pressure on the edge $r = a$. Solving the problem with ABAQUS version 6.7–3, we encountered difficulties when using elements with quadratic geometric order. Therefore we had to use 20 linear elements CAX8R/E over the thickness to obtain the converged solution. The resulting relative errors with respect to the analytical solution were: for the displacement $\epsilon_w \approx 4.3 \cdot 10^{-5}$, for the voltage $\epsilon_v \approx 2.1 \cdot 10^{-4}$.

9. Conclusions

The three-dimensional problem of the deformation of a plate splits asymptotically into a simple one-dimensional problem in the cross-section and a two-dimensional Kirchhoff's plate formulation with classical balance Eqs. (15), (20), conditions of compatibility (24), elastic relations (26), kinematic relations (35), and boundary conditions (70), (72). Additional terms in the equations appear due to the piezoelectric effect, as it is discussed in Section 7. Finding the leading terms in the solution of the three-dimensional problem requires the following steps to be performed:

1. The cross-section of the plate needs to be analyzed, producing the expressions of stiffnesses and generalized forces for the plate.
2. The plate problem needs to be solved either analytically or numerically.
3. With a two-dimensional plate solution at hand, we find the three-dimensional fields:
   (a) in-plane strains are computed from (23);
   (b) out-of-plane strains and in-plane stresses are found from (25) for the purely mechanical case (out-of-plane stresses appear only in the lower order terms); this step gets more complicated for piezoelectric plates, or when the stressed state in an edge layer needs to be determined;
   (c) leading order terms in the transversal and in-plane displacements are found from (33).

The fields, which are restored by the plate solution, satisfy the complete set of three-dimensional equations only partially. Thus, the original displacement–strain relationship will be fulfilled in the plane, but the strain in the thickness direction results from the minor terms in the expansion of the displacements. It is important that the determined leading order terms asymptotically approach the exact solution of the original problem when the thickness of the plate is decreasing. An experimental evidence of

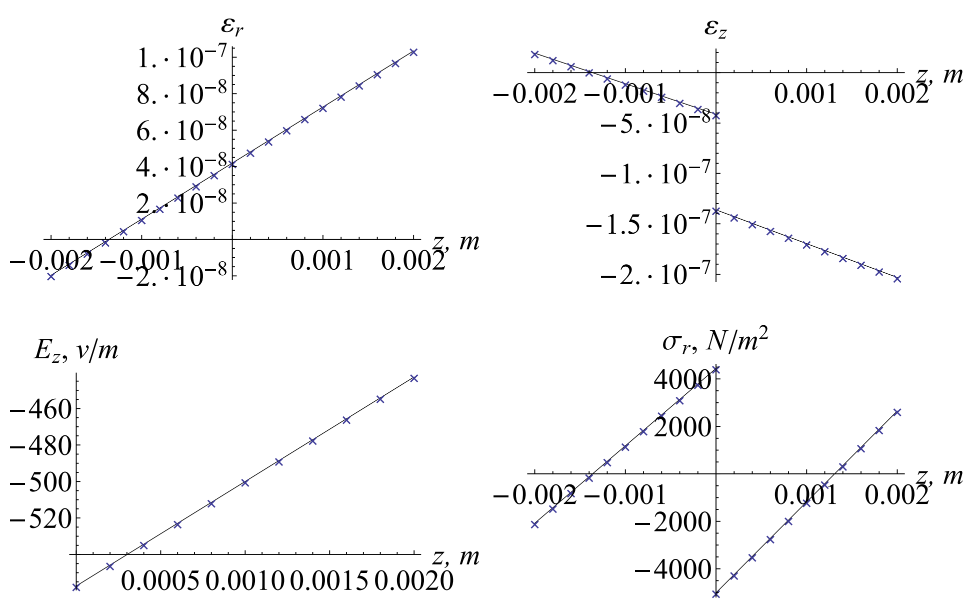

Fig. 3. Analytically restored distribution of three-dimensional fields over the thickness (lines) in comparison with nodal values in the finite element solution (crosses).

this fact is based on the comparison with the numerical solution of a sample problem, presented in Section 8. While for thin plates the results, obtained by means of finite element analysis, are very close to the analytical prediction based on the presented theory, the converged numerical solution required very fine meshes. For practical problems it can lead to an extremely high computational effort, as commercial finite element software like ABAQUS or ANSYS includes neither plate nor shell finite elements suitable for modeling of coupled electromechanical problems.

The practically important task of active control of deformable structures by means of piezoelectric sensors and actuators (see, e.g. Nader, 2008) has recently been studied for the case of measuring geometrically nonlinear deformations of rods (Krommer and Vetyukov, 2009) and shells (Vetyukov and Krommer, 2010b). Further work in this direction, including practical implementation of an experimental set up, requires a trustworthy modeling technique for the geometrically nonlinear behavior of thin-walled structures, equipped with piezoelectric sensors and actuators. This challenging problem can be solved with the help of a combined application of a direct approach, accomplished by the present results of asymptotic analysis of the three-dimensional problem, see Vetyukov and Krommer (2010a) for the first results.

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Appendix A. Parameters of the cross-section of a layered plate

Applying the procedure of Section 7.2 to the plate structure, considered in Section 8, we arrive at the following expressions of the coefficients in the constitutive relations (90):

$$A_1 = \frac{h\left(-C_5(C_1+E\nu)+\left(C_1C_5-C_3^2\right)\nu^2+C_3^2\right)}{2C_5(\nu^2-1)},$$
$$A_2 = \frac{h}{2}\left(C_2+\frac{E}{1+\nu}\right),\quad B_1=\frac{1}{8}h^2\left(\frac{C_3^2}{C_5}-C_1+\frac{E\nu}{1-\nu^2}\right),$$
$$B_2=\frac{h^2}{8}\left(-C_2+\frac{E}{1+\nu}\right),\quad D_2=\frac{h^3}{24}\left(C_2+\frac{E}{1+\nu}\right),$$
$$D_1=\frac{1}{96}h^3\left(\frac{\frac{(C_5e_{31}-C_3e_{33})^2}{C_5\epsilon_z+e_{33}^2}-4C_3^2+4C_1C_5}{C_5}+\frac{4E\nu}{1-\nu^2}\right),$$
$$p=-\frac{C_3e_{33}}{C_5}+e_{31},\quad m=\frac{h}{4}\left(\frac{C_3e_{33}}{C_5}-e_{31}\right),\quad c=-\frac{2\left(C_5\epsilon_z+e_{33}^2\right)}{C_5h}. \tag{97}$$

These parameters correspond to the middle layer of the plate $z = 0$ at the interface between the two materials. Young's modulus and Poisson's ratio of the structure are denoted as E and ν, coefficients C_i answer to the form of the stiffness tensor for the piezoelectric orthotropic material (27), $\epsilon_z = \boldsymbol{k}\cdot\boldsymbol{\epsilon}\cdot\boldsymbol{k}$, and $\boldsymbol{k}\cdot{}^3\mathbf{e} = e_{31}\mathbf{I}_\perp + e_{33}\boldsymbol{kk}$.

Numerical values of the material parameters for aluminium are standard, and for PZT-5A we use the values, given by Nader (2008) (SI system of units is implied):

$$E=7.1\cdot10^{10},\quad \nu=0.33,\quad C_1=7.54\cdot10^{10},\quad C_2=4.56\cdot10^{10},$$
$$C_3=7.52\cdot10^{10},\quad C_4=4.22\cdot10^{10},\quad C_5=11.1\cdot10^{10},$$
$$\epsilon_z=1700\epsilon_0,\quad \epsilon_0=8.854\cdot10^{-12},\quad e_{31}=-5.4,\quad e_{33}=15.8. \tag{98}$$

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