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Chromatic uniqueness of a family of K_4-homeomorphs☆

Yan-ling Peng

Department of Mathematics, University of Science and Technology of Suzhou, Suzhou,215009, Jiangsu, China

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Abstract

We discuss the chromaticity of one family of K_4-homeomorphs which has girth 7, and give sufficient and necessary condition for the graphs in the family to be chromatically unique.

Keywords: Chromatic polynomial; K_4-homeomorph; Chromatic uniqueness

1. Introduction

In this paper, we consider graphs which are simple. For such a graph G, let $P(G;\lambda)$ denote the chromatic polynomial of G. Two graphs G and H are chromatically equivalent, denoted by $G \sim H$, if $P(G;\lambda) = P(H;\lambda)$. A graph G is chromatically unique if for any graph H such that $H \sim G$, we have $H \cong G$, i.e., H is isomorphic to G.

A K_4-homeomorph is a subdivision of the complete graph K_4. Such a homeomorph is denoted by $K_4(\alpha, \beta, \gamma, \delta, \varepsilon, \eta)$ if the six edges of K_4 are replaced by the six paths of length $\alpha, \beta, \gamma, \delta, \varepsilon, \eta$, respectively, as shown in Fig. 1.

So far, the study of the chromaticity of K_4-homeomorphs with at least 2 paths of length 1 has been fulfilled (see [2,4,5,11]). Also the study of the chromaticity of K_4-homeomorphs which have girth 3, 4, 5 or 6 has been fulfilled. When referring to the chromaticity of K_4-homeomorphs which have girth 7, we know that only three types of K_4-homeomorphs which have girth 7 need to be solved, i.e. $K4(1, 2, 4, \delta, \varepsilon, \eta)$, $K4(3, 2, 2, \delta, \varepsilon, \eta)$ and $K4(1, 3, 3, \delta, \varepsilon, \eta)$. Because the length of this paper will be too long and some details cannot be left out, we study one type of them, that is the chromaticity of $K_4(1, 3, 3, \delta, \varepsilon, \eta)$ (as Fig. 2) in this paper. The chromaticity of the other two types will be given in other papers.

2. Auxiliary results

In this section we cite some known results used in what follows.

Proposition 1. *Let G and H are chromatically equivalent. Then*

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E-mail address: YanLingPeng35@hotmail.com.

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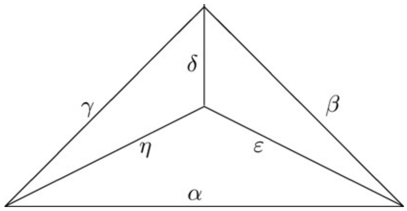

Fig. 1. $K_4(\alpha, \beta, \gamma, \delta, \varepsilon, \eta)$.

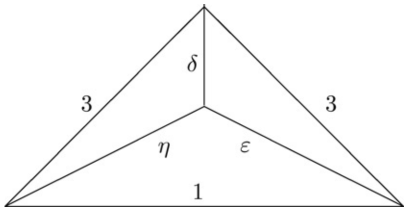

Fig. 2. $K_4(1, 3, 3, \delta, \varepsilon, \eta)$.

(1) $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$ *(see* [3]*);*
(2) *G and H have the same girth and same number of cycles with the length equal to their girth (see* [10]*);*
(3) *If G is a K_4-homeomorph, then H is a K_4-homeomorph as well (see* [1]*);*
(4) *If G and H are homeomorphic to K_4, then both the minimum values of parameters and the number of parameters equal to this minimum value of the graphs G and H coincide (see* [9]*).*

Proposition 2 (*Ren [8]*). *Let $G = K_4(\alpha, \beta, \gamma, \delta, \varepsilon, \eta)$ (see Fig. 1) when exactly three of $\alpha, \beta, \gamma, \delta, \varepsilon, \eta$ are the same. Then G is not chromatically unique if and only if G is isomorphic to $K_4(s, s, s-2, 1, 2, s)$ or $K_4(s, s-2, s, 2s-2, 1, s)$ or $K_4(t, t, 1, 2t, t+2, t)$ or $K_4(t, t, 1, 2t, t-1, t)$ or $K_4(t, t+1, t, 2t+1, 1, t)$ or $K_4(1, t, 1, t+1, 3, 1)$ or $K_4(1, 1, t, 2, t+2, 1)$, where $s \geq 3$, $t \geq 2$.*

Proposition 3 (*Peng [7]*). *Let K_4-homeomorphs $K_4(1, 3, 3, \delta, \varepsilon, \eta)$ and $K_4(3, 2, 2, \delta', \varepsilon', \eta')$ be chromatically equivalent, then $K_4(1, 3, 3, \delta, \varepsilon, \eta)$ is isomorphic to $K_4(3, 2, 2, \delta', \varepsilon', \eta')$.*

Proposition 4 (*Peng [6]*). *Let K_4-homeomorphs $K_4(1, 3, 3, \delta, \varepsilon, \eta)$ and $K_4(1, 3, 3, \delta', \varepsilon', \eta')$ be chromatically equivalent, then we have*

$$K_4(1, 3, 3, a-1, a, a+3) \sim K_4(1, 3, 3, a+1, a-1, a+2).$$

3. Main results

Lemma. *If $G \cong K_4(1, 3, 3, \delta, \varepsilon, \eta)$ and $H \cong K_4(1, 2, 4, \delta', \varepsilon', \eta')$, then we have*

(1) $P(G) = (-1)^{n+1}[r/(r-1)^2][-r^{n+1} + 2 + 2r + Q(G)]$, *where* $Q(G) = -2r^3 - 2r^4 - r^\delta - r^\varepsilon - r^\eta - r^{\varepsilon+1} - r^{\eta+1} + r^{\varepsilon+3} + r^{\eta+3} + r^{\varepsilon+4} + r^{\eta+4} + r^{\delta+6} + r^{\delta+\varepsilon+\eta}$
$r = 1 - \lambda$, *n is the number of vertices of G.*
(2) $P(H) = (-1)^{m+1}[r/(r-1)^2][-r^{m+1} + 2 + 2r + Q(H)]$, *where* $Q(H) = -r^2 - r^3 - r^4 - r^5 - r^{\delta'} - r^{\varepsilon'} - r^{\eta'} - r^{\varepsilon'+1} - r^{\eta'+1} + r^{\eta'+2} + r^{\varepsilon'+3} + r^{\varepsilon'+4} + r^{\eta'+5} + r^{\delta'+6} + r^{\delta'+\varepsilon'+\eta'}$
$r = 1 - \lambda$, *m is the number of vertices of H.*
(3) *If $P(G) = P(H)$, then $Q(G) = Q(H)$.*

Proof. (1) Let $r = 1 - \lambda$. From [9], we have the chromatic polynomial of K_4-homeomorph $K_4(\alpha, \beta, \gamma, \delta, \varepsilon, \eta)$ as follows

$$\begin{aligned} P(K_4(\alpha, \beta, \gamma, \delta, \varepsilon, \eta)) = {} & (-1)^{n+1}[r/(r-1)^2][(r^2 + 3r + 2) - (r+1)(r^\alpha + r^\beta + r^\gamma + r^\delta + r^\varepsilon + r^\eta) \\ & + (r^{\alpha+\delta} + r^{\beta+\eta} + r^{\gamma+\varepsilon} + r^{\alpha+\beta+\varepsilon} + r^{\beta+\delta+\gamma} + r^{\alpha+\gamma+\eta} + r^{\delta+\varepsilon+\eta} - r^{n+1})]. \end{aligned}$$

Then

$$\begin{aligned} P(G) &= P(K_4(1,3,3,\delta,\varepsilon,\eta)) \\ &= (-1)^{n+1}[r/(r-1)^2][(r^2+3r+2)-(r+1)(r^1+r^3+r^3+r^\delta+r^\varepsilon+r^\eta) \\ &\quad + (r^{\delta+1}+r^{\eta+3}+r^{\varepsilon+3}+r^{\varepsilon+4}+r^{\delta+6}+r^{\eta+4}+r^{\delta+\varepsilon+\eta}-r^{n+1})] \\ &= (-1)^{n+1}[r/(r-1)^2][-r^{n+1}+2r+2-2r^3-2r^4-r^\delta-r^\varepsilon-r^\eta-r^{\varepsilon+1}-r^{\eta+1}+r^{\varepsilon+3} \\ &\quad +r^{\eta+3}+r^{\varepsilon+4}+r^{\eta+4}+r^{\delta+6}+r^{\delta+\varepsilon+\eta}] \\ &= (-1)^{n+1}[r/(r-1)^2][-r^{n+1}+2r+2+Q(G)] \end{aligned}$$

where

$$Q(G) = -2r^3-2r^4-r^\delta-r^\varepsilon-r^\eta-r^{\varepsilon+1}-r^{\eta+1}+r^{\varepsilon+3}+r^{\eta+3}+r^{\varepsilon+4}+r^{\eta+4}+r^{\delta+6}+r^{\delta+\varepsilon+\eta}.$$

(2) We can handle this case in the same fashion as case (1), and get the result (2).

(3) If $P(G)=P(H)$, then it is easy to see that $Q(G)=Q(H)$. □

Theorem. *K_4-homeomorphs $K_4(1,3,3,\delta,\varepsilon,\eta)$ (see Fig. 2) which has exactly 1 path of length 1 and has girth 7 is not chromatically unique if and only if it is $K_4(1,3,3,a-1,a,a+3)$, $K_4(1,3,3,a+1,a-1,a+2)$, $K_4(1,3,3,2,b,b+2)$, $K_4(1,3,3,2,4,7)$, $K_4(1,3,3,2,5,8)$, $K_4(1,3,3,5,2,5)$, or $K_4(1,3,3,5,2,6)$, where $a>2$, $b\geq 2$.*

Proof. Let $G\cong K_4(1,3,3,\delta,\varepsilon,\eta)$ and $\min\{\delta,\varepsilon,\eta\}\geq 2$. If there is a graph H such that $P(H)=P(G)$, then from Proposition 1, we know that H is a K_4-homeomorph $K_4(\alpha',\beta',\gamma',\delta',\varepsilon',\eta')$ which has exactly 1 path of length 1, and the girth of H is 7. So H must be one of the following four types:

Type 1:

$$K_4(1,2,\gamma',2,\varepsilon',2)(\varepsilon'\geq 4,\gamma'\geq 4)$$

Type 2:

$$K_4(3,2,2,\delta',\varepsilon',\eta')(\delta'+\varepsilon'\geq 5,\varepsilon'+\eta'\geq 4,\delta'+\eta'\geq 5)$$

Type 3:

$$K_4(1,3,3,\delta',\varepsilon',\eta')(\delta'+\varepsilon'\geq 4,\varepsilon'+\eta'\geq 6,\delta'+\eta'\geq 4)$$

Type 4:

$$K_4(1,2,4,\delta',\varepsilon',\eta')(\delta'+\varepsilon'\geq 5,\varepsilon'+\eta'\geq 6,\delta'+\eta'\geq 3).$$

We now solve the equation $P(G)=P(H)$ to get all solutions.

If H has Type 1, then from Proposition 2, we know that H is chromatically unique. Since $G\sim H$, we have $G\cong H$. But it is obvious that G is not isomorphic to H. This is a contradiction.

If H has Type 2, then from Proposition 3, we know that G is isomorphic to H.

If H has Type 3, then from Proposition 4, we know that the solutions of the equation $P(G)=P(H)$ are

$$K_4(1,3,3,a-1,a,a+3)\sim K_4(1,3,3,a+1,a-1,a+2)$$

where $a>2$.

Suppose that H has Type 4, we solve the equation $Q(G)=Q(H)$. From the Lemma, we have

$$\begin{aligned} Q(G) &= -2r^3-2r^4-r^\delta-r^\varepsilon-r^\eta-r^{\varepsilon+1}-r^{\eta+1}+r^{\varepsilon+3}+r^{\eta+3}+r^{\varepsilon+4}+r^{\eta+4}+r^{\delta+6}+r^{\delta+\varepsilon+\eta} \\ Q(H) &= -r^2-r^3-r^4-r^5-r^{\delta'}-r^{\varepsilon'}-r^{\eta'}-r^{\varepsilon'+1}-r^{\eta'+1}+r^{\eta'+2}+r^{\varepsilon'+3}+r^{\varepsilon'+4}+r^{\eta'+5} \\ &\quad +r^{\delta'+6}+r^{\delta'+\varepsilon'+\eta'}. \end{aligned}$$

We denote the lowest remaining power by l.r.p. and the highest remaining power by h.r.p.. We can assume $\min\{\delta',\varepsilon',\eta'\}\geq 2$, $\varepsilon\leq\eta$, and we know that $\delta+\varepsilon+\eta=\delta'+\varepsilon'+\eta'$ (from Proposition 1). So we obtain the following after simplification:

$$\begin{aligned} Q(G) &: -r^3-r^4-r^\delta-r^\varepsilon-r^\eta-r^{\varepsilon+1}-r^{\eta+1}+r^{\varepsilon+3}+r^{\eta+3}+r^{\varepsilon+4}+r^{\eta+4}+r^{\delta+6} \\ Q(H) &: -r^2-r^5-r^{\delta'}-r^{\varepsilon'}-r^{\eta'}-r^{\varepsilon'+1}-r^{\eta'+1}+r^{\eta'+2}+r^{\varepsilon'+3}+r^{\varepsilon'+4}+r^{\eta'+5}+r^{\delta'+6}. \end{aligned}$$

Comparing the l.r.p. in $Q(G)$ with the l.r.p. in $Q(H)$, we have $\min\{\delta, \varepsilon\} = 2$. There are two cases to be considered.
Case A. If $\min\{\delta, \varepsilon\} = \delta = 2$, then we obtain the following after simplification:

$$Q(G) : -r^3 - r^4 - r^\varepsilon - r^\eta - r^{\varepsilon+1} - r^{\eta+1} + r^{\varepsilon+3} + r^{\eta+3} + r^{\varepsilon+4} + r^{\eta+4} + r^8$$
$$Q(H) : -r^5 - r^{\delta'} - r^{\varepsilon'} - r^{\eta'} - r^{\varepsilon'+1} - r^{\eta'+1} + r^{\eta'+2} + r^{\varepsilon'+3} + r^{\varepsilon'+4} + r^{\eta'+5} + r^{\delta'+6}.$$

By considering the h.r.p. in $Q(G)$, we have the h.r.p. in $Q(G)$ is 8 or $\eta + 4$. There are two cases to be considered.

Case 1. If the h.r.p. in $Q(G)$ is 8, from $Q(G) = Q(H)$, we have $\delta'+6 \le 8$. Since $\delta' \ge 2$, we get $\delta' = 2$. Comparing the l.r.p. in $Q(G)$ with the l.r.p. in $Q(H)$, we have $\delta' = \varepsilon = 2$. From $\delta = 2$, $\delta' = 2$ and $\delta + \varepsilon + \eta = \delta' + \varepsilon' + \eta'$, we get

$$\eta + 2 = \varepsilon' + \eta' \tag{1}$$

we obtain the following after simplification:

$$Q(G) : -2r^3 - r^4 - r^\eta - r^{\eta+1} + r^5 + r^6 + r^{\eta+3} + r^{\eta+4}$$
$$Q(H) : -r^5 - r^{\varepsilon'} - r^{\eta'} - r^{\varepsilon'+1} - r^{\eta'+1} + r^{\eta'+2} + r^{\varepsilon'+3} + r^{\varepsilon'+4} + r^{\eta'+5}.$$

Consider $-2r^3$ and $-r^4$ in $Q(G)$. Since $Q(G) = Q(H)$, there are two terms in $Q(H)$ which are equal to $-r^3$ and there is one term in $Q(H)$ which is equal to $-r^4$. So we have $\varepsilon' = \eta' = 3$ or $\varepsilon' = \eta' + 1 = 3$ or $\varepsilon' + 1 = \eta' = 3$.

If $\varepsilon' = \eta' = 3$, from (1), we get $\eta = 4$. Thus, $G \cong H$.

If $\varepsilon' = \eta' + 1 = 3$, from (1), we get $\eta = 3$, then $Q(G) \ne Q(H)$, a contradiction.

If $\varepsilon' + 1 = \eta' = 3$, from (1), we get $\eta = 3$, then $Q(G) \ne Q(H)$, a contradiction.

Case 2. If the h.r.p. in $Q(G)$ is $\eta + 4$, then

$$\eta + 4 > 8 \tag{2}$$
$$Q(G) : -r^3 - r^4 - r^\varepsilon - r^\eta - r^{\varepsilon+1} - r^{\eta+1} + r^{\varepsilon+3} + r^{\eta+3} + r^{\varepsilon+4} + r^{\eta+4} + r^8$$
$$Q(H) : -r^5 - r^{\delta'} - r^{\varepsilon'} - r^{\eta'} - r^{\varepsilon'+1} - r^{\eta'+1} + r^{\eta'+2} + r^{\varepsilon'+3} + r^{\varepsilon'+4} + r^{\eta'+5} + r^{\delta'+6}$$

By considering the h.r.p. in $Q(H)$, we have the h.r.p. in $Q(H)$ is $\varepsilon' + 4$ or $\eta' + 5$ or $\delta' + 6$. There are three cases to be considered.

Case 2.1. If the h.r.p. in $Q(H)$ is $\varepsilon' + 4$, since the h.r.p. in $Q(G)$ is $\eta + 4$, from $Q(G) = Q(H)$, we have $\eta + 4 = \varepsilon' + 4$. From $\delta = 2$ and $\delta + \varepsilon + \eta = \delta' + \varepsilon' + \eta'$, we get

$$\varepsilon + 2 = \delta' + \eta' \tag{3}$$

we obtain the following after simplification:

$$Q(G) : -r^3 - r^4 - r^\varepsilon - r^{\varepsilon+1} + r^{\varepsilon+3} + r^{\varepsilon+4} + r^8$$
$$Q(H) : -r^5 - r^{\delta'} - r^{\eta'} - r^{\eta'+1} + r^{\eta'+2} + r^{\eta'+5} + r^{\delta'+6}.$$

Consider $-r^3$ and $-r^4$ in $Q(G)$. It is due to $Q(G) = Q(H)$ that there are terms in $Q(H)$ which are equal to $-r^3$ and $-r^4$ respectively. So we have $\delta' + \eta' = 7$ (one of δ', η' is equal to 3), or $\delta' + \eta' + 1 = 7$ (one of δ', $\eta' + 1$ is equal to 3), or $\eta' + \eta' + 1 = 7$.

By $\varepsilon + 2 = \delta' + \eta'$ Eq. (3), It is easy to handle these cases in the same fashion as case 1, and we obtain $Q(G) \ne Q(H)$, a contradiction.

Case 2.2. If the h.r.p. in $Q(H)$ is $\eta' + 5$, we have

$$Q(G) : -r^3 - r^4 - r^\varepsilon - r^\eta - r^{\varepsilon+1} - r^{\eta+1} + r^{\varepsilon+3} + r^{\eta+3} + r^{\varepsilon+4} + r^{\eta+4} + r^8$$
$$Q(H) : -r^5 - r^{\delta'} - r^{\varepsilon'} - r^{\eta'} - r^{\varepsilon'+1} - r^{\eta'+1} + r^{\eta'+2} + r^{\varepsilon'+3} + r^{\varepsilon'+4} + r^{\eta'+5} + r^{\delta'+6}$$

since the h.r.p. in $Q(G)$ is $\eta + 4$, from $Q(G) = Q(H)$, we have $\eta + 4 = \eta' + 5$, that is

$$\eta = \eta' + 1. \tag{4}$$

Since the case of $\eta + 4 = \varepsilon' + 4$ has been discussed in Case 2.1, we can suppose $\eta' + 5 \ne \varepsilon' + 4$. Since the h.r.p. in $Q(H)$ is $\eta' + 5$, we have $\eta' + 5 > \varepsilon' + 4$ and $\eta' + 5 \ge \delta' + 6$, so

$$\eta' + 1 > \varepsilon' \tag{5}$$
$$\eta' \ge \delta' + 1 \tag{6}$$

after simplifying $Q(G)$ and $Q(H)$, we have

$$Q(G) : -r^3 - r^4 - r^{\varepsilon} - r^{\varepsilon+1} - r^{\eta+1} + r^{\varepsilon+3} + r^{\eta+3} + r^{\varepsilon+4} + r^8$$
$$Q(H) : -r^5 - r^{\delta'} - r^{\varepsilon'} - r^{\eta'} - r^{\varepsilon'+1} + r^{\eta'+2} + r^{\varepsilon'+3} + r^{\varepsilon'+4} + r^{\delta'+6}.$$

Consider $r^{\eta'+2}$ in $Q(H)$. It is due to $\eta' + 1 > \varepsilon'$ (from (5)) and $\eta' \geq \delta' + 1$ (from (6)) and $\eta + 4 > 8$ (from (2)) that $r^{\eta'+2}$ cannot be cancelled by the negative terms in $Q(H)$, so none of the negative terms in $Q(H)$ is equal to the term $-r^{\eta+1}$ in $Q(G)$ (by noting $\eta + 1 = \eta' + 2$ (from (4)). So $-r^{\eta+1}$ must be cancelled by the positive term in $Q(G)$. Therefore, $\eta + 1$ must be equal to one of $\varepsilon + 3$, $\varepsilon + 4$, 8 and $\eta' + 2$ must be equal to one of $\varepsilon + 3$, $\varepsilon + 4$, 8. So $\eta + 1 = \eta' + 2 = \varepsilon + 3 = 8$ or $\eta + 1 = \eta' + 2 = \varepsilon + 4 = 8$.

If $\eta + 1 = \eta' + 2 = \varepsilon + 3 = 8$, by $\delta = 2$ and $\delta + \varepsilon + \eta = \delta' + \varepsilon' + \eta'$. It is easy to get $Q(G) \neq Q(H)$, a contradiction.

If $\eta + 1 = \eta' + 2 = \varepsilon + 4 = 8$, since $\delta = 2$ and $\delta + \varepsilon + \eta = \delta' + \varepsilon' + \eta'$, we have $\delta' + \varepsilon' = 7$. After simplifying $Q(G)$ and $Q(H)$, we have

$$Q(G) : -r^3 - r^4 - r^4 + r^7 + r^{10}$$
$$Q(H) : -r^{\delta'} - r^{\varepsilon'} - r^{\varepsilon'+1} - r^6 + r^{\varepsilon'+3} + r^{\varepsilon'+4} + r^{\delta'+6}.$$

Comparing the l.r.p. in $Q(G)$ with the l.r.p. in $Q(H)$, by $\delta' + \varepsilon' = 7$, we have $\varepsilon' = 3$ and $\delta' = 4$. Thus we obtain the solution where G is isomorphic to $K_4(1, 3, 3, 2, 4, 7)$ and H is isomorphic to $K_4(1, 2, 4, 4, 3, 6)$. That is

$$K_4(1, 3, 3, 2, 4, 7) \sim K_4(1, 2, 4, 4, 3, 6).$$

Case 2.3. If the h.r.p. in $Q(H)$ is $\delta' + 6$, we have

$$Q(G) : -r^3 - r^4 - r^{\varepsilon} - r^{\eta} - r^{\varepsilon+1} - r^{\eta+1} + r^{\varepsilon+3} + r^{\eta+3} + r^{\varepsilon+4} + r^{\eta+4} + r^8$$
$$Q(H) : -r^5 - r^{\delta'} - r^{\varepsilon'} - r^{\eta'} - r^{\varepsilon'+1} - r^{\eta'+1} + r^{\eta'+2} + r^{\varepsilon'+3} + r^{\varepsilon'+4} + r^{\eta'+5} + r^{\delta'+6}$$

since the h.r.p. in $Q(G)$ is $\eta + 4$, from $Q(G) = Q(H)$, we have $\eta + 4 = \delta' + 6$, that is

$$\eta = \delta' + 2. \tag{7}$$

Since the case of $\eta + 4 = \varepsilon' + 4$ and the case of $\eta + 4 = \eta' + 5$ have been discussed in Case 2.1 and in Case 2.2 respectively, we can suppose $\delta' + 6 \neq \varepsilon' + 4$ and $\delta' + 6 \neq \eta' + 5$. Since the h.r.p. in $Q(H)$ is $\delta' + 6$, we have $\delta' + 6 > \varepsilon' + 4$ and $\delta' + 6 > \eta' + 5$, so

$$\delta' + 2 > \varepsilon' \tag{8}$$
$$\delta' + 1 > \eta' \tag{9}$$

after simplifying $Q(G)$ and $Q(H)$, we have

$$Q(G) : -r^3 - r^4 - r^{\varepsilon} - r^{\eta} - r^{\varepsilon+1} - r^{\eta+1} + r^{\varepsilon+3} + r^{\eta+3} + r^{\varepsilon+4} + r^8$$
$$Q(H) : -r^5 - r^{\delta'} - r^{\varepsilon'} - r^{\eta'} - r^{\varepsilon'+1} - r^{\eta'+1} + r^{\eta'+2} + r^{\varepsilon'+3} + r^{\varepsilon'+4} + r^{\eta'+5}.$$

Consider $r^{\eta+3}$ in $Q(G)$, it is due to $\varepsilon \leq \eta$ that $r^{\eta+3}$ can cancel none of the negative terms in $Q(G)$. Thus, $r^{\eta+3}$ must be equal to one of the terms in $Q(H)$. Since $\eta > \eta'$ and $\eta > \varepsilon'$ (noting Eqs. (7)–(9)), we have $r^{\eta+3} = r^{\varepsilon'+4}$ or $r^{\eta+3} = r^{\eta'+5}$.

Case 2.3.1. If $r^{\eta+3} = r^{\varepsilon'+4}$, then $\eta = \varepsilon' + 1$. Since $\delta + \varepsilon + \eta = \delta' + \varepsilon' + \eta'$ and $\delta = 2$, we have

$$\varepsilon + 3 = \delta' + \eta' \tag{10}$$

after simplifying $Q(G)$ and $Q(H)$, we have

$$Q(G) : -r^3 - r^4 - r^{\varepsilon} - r^{\varepsilon+1} - r^{\eta+1} + r^{\varepsilon+3} + r^{\varepsilon+4} + r^8$$
$$Q(H) : -r^5 - r^{\delta'} - r^{\varepsilon'} - r^{\eta'} - r^{\eta'+1} + r^{\eta'+2} + r^{\varepsilon'+3} + r^{\eta'+5}.$$

Consider $-r^3$ in $Q(G)$. Since $-r^3$ can cancel none of the positive terms in $Q(G)$, $-r^3$ must be equal to one of the terms in $Q(H)$. By $\eta > 4$ (from (2)) and $\eta = \varepsilon' + 1$, we have $\varepsilon' > 3$. So $-r^3 = -r^{\delta'}$ or $-r^3 = -r^{\eta'}$ or $-r^3 = -r^{\eta'+1}$.

If $-r^3 = -r^{\delta'}$, then $\delta' = 3$. From (10), we have $\varepsilon = \eta'$. From (7), we have $\eta = 5$. So from $\eta = \varepsilon' + 1$, we have $\varepsilon' = 4$. After simplifying, we have

$$Q(G) : -r^6 + r^{\varepsilon+3} + r^{\varepsilon+4} + r^8$$
$$Q(H) : -r^5 + r^{\eta'+2} + r^7 + r^{\eta'+5}.$$

It is easy to see that $\varepsilon = 3$ and $\eta' = 3$. Thus we obtain the solution where G is isomorphic to $K_4(1, 3, 3, 2, 3, 5)$ and H is isomorphic to $K_4(1, 2, 4, 3, 4, 3)$. That is

$$K_4(1, 3, 3, 2, 3, 5) \sim K_4(1, 2, 4, 3, 4, 3).$$

If $-r^3 = -r^{\eta'}$, then $\eta' = 3$. From (10), we have $\varepsilon = \delta'$. From (7), we have $\eta = \varepsilon + 2$. So from $\eta = \varepsilon' + 1$, we have $\varepsilon' = \varepsilon + 1$. After simplifying, we have $Q(G) = Q(H)$. Let $\varepsilon = b$. We obtain the solution where G is isomorphic to $K_4(1, 3, 3, 2, b, b + 2)$ and H is isomorphic to $K_4(1, 2, 4, b, b + 1, 3)$. That is

$$K_4(1, 3, 3, 2, b, b + 2) \sim K_4(1, 2, 4, b, b + 1, 3).$$

If $-r^3 = -r^{\eta'+1}$, then $\eta' = 2$. From (10), we have $\delta' = \varepsilon + 1$. From (7), we have $\eta = \varepsilon + 3$. So from $\eta = \varepsilon' + 1$, we have $\varepsilon' = \varepsilon + 2$. After simplifying, we have

$$Q(G) : -r^4 - r^{\varepsilon} + r^{\varepsilon+3} + r^8$$
$$Q(H) : -r^2 - r^5 - r^{\varepsilon'} + r^4 + r^{\varepsilon'+3} + r^7.$$

It is easy to say that $Q(G) \neq Q(H)$, this a contradiction.

Case 2.3.2. If $r^{\eta+3} = r^{\eta'+5}$, then $\eta = \eta' + 2$. Since $\delta + \varepsilon + \eta = \delta' + \varepsilon' + \eta'$ and $\delta = 2$, we have

$$\varepsilon + 4 = \delta' + \varepsilon' \tag{11}$$

after simplifying $Q(G)$ and $Q(H)$, we have

$$Q(G) : -r^3 - r^4 - r^{\varepsilon} - r^{\eta} - r^{\varepsilon+1} - r^{\eta+1} + r^{\varepsilon+3} + r^{\varepsilon+4} + r^8$$
$$Q(H) : -r^5 - r^{\delta'} - r^{\varepsilon'} - r^{\eta'} - r^{\varepsilon'+1} - r^{\eta'+1} + r^{\eta'+2} + r^{\varepsilon'+3} + r^{\varepsilon'+4}.$$

Consider $-r^3$ in $Q(G)$. Since $-r^3$ can cancel none of the terms in $Q(G)$, $-r^3$ must be equal to one of the terms in $Q(H)$. Since $\eta > 4$ (from (2)) and $\eta = \eta' + 2$, we have $\eta' > 2$. So we have $-r^3 = -r^{\delta'} = -r^{\eta'}$ (by $\eta = \eta' + 2$ and $\eta = \delta' + 2$ (from (7)) or $-r^3 = -r^{\varepsilon'}$ or $-r^3 = -r^{\varepsilon'+1}$.

If $-r^3 = -r^{\delta'} = -r^{\eta'}$, then $\delta' = \eta' = 3$. From (11), we have $\varepsilon' = \varepsilon + 1$. From (7), we have $\eta = 5$. After simplifying, we have

$$Q(G) : -r^{\varepsilon} - r^6 + r^{\varepsilon+3} + r^8$$
$$Q(H) : -r^3 - r^{\varepsilon'+1} + r^5 + r^{\varepsilon'+4}.$$

It is easy to see that $\varepsilon = 3$ and $\varepsilon' = 4$. Thus we obtain the solution where G is isomorphic to $K_4(1, 3, 3, 2, 3, 5)$ and H is isomorphic to $K_4(1, 2, 4, 3, 4, 3)$. That is

$$K_4(1, 3, 3, 2, 3, 5) \sim K_4(1, 2, 4, 3, 4, 3).$$

If $-r^3 = -r^{\varepsilon'}$, then $\varepsilon' = 3$. From (11), we have $\delta' = \varepsilon + 1$. From (7), we have $\eta = \varepsilon + 3$. So from $\eta = \eta' + 2$, we have $\eta' = \varepsilon + 1$. After simplifying, we have

$$Q(G) : -r^{\varepsilon} + r^8$$
$$Q(H) : -r^5 - r^{\eta'} - r^{\eta'+1} + r^{\eta'+2} + r^6 + r^7.$$

It is easy to see that $\varepsilon = 5$ and $\eta' = 6$. Thus we obtain the solution where G is isomorphic to $K_4(1, 3, 3, 2, 5, 8)$ and H is isomorphic to $K_4(1, 2, 4, 6, 3, 6)$. That is

$$K_4(1, 3, 3, 2, 5, 8) \sim K_4(1, 2, 4, 6, 3, 6).$$

If $-r^3 = -r^{\varepsilon'+1}$, then $\varepsilon' = 2$. From (11), we have $\delta' = \varepsilon + 2$. From (7), we have $\eta = \varepsilon + 4$. So from $\eta = \eta' + 2$, we have $\eta' = \varepsilon + 2$. After simplifying, we have

$$Q(G) : -r^4 - r^{\varepsilon} - r^{\varepsilon+1} - r^{\eta+1} + r^{\varepsilon+3} + r^8$$
$$Q(H) : -r^2 - r^{\delta'} - r^{\eta'} - r^{\eta'+1} + r^{\eta'+2} + r^6.$$

Comparing the l.r.p. in $Q(G)$ with the l.r.p. in $Q(H)$, we have $\varepsilon = 2$. So $\delta' = 4$, $\eta = 6$, $\eta' = 4$. It is easy to say that $Q(G) \neq Q(H)$, this a contradiction.

Case B. If $\min\{\delta, \varepsilon\} = \varepsilon = 2$, then we obtain the following after simplification:

$$Q(G) : -2r^3 - r^4 - r^{\delta} - r^{\eta} - r^{\eta+1} + r^5 + r^6 + r^{\eta+3} + r^{\eta+4} + r^{\delta+6}$$
$$Q(H) : -r^5 - r^{\delta'} - r^{\varepsilon'} - r^{\eta'} - r^{\varepsilon'+1} - r^{\eta'+1} + r^{\eta'+2} + r^{\varepsilon'+3} + r^{\varepsilon'+4} + r^{\eta'+5} + r^{\delta'+6}.$$

Consider $-r^5$ in $Q(H)$. There are two cases to be considered.

Case 1. If the term $-r^5$ in $Q(H)$ cannot be cancelled by the positive term in $Q(H)$ (which implies $\eta' + 2 \neq 5$ and $\varepsilon' + 3 \neq 5$), then none of the terms in $Q(H)$ is equal to the term r^5 in $Q(G)$. So, by $Q(G) = Q(H)$, there are two terms in $Q(G)$ which are equal to $-r^5$. Thus we have $-r^{\delta} = -r^{\eta} = -r^5$ or $-r^{\delta} = -r^{\eta+1} = -r^5$.

Case 1.1. If $-r^{\delta} = -r^{\eta} = -r^5$, from $\delta + \varepsilon + \eta = \delta' + \varepsilon' + \eta'$ and $\varepsilon = 2$, we have $\delta' + \varepsilon' + \eta' = 12$. Then we obtain the following after simplification:

$$Q(G) : -2r^3 - r^4 + r^8 + r^9 + r^{11}$$
$$Q(H) : -r^{\delta'} - r^{\varepsilon'} - r^{\eta'} - r^{\varepsilon'+1} - r^{\eta'+1} + r^{\eta'+2} + r^{\varepsilon'+3} + r^{\varepsilon'+4} + r^{\eta'+5} + r^{\delta'+6}.$$

Comparing the l.r.p. in $Q(G)$ with the l.r.p. in $Q(H)$, and by $\eta' + 2 \neq 5$, we have $\delta' = \varepsilon' = 3$. From $\delta' + \varepsilon' + \eta' = 12$, we get $\eta' = 6$. Thus we obtain the solution where G is isomorphic to $K_4(1, 3, 3, 5, 2, 5)$ and H is isomorphic to $K_4(1, 2, 4, 3, 3, 6)$. That is

$$K_4(1, 3, 3, 5, 2, 5) \sim K_4(1, 2, 4, 3, 3, 6).$$

Case 1.2. If $-r^{\delta} = -r^{\eta+1} = -r^5$, from $\delta + \varepsilon + \eta = \delta' + \varepsilon' + \eta'$ and $\varepsilon = 2$, we have $\delta' + \varepsilon' + \eta' = 11$. Then we obtain the following after simplification:

$$Q(G) : -2r^3 - 2r^4 + r^6 + r^7 + r^8 + r^{11}$$
$$Q(H) : -r^{\delta'} - r^{\varepsilon'} - r^{\eta'} - r^{\varepsilon'+1} - r^{\eta'+1} + r^{\eta'+2} + r^{\varepsilon'+3} + r^{\varepsilon'+4} + r^{\eta'+5} + r^{\delta'+6}.$$

Comparing the l.r.p. in $Q(G)$ with the l.r.p. in $Q(H)$, and by $\eta' + 2 \neq 5$, we have $\delta' = \varepsilon' = 3$. From $\delta' + \varepsilon' + \eta' = 11$, we get $\eta' = 5$. Thus $Q(G) \neq Q(H)$, a contradiction.

Case 2. If the term $-r^5$ in $Q(H)$ can be cancelled by the positive term in $Q(H)$, then $r^{\varepsilon'+3}$ or $r^{\eta'+2}$ can cancel $-r^5$ which implies $\varepsilon' = 2$ or $\eta' = 3$.

$$Q(G) : -2r^3 - r^4 - r^{\delta} - r^{\eta} - r^{\eta+1} + r^5 + r^6 + r^{\eta+3} + r^{\eta+4} + r^{\delta+6}$$
$$Q(H) : -r^5 - r^{\delta'} - r^{\varepsilon'} - r^{\eta'} - r^{\varepsilon'+1} - r^{\eta'+1} + r^{\eta'+2} + r^{\varepsilon'+3} + r^{\varepsilon'+4} + r^{\eta'+5} + r^{\delta'+6}.$$

Case 2.1. If $\varepsilon' = 2$, then we obtain the following after simplification:

$$Q(G) : -r^3 - r^4 - r^{\delta} - r^{\eta} - r^{\eta+1} + r^5 + r^{\eta+3} + r^{\eta+4} + r^{\delta+6}$$
$$Q(H) : -r^2 - r^{\delta'} - r^{\eta'} - r^{\eta'+1} + r^{\eta'+2} + r^{\eta'+5} + r^{\delta'+6}.$$

Comparing the l.r.p. in $Q(G)$ with the l.r.p. in $Q(H)$, we have $\delta = 2$ or $\eta = 2$.

Case 2.1.1. If $\delta = 2$, from $\delta + \varepsilon + \eta = \delta' + \varepsilon' + \eta'$ and $\varepsilon = 2$ and $\varepsilon' = 2$, we have

$$\eta + 2 = \delta' + \eta'. \tag{12}$$

After simplifying $Q(G)$ and $Q(H)$, we have

$$Q(G) : -r^3 - r^4 - r^{\eta} - r^{\eta+1} + r^5 + r^{\eta+3} + r^{\eta+4} + r^8$$
$$Q(H) : -r^{\delta'} - r^{\eta'} - r^{\eta'+1} + r^{\eta'+2} + r^{\eta'+5} + r^{\delta'+6}.$$

Consider $-r^3$ and $-r^4$ in $Q(G)$. It is due to $Q(G) = Q(H)$ that one of $-r^{\delta'}, -r^{\eta'}, -r^{\eta'+1}$ in $Q(H)$ is equal to $-r^3$ or $-r^4$. So we have $\delta' + \eta' = 7$ or $\delta' + \eta' + 1 = 7$ or $\eta' + \eta' + 1 = 7$.

If $\delta' + \eta' = 7$, by $\eta + 2 = \delta' + \eta'$ (Eq. (12)), we have $\eta = 5$, then $Q(G) \neq Q(H)$, a contradiction.

If $\delta' + \eta' + 1 = 7$, by $\eta + 2 = \delta' + \eta'$ (Eq. (12)), we have $\eta = 4$, then $Q(G) \neq Q(H)$, a contradiction.

If $\eta' + \eta' + 1 = 7$, by $\eta + 2 = \delta' + \eta'$ (Eq. (12)), we have $\eta = \delta' + 1$, then $Q(G) \neq Q(H)$, a contradiction.

Case 2.1.2. If $\eta = 2$, from $\delta + \varepsilon + \eta = \delta' + \varepsilon' + \eta'$ and $\varepsilon = 2$ and $\varepsilon' = 2$, we have

$$\delta + 2 = \delta' + \eta'. \tag{13}$$

After simplifying $Q(G)$ and $Q(H)$, we have

$$Q(G) : -2r^3 - r^4 - r^\delta + 2r^5 + r^6 + r^{\delta+6}$$
$$Q(H) : -r^{\delta'} - r^{\eta'} - r^{\eta'+1} + r^{\eta'+2} + r^{\eta'+5} + r^{\delta'+6}.$$

Consider $-2r^3$ and $-r^4$ in $Q(G)$. Since $Q(G) = Q(H)$, there are two terms in $Q(H)$ which are equal to $-r^3$ and there is one term in $Q(H)$ which is equal to $-r^4$. So we have $\delta' = \eta' = 3$. From $\delta + 2 = \delta' + \eta'$ Eq. (13), we have $\delta = 4$, then $Q(G) \neq Q(H)$, a contradiction.

Case 2.2. If $\eta' = 3$, then we obtain the following after simplification:

$$Q(G) : -r^3 - r^\delta - r^\eta - r^{\eta+1} + r^5 + r^6 + r^{\eta+3} + r^{\eta+4} + r^{\delta+6}$$
$$Q(H) : -r^{\delta'} - r^{\varepsilon'} - r^{\varepsilon'+1} + r^{\varepsilon'+3} + r^{\varepsilon'+4} + r^{\delta'+6} + r^8.$$

Consider $-r^3$ in $Q(G)$. It is due to $Q(G) = Q(H)$ that $-r^3 = -r^{\delta'}$ or $-r^3 = -r^{\varepsilon'}$ or $-r^3 = -r^{\varepsilon'+1}$. So we have $\delta' = 3$ or $\varepsilon' = 3$ or $\varepsilon' + 1 = 3$.

Case 2.2.1. If $\delta' = 3$, from $\delta + \varepsilon + \eta = \delta' + \varepsilon' + \eta'$ and $\varepsilon = 2$ and $\eta' = 3$, we have

$$\delta + \eta = \varepsilon' + 4. \tag{14}$$

After simplifying $Q(G)$ and $Q(H)$, we have

$$Q(G) : -r^\delta - r^\eta - r^{\eta+1} + r^5 + r^6 + r^{\eta+3} + r^{\eta+4} + r^{\delta+6}$$
$$Q(H) : -r^{\varepsilon'} - r^{\varepsilon'+1} + r^{\varepsilon'+3} + r^{\varepsilon'+4} + r^8 + r^9.$$

Consider $r^{\varepsilon'+4}$ in $Q(H)$. It is due to $Q(G) = Q(H)$ that $r^{\varepsilon'+4} = r^{\eta+3}$ or $r^{\varepsilon'+4} = r^{\eta+4}$ or $r^{\varepsilon'+4} = r^{\delta+6}$ or $r^{\varepsilon'+4} = r^6$.

If $r^{\varepsilon'+4} = r^{\eta+3}$, then $\eta = \varepsilon' + 1$, from $\delta + \eta = \varepsilon' + 4$ (Eq. (14)), we have $\delta = 3$. After simplifying $Q(G)$ and $Q(H)$, we have

$$Q(G) : -r^3 - r^{\eta+1} + r^5 + r^6 + r^{\eta+3} + r^{\eta+4} + r^9$$
$$Q(H) : -r^{\varepsilon'} + r^{\varepsilon'+3} + r^{\varepsilon'+4} + r^8 + r^9.$$

Comparing the l.r.p. in $Q(G)$ with the l.r.p. in $Q(H)$, we have $\varepsilon' = 3$. So $\eta = 4$. Thus, $G \cong H$.

If $r^{\varepsilon'+4} = r^{\eta+4}$, then $\varepsilon' = \eta$, from $\delta + \eta = \varepsilon' + 4$ (Eq. (14)), we have $\delta = 4$. After simplifying $Q(G)$ and $Q(H)$, we have $Q(G) \neq Q(H)$, this is a contradiction.

If $r^{\varepsilon'+4} = r^{\delta+6}$, from $\delta + \eta = \varepsilon' + 4$ (Eq. (14)), we have $\eta = 6$. After simplifying $Q(G)$ and $Q(H)$, we have

$$Q(G) : -r^\delta - r^7 + r^5 + r^{10}$$
$$Q(H) : -r^{\varepsilon'} - r^{\varepsilon'+1} + r^{\varepsilon'+3} + r^8.$$

Consider the term $-r^7$ in $Q(G)$. We have $-r^7 = -r^{\varepsilon'+1}$ or $-r^7 = -r^{\varepsilon'}$. If $\varepsilon' = 6$, then $\delta = 4$. Thus $Q(G) \neq Q(H)$, this is a contradiction. If $\varepsilon' = 7$, then $\delta = 5$. Thus we obtain the solution where G is isomorphic to $K_4(1, 3, 3, 5, 2, 6)$ and H is isomorphic to $K_4(1, 2, 4, 3, 7, 3)$. That is

$$K_4(1, 3, 3, 5, 2, 6) \sim K_4(1, 2, 4, 3, 7, 3).$$

If $r^{\varepsilon'+4} = r^6$, then $\varepsilon' = 2$, from $\delta + \eta = \varepsilon' + 4$ (Eq. (14)), we have $\delta + \eta = 6$. After simplifying $Q(G)$ and $Q(H)$, we have $Q(G) \neq Q(H)$, this is a contradiction.

Case 2.2.2. If $\varepsilon' = 3$, from $\delta + \varepsilon + \eta = \delta' + \varepsilon' + \eta'$ and $\varepsilon = 2$ and $\eta' = 3$, we have

$$\delta + \eta = \delta' + 4. \tag{15}$$

After simplifying $Q(G)$ and $Q(H)$, we have

$$Q(G): -r^{\delta} - r^{\eta} - r^{\eta+1} + r^5 + r^{\eta+3} + r^{\eta+4} + r^{\delta+6}$$
$$Q(H): -r^4 - r^{\delta'} + r^7 + r^8 + r^{\delta'+6}.$$

Consider the term $-r^4$ in $Q(H)$. We have $-r^4 = -r^{\delta}$ or $-r^4 = -r^{\eta}$ or $-r^4 = -r^{\eta+1}$.

If $-r^4 = -r^{\delta}$, then $\delta = 4$, from $\delta + \eta = \delta' + 4$ (Eq. (15)), we have $\eta = \delta'$. After simplifying $Q(G)$ and $Q(H)$, we have $\eta = 4$ and $\delta' = 4$. Thus $G \cong H$.

If $-r^4 = -r^{\eta}$, then $\eta = 4$, from $\delta + \eta = \delta' + 4$, we have $\delta = \delta'$. Thus $G \cong H$.

If $-r^4 = -r^{\eta+1}$, then $\eta = 3$, from $\delta + \eta = \delta' + 4$, we have $\delta = \delta' + 1$. After simplifying $Q(G)$ and $Q(H)$, we have $Q(G) \neq Q(H)$, this is a contradiction.

Case 2.2.3. If $\varepsilon' = 2$, from $\delta + \varepsilon + \eta = \delta' + \varepsilon' + \eta'$ and $\varepsilon = 2$ and $\eta' = 3$, we have

$$\delta + \eta = \delta' + 3. \tag{16}$$

After simplifying $Q(G)$ and $Q(H)$, we have

$$Q(G): -r^{\delta} - r^{\eta} - r^{\eta+1} + r^{\eta+3} + r^{\eta+4} + r^{\delta+6}$$
$$Q(H): -r^2 - r^{\delta'} + r^8 + r^{\delta'+6}.$$

Consider the term $-r^2$ in $Q(H)$. We have $-r^2 = -r^{\delta}$ or $-r^2 = -r^{\eta}$.

If $-r^2 = -r^{\delta}$, then $\delta = 2$, from $\delta + \eta = \delta' + 3$ (Eq. (16)), we have $\eta = \delta' + 1$. After simplifying $Q(G)$ and $Q(H)$, we have $Q(G) \neq Q(H)$, this is a contradiction.

If $-r^2 = -r^{\eta}$, then $\eta = 2$, from $\delta + \eta = \delta' + 3$ (Eq. (16)), we have $\delta = \delta' + 1$. After simplifying $Q(G)$ and $Q(H)$, we have $Q(G) \neq Q(H)$, this is a contradiction.

So far, we have solved the equation $P(G) = P(H)$ and got the solution as follows:

$$K_4(1,3,3,2,3,5) \sim K_4(1,2,4,3,4,3)$$
$$K_4(1,3,3,2,4,7) \sim K_4(1,2,4,4,3,6)$$
$$K_4(1,3,3,2,5,8) \sim K_4(1,2,4,6,3,6)$$
$$K_4(1,3,3,5,2,5) \sim K_4(1,2,4,3,3,6)$$
$$K_4(1,3,3,5,2,6) \sim K_4(1,2,4,3,7,3)$$
$$K_4(1,3,3,2,b,b+2) \sim K_4(1,2,4,b,b+1,3),$$

where $b \geq 2$.

The proof is completed. □

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