Stabilizability, Controllability, and Observability of Linear Continuous-time Systems Defined Over a Commutative Banach Algebra

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ABSTRACT

This paper is concerned with the controllability, observability, and stabilizability for linear continuous-time systems defined over a Hermitian commutative Banach* algebra. Practical applications of such systems can be found in the joint work of Kamen and Green. It is shown that, after Gelfand transformation, the abovementioned three properties of such systems are reduced to the corresponding ones in finite-dimensional linear system theory. In the process, we basically solve a problem posed by Kamen and Green.

1. INTRODUCTION AND PRELIMINARIES

In the finite-dimensional case, all the questions of controllability, observability, and stabilizability for linear systems have been reduced to simple forms by Wonham, Brocket, Wolovich, and others [1-4], but due to the complexity of infinite-dimensional systems (IDLS), all the abovementioned three questions of IDLS become manifold and difficult. Thus in the research area of IDLS, there have appeared many different approaches. Applying the Gelfand-transform technique, Kamen and Green specified the stabilizability of linear discrete-time systems over a commutative Banach algebra [5-7]. They also pointed out that the problem of stabilizability for related continuous-time systems remained unsolved [7]. In this paper, we solve this problem by analysing the continuity of the solutions of a continuous family of matrix Riccati equations. At the same time, we also study the controllability
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and observability of such linear systems. Our interest lies in the mathematical foundation of the theory; for the practical applications of such linear systems see References [5–7].

In what follows, we use the notions of Reference [7]. Let \( B \) denote a complex commutative Hermitian Banach* algebra with unit; that is, \( B \) as a commutative Banach algebra has a continuous involution \( a \rightarrow a^* \), and the spectrum of every self-adjoint element is real [8]. Given a positive integer \( n \), we shall denote by \( B^n \) the complex Banach space of all \( n \)-element vectors with entries in \( B \) and with norm

\[
\|x\|_2 = \left( \sum_{i=1}^{n} \|x_i\|^2 \right)^{1/2}
\]

where \( x_i \) is the \( i \)th component of the vector \( x \in B^n \). Given a positive integer pair \((m,n)\), we define \( M_{m \times n}(B) \) as the complex vector space consisting of all \( m \times n \) matrices with entries in \( B \), with natural addition and scalar multiplication, and with norm

\[
\|A\|_2 = \sup\{\|Ax\|_2 : \|x\|_2 \leq 1, x \in B^n\}
\]

for \( A \in M_{m \times n}(B) \). In particular, \( M_n(B) \) is defined to be \( M_{n \times n}(B) \). Then each element in \( M_{m \times n}(B) \) can be thought of as a bounded linear operator from the Banach space \( B^n \) to \( B^m \). Under this consideration, \( M_n(B) \) is a unital Banach subalgebra of \( L(B^n) \) (the operator algebra consisting of all bounded linear operators on \( B^n \)). Given \( A \in M_n(B) \), \( \sigma(A) \) is defined to be \( \sigma_{M_n(B)}(A) \), i.e., \( \sigma(A) \) represents the spectrum of \( A \) in \( M_n(B) \) [5]. For the related involution on \( M_{m \times n}(B) \), given \( L = (l_{ij}) \in M_{m \times n}(B) \), \( L^* \) is defined to be \( L^* = (l_{ji}^*) \) (see Reference [7]); with this involution, \( M_n(B) \) becomes a Hermitian Banach* algebra, which is generally not commutative.

Given \( G \in M_n(B) \), \( L \in M_{n \times m}(B) \), \( Q \in M_{r \times n}(B) \), the matrix triple \((G,L,Q)\) determines a linear continuous-time and bounded system, which has state space \( B^n \), input space \( B^m \), and output space \( B^r \), i.e.,

\[
\dot{x}(t) = Gx(t) + Lu(t),
\]

\[
y(t) = Qx(t),
\]

\[
x(0) = x_0 \in B^n,
\]

where \( x(t), u(t), y(t) \) are locally Lebesgue-Bochner integrable functions from
We denote the control subsystem of the system (1.1) by \((G, L)\) and the observed subsystem by \((G, Q)\). Now, we introduce the concepts of controllability, observability, and stabilizability. Our definitions are natural generalizations of corresponding ones in finite-dimensional system theory [1, 2]; in particular, our definition of stabilizability is the same as that of Kamen and Green.

**Definition 1.2** [9, 10]. Let \(G \in M_n(B)\), \(L \in M_{n \times m}(B)\), and \(Q \in M_{m \times n}(B)\). We say that the control system \((G, L)\) is exactly controllable if there exists a positive integer \(N\) such that \(B^n = \sum_{i=1}^{N} G^iLB^m\). The control system \((G, L)\) is said to be approximately controllable if \(B^n\) is the linear closure of \(\{G^kLB^m : k \geq 0\}\), in short, \(B^n = \bigvee_{k=0}^\infty G^kLB^m\). The observed system \((G, Q)\) is said to be exactly (approximately) observable if its dual control system \((G^*, Q^*)\) is exactly (approximately) controllable.

**Remark 1.** In Definition 1.2, when \(B = C\) (the complex field), our exact (approximate) controllability and exact (approximate) observability are respectively the controllability and observability in finite-dimensional system theory [1].

**Remark 2.** We know that \(B^n = \bigvee_{k=0}^\infty G^kLB^m\) if and only if given \(t_0 > 0\) and \(1 \leq p < \infty\), the operator \(W(t_0, p)\) from \(L^p(0, t_0; B^n)\) to \(B^n\) has dense range in \(B^n\), where \(W(t_0, p)\) is defined by \(u(\cdot) \in L^p(0, t_0; B^n) \rightarrow \int_0^{t_0} e^{G_sL}u(s)\, ds\). So the approximate controllability defined in Definition 1.2 is just the approximate controllability of the continuous-time control system (1.1); for this, see Reference [10].

**Definition 1.4.** The control system \((G, L)\) is said to be exponentially asymptotically stabilizable if there exists a matrix \(F \in M_{m \times n}(B)\) such that

\[
\lim_{t \to \infty} \|\exp[(G-LF)t]\| = 0.
\]

Here we record a proposition in Reference [5] for convenience and completeness.

**Lemma 1.5** [5, Proposition 1]. Let \(D \in M_n(B)\). Then \(\sigma(D) = \bigcup \{\sigma(\hat{D}(\varphi)) : \varphi \in X\} = \bigcup_{\varphi \in X} \{\lambda_1(\varphi), \lambda_2(\varphi), \ldots, \lambda_n(\varphi)\}\), where \(\sigma(\hat{D}(\varphi)) = \{\lambda_1(\varphi), \lambda_2(\varphi), \ldots, \lambda_n(\varphi)\}\) represents the set of eigenvalues of the \(n \times n\) scalar matrix \(\hat{D}(\varphi)\).
Let $X$ denote the carrier space of $B$. Given $b \in B$, let $\hat{b}$ denote the Gelfand transform of $b$, which is a continuous function on the compact Hausdorff space $X$. Then for a matrix $L = (l_{ij}) \in M_{n \times n}(B)$, the Gelfand transform of $L$ is defined to be $\hat{L} = (\hat{l}_{ij})_{mn}$; it can be considered as a continuous scalar matrix function on $X$.

It is an easy fact that $(\det L)(\varphi) = \det(\hat{l}_{ij}(\varphi))$ for any square matrix $L = (l_{ij})$ over $B$ and for any $\varphi \in X$, so by Lemma 1.5, the following facts are equivalent:

1. $L \in M_n(B)$ is invertible.
2. For each $\varphi \in X$, det$(\hat{l}_{ij}(\varphi)) \neq 0$.
3. det $L$ is invertible in $B$.

From this, our definition of the Gelfand transform on matrices over $B$ is reasonable.

2. CONTROLLABILITY AND OBSERVABILITY

**Lemma 2.1.** Suppose $A_k \in M_n(B)$, $k = 1, 2, \ldots, s$. If the $A_k$ satisfy $\bigcap_{k=1}^s \{ \varphi \in X : \det A_k(\varphi) = 0 \} = \emptyset$, then there exist $H_k \in M_n(B)$, $k = 1, 2, \ldots, s$, such that $\sum_{k=1}^s A_k H_k = I_n$, where $I_n$ is the $n \times n$ diagonal matrix whose entries in the main diagonal are the identity of $B$.

**Proof.** Let $Q = \sum_{k=1}^s A_k A_k^*$. We prove that $Q$ is invertible in $M_n(B)$. Otherwise, by Lemma 1.5, there exists $\varphi_0 \in X$ such that $Q(\varphi_0) = \sum_{k=1}^s A_k(\varphi_0)A_k(\varphi_0)^*$ is singular. But $Q(\varphi_0) > A_k(\varphi_0)A_k(\varphi_0)^* > 0$; thus $A_k(\varphi_0)$ is also singular for $k = 1, 2, \ldots, s$, i.e., $\varphi_0 \in \bigcap_{k=1}^s \{ \varphi \in X : \det A_k(\varphi) = 0 \}$, in contradiction with our assumption. Therefore $Q$ is invertible in $M_n(B)$, and the proof is finished with $H_k = A_k^* Q^{-1}$, $k = 1, 2, \ldots, s$.

**Lemma 2.2.** Given two natural numbers $p$ and $q$, with $p > q$, let $Q \in M_{p \times p}(B)$. Then the following conditions are equivalent:

1. There exists $H \in M_{p \times q}(B)$ such that $QH = I_q$.
2. For each $\varphi \in X$, $Q(\varphi)$ considered as a linear transformation from $C^p$ to $C^q$ is onto.
3. For each $\varphi \in X$, the rank of the scalar matrix $Q(\varphi)$ is $q$.

**Proof.** Obviously, (2) and (3) are equivalent, and (3) is implied by (1). We shall show that (3) implies (1). In fact, given $L \in M_{p \times p}(B)$, by Lemma 1.5, $\sigma(L) = \bigcup \{ \sigma(\hat{L}(\varphi)) : \varphi \in X \}$; hence the result is valid when $p = q$.

For the case $p > q$, the matrix $Q$ has $q$ rows and $p$ columns. We denote the $p$ columns by $Q_1, Q_2, \ldots, Q_p$, from left to right. If we choose columns
Let $i_1, i_2, \ldots, i_q$ of $Q$, $1 \leq i_1 < i_2 < \cdots < i_q \leq p$, then we can construct a $q \times q$ matrix

$$A_{i_1i_2\ldots i_q} = \begin{bmatrix} Q_{i_1} & Q_{i_2} & \cdots & Q_{i_q} \end{bmatrix}.$$ 

From (3),

$$\cap_{1 \leq i_1 < i_2 < \cdots < i_q \leq p} \{ \varphi \in X : \det A_{i_1i_2\ldots i_q}(\varphi) = 0 \} = \emptyset.$$ 

Thus by Lemma 2.1, there exist $L_{i_1i_2\ldots i_q} \in M_q(B)$, $1 \leq i_1 < i_2 < \cdots < i_q \leq p$, such that

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_q \leq p} A_{i_1i_2\ldots i_q} L_{i_1i_2\ldots i_q} = I_q. \quad (\ast)$$

We suppose the rows of $L_{i_1i_2\ldots i_q}$ are respectively

$L^1_{i_1i_2\ldots i_q}$, $L^2_{i_1i_2\ldots i_q}$, \ldots, $L^q_{i_1i_2\ldots i_q}$

(ordered from top to bottom in $L_{i_1i_2\ldots i_q}$). Let

$$H_{i_1i_2\ldots i_q} \in M_{p \times q}(B) = \begin{bmatrix} 0 & \cdots & 0 \\ L^1_{i_1i_2\ldots i_q} & \cdots & L^q_{i_1i_2\ldots i_q} \end{bmatrix}$$

with

- $i_1$th row
- $i_2$th row
- $i_q$th row
Then by \((*)\)

\[
\sum_{1 \leq i_1 < i_2 < \cdots < i_q \leq p} QH_{i_1i_2\ldots i_q} = I_q.
\]

Assume

\[
H = \sum_{1 \leq i_1 < i_2 < \cdots < i_q \leq p} H_{i_1i_2\ldots i_q}.
\]

Therefore \(H \in M_{p \times q}(B)\) and \(QH = I_q\).

Combined with left invertibility, Lemma 2.2 has the following dual result.

**Lemma 2.2'.** Given two natural numbers \(n\) and \(s\), \(n \geq s\), and \(P \in M_{n \times s}(B)\), then the following conditions are equivalent:

1. There is \(Q \in M_{s \times n}(B)\) such that \(QP = I_s\).
2. For each \(\varphi \in X\), \(\hat{P}(\varphi)\) considered as a linear transformation from \(C^s\) to \(C^n\) is one to one.
3. For each \(\varphi \in X\), the rank of the scalar matrix \(\hat{P}(\varphi)\) is \(s\).

**Theorem 2.3.** Given \(G \in M_n(B)\), \(L \in M_{n \times m}(B)\), the following conditions are equivalent:

1. The control system \((G, L)\) is exactly controllable.
2. The control system \((G, L)\) is approximately controllable.
3. For each \(\varphi \in X\), the finite-dimensional linear control system \((\hat{G}(\varphi), \hat{L}(\varphi))\) is controllable [1].
4. For each complex number \(\lambda\), there exists \(T_\lambda \in M_{(n+m) \times n}(B)\) such that \((\lambda I_n - G, L)T_\lambda = I_n\).
5. There exist matrices \(K_i \in M_{m \times n}(B)\) for \(i = 0, 1, 2, \ldots, n-1\) such that \(\sum_{i=0}^{n-1} G^i LK_i = I_n\).
6. For each \(\varphi \in X\), the \(n \times (mn)\) scalar matrix \([\hat{L}(\varphi) \hat{G}(\varphi)\hat{L}(\varphi) \cdots \hat{G}^{(n-1)}(\varphi)\hat{L}(\varphi)]\) has rank \(n\).

**Proof.** From Lemma 2.2 and References [1, 9], we see that (3), (4), (5), and (6) are equivalent; also, (1) \(\Rightarrow\) (2) and (5) \(\Rightarrow\) (1) are trivial. So it remains to prove that (2) \(\Rightarrow\) (3). In fact, if the control system \((G, L)\) is approximately controllable, then from Definition 1.2, \(V_{k=0}^\infty G^k LB^m = B^n\). Identifying \(M_{m \times 1}(B) [M_{n \times 1}(B)]\) with \(B^m [B^n]\) and finding the Gelfand transform, we deduce that \(V_{k=0}^\infty G^k(\varphi)\hat{L}(\varphi)C^m = C^n\) for each \(\varphi \in X\); hence the finite-
dimensional linear control system \((\hat{G}(\varphi), \hat{L}(\varphi))\) is controllable for each \(\varphi \in X\) [1].

Corresponding to Lemma 2.2', on the observability of such systems, Theorem 2.3 has following dual result:

**Theorem 2.3.** Given \(G \in M_n(B), Q \in M_{m \times n}(B)\), the following conditions are equivalent:

1. The observed system \((G, Q)\) is exactly observable.
2. The observed system \((G, Q)\) is approximately observable.
3. For each \(\varphi \in X\), the finite-dimensional observed system \((\hat{G}(\varphi), \hat{L}(\varphi))\) is observable.
4. For each complex number \(\lambda\), there is \(H_\lambda \in M_{n \times (n + m)}(B)\) such that
   \[
   H_\lambda \begin{bmatrix} \lambda I_n - G \\ Q \end{bmatrix} = I_n.
   \]
5. There exist \(K_i \in M_{n \times m}(B)\) for \(i = 0, 1, \ldots, n - 1\) such that \(\sum_{i=0}^{n-1} K_i Q G^i = I_n\).
6. For each \(\varphi \in X\), the \((nm) \times n\) scalar matrix
   \[
   \begin{bmatrix}
   \hat{Q}(\varphi) \\
   \hat{Q}(\varphi) \hat{G}(\varphi) \\
   \vdots \\
   \hat{Q}(\varphi) \hat{G}^{(n-1)}(\varphi)
   \end{bmatrix}
   \]
   has rank \(n\).

Remark 1. Generally speaking, exact controllability (observability) implies approximate controllability (observability) for ordinary bounded linear systems defined over Banach spaces, but the converse is usually untrue. However, for the bounded linear systems defined over a commutative Hermitian Banach* algebra, the exact property and the approximate property are congruent; see Theorems 2.3 and 2.3'.

Remark 2. For the linear systems discussed in this paper, the problems of controllability and observability have been reduced to the corresponding ones in finite-dimensional system theory [1]. The latter affords many simple criteria; thus Theorems 2.3 and 2.3' are of practical value.
3. STABILIZABILITY

**Theorem 3.1.** Given $G \in M_n(B)$ and $L \in M_{n \times m}(B)$, the linear continuous-time control system determined by $(G, L)$, i.e.,
\[
\dot{x}(t) = Gx(t) + Lu(t), \quad t > 0,
\]
\[
x(0) = x_0,
\]
is exponentially asymptotically stabilizable if and only if for each $\varphi \in X$, the finite-dimensional control system $(\hat{G}(\varphi), \hat{L}(\varphi))$ is exponentially asymptotically stabilizable [1].

In order to prove Theorem 3.1, it is necessary to establish the following basic facts:

1. If $\Gamma$ is the Gelfand transform from $B$ to $C(X)$, then $\Gamma(B)$ is a dense *-subalgebra of $C(X)$.

   Clearly $\Gamma$ is a *-homomorphism and $\Gamma(B)$ separates the points of $X$; thus by the Stone-Weierstrass theorem, $\Gamma(B)$ is a dense *-subalgebra of $C(X)$.

2. For each finite-dimensional control system $(A, S)$, where $A$ is an $n \times n$ scalar matrix and $S$ is an $n \times m$ scalar matrix, $(A, S)$ is exponentially asymptotically stabilizable if and only if the algebraic matrix Riccati equation
   \[
   A^*P + PA - S^*PS + P^*P + I_n = 0
   \]
   has a unique positive definite solution $P \in M_n(C)$ [1], where $I_n$ is the identity scalar matrix of order $n$.

3. Given a compact Hausdorff space $J$, suppose $A : J \to M_n(C)$ and $S : J \to M_{n \times m}(C)$ are continuous scalar-matrix-valued functions on $J$, and for each $\omega \in J$, the finite-dimensional linear control system $(A(\omega), S(\omega))$ is exponentially asymptotically stabilizable. Then the algebraic Riccati equation
   \[
   A^*(\omega)P(\omega) + P(\omega)A(\omega) - S(\omega)^*S(\omega)P(\omega) + I_n = 0 \quad (\omega \in J)
   \]
   has a continuous positive definite solution $P(\omega) \in M_n(C)$ on $J$.

   This follows easily from fact (2) and Theorem 3.6 in Reference [11].

4. Suppose $U$ is a unital Banach algebra, $a \in U$, and the spectrum of $a$ is $\sigma(a) \subset C_- = \{ \lambda \in C : \text{Re} \lambda < 0 \}$. Then $\lim_{t \to +\infty} e^{at} = 0$.

   In fact, $\sigma(a) \subset \{ \lambda \in C : \text{Re} \lambda < 0 \}$; thus by the spectral mapping theorem in Banach-algebra theory [12], $\sigma(e^{at}) \subset \{ \lambda \in C : |\lambda| < 1 \}$. Thus the spectral
radius of \( e^a \) is

\[
\gamma_a(e^a) = \lim_{n \to \infty} \| (e^a)^n \|^{1/n} = \lim_{n \to \infty} \| e^{na} \|^{1/n} < 1,
\]

so there is \( 0 < l < 1 \) and a natural number \( N \) such that \( \| e^{na} \|^{1/n} < l \) whenever \( n > N \). Therefore \( \| e^{na} \| < l^n = e^{n \ln l} \).

For any real number \( t \), whenever \( t > N \),

\[
e^{ta} = e^{tE_0} e^{(t-\ln l)A}.
\]

Let \( \nu = -\ln l > 0 \), and \( M = \max_{0 < t < 1} \| e^{ta} \| \). Then \( \| e^{ta} \| \leq \| e^{tE_0} \| M < e^{-\nu t} M \), so \( \lim_{t \to \infty} \| e^{ta} \| = 0 \).

**Proof of Theorem 3.1.** The necessity obviously follows from Lemma 1.5 and Reference [1]. For sufficiency, for each \( \phi \in X \), the finite-dimensional control system \( (\hat{G}(\phi), \hat{L}(\phi)) \) is exponentially asymptotically stabilizable; moreover, \( G \) and \( L \) are continuous scalar-matrix-valued functions on the compact Hausdorff space \( X \). Thus, from fact (3) above, the Riccati equation

\[
\hat{G}^*(\phi) P(\phi) + P(\phi) \hat{G}(\phi) - P(\phi) \hat{L}(\phi) \hat{L}^*(\phi) P(\phi) + 1_n = 0 \quad (\phi \in X)
\]

has a continuous positive definite solution \( P(\phi) \in M_n(C), \phi \in X \), i.e.,

\[
\{ \hat{G}(\phi) - \hat{L}(\phi) \hat{L}^*(\phi) P(\phi) \}^* P(\phi) + P(\phi) \{ \hat{G}(\phi) - \hat{L}(\phi) \hat{L}^*(\phi) P(\phi) \} = -1_n - P(\phi) \hat{L}(\phi) \hat{L}^*(\phi) P(\phi).
\]

Therefore, for every \( \phi \in X \), \( \sigma(\hat{G}(\phi) - \hat{L}(\phi) \hat{L}^*(\phi) P(\phi)) \subset C_- = \{ \lambda \in C : \Re \lambda < 0 \} \) [1]. We can consider \( \hat{G} - \hat{L} \hat{L}^* P \) as an \( n \times n \) matrix over \( C(X) \). Then \( \sigma(G - LL^* P) \subset C_- \) (see Lemma 1.5). By the upper semicontinuity of the spectrum [12] and fact (1) above, there exists an \( n \times n \) matrix \( H \) over \( B \) such that \( H \in M_n(B) \) and \( \sigma(\hat{G} - \hat{L} \hat{L}^* \hat{H}) \subset C_- \). Again using Lemma 1.5, we have \( \sigma(G - LL^* H) = \sigma(\hat{G} - \hat{L} \hat{L}^* H) \subset C_- \); hence when \( t \to \infty \), \( \| \exp(G - LL^* H) t \| \) approaches zero [see fact (4)].

**Corollary 3.2.** If the control system \( (G, L) \) is approximately controllable, then it is exponentially asymptotically stabilizable.

**Corollary 3.3.** The control system \( (G, L) \) is exponentially asymptotically stabilizable if and only if for each \( \lambda \in C \) with \( \Re \lambda \geq 0 \) there is an \( m \times n \) matrix \( K_\lambda \) over \( B \) such that \( \lambda I_n - (G + L K_\lambda) \) is invertible in \( M_n(B) \).

**REFERENCES**


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