Abstract

In this paper, we are concerned with the problem of boundedness of solutions for the following nonlinear $p$-Laplacian:

\begin{equation}
(\phi_p(x'))' + (p-1)n^p\phi_p(x) + f(x)\phi_p(x') + g(x) = e(t),
\end{equation}

where $\phi_p(u) = |u|^{p-2}u,$ $p \geq 2,$ for some $n \in \mathbb{N},$ $f, g \in C^\infty(R)$ are odd functions, $e(t) \in C^\infty(R)$ is odd and $2\pi_p$-periodic.

In the early 1960s, Littlewood [10] suggested to study the boundedness of solutions of the second order differential equation.

1. Introduction and main results

In this paper, we consider the problem of boundedness and unboundedness of solutions for the nonlinear $p$-Laplacian

\begin{equation}
(\phi_p(x'))' + (p-1)n^p\phi_p(x) + f(x)\phi_p(x') + g(x) = e(t) \quad (t = d/dt),
\end{equation}

where $\phi_p(u) = |u|^{p-2}u,$ $p \geq 2$ is a constant, $n \in \mathbb{N},$ $f, g \in C^\infty(R)$ are odd functions, $e(t) \in C^\infty(R)$ is odd, $2\pi_p$-periodic and $\pi_p = 2\pi/(p \sin(\pi/p)).$

In the early 1960s, Littlewood [10] suggested to study the boundedness of solutions of the second order differential equation.
\[ x'' + g(x) = e(t), \quad (2) \]

where \( e(t) \in C^\infty(R) \) is 2\( \pi \)-periodic and \( g \) satisfies one of the following two conditions:

(a) superlinear case: \( g(x)/x \to +\infty \) as \( |x| \to \infty \);
(b) sublinear case: \( \text{sgn}(x) \cdot g(x) \to +\infty, \ g(x)/x \to 0 \) as \( |x| \to \infty \).

The first positive result in superlinear case is due to Morris [14], who proved that every solution of

\[ x'' + 2x^3 = e(t) \]

is bounded, where \( e(t) \in C^0(S^1) \). Later, several authors extended his result to more general superlinear cases (see [4,9,11,15,18]). In 1999, Küpper and You [8] proved the boundedness of solutions for the following sublinear equation:

\[ x'' + \varphi_{\alpha_0}(x) = e(t), \]

where \( 1 < \alpha_0 < 2, e(t) \in C^\infty(R) \) is 2\( \pi \)-periodic.

If \( g(x) \) satisfies semilinear condition \( 0 < k \leq g(x)/x \leq K < +\infty \) for \( x \neq 0 \), then Eq. (2) is called semilinear. The boundedness problem in the semilinear case is quite different and very delicate. For example, the linear equation

\[ x'' + n^2 x = \cos nt, \quad n \in N, \]

has no bounded solutions. Another interesting example was constructed by Ding [5]. He proved that the equation

\[ x'' + n^2 x + \arctan x = 4 \cos nt, \quad n \in N, \quad (3) \]

has no bounded solutions. More results on this case can be seen in [1,12–14,16].

Recently, Kunze et al. [7] considered the following second order differential equation:

\[ x'' + n^2 x + f(x)x' + g(x) = e(t), \quad n \in N, \quad (4) \]

where \( g(x) \in C^\infty(R) \) is odd and bounded and the limit \( \lim_{x \to +\infty} g(x) = g(+\infty) \) exists, \( f(x) \in C^\infty(R) \) is odd, \( e(t) \in C^\infty(R) \) is 2\( \pi \)-periodic, which is a special case of (1) for \( p = 2 \). They obtained boundedness and unboundedness conditions for solutions of (4) under the assumption

\[ \left| \frac{d^k F(x)}{dx^k} \right| \leq M, \quad 0 \leq k \leq 6, \quad x \in R, \quad (5) \]

for some constant \( M > 0 \), where \( F(x) = \int_0^x f(s) \, ds \).

They also asked whether the assumption (5) is sharp.

Since the publishing of pioneering work of Del Pino et al. [3], a lot of research results have been published in the research fields such as the existence, uniqueness and multiplicity, oscillation and nonoscillation properties of the solutions of the following \( p \)-Laplacian:

\[ \left( \varphi_p(x') \right)' + f(t, x, x') = 0, \quad (6) \]

for example, we refer to the papers [2,6] and references therein. Since both in practice and theory, Eq. (6) plays an important role in studying the properties of solutions of second
order nonlinear operators. But very few results on the boundedness of all the solutions of (6) are obtained so far.

For example, one could ask the boundedness of all the solutions of the following $p$-Laplacian problem:

$$\left(\varphi_p(x')\right)' + f(x)\varphi_p(x') + \varphi_p(x) + 2\arctan x = \sin\left(\frac{\pi t}{\pi p}\right),$$

where $f(x) = x/(1 + x^2)^{p/(p+1)}$.

In [19], the author of this paper generalizes Liu’s [12] results for linear equation to nonlinear equation of the form of (6) for $p \neq 2$. In the case of $p = 2$, after polar coordinates transformation, Eq. (4) is transformed into a reversible perturbation of integrable planar system. Hence a variant of Moser twist theorem for reversible system [17] can be used in this case. Under suitable assumptions, the Poincaré map that results after a suitable series of transformations will be reversible and guarantees the existence of invariant curves. Hence, one obtains boundedness results for Eq. (4) under some reasonable assumptions. After careful examination of the processes conducted in linear second order operator, we find, by introducing generalized polar coordinates transformation, that Eq. (1) can be transformed into a similar form of planar system, on which the variant of Moser twist theorem for reversible system is also applicable. Hence one can also obtain the boundedness results for all the solutions for a special kind of nonlinear second order operators, that is, the $p$-Laplacian operator. The main difference of the method used in this paper with the method used in [7] for the case $p = 2$ is the introduction of the generalized polar coordinates transformation, which reduced to normal polar coordinates transformation in the standard case $p = 2$. Moreover, the result of this paper can be used as an example to investigate the boundedness of solutions of general second nonlinear operators. At the end of this paper, we shall also give an example, which cannot be covered by the result of [7] if $p \neq 2$.

In this paper, we denote by $C > 0$ a universal positive constant without regarding its value. We assume also

(H$_1$) $f, g \in C^\infty(R)$ are odd functions, $e \in C^\infty(R)$ is odd and $2\pi p$-periodic;

(H$_2$) $g$ is bounded and $\lim_{x \to +\infty} g(x) =: g(+\infty)$ exists. $g$ satisfies the estimate

$$\lim_{|x| \to \infty} x^6 \frac{d^6 g(x)}{dx^6} = 0;$$

(H$_3$) for $0 \leq k \leq 6$ and $x \neq 0$, there exist constants $C > 0$, $\delta_0 > 0$ such that

$$\left| x^k \frac{d^k f(x)}{dx^k} \right| \leq C|x|^{-\delta_0}. $$

The main results of this paper are

**Theorem 1.** Assume (H$_1$)–(H$_3$) hold. If in addition for all $\tau \in S^1$,

$$4B\left(\frac{2}{p}, 1 - \frac{1}{p}\right)|g(+\infty)| > \left| \int_0^{2\pi} S\left(\frac{\theta}{n}\right) e\left(\tau + \frac{\theta}{n}\right) d\theta \right|,$$

then...
where $B(r,s) = \int_0^1 t^{r-1} (1-t)^{s-1} \, dt$ is the Beta function for $r > 0$ and $s > 0$ and $S(t)$ is the unique solution for the following $p$-Laplacian initial value problem:

$$
(\varphi_p(u'))' + (p-1)\varphi_p(u) = 0,
$$

$$
u(0) = 0, \quad u'(0) = 1.
$$

Then every solution $x$ of (1) is bounded, i.e., $x(t)$ is defined in $(-\infty, +\infty)$ and

$$
\sup_{t \in \mathbb{R}} (|x(t)| + |x'(t)|) < +\infty.
$$

**Remark 1.** (a) The assumption (H1) can be relaxed to require that $f$, $g$ and $e \in C^0_0$ for some $n_0 \in \mathbb{N}$ sufficiently large.

(b) Since $g$ is odd, (H2) implies that the limit $\lim_{x \to -\infty} g(x) =: g(-\infty) = -g(+\infty)$ exists.

(c) (H2) and l’Hospital’s rule implies that for $1 \leq k \leq 6$,

$$
\lim_{|x| \to \infty} x^k \frac{d^k g(x)}{dx^k} = 0
$$

and

$$
|x^k \frac{d^k g(x)}{dx^k}| \leq C, \quad 0 \leq k \leq 6.
$$

(d) In case $p = 2$, by changing Eq. (4) into the polar coordinates form given by [7], one can verify that the condition (H3) can be replaced by the following weaker conditions:

$$
|x^k \frac{d^k F(x)}{dx^k}| \leq C|x|^{1-\delta}, \quad 0 \leq k \leq 6, \quad x \in \mathbb{R}.
$$

Since (5) means that $f$ decays as $x^{-\alpha}$ with $\alpha \geq 1$ whereas in (13) it is valid for all $\alpha > 0$. Therefore Theorem 1 generalizes and improves the corresponding theorem in [7] and a negative answer to the question proposed in [7] is given in case $p = 2$. But I do not know if the results of this paper is sharp or not.

### 2. Generalized polar coordinates transformation

In this section, we first change Eq. (1) into planar system by using a generalized polar transformation and then give some approximation expression for the solutions.

If we introduce a new variable $y = \varphi_p(x')$ and let $q = p/(p - 1)$, then Eq. (1) is equivalent to the planar system

$$
x' = \varphi_q(y),
$$

$$
y' = -(p-1)n^p\varphi_p(x) - f(x)y - g(x) + e(t).
$$

We consider first the auxiliary equation

$$
(\varphi_p(u'))' + (p-1)\varphi_p(u) = 0,
$$

$$
u(0) = 0, \quad u'(0) = 1.
$$
Let \( C(t) = \varphi_p(S'(t)) \); then by the result in [2], it is not difficult to verify that \( S(t) \) and \( C(t) \) are \( 2\pi_p \)-periodic and satisfy the following conditions:

(i) \( S(t) \) is odd and \( C(t) \) is even;

(ii) \( S(t) \in C^2(\mathbb{R}), \quad C(t) \in C^1(\mathbb{R}), \quad S'(t) = \varphi_q(C(t)), \quad C'(t) = -(p - 1)\varphi_p(S(t)); \quad (16) \)

(iii) \( |C(t)|^q + |S(t)|^p \equiv 1, \quad \forall t \in \mathbb{R}. \quad (17) \)

If \( p = 2 \), then the function \( S \) and \( C \) reduce to the standard \( \sin \) and \( \cos \) functions. Hence \( S \) and \( C \) are generalization of the well-know \( 2\pi \)-periodic \( \sin \) and \( \cos \) functions.

Now we define the generalized polar coordinates \((r, \theta)\) with \( r > 0 \) and \( \theta \) \((\text{mod} \ 2\pi_p)\) by the mapping \( M: (r, \theta) \rightarrow (x, y) \),

\[
M: x = x(r, \theta) = d_0^{1/p} r^{1/p} S\left(\frac{\theta}{n}\right), \quad y = y(r, \theta) = d_0^{1/q} r^{1/q} C\left(\frac{\theta}{n}\right),
\]

with \( d_0 = q/n \).

Then it is not difficult to verify by applying (16) and (17) that the map \( M \) is symplectic in the sense:

\[
\det \begin{pmatrix} x_r & x_\theta \\ y_r & y_\theta \end{pmatrix} = 1. \quad (19)
\]

In the new coordinates \((r, \theta)\), by using (17), one can verify that system (14) becomes

\[
\frac{dr}{dt} = n d_0^{1/p} r^{1/p} \varphi_q\left(C(n\theta)\right)\left(e(t) - g(x)\right) - n d_0 r |C(n\theta)|^q f(x),
\]

\[
\frac{d\theta}{dt} = n - d_0^{1/p} r^{-1/q} S(n\theta)\left(e(t) - g(x)\right) + \frac{d_0}{p} S(n\theta)C(n\theta) f(x), \quad (20)
\]

with \( x = d_0^{1/p} r^{1/p} S(n\theta) \).

Moreover, it is easy to verify that system (20) is reversible with respect to the involution, \( G: (r, \theta) \rightarrow (r, -\theta) \). (The definition of reversible system is given by Appendix A at the end of this paper.)

From assumption \((H_3)\),

\[
\left| \frac{x^k d^k f(x)}{d x^k} \right| \leq C\left| x \right|^{-\delta_0}, \quad x \neq 0, \quad 0 \leq k \leq 6. \quad (21)
\]

We can divide the case \( \delta_0 > 0 \) into two subcases:

(i) \( \delta_0 \geq p - 1 \);

(ii) \( \delta_0 < p - 1 \).

Since \( \delta_0 \geq p - 1 \) in (21) can be replaced by \( \delta_0 = p - 1 \), we can assume \( \delta_0 = p - 1 \) in case (i). In \((r, \theta)\) coordinates, (21) reads

\[
\left| \frac{d^k f(x)}{d x^k} \right| \leq C r^{-(p-1)/p-k/p}, \quad 0 \leq k \leq 6. \quad (22)
\]
Definition 1. Let $r > 0$; we say a function $h = h(r, \theta, t)$ is $O_n(r^{-\delta})$, if for some $\delta > 0$, $h$ is smooth in $(r, t)$, continuous in $\theta$, periodic in $\theta$ and $t$, and satisfies
\[
\left| r^{k+m} \frac{\partial^{k+m}}{\partial r^k \partial t^m} h(r, \theta, t) \right| \leq C, \quad 0 \leq k + m \leq n,
\]
for all $(r, \theta, t)$. We say a function $h$ is $o_n(r^{-\delta})$, if it is smooth in $(r, t)$, continuous in $\theta$, periodic in $\theta$ and $t$, and satisfies
\[
\lim_{r \to +\infty} r^{k+m} \frac{\partial^{k+m}}{\partial r^k \partial t^m} h(r, \theta, t) = 0, \quad 0 \leq k + m \leq n,
\]
uniformly in $(\theta, t)$.

In case $\delta = 0$, we denote by $O_n(1)$ and $o_n(1)$, respectively.

First, we prove Theorem 1 for the case $\delta_0 = p - 1$. From (21) and (22), for $r \gg 1$, system (20) can be written as
\[
\frac{dr}{d\theta} = d_0^{1/p} r^{1/p} \varphi_q(C(n\theta))(e(t) - g(x)) - d_0 rf(x)\left| C(n\theta) \right|^q + O_6(r^{-\delta_1}),
\]
\[
\frac{dt}{d\theta} = \frac{1}{n} \left[ 1 + d_0^{1/p} r^{-1/q} S(n\theta)(e(t) - g(x)) - \frac{d_0}{np} S(n\theta) C(n\theta) f(x) \right]
\]
\[
+ O_6(r^{-2/q}),
\]
where $\delta_1 = 1/q - 1/p = (p - 2)/p \geq 0$.

From Lemma A.1 in Appendix A, system (23) is still reversible with respect to the involution $G : (r, t) \to (r, -t)$.

In order to apply a twist theorem to prove the existence of invariant curves for the Poincaré map of (23), we need to further transform (23) into near integrable system. To this end, the following lemma will be useful.

Lemma 1. Define a transformation $T_1 : (r, t) \to (\rho, t)$ with
\[
T_1 : \rho = r + S_1(r, \theta),
\]
\[
S_1(r, \theta) = \int_0^\theta \left[ d_0^{1/p} r^{1/p} \varphi_q(C(nu))g(x) + d_0 rf(x)\left| C(nu) \right|^q \right] du,
\]
where $x = d_0^{1/p} r^{1/p} S(nu)$. Then under the transformation $T_1$, system (23) is of the form
\[
\frac{d\rho}{d\theta} = d_0^{1/p} \rho^{1/p} \varphi_q(C(n\theta))e(t) + O_6(\rho^{-\delta_1}),
\]
\[
\frac{dt}{d\theta} = \frac{1}{n} \left[ 1 + d_0^{1/p} \rho^{-1/q} S(n\theta)(e(t) - g(x)) - \frac{d_0}{np} S(n\theta) C(n\theta) f(x) \right]
\]
\[
+ O_6(\rho^{-2/q}).
\]
Moreover, the new system (24) is reversible with respect to the involution $G : (\rho, t) \to (\rho, -t)$. 
Proof. Since $f$ and $g$ are odd, we have $S_1(r, 2\pi p) = 0$, hence $S_1(r, \theta + 2\pi p) = S_1(r, \theta)$ for all $\theta \in S^1$. Since $g$ is bounded and $|f(x)| \leq C r^{-1/q}$, we obtain $S_1 \in O_6(r^{1/p})$. In particular,

$$
\left| \frac{\partial S_1}{\partial r}(r, \theta) \right| \leq C r^{-1/q}
$$

which implies that $r \to \rho$ is a diffeomorphism from some $[r_0, +\infty)$ to some $[\rho_0, +\infty)$ and $\rho \equiv r$ for $r \gg 1$. This implies (24) is a direct consequence of the transformation $T_1$. From $S_1(r, -\theta) = S_1(r, \theta)$, we apply Lemma A.1 to obtain the last statement of Lemma 1. □

Define an average

$$J(\rho) = \frac{d_0^{1/p}}{2\pi_p pn^2 \rho^{1/q}} \int_0^{2\pi_p} S(n\theta) g(x) d\theta, \quad x = d_0^{1/p} \rho^{1/p} S(n\theta). \tag{25}$$

The following lemma gives the estimate of $J(\rho)$ for $\rho \gg 1$.

**Lemma 2.** For $0 \leq k \leq 6$, we have

$$
\lim_{\rho \to +\infty} \rho^{k+1/q} \frac{d^k J(\rho)}{d \rho^k} = (-1)^k \left[ k - 1 + \frac{1}{q} \right] \frac{2d_0^{1/p} B(2/p, 1 - 1/p)}{pn^2 \pi_p} g(+\infty), \tag{26}
$$

where $[k - 1 + 1/q]! = (k - 1 + 1/q)(k - 2 + 1/q) \cdots 1/q$.

The proof of Lemma 2 will be given in Appendix A at the end of this paper.

**Lemma 3.** If we make a further transformation $T_2: (\rho, t) \to (\rho, \tau)$, where $\tau = t + S_2(\rho, \theta)$ with

$$
S_2(\rho, \theta) = \frac{1}{n} \int_0^\theta \left[ d_0^{1/p} \rho^{-1/q} S(nu) g(x) - J(\rho) + \frac{d_0}{pn} S(nu) C(nu) f(x) \right] du,
$$

$0 \leq \theta \leq 2\pi_p,$

where $x = d_0^{1/p} \rho^{1/p} S(nu)$. Then system (24) is transformed into

$$
\frac{d\rho}{d\theta} = \frac{d_0^{1/p}}{n^2} \rho^{1/q} \varphi_{\rho_q} (C(n\theta)) e(\tau) + O_6(\rho^{-5}),
$$

$$
\frac{d\tau}{d\theta} = \frac{1}{n} - J(\rho) + \frac{d_0^{1/p}}{n^2} \rho^{-1/q} S(n\theta) e(\tau) + O_5(\rho^{-2/q}). \tag{27}
$$

Moreover, system (27) is reversible with respect to the involution $G: (\rho, \tau) \to (\rho, -\tau)$.

The proof of Lemma 3 is similar to that of Lemma 2, so we omit it.
Lemma 4. Define a function

$$
\eta(\rho) = J(\rho) - \frac{2d_0^{1/p} B(2/p, 1 - 1/p) g(+\infty)}{pn\pi_p \rho^{-1/q}}.
$$

(28)

Then we have

$$
\eta(\rho) = o_6(\rho^{-1/q}).
$$

(29)

Proof. Write

$$
2d_0^{1/p} B(2/p, 1 - 1/p) g(+\infty)/pn\pi_p = C_0,
$$

then for $0 \leq k \leq 6$,

$$
\frac{d^k}{d\rho^k} \eta(\rho) = \frac{d^k}{d\rho^k} J(\rho) - C_0(-1)^k \left(k - 1 + \frac{1}{q}\right) \left(k - 2 + \frac{1}{q}\right) \frac{1}{q} \rho^{-(k+1/q)}
$$

which approaches to 0 as $\rho \to +\infty$ by (26) in Lemma 2. □

By the definition of $\eta(\rho)$, we have

$$
J(\rho) = C_0 \rho^{-1/q} + \eta(\rho) = C_0 \rho^{-1/q} + o_6(\rho^{-1/q}).
$$

Now, system (27) is of the form

$$
\begin{align*}
\frac{d\rho}{d\theta} &= d_0^{1/p} \rho^{1/p} \phi(\cdot(C(n\theta)) e(\tau) + O_6(\rho^{-1/q}),
\frac{d\tau}{d\theta} &= \frac{1}{n} - C_0 \rho^{-1/q} + \frac{d_0^{1/p}}{pn^2} \rho^{-1/q} S(n\theta) e(\tau) + o_5(\rho^{-1/q}).
\end{align*}
$$

(30)

Moreover, system (30) is reversible with respect to the involution $G : (\rho, \tau) \to (\rho, -\tau)$.

3. Further transformations

Let $\lambda > 1$ be a fixed parameter that will be determined later, $u$ be a new variable varying in the closed interval $[1/\lambda, \lambda]$ and a small positive parameter $\varepsilon$ by the formula

$$
\rho = u/\varepsilon^p, \quad u \in [1/\lambda, \lambda].
$$

Obviously, $\rho \gg 1$ if and only if $\varepsilon \ll 1$. In the new variable $(u, \tau)$, system (30) can be written in the form

$$
\begin{align*}
\frac{du}{d\theta} &= d_0^{1/p} \phi(\cdot C(n\theta)) e(\tau) \varepsilon^{p-1} u^{1/p} + O_6(\varepsilon^{5/p+p}),
\frac{d\tau}{d\theta} &= \frac{1}{n} - \left[ C_0 - \frac{d_0^{1/p}}{pn^2} S(n\theta) e(\tau) \right] \varepsilon^{p-1} u^{1/p-1} + o_5(\varepsilon^{p-1}).
\end{align*}
$$

(31)

By (31), it follows that if $\varepsilon \ll 1$, the solution $(u(\theta, u_0, \tau_0), \tau(\theta, u_0, \tau_0))$ with the initial condition $(u(0, u_0, \tau_0), \tau(0, u_0, \tau_0)) = (u_0, \tau_0)$ exists for $\theta \in [0, 4\pi_p]$ and $(u_0, \tau_0) \in [1/\lambda, \lambda] \times [0, 2\pi_p]$.

Moreover,

$$
0 < \frac{1}{2\lambda} \leq u(\theta, u_0, \tau_0) \leq 2\lambda, \quad \forall \theta \in [0, 4\pi_p].
$$
From (31), the solution \((u(\theta, u_0, \tau_0), \tau(\theta, u_0, \tau_0))\) has the following expression:

\[
\begin{align*}
u(\theta, u_0, \tau_0) &= u_0 + \varepsilon^{p-1} F_1(\theta, u_0, \tau_0, \varepsilon), \\
\tau(\theta, u_0, \tau_0) &= \tau_0 + \frac{1}{n} \theta + \varepsilon^{p-1} F_2(\theta, u_0, \tau_0, \varepsilon),
\end{align*}
\] (32)

with \(F_1(\theta, u_0, \tau_0, 0) = F_2(\theta, u_0, \tau_0, 0) = 0\).

Denote the Poincaré map of (31) by \(P\). Then

\[
P(u_0, \tau_0) = (u_1, \tau_1) = \left( u_0 + \varepsilon^{p-1} F_1(2\pi p, u_0, \tau_0, \varepsilon), \\
\tau_0 + \frac{2\pi p}{n} + \varepsilon^{p-1} F_2(2\pi p, u_0, \tau_0, \varepsilon) \right).
\]

It is evident that this map is well defined in the region \([1/\lambda, \lambda] \times S^1\), if \(\varepsilon \ll 1\).

Since \((u(\theta, u_0, \tau_0), \tau(\theta, u_0, \tau_0))\) is the solution of (31), we have

\[
\frac{dF_1}{d\theta} = \varphi_q \left( C(n\theta) \right) \left( u_0 + \varepsilon^{p-1} F_1 \right) + \varepsilon O_6(1),
\]

\[
\frac{dF_2}{d\theta} = \Gamma(\theta, \tau) \left( u_0 + \varepsilon^{p-1} F_1 \right)^{1/p-1} + o_5(1),
\] (33)

where

\[
\Gamma(\theta, \tau) = - \left[ C_0 - \frac{d_0^{1/p}}{pn^2} S(n\theta) e(\tau) \right].
\]

As in [4], one can show that

\[
F_1(\theta, u_0, \tau_0) = \varepsilon O_6(1), \quad F_2(\theta, u_0, \tau_0) = o_5(1)
\]

which, together with (32), implies that

\[
u(\theta, u_0, \tau_0) = u_0 + \varepsilon^p O_6(1), \quad \tau(\theta, u_0, \tau_0) = \tau_0 + \frac{1}{n} \theta + \varepsilon^{p-1} o_5(1).
\] (34)

It follows from (33) that

\[
F_1(2\pi p, u_0, \tau_0) = d_0^{1/p} \int_0^{2\pi p} \varphi_q \left( C(n\theta) \right) e(\tau) d\theta \cdot u_0^{1/p-1} + \varepsilon O_6(1),
\]

\[
F_2(2\pi p, u_0, \tau_0) = \int_0^{2\pi p} \Gamma(\theta, \tau) d\theta \cdot u_0^{1/p-1} + o_5(1).
\]

From (34), we have further

\[
F_1(2\pi p, u_0, \tau_0) = d_0^{1/p} \int_0^{2\pi p} \varphi_q \left( C(n\theta) \right) e \left( \tau_0 + \frac{1}{n} \theta \right) d\theta \cdot u_0^{1/p-1} + \varepsilon O_6(1),
\]

\[
F_2(2\pi p, u_0, \tau_0) = \left[ -C_0 2\pi p + \int_0^{2\pi p} \frac{d_0^{1/p}}{pn^2} S(n\theta) e \left( \tau_0 + \frac{1}{n} \theta \right) d\theta \right] u_0^{1/p-1} + o_5(1).
\]
Now we obtain an expression of $P$ as
\[ P: \begin{cases} 
\tau_1 = \tau_0 + \frac{2\pi p}{n} + \varepsilon^{p-1}l_1(u_0, \tau_0) + \varepsilon^{p-1}o_5(1), \\
u_1 = u_0 + \varepsilon^{p-1}l_2(u_0, \tau_0) + \varepsilon^p O_6(1)
\end{cases} \tag{35} \]
for $(u_0, \tau_0) \in \left[1/\lambda, \lambda\right] \times S^1$, where
\[ l_1(u_0, \tau_0) = \left[-C_0 2\pi p + \int_0^{2\pi p} \frac{d_0}{pn^2} S(n\theta) e^{\left(\tau_0 + \frac{1}{n} \theta\right)} d\theta\right] u_0^{1/p-1}, \]
\[ l_2(u_0, \tau_0) = d_0^{1/p} \int_0^{2\pi p} \varphi_q(C(n\theta)) e^{\left(\tau_0 + \frac{1}{n} \theta\right)} d\theta u_0^{1/p}. \tag{36} \]

By Lemma A.1, the map $P$ is reversible with respect to the involution $G: (u_0, \tau_0) \to (u_0, -\tau_0)$, since system (31) is.

4. Proof of Theorem 1 in the case $\delta_0 \geq p - 1$

Now we are in a position to prove Theorem 1 in case $\delta_0 \geq p - 1$. To this end, it is sufficient to assume $\delta_0 = p - 1$ and to show that for every $\varepsilon \ll 1$, the Poincaré map $P$ of (40) has an invariant closed curve in the annulus $[1/\lambda, \lambda] \times S^1$ which surrounds the circle $u = 1/\lambda$. However, in its standard version, Moser’s twist theorem is concerned with a map of the form
\[ \tau_1 = \tau_0 + \omega + \delta u_0 + \cdots, \]
\[ u_1 = u_0 + \cdots, \]
where $\omega$ is a fixed number, $\delta > 0$ is a small parameter and the remaining terms are of order $o_5(\delta)$ as $\delta \to 0^+$. But the map $P$ defined by (35)-(36) does not meet all the conditions of Moser’s theorem except in the case
\[ \int_0^{2\pi p} \varphi_q(C(n\theta)) e^{\left(\tau_0 + \frac{1}{n} \theta\right)} d\theta \equiv 0 \quad \text{for all } \tau_0 \in S^1. \]

Fortunately, there is a variant of Moser’s theorem for reversible system which allows us to prove the existence of invariant curves for $P$.

Let $A = [a, b] \times S^1$ be a finite cylinder with universal cover $\tilde{A} = [a, b] \times R$. Consider the map
\[ \tilde{f}: \tilde{A} \to R \times S^1. \]
Suppose that
\[ f: \tilde{A} \to R \times R \]
is a lift of $\tilde{f}$ having the form $(u_1, \tau_1) = f(u_0, \tau_0)$, with
\[ \tau_1 = \tau_0 + 2\pi \rho + \varepsilon l_1(u_0, \tau_0) + \varepsilon g_1(u_0, \tau_0), \]
\[ u_1 = u_0 + \varepsilon l_2(u_0, \tau_0) + \varepsilon g_2(u_0, \tau_0), \]  
and the functions \( l_1, l_2, g_1 \) and \( g_2 \) satisfying \( l_2, g_1, g_2 \in C^5(\bar{A}) \) and
\[ l_1 \in C^6(A), \quad l_1 > 0, \quad \frac{\partial l_1}{\partial u_0} > 0 \text{ in } \bar{A}. \]  
(37)

In addition, we assume that there exists a function \( I : \bar{A} \to \mathbb{R} \) which is \( 2\pi \rho \)-periodic and even in \( \tau_0 \) such that
\[ I \in C^6(\bar{A}), \quad \frac{\partial I}{\partial u_0}(u_0, \tau_0) > 0, \]  
and
\[ l_1 \frac{\partial I}{\partial \tau_0} + l_2 \frac{\partial I}{\partial u_0} = 0 \]  
for all \((u_0, \tau_0) \in \bar{A}.

Let
\[ \bar{I}(u_0) = \max_{\tau_0} I(u_0, \tau_0), \quad \underline{I}(u_0) = \min_{\tau_0} I(u_0, \tau_0), \quad u_0 \in [a, b], \tau_0 \in S^1. \]

Then we have the following variant of Moser’s twist theorem for reversible system.

**Lemma 5** [7, 17, 19]. Assume there exists a function \( I : \bar{A} \to \mathbb{R} \) which satisfies (39)–(40) and there are numbers \( \bar{a} \) and \( \bar{b} \) with \( a < \bar{a} < \bar{b} < b \) and
\[ \bar{I}(\bar{a}) < \underline{I}(\bar{a}) < \bar{I}(\bar{b}) < \underline{I}(\bar{b}) < I(b). \]  
(41)

Then there exist \( \delta > 0 \) and \( \Delta > 0 \) such that if the conditions (37) and (38) hold for a lift \( f \) of \( \bar{f} \) such that additionally
\[ \|g_1\|_{C^5(A)} + \|g_2\|_{C^5(A)} < \delta \quad \text{and} \quad \delta < \Delta \]
and let \( f \) be reversible with respect to the involution \( G : (u_0, \tau_0) \mapsto (u_0, -\tau_0) \), i.e.,
\[ f^{-1} = G \circ f \circ G \quad \text{in } \bar{A}. \]

Then \( \tilde{f} \) has an invariant curve \( \Gamma \). Denote \( \mu(\Gamma, \varepsilon) \in S^1 \) the rotation number of \( \tilde{f} \); then
\[ \lim_{\varepsilon \to 0^+} \mu(\Gamma, \varepsilon) = 0. \]

**Remark 2.** The change of variables \( \bar{\tau} = -\tau, \bar{u} = u \), shows that condition (38) can be replaced by
\[ l_1 \in C^6(A), \quad l_1 < 0, \quad \frac{\partial l_1}{\partial u_0} < 0 \text{ in } A. \]  
(42)

In the rest of this paper, we assume \( n = 1 \) for simplicity, otherwise, we can consider the map
\[ P^n = P \circ P \circ P \circ \cdots \circ P. \]
In this case, under a further transformation
\[ \tau = \tau, \quad v = u^{-1}, \quad \delta = \varepsilon^{p-1}, \]
the Poincaré map \( P \) is of the form
\[
P: \begin{cases} 
\tau_1 = \tau_0 + 2\pi_p \delta \bar{l}_1(v_0, \tau_0) + \delta h_1(v_0, \tau_0, \delta), \\
v_1 = v_0 + \delta \bar{l}_2(v_0, \tau_0) + \delta h_2(u_0, \tau_0, \delta),
\end{cases}
\]
where \((v_0, \tau_0) \in [1/\lambda, \lambda] \times S^1, \quad \bar{l}_1(v_0, \tau_0) = -C_0 \frac{d_{1/p}}{p} \int_0^{2\pi_p} S(\theta) e^{\tau_0 + \theta} d\theta v_0^{1-1/p}, \quad \bar{l}_2(v_0, \tau_0) = -d_{1/p} \int_0^{2\pi_p} \varphi_q(C(\theta)) e^{\tau_0 + \theta} d\theta v_0^{2-1/p},
\]
and the functions \( h_1, h_2 \) are of order \( o_5(1) \).

Define a function
\[
\mu(\tau_0) = C_0 2\pi_p - \frac{d_{1/p}}{p} \int_0^{2\pi_p} S(\theta) e^{\tau_0 + \theta} d\theta.
\]
Then \(|\mu(\tau_0)| > 0 \) by (7) (and we can assume \( \mu(\tau_0) > 0 \) for all \( \tau_0 \in S^1 \), then
\[
\mu'(\tau_0) = -\frac{d_{1/p}}{p} \int_0^{2\pi_p} S(\theta) e^{\tau_0 + \theta} d\theta
\]
\[
= -\frac{d_{1/p}}{p} S(\theta) e^{\tau_0 + \theta} \bigg|_0^{2\pi_p} + \frac{d_0}{p} \int_0^{2\pi_p} \varphi_q(C(\theta)) e^{\tau_0 + \theta} d\theta
\]
\[
= -\frac{d_{1/p}}{p} \int_0^{2\pi_p} \varphi_q(C(\theta)) e^{\tau_0 + \theta} d\theta.
\]
Therefore, \( \tilde{l}_1 \) and \( \tilde{l}_2 \) can be written as
\[
\tilde{l}_1(v_0, \tau_0) = -\mu(\tau_0) v_0^{1-1/p},
\]
\[
\tilde{l}_2(v_0, \tau_0) = -p \mu'(\tau_0) v_0^{2-1/p}.
\]
Then \( \tilde{l}_1 \in C^6(A) \) and \( \tilde{l}_1(\partial \tilde{l}_1/\partial u_0) > 0 \).

Define
\[
I(v_0, \tau_0) = -\left( \tilde{l}_1(v_0, \tau_0) \right)^{-1} v_0.
\]
Then
\[ \frac{\partial I}{\partial v_0} = \frac{v_0^{1/p - 1}}{p \mu(\tau_0)} > 0 \]
and \( I \) satisfies
\[ \bar{I}_1 \frac{\partial I}{\partial \tau_0} + \bar{I}_2 \frac{\partial I}{\partial v_0} = 0 \text{ in } A. \]
Let
\[ \Delta_1 = \min \mu(\tau_0), \quad \Delta_2 = \max \mu(\tau_0). \]
Then \( \Delta_2 \geq \Delta_1 > 0 \). Define \( \lambda \) by
\[ \lambda = \left( \frac{2 \Delta_2}{\Delta_1} \right)^{3/2} > 1 \]
and
\[ a = \lambda^{-1}, \quad \bar{a} = \lambda^{-1/3}, \quad \bar{b} = \lambda^{1/3}, \quad b = \lambda. \]
Then it is easy to see that
\[ \bar{I}(a) < I(\bar{a}) \leq \bar{I}(\bar{b}) < \bar{I}(b). \]
Thus the have showed that the map \( P \) satisfies all the conditions of Lemma 6. Consequently, the Poincaré map \( P \) has an invariant closed curve diffeomorphic to \( v_0 = \text{const} \) for all small \( \varepsilon > 0 \). This completes the proof of Theorem 1 for the case \( \delta_0 \geq p - 1 \).

5. Proof of Theorem 1 in the case \( 0 < \delta_0 < p - 1 \)

In this section, we introduce a series of transformations such that the transformed system of (20) satisfies the conditions of Theorem 1 for case \( \delta_0 \geq p - 1 \). Then the proof of Theorem 1 for the case \( \delta_0 \geq p - 1 \) applies.

Consider system (20) with \( n = 1 \) and denote
\[ h_1(r, \theta, t) = d_0^{1/p} r^{1/p} q_0(C(\theta))(e(t) - g(x)), \quad h_2(r, \theta) = -d_0 |C(\theta)|^q f(x), \]
\[ h_3(r, \theta, t) = -\frac{d_0^{1/p}}{p} r^{-1/q} S(\theta)(e(t) - g(x)), \quad h_4(r, \theta) = \frac{d_0}{p} S(\theta) C(\theta) f(x). \]
Then system (20) is of the form
\[ \frac{dr}{dt} = h_1(r, \theta, t) + h_2(r, \theta), \]
\[ \frac{d\theta}{dt} = 1 + h_3(r, \theta, t) + h_4(r, \theta), \]  
\[ (43) \]
and \( h_1(r, -\theta, t) = -h_1(r, \theta, t), \quad h_2(r, -\theta) = -h_2(r, \theta), \quad h_3(r, -\theta, t) = -h_3(r, \theta, t), \quad h_4(r, -\theta) = h_4(r, \theta), \) which means that system (43) is reversible with respect to the involution \( G : (r, \theta) \rightarrow (r, -\theta) \). Moreover, by \( |f(x)| \leq C|x|^{-\delta_0} \leq Cr^{-\delta_0/p} \), we have
\[ h_1 \in \mathcal{O}(r^{1/p}), \quad h_2 \in \mathcal{O}(r^{1-\delta_0/p}), \quad h_3 \in \mathcal{O}(r^{-1/q}), \quad h_4 \in \mathcal{O}(r^{-\delta_0/p}). \]  
\[ (44) \]
Lemma 6. There exists a $G$-invariant diffeomorphism $\psi_1$, having the form
\[ \psi_1 : r = \rho + U_1(\rho, \theta), \quad \theta = \theta, \]
where $U_1(\rho, \theta)$ satisfies $U_1(\rho, \theta) \in O_6(\rho^{1-e_0})$, with $e_0 = \delta_0 / p > 0$. Under this transformation, system (43) is transformed into
\[ \frac{d\rho}{dt} = \tilde{h}_1(\rho, \theta, t) + \tilde{h}_2(\rho, \theta), \]
\[ \frac{d\theta}{dt} = 1 + \tilde{h}_3(\rho, \theta, t) + \tilde{h}_4(\rho, \theta), \]
for system (45)
where $\tilde{h}_1, \tilde{h}_3, \tilde{h}_4$ satisfy (44) and $\tilde{h}_2 \in O_6(\rho^{1-2e_0})$. Moreover
\[ \tilde{h}_1(\rho, -\theta, -t) = -\tilde{h}_1(\rho, \theta, t), \quad \tilde{h}_2(\rho, -\theta) = -\tilde{h}_2(\rho, \theta), \]
\[ \tilde{h}_3(\rho, -\theta, -t) = \tilde{h}_3(\rho, \theta, t), \quad \tilde{h}_4(\rho, -\theta) = -\tilde{h}_4(\rho, \theta). \]

Proof. Define a transformation $\Phi_1$ by
\[ \Phi_1 : \rho = r + V_1(r, \theta), \quad \theta = \theta, \]
where
\[ V_1(r, \theta) = -\int_0^\theta h_2(r, s) \, ds. \]
Since $h_2(r, -\theta) = -h_2(r, \theta)$ and $h_2 \in O_6(\rho^{1-e_0})$, we obtain
(i) $V(r, -\theta) = V(r, \theta)$;
(ii) the transformation $\Phi_1$ is diffeomorphism and satisfies
\[ V_1(r, \theta) \in O_6(\rho^{1-e_0}). \]
Let $\psi_1 = \Phi_1^{-1} : r = \rho + U_1(\rho, \theta), \theta = \theta$. Then $U_1$ satisfies $U_1(\rho, -\theta) = U_1(\rho, \theta)$ and
\[ U_1(\rho, \theta) \in O_6(\rho^{1-e_0}). \]
(The proof of the above properties of $U_1$ is similar to that of Lemma 3.1 in Ref. [11].)
Therefore, the transformation $\psi_1$ is $G$-invariant diffeomorphism and which implies that transformed system (45) is reversible with respect to the involution $G : (\rho, \theta) \rightarrow (\rho, -\theta)$.
Under the transformation $\psi_1$, $\tilde{h}_1$ is of the following form ($i = 1, 2, 3, 4$):
\[ \tilde{h}_1(\rho, \theta, t) = h_1(r, \theta, t) + \tilde{h}_1(\rho, \theta, t), \]
\[ \tilde{h}_1(\rho, \theta, t) = -\left( \int_0^\theta \frac{\partial h_2}{\partial r}(r, s) \, ds \right) h_1 - h_2 h_3, \]
\[ \tilde{h}_2(\rho, \theta) = -\left[ \left( \int_0^\theta \frac{\partial h_2}{\partial r}(r, s) \, ds \right) h_2 + h_2 h_4 \right], \]
\[ \tilde{h}_3(\rho, \theta, t) = h_3(r, \theta, t), \quad \tilde{h}_4(\rho, \theta) = h_4(r, \theta), \]
for (48)
where \( r = \rho + U_1(\rho, \theta) \).

Since \( h_1 \in O_6(r^{1/p}) \), \( \tilde{h}_1 \in O_6(r^{1/p-\varepsilon_0}) \), the dominated term of \( \tilde{h}_1 \) is still \( h_1 \). Moreover, it is easy to see \( \tilde{h}_2 \in O_6(r^{1-2\varepsilon_0}) \). The statement of (46) is a direct consequence of the properties of \( h_i, i = 1, 2, 3, 4 \), and (48). \( \Box \)

Applying the transformation successively \( j \) times with \( j \geq [\frac{(p-1)}{\delta_0}] \), where \([A]\) stands for the integer part of \( A \). After these transformations, \( h_1 \) and \( h_4 \) remain unchanged and \( \tilde{h}_1 = h_1 \), \( \tilde{h}_1 = \tilde{h}_1 \) with \( h_1 \in O_6(\rho^{1/p}) \), \( \tilde{h}_1 \in O_6(r^{1/p-\varepsilon_0}) \) with \( 1/p - j\varepsilon_0 \leq 0 \), \( \tilde{h}_2 \in O_6(\rho^{1/p}). \) Moreover, the transformed system is still reversible with respect to the involution \( G: (\rho, \theta) \mapsto (\rho, -\theta) \).

**Lemma 7.** There exists a \( G \)-invariant diffeomorphism \( \psi_2 \), having the form

\[
\psi_2 : \rho = \rho, \quad \theta = \sigma + U_2(\rho, \sigma),
\]

where \( U_2(\rho, \sigma) \) satisfies \( U_2(\rho, \sigma) \in O_6(\rho^{-\varepsilon_0}) \) and \( U_2(\rho, -\sigma) = -U_2(\rho, \sigma) \). Under this transformation, system (45) is transformed into

\[
\begin{align*}
\frac{d\rho}{dt} &= h_1(\rho, \sigma, t) + h_2(\rho, \sigma), \\
\frac{d\sigma}{dt} &= 1 + h_3(\rho, \sigma, t) + h_4(\rho, \sigma),
\end{align*}
\]

where \( h_1(\rho, \sigma, t) = h_1(\rho, \theta, t), h_2(\rho, \sigma) = h_2(\rho, \theta) \), with \( \theta = \sigma + U_2(\rho, \sigma) \), \( h_3(\rho, \sigma, t) = h_3(\rho, \theta, t) + h_3(\rho, \theta, t) \) with \( h_3 \in O_6(\rho^{-1/q}), h_4(\rho, \sigma, t) = h_4(\rho, \sigma, t) \) with \( h_4 \in O_6(\rho^{-2\varepsilon_0}) \). Moreover, \( h_i, i = 1, 2, 3, 4 \), satisfies (46), which implies that system (49) is reversible with respect to the involution \( G: (\rho, \sigma) \mapsto (\rho, -\sigma) \).

**Proof.** Let \( \Phi_2 : \rho = \rho, \sigma = \theta + V_2(\rho, \theta) \) with

\[
V_2(\rho, \theta) = -\int_0^\theta h_4(\rho, s) \, ds
\]

and \( \psi_2 = \Phi_2^{-1} \). Then similar to the proof of Lemma 6, we can prove the statement of Lemma 7. We omit the details. \( \Box \)

By applying transformation \( \psi_2 \) \( j \) times with \( j \geq [\frac{(p-1)}{\delta_0}] \), we obtain a reversible system of the form (49) with \( \tilde{h}_1 = h_1 + h_3, \tilde{h}_3 \in O_6(\rho^{-1/q}), h_4, h_4 = h_1, h_2 = h_2 \).

Proof of Theorem 1 in the case \( 0 < \delta_0 < p - 1 \).

From Lemmas 6 and 7, after at most \( 2j = 2[(p-1)/\delta_0] \) times \( G \)-invariant transformations, we obtain a system of the form

\[
\begin{align*}
\frac{d\rho}{dt} &= h_1(\rho, \sigma, t) + \tilde{h}_2(\rho, \theta, t), \\
\frac{d\theta}{dt} &= 1 + h_3(\rho, \theta, t) + h_4(\rho, \theta, t),
\end{align*}
\]

(50)
with $\tilde{h}_2 \in O_6(\rho^{1/p})$, $\tilde{h}_4 \in O_6(\rho^{-1/q})$ and system (50) is reversible with respect to the involution $G: (\rho, \theta) \rightarrow (\rho, -\theta)$. Therefore all conditions for the case $\delta_0 \geq p - 1$ are satisfied and Theorem 1 is thus proved.

**Remark 3.** Let $p = 2$; then $B(1, 1/2) = 2$ and (7) reduces to

$$4|g(+\infty)| > \left| \int_0^{2\pi} \sin \theta e(\tau + \theta) d\theta \right| = \left| \cos \tau \int_0^{2\pi} \sin \theta e(\tau) d\theta \right| \quad \text{for all } \tau \in [0, 2\pi]$$

which is the condition (3) of [7].

**Example 1.** All solutions of

$$x'' + \frac{xx'}{(1 + x^2)^{3/4}} + n^2 x + 2 \arctan x = \sin t$$

are bounded.

**Example 2.** Consider Eq. (7) in Section 1:

$$\left(\varphi_p(x')\right)' + f(x)\varphi_p(x') + \varphi_p(x) + 2 \arctan x = \sin \left(\frac{\pi t}{\pi p}\right),$$

where $p > 1$, $f(x) = x/(1 + x^2)^{p/(p+1)}$.

Since $\min_{r,s>0} B(r,s) = B(1, 1) = 1$, it follows from $|S(t)| \leq 1$, $|\sin t| \leq 1$ that (6) and (7) hold. Hence all the solutions of (7) are bounded according to Theorem 1.

**Appendix A**

**Definition A.1.** Let $\Omega \subset \mathbb{R}^n$ be an open set, $X : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^n$ be continuous and $T$-periodic in the last variable. The system

$$x' = X(x, t) \quad (A.1)$$

is called a reversible system if there exists an involution $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$ (i.e., $G$ is a $C^1$-diffeomorphism such that $G^2 = I_{\mathbb{R}^n}$), with $G(\Omega) = \Omega$ and such that

$$DG(Gx) \cdot X(Gx, -t) = -X(x, t), \quad x \in \Omega, \ t \in \mathbb{R}, \quad (A.2)$$

where $DG$ stands for the Jacobian of $G$.

**Definition A.2.** Let $A : \Omega \rightarrow \mathbb{R}^n$ be a homeomorphism onto its image and let $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a homeomorphism with $G^2 = I_{\mathbb{R}^n}$. $A$ is called reversible with respect to $G$ on a set $D \subset \Omega \cap \Lambda(\Omega)$, with $G(D) = D$, if

$$A^{-1} = GAG, \quad \forall x \in D.$$
Definition A.3. Assume that \( T(\cdot,t) \) is an invertible transformation of \( \Omega \) for every fixed \( t \), and \( G \) is an involution of \( \mathbb{R}^n \) with \( G(\Omega) = \Omega \). Then \( T \) is called \( G \)-invariant if

\[
G(T(x,t)) = T(Gx,-t), \quad x \in \Omega, \ t \in \mathbb{R}.
\]  

(A.3)

Lemma A.1. Suppose that system (A.1) is reversible with respect to an involution \( G \). If a transformation \( T(\cdot,t) : \Omega \to \mathbb{R}^n \) is \( G \)-invariant and \( C^1 \) in \( x \) and \( t \), then the transformed system of (A.3) under \( T \) is also reversible with respect to \( G \).

Remark A.1. Let \( X = (X_1, X_2) \). Then for involution \( G_1(x,y) = (-x,y) \) and \( G_2(x,y) = (x,-y) \) for \( (x,y) \in \mathbb{R}^2 \). The condition (A.2) reads as

\[
X_1(-x,y,-t) = X_1(x,y,t), \quad X_2(-x,y,-t) = -X_2(x,y,t) \quad \text{for} \ G_1,
\]

and

\[
X_1(x,-y,-t) = -X_1(x,y,t), \quad X_2(x,-y,-t) = -X_2(x,y,t) \quad \text{for} \ G_2.
\]

Let \( G_2 : (x,y) \to (x,-y) \). Then (A.3) reads as

\[
T_1(x,-y,-t) = T_1(x,y,t), \quad T_2(x,-y,-t) = -T_2(x,y,t),
\]

where \( T = (T_1, T_2) \).

Proof of Lemma 2. Since \( S(t) = \sin_p t \) is the unique solution of the initial value problem

\[
(\varphi_p(u'))' + (p-1)\varphi_p(u) = 0, \quad u(0) = 0, \ u'(0) = 1,
\]

(A.5)

we obtain from (25) and using the periodicity of \( S \),

\[
\lim_{\rho \to +\infty} \rho^{1/p} f(\rho) = \frac{1}{2\pi_p} \int_0^{2\pi_p/n} S(n\theta)g(x) d\theta
\]

\[
= \frac{1}{2\pi_p} \int_0^{\pi_p/n} S(n\theta)g(x) d\theta + \frac{2\pi_p/n}{\pi_p/n} \int_{\pi_p/n}^{2\pi_p/n} S(n\theta)g(x) d\theta
\]

\[
= \frac{1}{2\pi_p} \int_0^{\pi_p/n} g(+\infty) \int_0^\pi \sin_p(nt) dt - g(-\infty) \int_0^\pi \sin_p(nt) dt
\]

\[
= \frac{1}{\pi_p} \int_0^{\pi_p/n} g(+\infty) \int_0^\pi \sin_p t dt.
\]

Next, we calculate \( \int_0^{\pi_p/n} \sin_p t dt \). Multiplying (A.4) by \( u'(t) \) and integrating from 0 to \( t, t \in [0, \pi_p/2] \), one obtains by using (A.5),

\[
|u'(t)|^p + |u(t)|^p = 1, \quad t \in \left[0, \frac{\pi_p}{2}\right].
\]

(A.6)
From \( \sin_p(\pi_p - t) = \sin_p t \), we obtain \( u'(\pi_p / 2) = 0 \) and \( u(\pi_p / 2) = \max_{t \in [0, \pi_p]} u(t) \).

It follows from (A.6) that

\[
\frac{du}{(1 - u^p)^{1/p}} = dt, \quad t \in \left[ 0, \frac{\pi_p}{2} \right].
\]

Hence

\[
\int_0^{\pi_p} \sin_p t \, dt = 2 \int_0^{\pi_p/2} \sin_p t \, dt = 2 \int_0^1 \frac{u \, du}{(1 - u^p)^{1/p}} \quad (u^p = v)
\]

\[
= 2 \int_0^1 \frac{1}{p} v^{2/p - 1} (1 - v)^{-1/p} \, dv = \frac{2}{p} B \left( \frac{2}{p}, 1 - \frac{1}{p} \right),
\]

where \( B(r, s) = \int_1^0 t^{r-1} (1-t)^{s-1} \, dt \) for \( r > 0 \) and \( s > 0 \).

Therefore

\[
\lim_{\rho \to +\infty} \rho^{1/q} J(\rho) = \frac{2 \alpha_0^{1/p} B(2/p, 1 - 1/p)}{\rho \pi_p} g(+\infty). \quad (A.7)
\]

It is not difficult to verify by induction that for \( k \geq 1 \),

\[
\rho^{k+1/q} \frac{d^k J(\rho)}{d\rho^k} = - \left( k - 1 + \frac{1}{q} \right) \rho^{k-1+1/q} \frac{d^{k-1} J(\rho)}{d\rho^{k-1}} + c_1 \int_0^{2\pi_p} S(n\theta) \frac{dg(x)}{dx} x \, d\theta
\]

\[
+ c_2 \int_0^{2\pi_p} S(n\theta) \frac{d^2 g(x)}{dx^2} x^2 \, d\theta + \ldots + c_k \int_0^{2\pi_p} S(n\theta) \frac{d^k g(x)}{dx^k} x^k \, d\theta,
\]

where \( c_1, c_2, \ldots, c_k \) are constants.

From the assumption \( x^6(d^6 g(x)/dx^6) \to 0 \) and (c) of Remark 1, \( x^k(d^k g(x)/dx^k) \to 0 \) as \( |x| \to \infty \), \( 1 \leq k \leq 5 \), we have for \( \rho \gg 1 \) and \( 1 \leq k \leq 6 \),

\[
\rho^{k+1/q} \frac{d^k J(\rho)}{d\rho^k} = - \left( k - 1 + \frac{1}{q} \right) \rho^{k-1+1/q} \frac{d^{k-1} J(\rho)}{d\rho^{k-1}} + o(1)
\]

\[
= (-1)^k \left( k - 1 + \frac{1}{q} \right) \left( k - 2 + \frac{1}{q} \right) \ldots \left( 1 + \frac{1}{q} \right) \rho^{1/q} J(\rho)
\]

\[
+ o(1). \quad (A.8)
\]

Now (26) follows from (A.7) and (A.8). \( \square \)
References