Note on an Iyengar type inequality

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Abstract


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In 1938 Iyengar proved the following inequality in [1]:

**Theorem 1.** Let function $f$ be differentiable on $[a, b]$ and $|f'(x)| \leq M$. Then

$$ \left| \frac{1}{b-a} \int_a^b f(x) \, dx - \frac{f(a) + f(b)}{2} \right| \leq \frac{M(b-a)}{4} - \frac{(f(b) - f(a))^2}{4M(b-a)}. $$

(1)

Through the years, Iyengar’s inequality (1) has been generalized in various ways. Set

$$ I = \int_a^b f(x) \, dx - \frac{1}{2}(b-a)(f(a) + f(b)) + \frac{1}{8}(b-a)^2(f'(b) - f'(a)). $$

In [2] and [3], the following Iyengar type inequality was obtained:

**Theorem 2.** Let $f \in C^2[a, b]$ and $|f''(x)| \leq M$. Then

$$ |I| \leq \frac{M}{24}(b-a)^3 - \frac{|\Delta|^3}{24M^2}, $$

(2)

where $\Delta = f'(a) - 2f'(a + \frac{b-a}{2}) + f'(b)$.
Theorem 3. Let \( f \in C^2[a, b] \) and \( |f''(x)| \leq M \). Then
\[
|I| \leq \frac{M(b - a)^3}{24} - \frac{(b - a)^2 \Delta_1^2}{8[M(b - a) + f'(a) - f'(b)]}
\]
where \( \Delta_1 = f'(a) - 2\frac{f(b) - f(a)}{b-a} + f'(b) \).

He also showed that, for some classes of functions, inequality (3) gives better estimations than inequality (2).

In this work, under same assumptions, we will prove another Iyengar type inequality and show that it is always better than (2) and better than (3) for the same class of functions for which (3) is better than (2).

For the proof of our result we will use the Hayashi modification of Steffensen’s well-known inequality, so we will state it first (see [5]).

Theorem 4. Let \( F : [a, b] \to \mathbf{R} \) be a decreasing function and \( G : [a, b] \to \mathbf{R} \) an integrable function such that \( 0 \leq G(x) \leq A \) for each \( x \in [a, b] \). Then
\[
A\int_{b-\lambda}^{b} F(x) \, dx \leq \int_{a}^{b} F(x)G(x) \, dx \leq A\int_{a}^{a+\lambda} F(x) \, dx,
\]
where \( \lambda = \frac{1}{A} \int_{a}^{b} G(x) \, dx \).

Now we state our main result:

Theorem 5. Let \( f \in C^2[a, b] \) and \( |f''(x)| \leq M \). Then
\[
\frac{M(b - a)^3}{24} + \frac{M}{3} \left( \lambda_a^3 + \lambda_b^3 \right) \leq I
\]
\[
\leq \frac{M(b - a)^3}{24} - \frac{M}{3} \left[ \left( \frac{b-a}{2} - \lambda_a \right)^3 + \left( \frac{b-a}{2} - \lambda_b \right)^3 \right],
\]
where
\[
\lambda_a = \frac{1}{2M} \left( f' \left( \frac{a+b}{2} \right) - f'(a) \right) + \frac{b-a}{4},
\]
\[
\lambda_b = \frac{1}{2M} \left( f'(b) - f' \left( \frac{a+b}{2} \right) \right) + \frac{b-a}{4}.
\]

Proof. Set \( \Theta = (a+b)/2 \), \( F(x) = (x - \Theta)^2/2, G(x) = f''(x) + M \). We have: \( 0 \leq G(x) \leq 2M \) for each \( x \in [a, b] \), so \( G(x) \) satisfies the conditions of Theorem 4. It is clear that function \( F(x) \) is decreasing on \( [a, \Theta] \) and increasing on \( [\Theta, b] \). Therefore, inequality (4) is in that case reversed.

So, on \( [a, \Theta] \), inequality (4) gives us
\[
2M \int_{\Theta-\lambda_a}^{\Theta} F(x) \, dx \leq \int_{a}^{\Theta} F(x)G(x) \, dx \leq 2M \int_{a}^{a+\lambda_a} F(x) \, dx
\]
i.e.,
\[
\frac{M}{3} \lambda_a^3 \leq \int_{a}^{\Theta} F(x)G(x) \, dx \leq \frac{M}{3} \left[ \left( \frac{b-a}{2} \right)^3 - \left( \frac{b-a}{2} - \lambda_a \right)^3 \right]
\]
where \( \lambda_a = \frac{1}{2M} \int_{a}^{\Theta} G(x) \, dx \) as is defined in (6).

On \( [\Theta, b] \), we have
\[
2M \int_{\Theta}^{\Theta+\lambda_b} F(x) \, dx \leq \int_{\Theta}^{b} F(x)G(x) \, dx \leq 2M \int_{b-\lambda_b}^{b} F(x) \, dx,
\]
Proof.

for each \( x \)

\[
\frac{M}{3} \lambda_b^3 \leq \int_{\Theta}^b F(x)G(x) \, dx \leq \frac{M}{3} \left[ \left( \frac{b-a}{2} \right)^3 - \left( \frac{b-a}{2} - \lambda_b \right)^3 \right] \tag{9}
\]

where \( \lambda_b = \frac{1}{M} \int_{\Theta}^b G(x) \, dx \) as is defined in (7).

Addition of (8) and (9) produces inequality (5). That is,

\[
\int_a^b F(x)G(x) \, dx = I + \frac{M}{24} (b-a)^3.
\]

The proof of this theorem is thus complete. \( \square \)

**Remark 1.** Inequality (5) should be compared with results obtained in [6].

In [7], Vasić and Pećarić derived necessary and sufficient conditions for extension of Steffensen’s inequality. They proved that (4) holds if and only if function \( G \) satisfies

\[
0 \leq \int_x^b G(t) \, dt \leq A(b-x) \quad \text{and} \quad 0 \leq \int_a^x G(t) \, dt \leq A(x-a). \tag{10}
\]

Using this, we can obtain inequality (5) under a weaker condition on function \( f \).

**Theorem 6.** Assume \( f \in C^1[a, b] \) and

\[
|f'(x) - f'(a)| \leq M(x-a), \quad |f'(b) - f'(x)| \leq M(b-x) \tag{11}
\]

for each \( x \in [a, b] \), then we have (5).

**Proof.** Using (11) it is easy to see that function \( G(t) = f'''(t) + M \) satisfies both conditions in (10), so we can apply Hayashi’s modification of Steffensen’s inequality and from there we get our statement. \( \square \)

Next, we will compare estimations in (2) and (5). Define

\[
H(x) = \frac{M}{3} \left( \frac{a+b}{2} - x \right)^3, \quad \text{for } x \in [a, b].
\]

Now we can write (2) and (5) in the following form:

\[
|I| \leq \frac{M(b-a)^3}{24} - |H(a+\lambda)|, \tag{12}
\]

\[
H(\Theta - \lambda_a) - H(\Theta + \lambda_b) - \frac{M(b-a)^3}{24} \leq I \leq \frac{M(b-a)^3}{24} - H(a + \lambda_a) + H(b - \lambda_b), \tag{13}
\]

where \( \lambda = \lambda_a - \lambda_b + \frac{b-a}{2} \).

\( H(x) \) is decreasing, \( H(\Theta) = 0 \) and \( a + \lambda_a \leq \Theta \leq b - \lambda_b, 0 \leq \lambda_a, \lambda_b \leq \frac{b-a}{2} \). Also, \( a + \lambda_a \leq a + \lambda \leq b - \lambda_b \).

Assume first \( \lambda_a \leq \lambda_b \). Then \( H(a+\lambda) \geq 0 \) and

\[
H(a+\lambda) \leq H(a + \lambda_a) \leq H(a + \lambda_a) - H(b - \lambda_b),
\]

since \( H(b - \lambda_b) \leq 0 \). Suppose \( \lambda_a \geq \lambda_b \). Then \( H(a+\lambda) \leq 0 \) and

\[
H(a + \lambda) \geq H(b - \lambda_b) \geq H(b - \lambda_b) - H(a + \lambda_a),
\]

since \( H(a + \lambda_a) \geq 0 \). The proof that the lower bound in (13) is also better is analogous: just notice that \( |H(b - \lambda)| = |H(a + \lambda)| \).

We will finish this work by giving some classes of functions for which (5) gives better estimates than (3). We claim that

\[
\frac{(b-a)^2 \Delta^2}{8[M(b-a) + f''(a) - f''(b)]} \leq \frac{M}{3} \left[ \left( \frac{b-a}{2} - \lambda_a \right)^3 + \left( \frac{b-a}{2} - \lambda_b \right)^3 \right] \tag{14}
\]
for \( f(x) = x^n, n \geq 5 \) on \([0, 1]\). Those inequalities now become
\[
\frac{n-2}{8n} \leq \frac{n(n-1)}{3} \left[ \left( \frac{1}{4} - \frac{1-2^{1-n}}{2(n-1)} \right)^3 + \left( \frac{1}{4} - \frac{2^{1-n}}{2(n-1)} \right)^3 \right].
\]
(16)

\[
\frac{n-2}{8n} \leq \frac{n(n-1)}{3} \left[ \left( \frac{1}{4} + \frac{1-2^{1-n}}{2(n-1)} \right)^3 + \left( \frac{1}{4} + \frac{2^{1-n}}{2(n-1)} \right)^3 \right].
\]
(17)

Routine calculation shows that (16) is valid for \( n \geq 5 \) (for \( n = 2 \) we get equality) and (17) is valid for \( n \geq 2 \). Thus, we have shown that (5) is better than (3) for the same class of functions for which (3) is better than (2).

Further, with no loss in generality, we can consider functions on \([0, 1]\) such that \( f(0) = f'(0) = 0 \) and \(|f''(x)| \leq 1\). Inequalities (14) and (15) turn into
\[
\frac{2f(1) - f'(1)^2}{1 - f'(1)} \leq \frac{1}{24} \left[ \left( 1 - 2f'\left( \frac{1}{2} \right) \right)^3 + \left( 1 - 2f'(1) + 2f'\left( \frac{1}{2} \right) \right)^3 \right].
\]
(18)

\[
\frac{2f(1) - f'(1)^2}{1 - f'(1)} \leq \frac{1}{24} \left[ \left( 1 + 2f'\left( \frac{1}{2} \right) \right)^3 + \left( 1 + 2f'(1) - 2f'\left( \frac{1}{2} \right) \right)^3 \right].
\]
(19)

When \( f(1) = f'(1) = 0 \), or, more generally, when \( 2f(1) = f'(1) \) and \( f'(1) \neq 1 \), (5) gives better estimations than (3), since the right hand sides of (18) and (19) are obviously positive. If we take \( f'(1/2) = t, 0 \leq t \leq 1/2 \), when \( f'(1) = 0 \), (18) and (19) reduce to
\[
4f^2(1) \leq t^2 + 1/12.
\]
(20)

Maximizing the left hand side of (20) using a continuous piecewise linear function with \(|f''(x)| = 1\) (where \( f'' \) exists), (20) will follow if
\[
(t^2 - t - 1/4)^2 \leq t^2 + 1/12.
\]
(21)

Using Wolfram’s Mathematica 5.0, we see that the approximate solutions on \([0, 1/2]\) of the equation in (21) are \( t_1 = 0.044 \) and \( t_2 = 0.395 \), so for \( 0 \leq t \leq t_1 \) or \( t_2 \leq t \leq 1/2 \), (5) is always better than (3). For \( t_1 < t < t_2 \), (3) may give better estimations.

References