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Evolution equations for abstract differential operators

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Abstract

We study in this paper the wellposedness and regularity of solutions of evolution equations associated with abstract differential operators on a Banach space. The results can be applied to many partial differential equations on different function spaces.

Keywords: Evolution equation; Evolution family; Partial differential equation; Functional calculus

1. Introduction

Let $P(t,\xi) = \sum_{|\mu|\leqslant m} a_\mu(t)\xi^\mu$ be a polynomial of $\xi \in \mathbf{R}^n$, where $a_\mu \in C([0,T],\mathbf{C})$ for $|\mu| \leqslant m$. Corresponding to this polynomial, we introduce an abstract differential operator as follows: $P(t,A) = \sum_{|\mu|\leqslant m} a_\mu(t)A^\mu$ with maximal domain, where $A^\mu = A_1^{\mu_1}\cdots A_n^{\mu_n}$ and iA_j $(1 \leqslant j \leqslant n)$ are commuting generators of bounded C_0-groups on a Banach space X. This allows us to avoid

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the troubles caused by different function spaces and can be applied extensively (cf. [4,6]). This paper is concerned with the inhomogeneous evolution equation

$$u'(t) = P(t, A)u(t) + f(t), \quad 0 < t \leqslant T,\ u(0) = x, \tag{1}$$

on X, where $f \in C([0,T], X)$. A function $u:[0,T] \to X$ is called a solution of (1), if $u \in C([0,T], X) \cap C^1((0,T], X)$ and (1) is satisfied.

It is well known that the wellposedness of (1) depends on the construction of an evolution family for homogeneous evolution equation (1) (i.e., $f \equiv 0$). We emphasize that the domain of $P(t, A)$ may depend on t. In the case where $P(t, \xi)$ is strongly elliptic for every $t \in [0, T]$, some authors have studied how to construct the evolution family (see, e.g., [12,14]). Recently, motivated by regularized semigroups (cf. [4]), people paid attention to constructing a regularized evolution family for elliptic, even nonelliptic $P(t, \xi)$ [3,4,6,7,13,14].

The purpose of this paper is to extend these results to more general situations. We construct in Section 2 evolution families and regularized evolution families for strongly elliptic and some nonelliptic cases, respectively. Particularly, some regularity results of these families are contained. Our main results are stated in Section 3, which include the wellposedness as well as the regularity of solutions of (1). The last section deals with the application to partial differential equations (PDEs).

Throughout the paper, $B(X)$ will be the space of bounded linear operators on X, $\mathcal{S}(\mathbf{R}^n)$ (respectively $C_c^\infty(\mathbf{R}^n)$) the space of rapidly decreasing functions (respectively C^∞-functions with compact support) on $\mathbf{R}^n$, and $H(\Sigma, X)$ the set of analytic functions from Σ into X. By $\mathcal{D}(B)$, $\mathcal{R}(B)$, and $\rho(B)$ we denote the domain, range, and resolvent set of the operator B, respectively. We also denote by $B(A^\infty)$ the Fréchet space

$$\left\{B : X \to \bigcap_{\mu \in \mathbf{N}_0^n} \mathcal{D}\big(A^\mu\big);\ A^\mu B \in B(X) \text{ for } \mu \in \mathbf{N}_0^n\right\}$$

with the family of seminorms $\|B\|_\mu := \|A^\mu B\|$, where $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$.

We now introduce a functional calculus for iA_j $(1 \leqslant j \leqslant n)$. Let $\mathcal{F}$ denote the Fourier transform, i.e., $(\mathcal{F}u)(r) = \int_{\mathbf{R}^n} u(s)e^{-i(s,r)}\,ds$. If $u \in \mathcal{F}L^1(\mathbf{R}^n)$, then there exists a unique L^1-function $\mathcal{F}^{-1}u$ (i.e., the inverse Fourier transform of u in the distributional sense) such that $u = \mathcal{F}(\mathcal{F}^{-1}u)$. We define $u(A) \in B(X)$ by

$$u(A)x = \int_{\mathbf{R}^n} (\mathcal{F}^{-1}u)(\xi)e^{-i(\xi, A)}x\,d\xi \quad \text{for } x \in X. \tag{2}$$

It is known that $\mathcal{F}L^1(\mathbf{R}^n)$ is a Banach algebra under pointwise multiplication and addition with norm $\|u\|_{\mathcal{F}L^1} := \|\mathcal{F}^{-1}u\|_{L^1}$, and $u \mapsto u(A)$ is an algebra homomorphism from $\mathcal{F}L^1(\mathbf{R}^n)$ into $B(X)$ with $\|u(A)\| \leqslant M\|u\|_{\mathcal{F}L^1}$ for some constant $M > 0$.

Finally, let $|A|^2 = \sum_{j=1}^n A_j^2$ and $(1+|A|^2)^{-\alpha/2}$ ($\alpha \in \mathbf{R}$) be defined as fractional powers. Then $(1+|A|^2)^{-\alpha/2} \in B(X)$ for $\alpha > 0$. $Y_\alpha := \mathcal{D}((1+|A|^2)^{\alpha/2})$ ($\alpha \geqslant 0$) will be a Banach space with graph norm $\|x\|_\alpha := \|(1+|A|^2)^{\alpha/2}x\|$. Moreover, we denote by M a general positive constant.

2. Evolution families

In this section, let Σ, $\widetilde{\Sigma}$ be some convex neighborhoods of $[0,T]$ in $\mathbf{C}$. We write $\Omega = \{(t,s) \in \mathbf{R}\times\mathbf{R};\ 0 \leqslant s < t \leqslant T\}$ and $\Sigma_\theta = \{(t,s) \in \Sigma\times\Sigma;\ t \neq s,\ |\arg(t-s)| < \theta\}$, where $\theta \in (0,\pi/2]$, and denote by $\overline{\Omega}$ (respectively $\overline{\Sigma}$) the closure of Ω (respectively Σ). In the sequel, except in Proposition 4 and Theorem 4, we always assume that $P(t,\xi) = \sum_{|\mu|\leqslant m} a_\mu(t)\xi^\mu$ with $a_\mu \in C[0,T]$ ($|\mu| \leqslant m$). For fixed $t \in [0,T]$, $P(t,\xi)$ is said to be strongly elliptic if $\sum_{|\mu|=m} \operatorname{Re} a_\mu(t)\xi^\mu < 0$ for $\xi \neq 0$.

Let $C \in B(X)$ be injective. A two parameter family $U(t,s) \in B(X)$, $(t,s) \in \overline{\Omega}$, is called a C-regularized evolution family if $U(t,r)U(r,s) = U(t,s)C$ for $0 \leqslant s \leqslant r \leqslant t \leqslant T$, $U(t,t) = C$ for $0 \leqslant t \leqslant T$, and $U(\cdot,\cdot)x \in C(\overline{\Omega}, X)$ for $x \in X$. In the case $C = I$, $(U(t,s))_{(t,s)\in\overline{\Omega}}$ is called an evolution family.

Proposition 1. *Let $P(t,\xi)$ be strongly elliptic for every $t \in [0,T]$. Then there exists a unique evolution family $(U(t,s))_{(t,s)\in\overline{\Omega}}$ such that*:

(a) *$U(\cdot,\cdot) \in C^1(\Omega, B(A^\infty))$, $\frac{\partial}{\partial t}U(t,s) = P(t,A)U(t,s)$ and $\frac{\partial}{\partial s}U(t,s) = -P(s,A)U(t,s)$ for $(t,s) \in \Omega$.*
(b) *$a_\mu \in C^j[0,T]$ ($|\mu| \leqslant m$) for some $j \in \mathbf{N}$ implies $U(\cdot,\cdot) \in C^{j+1}(\Omega, B(A^\infty))$. In particular $a_\mu \in C^\infty[0,T]$ ($|\mu| \leqslant m$) implies $U(\cdot,\cdot) \in C^\infty(\Omega, B(A^\infty))$.*
(c) *$a_\mu \in H(\widetilde{\Sigma})$ ($|\mu| \leqslant m$) for some $\widetilde{\Sigma}$ implies $U(\cdot,\cdot) \in H(\Sigma_\theta, B(A^\infty))$ for some Σ_θ.*

Proof. This is a consequence of [14, Theorem 4.1]. Here, we give a different proof of (c).

By the assumptions on $P(t,\xi)$ we have $\sup\{\operatorname{Re} P(t,\xi);\ \xi \in \mathbf{R}^n,\ t \in \Sigma\} < \infty$ for some Σ with $\overline{\Sigma} \subseteq \widetilde{\Sigma}$. Let $t,\ s \in \Sigma$ with $\operatorname{Re} t > \operatorname{Re} s$ and $\operatorname{Im} t = \operatorname{Im} s$. Then, by the strong ellipticity of $P(t,\xi)$, there exist constants δ, $L > 0$ such that

$$\operatorname{Re}\int_s^t P(\tau,\xi)\,d\tau = \int_{\operatorname{Re} s}^{\operatorname{Re} t} \operatorname{Re} P(\tau + i\operatorname{Im} t, \xi)\,d\tau$$
$$\leqslant -\delta|\xi|^m \operatorname{Re}(t-s) \quad \text{for } |\xi| \geqslant L.$$

Also, there exists $M_1 > 0$ (without loss of generality, $M_1 \geqslant \sup_{t\in\Sigma} \operatorname{Re} t$) such that

$$\left|D^\nu \xi^\mu\right| \leqslant \begin{cases} M_1^l |\xi|^{ml-|\nu|} & \text{for } |\xi| \geqslant L, \\ M_1^l & \text{for } |\xi| < L, \end{cases}$$

where $|\mu| \leqslant ml$, $l \in \mathbf{N}$, and $|\nu| \leqslant [\frac{n}{2}] + 1$ $(\nu \in \mathbf{N}_0^n)$. Thus the same method as in the proof of [14, (3.7)] leads to

$$\left\|v_{t,s}^\mu\right\|_{\mathcal{F}L^1} \leqslant Ml!\left(\frac{M_1}{\operatorname{Re}(t-s)}\right)^l \quad \text{for } |\mu| \leqslant ml \text{ and } l \in \mathbf{N}_0, \tag{3}$$

where $v_{t,s}^\mu(\xi) = \xi^\mu \exp\{\int_s^t P(\tau,\xi)\,d\tau\}$ for $\xi \in \mathbf{R}^n$ and $\mu \in \mathbf{N}_0^n$. We note that

$$I_k \equiv \sum_{j=0}^k \frac{k!}{j!} 2^j (l+j)! = 2^k (l+k)! + kI_{k-1} \quad \text{for } k \in \mathbf{N}.$$

Thus by induction on k

$$\sum_{j=0}^k \frac{k!}{j!} 2^j (l+j)! \leqslant 2^{k+1}(l+k)! \quad \text{for } k,\ l \in \mathbf{N}_0.$$

Also note that, by the Cauchy estimate, there exists a constant $M_2 > 0$ such that

$$\sum_{|\mu|\leqslant m} \left|a_\mu^{(k)}(t)\right| \leqslant k! M_2^{k+1} \quad \text{for } t \in \Sigma \text{ and } k \in \mathbf{N}_0.$$

We now show by induction on k that

$$\left\|\left(\frac{\partial}{\partial t}\right)^k v_{t,s}^\mu\right\|_{\mathcal{F}L^1} \leqslant M(2M_2)^k (l+k)!\left(\frac{M_1}{\operatorname{Re}(t-s)}\right)^{l+k} \tag{4}$$

for $|\mu| \leqslant ml$ and $l, k \in \mathbf{N}_0$. When $k = 0$, the claim follows from (3). If (4) is true for k then (since $M_1 \geqslant \sup_{t\in\Sigma} \operatorname{Re} t$)

$$\begin{aligned}
\left\|\left(\frac{\partial}{\partial t}\right)^{k+1} v_{t,s}^\mu\right\|_{\mathcal{F}L^1} &= \left\|\left(\frac{\partial}{\partial t}\right)^{k} \left(P(t,\cdot)v_{t,s}^\mu\right)\right\|_{\mathcal{F}L^1} \\
&\leqslant \sum_{k_1+k_2=k} \binom{k}{k_1} \sum_{|\nu|\leqslant m} \left|a_\nu^{(k_1)}(t)\right| \cdot \left\|\left(\frac{\partial}{\partial t}\right)^{k_2} v_{t,s}^{\nu+\mu}\right\|_{\mathcal{F}L^1} \\
&\leqslant \sum_{k_1+k_2=k} \binom{k}{k_1} k_1! M_2^{k_1+1} M(2M_2)^{k_2}(l+k_2+1)!\left(\frac{M_1}{\operatorname{Re}(t-s)}\right)^{l+k_2+1} \\
&\leqslant M(2M_2)^{k+1}(l+k+1)!\left(\frac{M_1}{\operatorname{Re}(t-s)}\right)^{l+k+1}
\end{aligned}$$

for $|\mu| \leqslant ml$ and $l \in \mathbf{N}_0$, as desired. From (4) and $\binom{l+k}{k} \leqslant 2^{l+k}$ we obtain

$$\left\|\left(\frac{\partial}{\partial t}\right)^k v_{t,s}^\mu\right\|_{\mathcal{F}L^1} \leqslant Mk!l!\left(\frac{4M_1M_2}{\operatorname{Re}(t-s)}\right)^{l+k} \quad \text{for } |\mu| \leqslant ml \text{ and } l, k \in \mathbf{N}_0.$$

Then, for fixed $t \in \Sigma$, $s \mapsto v^{\mu}_{t,s}$ can be extended analytically to $\{\tau \in \Sigma;\ |\arg(t-\tau)| < \theta\}$, where $\theta = \arctan(4M_1M_2)^{-1}$. Similarly, $t \mapsto v^{\mu}_{t,s}$ can also be extended analytically to $\{\tau \in \Sigma;\ |\arg(\tau - s)| < \theta\}$ for fixed $s \in \Sigma$. It hence follows from Hartogs' theorem (see, e.g., [2]) that the function $(t,s) \mapsto v^{\mu}_{t,s}$ is in $H(\Sigma_\theta, \mathcal{F}L^1(\mathbf{R}^n))$, and so $(t,s) \mapsto v^{\mu}_{t,s}(A)$ is in $H(\Sigma_\theta, B(X))$. Let $(U(t,s))_{(t,s)\in\overline{\Omega}}$ be the unique evolution family satisfying (a). Then

$$A^{\mu}U(t,s) = v^{\mu}_{t,s}(A) \quad \text{for } (t,s) \in \Sigma_\theta$$

(cf. the proof of [14, Theorem 4.1]), and therefore $U(\cdot,\cdot) \in H(\Sigma_\theta, B(A^\infty))$. $\square$

Proposition 1 improves [6, Theorem 5.1] in several aspects. First, we do not assume that the coefficients of $P(t,\xi)$ are real valued. Second, the conclusion

$$U(t,s)x \in \bigcap_{0\leqslant r\leqslant T} \mathcal{D}\big(\overline{P(r,A)}\big) \quad \text{for } (t,s) \in \Omega \text{ and } x \in X$$

in [6] is sharpened by $U(\cdot,\cdot) \in C^1(\Omega, B(A^\infty))$. Finally, the regularity of $(U(t,s))_{(t,s)\in\overline{\Omega}}$ (i.e., (b) and (c)) was not discussed in [6].

The subsequent two propositions are essentially due to [14], which will be used in the next section.

Proposition 2. *Let there exist constants* δ, $L > 0$ *and* $r \in (0, m-1]$ *such that*

$$\operatorname{Re} P(t,\xi) \leqslant -\delta|\xi|^r \quad \textit{for } |\xi| \geqslant L \textit{ and } t \in [0,T]. \tag{5}$$

Then there exists a unique C*-regularized evolution family* $(U(t,s))_{(t,s)\in\overline{\Omega}}$*, where* $C = (1+|A|^2)^{-m\alpha/2}$ *with* $\alpha > \frac{n(m-r)}{2m}$*, such that the conclusions* (a) *and* (b) *of Proposition* 1 *are still true.*

A polynomial $P(\xi)$ is called to be r-coercive, if $|P(\xi)|^{-1} = \mathrm{O}(|\xi|^{-r})$ as $|\xi| \to \infty$. Thus, the estimate (5) means that $\operatorname{Re} P(t,\xi)$ is bounded above and r-coercive, uniformly for $t \in [0,T]$.

Proposition 3. *Let* $\sup\{\operatorname{Re} P(t,\xi);\ \xi \in \mathbf{R}^n,\ t \in [0,T]\} < \infty$*. Then there exists a unique* C*-regularized evolution system* $(U(t,s))_{(t,s)\in\overline{\Omega}}$*, where* $C = (1+|A|^2)^{-m\alpha/2}$ *with* $\alpha > \frac{n}{2}$*, such that*:

(a) $U(t,s): Y_\beta \to Y_\gamma$ *for* $0 \leqslant \gamma < \beta + m(\alpha - \frac{n}{2})$ *and* $(t,s) \in \overline{\Omega}$*. In particular,* $U(t,s): Y_\beta \to Y_\beta$ *for* $\beta \geqslant 0$ *and* $(t,s) \in \overline{\Omega}$.
(b) $U(t,s): Y_\beta \to \mathcal{D}(P(r,A))$ *for* $\beta > m(1-\alpha+\frac{n}{2})$, $(t,s) \in \overline{\Omega}$*, and* $r \in [0,T]$*. In particular,* $U(t,s): Y_m \to \mathcal{D}(P(r,A))$ *for* $(t,s) \in \overline{\Omega}$ *and* $r \in [0,T]$.
(c) *For* $(t,s) \in \overline{\Omega}$ *and* $x \in Y_\beta$ $(\beta \geqslant m)$, $U(\cdot,\cdot)x \in C^1(\overline{\Omega}, X)$,

$$\frac{\partial}{\partial t}U(t,s)x = P(t,A)U(t,s)x, \quad \text{and}$$
$$\frac{\partial}{\partial s}U(t,s)x = -P(s,A)U(t,s)x. \tag{6}$$

Proposition 3 improves [6, Theorem 5.3]. In the general case, i.e., $P(t,A)$ $(0 \leqslant t \leqslant T)$ is replaced by a family $A(t)$ $(0 \leqslant t \leqslant T)$ of closed operators on X, similar results were given by deLaubenfels [3, Theorem 6.3] and Tanaka [13, Theorem 2.1]. But Proposition 3 cannot be deduced from them. In fact, even in the case when $A(t) = P(t,D)$ $(0 \leqslant t \leqslant T)$, it is possible to yield a large value α [3, Example 6.4], or the ellipticity of $P(t,\xi)$ is required [13, Theorem 3.4].

In the case $m = 1$, we can directly construct the evolution family.

Proposition 4. *Let*

$$P(t,\xi) = \sum_{j=1}^{n} i a_j(t)\xi_j + a_0(t) \quad \text{for } \xi = (\xi_1,\dots,\xi_n) \in \mathbf{R}^n, \tag{7}$$

where $a_j \in C([0,T],\mathbf{R})$ $(1 \leqslant j \leqslant n)$ *and* $a_0 \in C[0,T]$. *Then there exists a unique evolution family* $(U(t,s))_{(t,s)\in\overline{\Omega}}$ *such that* $U(t,s): D \to D$ *and* (6) *hold for* $(t,s) \in \overline{\Omega}$ *and* $x \in D$, *where* $D = \bigcap_{j=1}^{n} \mathcal{D}(A_j)$.

Proof. By the assumption on A we can define

$$U(t,s) = \exp\left\{ \sum_{j=1}^{n} \int_s^t a_j(\tau)\,d\tau\, iA_j + \int_s^t a_0(\tau)\,d\tau \right\} \quad \text{for } (t,s) \in \overline{\Omega}.$$

Now one can easily check from the properties of C_0-groups that $(U(t,s))_{(t,s)\in\overline{\Omega}}$ satisfies the desired conclusions. The uniqueness of $(U(t,s))_{(t,s)\in\overline{\Omega}}$ can be shown by a standard method (cf. the proof of [6, Corollary 5.4]). $\square$

For general polynomials with time-dependent coefficients we have the following result.

Proposition 5. *For any polynomial* $P(t,\xi)$, *there exists a two parameter family* $(U(t,s))_{t,s\in[0,T]} \subset B(X)$ *such that*:

(a) *There exists an injective* $C \in B(X)$ *such that* $U(t,r)U(r,s) = U(t,s)C$ *and* $U(t,t) = C$ *for* $t,r,s \in [0,T]$.
(b) *Proposition* 1(a) *with* Ω *replaced by* $[0,T] \times [0,T]$ *holds. In particular,* $(U(t,s))_{t,s\in\overline{\Omega}}$ *is a* C*-regularized evolution family.*
(c) *If there exists* Σ *such that* $a_\mu \in H(\Sigma)$ *for* $|\mu| \leqslant m$, *then* $U(\cdot,\cdot) \in H(\Sigma \times \Sigma, B(A^\infty))$.
(d) *If* a_μ $(|\mu| \leqslant m)$ *are all entire functions, then* $(U(t,s))_{t,s\in[0,T]}$ *can be extended to an entire* $B(A^\infty)$*-valued function* $(U(t,s))_{t,s\in\mathbf{C}}$.

Proof. Define

$$u_{t,s}(\xi) = \exp\left\{-|\xi|^{2m} + \int_s^t P(\tau,\xi)\,d\tau\right\} \quad \text{for } t,s \in [0,T].$$

Since $u_{t,s} \in \mathcal{S}(\mathbf{R}^n) \subset \mathcal{F}L^1(\mathbf{R}^n)$ $(t,s \in [0,T])$ we can define $U(t,s) = u_{t,s}(A)$ $(t,s \in [0,T])$ and $C = U(0,0)$. Then (a) follows from the property of the algebra homomorphism of (2). The proof of (b) is the same as the one of [14, Theorem 4.1(a)] with Ω replaced by $[0,T] \times [0,T]$. Finally, it is not difficult to show, by the condition of (c) (respectively (d)), that for every $\mu \in \mathbf{N}_0^n$, the function $(t,s) \mapsto \xi^\mu u_{t,s}(\xi)$ is in $H(\Sigma \times \Sigma, \mathcal{F}L^1(\mathbf{R}^n))$ (respectively $H(\mathbf{C} \times \mathbf{C}, \mathcal{F}L^1(\mathbf{R}^n))$). Thus we conclude (c) (respectively (d)). □

3. Evolution equations

The purpose of this section is to treat the inhomogeneous evolution equation (1). Let $C^\beta(J,X)$ $(0 < \beta < 1)$ be the space of Hölder continuous functions, $C^{j+\beta}(J,X) = \{f \in C^j(J,X);\ f^{(j)} \in C^\beta(J,X)\}$ $(j \in \mathbf{N}_0)$, and $C^0(J,X) = C(J,X)$, where J is an interval in $\mathbf{R}$. For injective $C \in B(X)$, we denote by $[\mathcal{R}(C)]$ the Banach space $(\mathcal{R}(C), \|C^{-1}\cdot\|)$. Moreover, Σ (respectively Σ') will denote some convex neighborhood of $[0,T]$ (respectively $(0,T]$) in $\mathbf{C}$.

Theorem 1. *Let $P(t,\xi)$ be strongly elliptic for every $t \in [0,T]$, and suppose there exist $j \in \mathbf{N}_0$, $\beta \in (0,1)$ such that $a_\mu \in C^{j+\beta}[0,T]$ $(|\mu| \leqslant m)$ and $f \in C^{j+\beta}([0,T],X)$. Then for every $x \in X$, (1) has a unique solution $u \in C([0,T],X) \cap C^{j+1+\gamma}([\delta,T],X)$ for $\delta \in (0,T)$ and $\gamma \in (0,\beta)$, such that*

$$\|u(t)\| \leqslant M\Big(\|x\| + \sup_{0\leqslant s\leqslant t}\|f(s)\|\Big) \quad \text{for } t \in [0,T]. \tag{8}$$

In particular, $a_\mu \in C^\infty[0,T]$ $(|\mu| \leqslant m)$ and $f \in C^\infty([0,T],X)$ imply $u \in C^\infty((0,T],X)$. Moreover, if there exists Σ such that $a_\mu \in H(\Sigma)$ $(|\mu| \leqslant m)$ and $f \in H(\Sigma,X)$, then $u \in H(\Sigma',X)$.

Proof. When $j = 0$, by the same argument as in the proof of [6, Corollary 5.2] we can deduce that (1) has a unique solution u given by

$$u(t) = U(t,0)x + \int_0^t U(t,s)f(s)\,ds \quad \text{for } t \in [0,T] \tag{9}$$

where $(U(t,s))_{(t,s)\in\overline{\Omega}}$ is the evolution family provided by Proposition 1. Thus, (8) follows from (9).

When $j > 0$, by Proposition 1(b) one has $U(\cdot,\cdot) \in C^{j+1}(\Omega, B(A^\infty))$. Let $0 < \tau \leqslant t \leqslant T$. Since Proposition 1(a) implies that

$$\left(\frac{\partial}{\partial t}\right)^{l+1} U(t,t-\tau) = \left(\frac{\partial}{\partial t}\right)^{l} \left\{\left[P(t,A) - P(t-\tau,A)\right]U(t,t-\tau)\right\}$$
$$= \sum_{k=0}^{l} \binom{l}{k} \left[P^{(k)}(t,A) - P^{(k)}(t-\tau,A)\right]\left(\frac{\partial}{\partial t}\right)^{k-l} U(t,t-\tau),$$

a simple induction shows that $(\frac{\partial}{\partial t})^l U(t,t-\tau)$ $(l \leqslant j+1)$ is a sum of terms of the form

$$\text{const.}\left[P(t,A) - P(t-\tau,A)\right]^{k_1} \cdots \left[P^{(l-1)}(t,A) - P^{(l-1)}(t-\tau,A)\right]^{k_l} \\ \times U(t,t-\tau), \tag{10}$$

where $k_1 + 2k_2 + \cdots + lk_l = l$. By (10) and the binomial formula we find that $\{(\frac{\partial}{\partial t})^l U(t,t-\tau)\}|_{\tau=t}$ $(l \leqslant j+1)$ is a sum of terms of the form

$$\text{const.}\left[P(t,A)\right]^{p_1} \cdots \left[P^{(l-1)}(t,A)\right]^{p_l} Q(A)U(t,0), \tag{11}$$

where $Q(\xi)$ is a polynomial of degree qm for some $q \in \mathbf{N}_0$ and $p_1 + 2p_2 + \cdots + lp_l + q \leqslant l$. Therefore, based on the method of proof of (10), one deduces further from (11) that $(\frac{\partial}{\partial t})^p \{[(\frac{\partial}{\partial t})^l U(t,t-\tau)]|_{\tau=t}\}$ $(p + l \leqslant j+1)$ is a sum of terms of the form

$$\text{const.}\left[P(t,A)\right]^{q_l} \cdots \left[P^{(p+l-1)}(t,A)\right]^{q_{p+l}} Q(A)U(t,0), \tag{12}$$

where $q_1 + 2q_2 + \cdots + (p+l)q_{p+l} + q \leqslant p + l$.

We now turn to estimate

$$\left(\frac{\partial}{\partial t}\right)^{l} U(t,t-\tau) \quad (l \leqslant j+1)$$

and

$$\left(\frac{\partial}{\partial t}\right)^{p} \left\{\left[\left(\frac{\partial}{\partial t}\right)^{l} U(t,t-\tau)\right]\Big|_{\tau=t}\right\} \quad (p + l \leqslant j+1).$$

If $l \leqslant j$ or $l = j+1$ with $k_{j+1} = 0$ then, by $a_\mu \in C^{j+\beta}[0,T]$ $(|\mu| \leqslant m)$ and (12),

$$\left\|\left[P(t,A) - P(t-\tau,A)\right]^{k_l} \cdots \left[P^{(l-1)}(t,A) - P^{(l-1)}(t-\tau,A)\right]^{k_l} \\ \times U(t,t-\tau)\right\| \leqslant M \sum_{|\mu| \leqslant mk_l'} \tau^{k_l'} \left\|A^\mu U(t,t-\tau)\right\| \leqslant M \tag{13}$$

where $k_l' = k_1 + \cdots + k_l$. If $l = j+1$ with $k_{j+1} > 0$ then (10) consists of the term

$$\text{const.}\left[P^{(l-1)}(t,A) - P^{(l-1)}(t-\tau,A)\right]U(t,t-\tau). \tag{14}$$

From this, the same reasons as above lead to

$$\left\|\left[P^{(l-1)}(t,A) - P^{(l-1)}(t-\tau,A)\right]U(t,t-\tau)\right\| \leqslant M\tau^{\beta-1}.$$

Summarizing these estimates we obtain

$$\left\| \left(\frac{\partial}{\partial t}\right)^{l} U(t, t-\tau)\right\| \leqslant \begin{cases} M & \text{for } l \leqslant j, \\ M\tau^{\beta-1} & \text{for } l = j+1. \end{cases} \tag{15}$$

To obtain the second estimate we note that by (12)

$$\begin{aligned} &\left\| \left[P(t,A)\right]^{q_l} \cdots \left[P^{(p+l-1)}(t,A)\right]^{q_{p+l}} Q(A)U(t,0)\right\| \\ &\quad \leqslant M \sum_{|\mu| \leqslant m(p+l)} \left\| A^{\mu} U(t,0)\right\| \leqslant M t^{-(p+l)}. \end{aligned}$$

Consequently,

$$\begin{aligned} &\left\| \left(\frac{\partial}{\partial t}\right)^{p} \left\{ \left[\left(\frac{\partial}{\partial t}\right)^{l} U(t, t-\tau)\right] \Big|_{\tau=t} \right\} \right\| \leqslant M t^{-(p+l)} \\ &\quad \text{for } p + l \leqslant j + 1. \end{aligned} \tag{16}$$

To prove the desired conclusion we have to give a representation for $u^{(j+1)}(t)$ $(0 < t \leqslant T)$. Since it is easy to deal with $U(\cdot\,,0)x$ for $x \in X$ we first consider the term $v(t) := \int_0^t U(t,s) f(s)\, ds$ for $t \in (0, T]$. In fact, we will show by induction that

$$\begin{aligned} v^{(j+1)}(t) = {} & \sum_{k=0}^{j} \left(\frac{\partial}{\partial t}\right)^{j-k} \sum_{l=0}^{k} \binom{k}{l} \left\{ \left(\frac{\partial}{\partial t}\right)^{l} U(t, t-\tau)\right\} \Big|_{\tau=t} f^{(k-l)}(0) \\ & + \int_0^t \sum_{k=1}^{j+1} \binom{j+1}{k} \left\{ \left(\frac{\partial}{\partial t}\right)^{k} U(t, t-\tau)\right\} f^{(j-k+1)}(t-\tau)\, d\tau \\ & + \int_0^t P(\tau, A) U(t,\tau)\left[f^{(j)}(\tau) - f^{(j)}(t)\right] d\tau \\ & + U(t,0)\left[f^{(j)}(t) - f^{(j)}(0)\right]. \end{aligned} \tag{17}$$

Here we note that, by (15)–(16) and the assumption on f, all terms on the right-hand side of (17) are well defined.

When $j = 0$, from the proof of [6, Corollary 5.2] and integration by parts we obtain

$$\begin{aligned} v'(t) &= f(t) + \int_0^t P(t,A) U(t,\tau) f(\tau)\, d\tau \\ &= f(t) + \int_0^t \left[P(t,A) - P(\tau,A)\right] U(t,\tau) f(\tau)\, d\tau \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t P(\tau,A)U(t,\tau)\big[f(\tau)-f(t)\big]\,d\tau+\int_0^t P(\tau,A)U(t,\tau)f(t)\,d\tau\\
&=U(t,0)f(0)+\int_0^t\big[P(t,A)-P(t-\tau,A)\big]U(t,t-\tau)f(t-\tau)\,d\tau\\
&\quad+\int_0^t P(\tau,A)U(t,\tau)\big[f(\tau)-f(t)\big]\,d\tau\\
&\quad+U(t,0)\big[f(t)-f(0)\big], \qquad (18)
\end{aligned}
$$

i.e., (17) is true for $j=0$. If (17) with j replaced by $j-1$ is still true, differentiation yields

$$
\begin{aligned}
v^{(j+1)}(t)&=\sum_{k=0}^{j-1}\left(\frac{\partial}{\partial t}\right)^{j-k}\sum_{l=0}^{k}\binom{k}{l}\left\{\left(\frac{\partial}{\partial t}\right)^{l}U(t,t-\tau)\right\}\bigg|_{\tau=t}f^{(k-l)}(0)\\
&\quad+\sum_{k=1}^{j}\binom{j}{k}\left\{\left(\frac{\partial}{\partial t}\right)^{k}U(t,t-\tau)\right\}\bigg|_{\tau=t}f^{(j-k)}(0)\\
&\quad+\int_0^t\sum_{k=1}^{j}\binom{j}{k}\left\{\left(\frac{\partial}{\partial t}\right)^{k+1}U(t,t-\tau)\right\}f^{(j-k)}(t-\tau)\,d\tau\\
&\quad+\int_0^t\sum_{k=1}^{j}\binom{j}{k}\left\{\left(\frac{\partial}{\partial t}\right)^{k}U(t,t-\tau)\right\}f^{(j-k+1)}(t-\tau)\,d\tau\\
&\quad+\int_0^t\left\{\frac{\partial^2}{\partial t\partial\tau}U(t,t-\tau)\right\}\big[f^{(j-1)}(t-\tau)-f^{(j-1)}(t)\big]\,d\tau\\
&\quad+\int_0^t P(t-\tau,A)U(t,t-\tau)\big[f^{(j)}(t-\tau)-f^{(j)}(t)\big]\,d\tau\\
&\quad+\big[P(t,A)-P(0,A)\big]U(t,0)\big[f^{(j-1)}(t)-f^{(j-1)}(0)\big]\\
&\quad+U(t,0)f^{j}(t)\\
&:=I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8.
\end{aligned}
$$

Obviously, $I_1 + I_2 + U(t,0)f^{(j)}(0)$ (respectively I_6, $I_8 - U(t,0)f^{(j)}(0)$) is the first (respectively the third, the fourth) term on the right-hand side of (17). Noting that Proposition 1(b) and integration by parts lead to

$$I_5 = -I_7 + \int_0^t \left\{\frac{\partial}{\partial t}U(t,t-\tau)\right\} f^{(j)}(t-\tau)\,d\tau,$$

and also noting the fact $\binom{j}{k-1} + \binom{j}{k} = \binom{j+1}{k}$, one easily checks that $I_3 + I_4 + I_5 + I_7$ is exactly the second term on the right-hand side of (17), and thus the desired result follows.

We are now in the position to show $v^{(j+1)} \in C^\gamma([\delta,T],X)$. Fix $\delta \in (0,T)$ and $\gamma \in (0,\beta)$. We first consider the case $j = 0$. It is not difficult to show by (18) that

$$\begin{aligned}
v'(t) - v'(s) = {} & \int_s^t \big[P(t,A) - P(\tau,A)\big]U(t,\tau)f(\tau)\,d\tau \\
& + \int_0^s \big[P(t,A) - P(s,A)\big]U(t,\tau)f(\tau)\,d\tau \\
& + \int_0^s \big[P(s,A) - P(\tau,A)\big]\big[U(t,\tau) - U(s,\tau)\big]f(\tau)\,d\tau \\
& + \int_0^s P(\tau,A)\big[U(t,\tau) - U(s,\tau)\big]\big[f(\tau) - f(s)\big]\,d\tau \\
& + \int_s^t P(\tau,A)U(t,\tau)\big[f(\tau) - f(t)\big]\,d\tau \\
& + \big[U(s,0) - U(t,0)\big]f(t) \\
& + \big[U(t,s) - U(t,0) + U(s,0) - I\big]\big[f(s) - f(t)\big] \\
:= {} & \sum_{k=0}^{6} g_k(t,s) \quad \text{for } \delta \leqslant s \leqslant t \leqslant T.
\end{aligned}$$

Hence, from $a_\mu \in C^\beta[0,T]$ ($|\mu| \leqslant m$) and $f \in C^\beta([0,T],X)$ we can deduce that $\|g_k(t,s)\| \leqslant M(t-s)^\beta$ $(k = 0,4,6)$, $\|g_k(t,s)\| \leqslant M(t-s)^\gamma$ $(k = 1,2,3)$, and $\|g_5(t,s)\| \leqslant M(t-s)$ for $\delta \leqslant s < t \leqslant T$. Summarizing these estimates one has $v' \in C^\gamma([\delta,T],X)$. By a careful observation of (17) we find that $v' \in C^\gamma([\delta,T],X)$ implies $g_k \in C^\gamma([\delta,T],X)$ $(k = 7,8,9)$, where

$$g_7(t) = \int_0^t \left[\frac{\partial}{\partial t} U(t, t-\tau)\right] f(t-\tau)\, d\tau,$$

$$g_8(t) = \int_0^t P(\tau, A) U(t, \tau)\big[f(\tau) - f(t)\big]\, d\tau, \quad \text{and}$$

$$g_9(t) = U(t, 0)\big[f(t) - f(0)\big].$$

Return now to (17). We will denote by $f_k(t)$ $(k = 1, 2, 3, 4)$ the four terms on the right-hand side of (17) in proper order, and write further

$$\begin{aligned} f_2(t) &= \int_0^t \left\{\left(\frac{\partial}{\partial t}\right)^{j+1} U(t, t-\tau)\right\} f(t-\tau)\, d\tau \\ &\quad + \int_0^t \sum_{k=2}^{j} \binom{j+1}{k} \left\{\left(\frac{\partial}{\partial t}\right)^{k} U(t, t-\tau)\right\} f^{(j-k+l)}(t-\tau)\, d\tau \\ &\quad + \int_0^t (j+1) \left\{\frac{\partial}{\partial t} U(t, t-\tau)\right\} f^{(j)}(t-\tau)\, d\tau \\ &:= f_5(t) + f_6(t) + f_7(t). \end{aligned}$$

Then (15) and (16) imply $f_1,\ f_6 \in C^1([\delta, T], X)$. Since f_3 (respectively f_4, f_7) is exactly g_8 (respectively g_9, g_7) in which f is replaced by $f^{(j)}$ one obtains $f_k \in C^\gamma([\delta, T], X)$ $(k = 3, 4, 7)$. To show the same conclusion for f_5 we denote by $V(t, \tau, k_{j+1})$ the term (10) with $l = j + 1$. If $k_{j+1} = 0$ then, by (13), $\|V(t, \tau, k_{j+1})\| \leqslant M$ for $0 < \tau \leqslant t \leqslant T$. Also, it is similar to (15) (with $l = j + 1$) to show $\|\frac{\partial}{\partial t} V(t, \tau, k_{j+1})\| \leqslant M\tau^{\beta-1}$ for $0 < \tau \leqslant t \leqslant T$. Therefore we have

$$t \mapsto \int_0^t V(t, \tau, k_{j+1}) f(t-\tau)\, d\tau \in C^1\big([\delta, T], X\big) \quad \text{for } k_{j+1} = 0. \tag{19}$$

If $k_{j+1} > 0$ then $V(t, \tau, k_{j+1})$ is exactly the term (14) and thus, by the result on g_7,

$$t \mapsto \int_0^t V(t, \tau, k_{j+1}) f(t-\tau)\, d\tau \in C^\gamma\big([\delta, T], X\big) \quad \text{for } k_{j+1} > 0. \tag{20}$$

Combining (19) and (20) we obtain $f_5 \in C^\gamma([\delta, T], X)$. Hence the proof of $v^{(j+1)} \in C^\gamma([\delta, T], X)$ is complete. Since similar methods as in the proof of (19) and (20) yield $t \mapsto (\frac{d}{dt})^{j+1} U(t, 0)x \in C^\gamma([\delta, T], X)$ for $x \in X$, the desired result follows from (9). Moreover, the conclusion about the analyticity of the solution u is a direct consequence of Proposition 1(c) and (9). $\square$

Theorem 1 improves [6, Corollary 5.2], in which $a_\mu(t)$ ($|\mu| = m$) is real valued. In particular, some regularity for the solution of (1) was shown in Theorem 1, but there were none in [6]. Only a few results on higher order differentiability of solutions of nonautonomous evolution equations are known (see [10,12]). However, Theorem 1 cannot be deduced from the corresponding theorems in [10,12]. Indeed, $\mathcal{D}(P(t, A))$ independent of t was assumed in [10], while this is not satisfied in Theorem 1. Although it is allowable that $\mathcal{D}(P(t, A))$ depends on t in [12], a stronger regularity condition on the coefficients a_μ ($|\mu| \leqslant m$) must be satisfied (cf. remarks after Corollary 1 below).

Theorem 2. *Let* $P(t,\xi)$ *satisfy* (5), *and let there exist* $\beta \in [0, 1]$ *such that* $a_\mu \in C^\beta[0, T]$ ($|\mu| \leqslant m$) *and* $f \in C^\beta([0, T], Y_\gamma)$, *where*

$$\gamma = m\alpha + m - r\beta - r \quad \textit{for some } \alpha > \frac{n(m-r)}{2m}. \tag{21}$$

Then, for every $x \in Y_{m\alpha}$, (1) *has a unique solution* u *such that*

$$\|u(t)\| \leqslant M\Big(\|x\|_{m\alpha} + \sup_{0\leqslant s\leqslant t} \|f(s)\|_{m\alpha-r}\Big) \quad \textit{for } t \in [0, T]. \tag{22}$$

Proof. Let $(U(t,s))_{(t,s)\in\overline{\Omega}}$ be the C-regularized evolution family provided by Proposition 2, where $C = (1 + |A|^2)^{-m\alpha/2}$. Then for every $x \in Y_{m\alpha}$, $w := U(\cdot\,, 0)C^{-1}x$ is a solution of (1) (with $f \equiv 0$) and satisfies

$$\|w(t)\| \leqslant M\|x\|_{m\alpha} \quad \text{for } t \in [0, T]. \tag{23}$$

Now, choose $\alpha' \in (\frac{n(m-r)}{2m}, \alpha)$ such that $\alpha - \alpha' < \frac{r}{m}$, and define

$$v_{t,s} \equiv \big(1 + |\cdot|^2\big)^{-(m\alpha-r)/2} \exp\left\{\int_s^t P(\tau,\cdot)\,d\tau\right\} \quad \text{for } (t,s) \in \overline{\Omega}.$$

Then, similarly to the proof of [14, Theorem 3.1], one has by (5)

$$\begin{aligned}\big|D^\nu v_{t,s}(\xi)\big| &\leqslant M|\xi|^{(m-r-1)|\nu|-(m\alpha-r)} \exp\big\{-\delta|\xi|^r(t-s)\big\}\\ &\leqslant M(t-s)^{m(\alpha-\alpha')/r-1}|\xi|^{(m-r-1)|\nu|-m\alpha'}\end{aligned}$$

for $(t,s) \in \Omega$, $|\xi| \geqslant L$, and $|\nu| \leqslant [\frac{n}{2}] + 1$ ($\nu \in \mathbf{N}_0^n$), where we note that $m(\alpha - \alpha')/r - 1 < 0$. It follows therefore from [14, Lemma 1.1(c)] that $v_{t,s} \in \mathcal{F}L^1(\mathbf{R}^n)$ and

$$\|v_{t,s}\|_{\mathcal{F}L^1} \leqslant M(t-s)^{m(\alpha-\alpha')/r-1} \quad \text{for } |\mu| \leqslant m \text{ and } (t,s) \in \Omega, \tag{24}$$

which implies that $v(t) := \int_0^t v_{t,s}(A)C_1^{-1}f(s)\,ds$ $(0 < t < T)$ exists and is in $C([0,T],X)$, where $C_1 = (1+|A|^2)^{-(m\alpha - r)/2}$.

On the other hand, define

$$v_{t,s}^{\mu}(\xi) = \xi^{\mu}\bigl(1+|\xi|^2\bigr)^{-\gamma/2}\exp\left\{\int_s^t P(\tau,\xi)\,d\tau\right\}$$

for $\mu \in \mathbf{N}_0^n$, $(t,s) \in \overline{\Omega}$, and $\xi \in \mathbf{R}^n$. Then, the same argument as in the proof of (24) yields that $v_{t,s}^{\mu} \in \mathcal{F}L^1(\mathbf{R}^n)$ and

$$\bigl\|v_{t,s}^{\mu}\bigr\|_{\mathcal{F}L^1} \leqslant M(t-s)^{(-m+\gamma-m\alpha')/r} \quad \text{for } |\mu| \leqslant m \text{ and } (t,s) \in \Omega.$$

Combining our assumptions with this leads to

$$\begin{aligned}
&\bigl\|P(t,A)v_{t,s}(A)C_1^{-1}\bigl(f(t)-f(s)\bigr)\bigr\| \\
&\quad\leqslant M\sum_{|\mu|\leqslant m}\bigl\|v_{t,s}^{\mu}(A)\bigr\|\cdot\bigl\|f(t)-f(s)\bigr\|_{\gamma} \\
&\quad\leqslant M(t-s)^{(-m+\gamma-m\alpha')/r+\beta} \\
&\quad= M(t-s)^{m(\alpha-\alpha')/r-1} \quad \text{for } (t,s) \in \Omega
\end{aligned}$$

and

$$\begin{aligned}
&\bigl\|\bigl(P(t,A)-P(s,A)\bigr)v_{t,s}(A)C_1^{-1}f(t)\bigr\| \\
&\quad\leqslant M(t-s)^{\beta}\sum_{|\mu|\leqslant m}\bigl\|v_{t,s}^{\mu}(A)\bigr\|\cdot\bigl\|f(t)\bigr\|_{\gamma} \\
&\quad\leqslant M(t-s)^{m(\alpha-\alpha')/r-1} \quad \text{for } (t,s) \in \Omega.
\end{aligned}$$

Therefore, by integration by parts, one has (cf. (18))

$$\begin{aligned}
v'(t) &= f(t) + \int_0^t P(t,A)v_{t,s}(A)C_1^{-1}f(s)\,ds \\
&= v_{t,0}(A)C_1^{-1}f(t) + \int_0^t P(t,A)v_{t,s}(A)C_1^{-1}\bigl(f(s)-f(t)\bigr)\,ds \\
&\quad + \int_0^t \bigl(P(t,A)-P(s,A)\bigr)v_{t,s}(A)C_1^{-1}f(t)\,ds,
\end{aligned}$$

i.e., $v \in C^1((0,T],X)$. Also,

$$\int_0^t P(t,A)v_{t,s}(A)C_1^{-1}f(s)\,ds = P(t,A)\int_0^t v_{t,s}(A)C_1^{-1}f(s)\,ds.$$

Thus v is a solution of (1) (with $x = 0$) and satisfies, by (24),

$$\|v(t)\| \leqslant M \sup_{0\leqslant s\leqslant t} \|f(s)\|_{m\alpha-r} \quad \text{for } t \in [0, T]. \tag{25}$$

Therefore $u := w + v$ is a solution of (1), while (22) follows from (23) and (25).

If u_1 is also a solution of (1), then from Proposition 2 one deduces that $\frac{\partial}{\partial s}[U(t,s)(u(s) - u_1(s))] = 0$ for $(t,s) \in \Omega$. Integrating this from $s = 0$ to $s = t$ yields that $C(u(t) - u_1(t)) = 0$, i.e., $u(t) = u_1(t)$ for $t \in [0, T]$. □

First, from the proof of Theorem 2 one sees that it is also true for $r = m$. Next, in Theorem 2 the index β indicates the degree of regularity of a_μ and f on the time-variable. Because $f(t) \in Y_\gamma$ $(0 \leqslant t \leqslant T)$, the index γ indicates the degree of regularity of f on the space-variable, while (21) showed the relationship between these two indices. Finally, in the case $\beta = 1$ the condition (21) can be rewritten as $\gamma > \frac{n}{2}(m - r) + m - 2r$. In particular, when $r > m - \frac{2m}{n+4}$ we can choose $\gamma = 0$.

In the subsequent theorem, we will improve Theorem 4.6 and Corollary 5.4 in [6].

Theorem 3. *Let* $\sup\{\operatorname{Re} P(t,\xi);\ \xi \in \mathbf{R}^n,\ t \in [0,T]\} < \infty$, *and let* $f \in C([0,T], Y_{m(\alpha+1)})$, *where* $\alpha > n/2$. *Then, for every* $x \in Y_{m(\alpha+1)}$, (1) *has a unique solution* $u \in C([0,T], Y_m) \cap C^1([0,T], X)$ *satisfying* (22) (*with* $r = 0$) *and*

$$\|u(t)\|_m \leqslant M\Big(\|x\|_{m(\alpha+1)} + \sup_{0\leqslant s\leqslant t} \|f(s)\|_{m(\alpha+1)}\Big) \quad \textit{for } t \in [0, T]. \tag{26}$$

Proof. Let $(U(t,s))_{(t,s)\in\overline{\Omega}}$ be the C-regularized evolution family provided by Proposition 3, and define

$$u(t) = U(t,s)C^{-1}x + \int_0^t U(t,s)C^{-1}f(s)\,ds \quad \text{for } t \in [0, T]. \tag{27}$$

Then, by our assumptions and Proposition 3, one sees that $u \in C^1([0,T], X)$ and

$$\begin{aligned} u'(t) &= P(t,A)U(t,s)C^{-1}x + f(t) + \int_0^t P(t,A)U(t,s)C^{-1}f(s)\,ds \\ &= P(t,A)U(t,s)C^{-1}x + f(t) + P(t,A)\int_0^t U(t,s)C^{-1}f(s)\,ds. \end{aligned} \tag{28}$$

Thus u is a solution of (1), and (22) (with $r = 0$) follows immediately from (27). The rest of the proof is the same as in that of [6, Corollary 5.4]. □

It is obvious from the proof that the assumption $f \in C([0,T], Y_{m(\alpha+1)})$ in Theorem 3 can be replaced by the weaker one: $f \in C([0,T], Y_{m\alpha}) \cap L^1([0,T], Y_{m(\alpha+1)})$. In this case, (26) is of the form

$$\left\|u(t)\right\|_m \leqslant M\left(\|x\|_{m(\alpha+1)} + \int_0^t \left\|f(s)\right\|_{m(\alpha+1)} ds\right) \quad \text{for } t \in [0,T].$$

Theorem 4. *Let $P(t,\xi)$ be given by* (7), *and let $f \in C([0,T],[D])$, where $[D]$ means $D := \bigcap_{j=1}^n \mathcal{D}(A_j)$, made into a Banach space with the graph norm $\|x\|_D := \|x\| + \sum_{j=1}^n \|A_j x\|$. Then, for every $x \in D$,* (1) *has a unique solution $u \in C([0,T],[D]) \cap C^1([0,T],X)$ satisfying* (8) *and*

$$\left\|u(t)\right\|_D \leqslant M\Big(\|x\|_D + \sup_{0\leqslant s\leqslant t} \left\|f(s)\right\|_D\Big) \quad \textit{for } t \in [0,T]. \tag{29}$$

Proof. Let u be defined by (9), in which $(U(t,s))_{(t,s)\in\overline{\Omega}}$ is the evolution family provided by Proposition 4. Then it follows from our assumptions and Proposition 4 that $u \in C^1([0,T],X)$ and (28) (with $C = I$) is true. Thus u is a solution of (1), while (8), (29), $u \in C([0,T],[D])$, and the uniqueness of u are all consequences of the representation (9). □

We remark that when iA_j $(1 \leqslant j \leqslant n)$ are commuting generators of contraction semigroups Theorem 4 follows from [5, Section 13.2].

Theorem 5. *Let $f \in C([0,T],[\mathcal{R}(C)])$, where C is defined as in the proof of Proposition* 5. *Then, for every $x \in \mathcal{R}(C)$,* (1) *has a unique solution $u \in C^1([0,T],X)$, such that*

$$\left\|u(t)\right\| \leqslant M\Big(\left\|C^{-1}x\right\| + \sup_{0\leqslant s\leqslant t} \left\|C^{-1}f(s)\right\|\Big) \quad \textit{for } t \in [0,T]. \tag{30}$$

Moreover, if in addition a_μ $(|\mu| \leqslant m)$ and f are all entire functions, then so is the solution u.

Proof. Let u be defined by (27), in which $(U(t,s))_{t,s\in[0,T]}$ is the two parameter family provided by Proposition 5. Then $u \in C^1([0,T],X)$ and (28) follows easily from our assumptions and Proposition 5. Thus u is a solution of (1), while (30) and the uniqueness of u follow from the representation (27). The remaining statement can be obtained by (27) and Proposition 5(d). □

4. Applications to PDEs

This section is concerned with the following PDE

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = \displaystyle\sum_{|\mu|\leqslant m} a_\mu(t)D^\mu u(t,x) + f(t,x) \\ \quad \text{for } (t,x) \in (0,T] \times \mathbf{R}^n, \\ u(0,x) = u_0(x) \quad \text{for } x \in \mathbf{R}^n, \end{cases} \tag{31}$$

on some function space X on which translations are uniformly bounded and strongly continuous. Then the results in Section 3 can be applied to (31) (i.e., take $iA_j = iD_j := \partial/\partial x_j$), immediately. X can be chosen as, for example, $L^p(\mathbf{R}^n)$, $L^p([0,1]^n)$ $(1 \leqslant p < \infty)$, or one of the following spaces of continuous functions:

$$\begin{aligned} &\bigl\{f \in C\bigl(\mathbf{R}^n\bigr);\ f \text{ is bounded and uniformly continuous}\bigr\}, \\ &\bigl\{f \in C\bigl(\mathbf{R}^n\bigr);\ \lim_{|x|\to\infty} f(x) = 0\bigr\}, \\ &\bigl\{f \in C\bigl(\mathbf{R}^n\bigr);\ f(x) \text{ exists as } |x| \to \infty\bigr\}, \\ &\bigl\{f \in C\bigl(\mathbf{R}^n\bigr);\ f \text{ is 1-periodic}\bigr\}, \\ &\bigl\{f \in C\bigl(\mathbf{R}^n\bigr);\ f \text{ is almost periodic}\bigr\}, \\ &\bigl\{f \in C\bigl([0,1]^n\bigr);\ f|_{x_j=0} = f|_{x_j=1} = 0\bigr\}, \\ &\bigl\{f \in C\bigl([0,1]^n\bigr);\ f|_{x_j=0} = f|_{x_j=1}\bigr\} \end{aligned}$$

with sup-norms.

Let $W^{\alpha,X}(\mathbf{R}^n)$ $(\alpha \geqslant 0)$ be the completion of $\mathcal{S}(\mathbf{R}^n)$ under the norm

$$\|u\|_{\alpha,X} \equiv \|u\|_X + \bigl\|\mathcal{F}^{-1}\bigl(\bigl(1+|\cdot|^2\bigr)^{\alpha/2}\mathcal{F}u\bigr)\bigr\|_X \quad \text{for } u \in \mathcal{S}\bigl(\mathbf{R}^n\bigr).$$

When $X = L^p(\mathbf{R}^n)$ $(1 \leqslant p < \infty)$ and $\alpha \geqslant 0$, $W^{\alpha,p}(\mathbf{R}^n) \equiv W^{\alpha,X}(\mathbf{R}^n)$ is the so-called Bessel potential space. From [7, Lemma 2.1] we have that $\{\lambda \in \mathbf{C};\ \operatorname{Re}\lambda > 0\} \subseteq \rho(\Delta)$ and

$$(1-\Delta)^{-\alpha/2}W^{\beta,X}(\mathbf{R}^n) = W^{\alpha+\beta,X}(\mathbf{R}^n) \quad \text{for } \alpha, \beta \geqslant 0.$$

In particular, when $-|A|^2 = \Delta$, $Y_\alpha = W^{\alpha,X}(\mathbf{R}^n)$ for $\alpha \geqslant 0$. Moreover, we define

$$n_X \begin{cases} = n\bigl|\frac{1}{2} - \frac{1}{p}\bigr| & \text{if } X = L^p \ (1 < p < \infty), \\ > n/2 & \text{if } X = L^1 \text{ or the above space of continuous functions,} \end{cases}$$

and $n_p = n_X$ for $X = L^p(\mathbf{R}^n)$. Thus the following result holds.

Corollary 1. *Let $P(t,\xi) = \sum_{|\mu|\leqslant m} a_\mu(t)\xi^\mu$ with $a_\mu \in C[0,T]$ $(|\mu| \leqslant m)$.*

(a) *If $P(t,\xi)$ is strongly elliptic for every $t \in [0,T]$, and if there exist $j \in \mathbf{N}_0$ and $\beta \in (0,1)$ such that $a_\mu \in C^{j+\beta}[0,T]$ $(|\mu| \leqslant m)$ and $f \in C^{j+\beta}([0,T],X)$,* *then for every $x \in X$, (31) has a unique solution $u \in C^{j+1+\gamma}([\delta, T], X)$ for $\delta \in (0, T)$ and $\gamma \in (0, \beta)$, such that*

$$\|u(t,\cdot)\|_X \leqslant M\Big(\|u_0\|_X + \sup_{0\leqslant s\leqslant t}\|f(s,\cdot)\|_X\Big) \quad \textit{for } t \in [0, T]. \tag{32}$$

Moreover, if in addition $a_\mu \in H(\Sigma)$ ($|\mu| \leqslant m$) and $f \in H(\Sigma, X)$, then $u \in H(\Sigma', X)$.

(b) *If $P(t, \xi)$ satisfies (5) for some $r \in (0, m]$, and if there exists $\beta \in [0, 1]$ such that $a_\mu \in C^\beta[0, T]$ ($|\mu| \leqslant m$) and $f \in C^\beta([0, T], W^{\gamma,X}(\mathbf{R}^n))$, where $\gamma > (n_X + 1)(m - r) - r\beta$, then for every $u_0 \in W^{n_X(m-r),X}(\mathbf{R}^n)$, (31) has a unique solution u such that*

$$\|u(t,\cdot)\|_X \leqslant M\Big(\|u_0\|_{n_X(m-r),X} + \sup_{0\leqslant s\leqslant t}\|f(s,\cdot)\|_{\gamma-m+r\beta,X}\Big) \quad \textit{for } t \in [0, T]. \tag{33}$$

(c) *If $\sup\{\mathrm{Re}\, P(t, \xi);\ \xi \in \mathbf{R}^n,\ t \in [0, T]\} < \infty$, and if $f \in C([0, T], W^{\gamma,X}(\mathbf{R}^n))$, where $\gamma > m(n_X + 1)$, then for every $u_0 \in W^{m(n_X+1),X}(\mathbf{R}^n)$, (31) has a unique solution*

$$u \in C\big([0, T], W^{m,X}(\mathbf{R}^n)\big) \cap C^1\big([0, T], X\big), \tag{34}$$

such that

$$\|u(t,\cdot)\|_{m,X} \leqslant M\Big(\|u_0\|_{m(n_X+1),X} + \sup_{0\leqslant s\leqslant t}\|f(s,\cdot)\|_{\gamma,X}\Big) \quad \textit{for } t \in [0, T]. \tag{35}$$

(d) *If $f \in C([0, T], C_c^\infty(\mathbf{R}^n))$ then there exists a dense subspace $\mathcal{D}$, which contains $C_c^\infty(\mathbf{R}^n)$, such that for every $u_0 \in \mathcal{D}$, (31) has a unique solution $u \in C^1([0, T], X)$. In particular $u_0 \in C_c^\infty(\mathbf{R}^n)$ implies $u \in C^1([0, T], C^\infty(\mathbf{R}^n))$. Moreover, if in addition a_μ ($|\mu| \leqslant m$) and f are all entire, then so is the solution u.*

Corollary 1(a) and (d) follow from Theorems 1 and 5, respectively. When X is a space of continuous functions or $L^1(\mathbf{R}^n)$, Corollary 1(b) and (c) follow Theorems 2 (also see its remark) and 3, respectively. When $X = L^p(\mathbf{R}^n)$ ($1 < p < \infty$), Corollary 1(b) and (c) can be deduced by modifying the proofs of Theorem 2 and 3, respectively. The main points are using the Riesz–Thorin convexity theorem and a multiplier theorem [8, Theorem G], as well as noting $u(D)\phi = \mathcal{F}^{-1}(u\mathcal{F}\phi)$ for $u \in \mathcal{F}L^1(\mathbf{R}^n)$ and $\phi \in \mathcal{S}(\mathbf{R}^n)$.

Corollary 1(a) improves [6, Theorem 5.5]. If $\mathcal{D}(P(t, D))$ is independent of t then, as seen in [10, Section 3], Corollary 1(a) can be deduced from [10, Theorem 1]. This is not possible in the general case. To illustrate the assumptions in Corollary 1(a) to be weaker, we will use Theorem 2 in [12] to gain the solution $u \in C^{j+1+\gamma}([\delta, T], X)$. To this end, we choose $\omega > \sup\{\mathrm{Re}\, P(t, \xi);\ \xi \in \mathbf{R}^n,$

$t \in [0, T]\}$. From [12, Theorem 2] it is necessary to guarantee the following condition

$$\left(\omega - P(\cdot, D)\right)^{-1} \in C^{j+1+\beta}\left([0, T], B(X)\right) \quad \text{for some } \beta \in (\gamma, 1). \tag{36}$$

A careful computation shows that the assumption on a_μ has to take the form $a_\mu \in C^{j+1+\beta}[0, T]$ ($|\mu| \leqslant m$). It is not sufficient for (36) to suppose only $a_\mu \in C^{j+\beta}[0, T]$ ($|\mu| \leqslant m$). The other conditions of [12, Theorem 2] are implied by that of Corollary 1(a). Thus to obtain the claim by [12, Theorem 2], a stronger assumption, i.e., $a_\mu \in C^{j+1+\beta}[0, T]$ ($|\mu| \leqslant m$) is necessary.

Corollary 1(c) improves [13, Theorem 3.4] and, in the case $X = L^p(\mathbf{R}^n)$ ($1 < p < \infty$), [7, Corollary 3.2]. Moreover, by a careful observation of the proof of Theorem 3 we find that, corresponding to Corollary 1(c), the following result on the so-called strong solution of (31) is true.

Corollary 2. *Let $P(t, \xi) = \sum_{|\mu| \leqslant m} a_\mu(t)\xi^\mu$ with $a_\mu \in L^\infty(0, T)$ ($|\mu| \leqslant m$), and suppose there exists $\omega \in \mathbf{R}$ such that $\sup_{\xi \in \mathbf{R}^n} \operatorname{Re} P(t, \xi) \leqslant \omega$ a.e. on $[0, T]$. If $f \in L^1([0, T], W^{\alpha, X}(\mathbf{R}^n))$ where $\alpha > m(n_X + 1)$, then for every $u_0 \in W^{m(n'_X+1), X}(\mathbf{R}^n)$, (31) has a unique strong solution u (i.e., u is differentiable a.e. on $[0, T]$, $u' \in L^1([0, T], X)$, and u satisfies (31) a.e. on $[0, T]$).*

We now turn to consider (31) with constant coefficients, i.e., $a_\mu(t) \equiv a_\mu$ ($|\mu| \leqslant m$). First, we note that an improvement of Corollary 1(a) can be obtained. More precisely, we can choose $\gamma = \beta$ in Corollary 1(a). In fact, this follows immediately from [14, Theorem 2.2] and the following general result (cf. [9]).

Lemma 1. *Let B be the generator of an analytic semigroup on a Banach space X, and let $f \in C^{j+\beta}([0, T], X)$ for some $j \in \mathbf{N}_0$ and $\beta \in (0, 1)$. Then for every $x \in X$, the inhomogeneous Cauchy problem*

$$u'(t) = Bu(t) + f(t), \quad 0 < t \leqslant T,\ u'(0) = x,$$

has a unique solution $u \in C^{j+1+\beta}([\delta, T], X)$ for $\delta > 0$.

Next, we can give the higher order differentiability of the solution in Corollary 1(b). Indeed, this can be deduced from the following result.

Lemma 2. *Let $P(\xi) = \sum_{|\mu| \leqslant m} a_\mu \xi^\mu$ ($\xi \in \mathbf{R}^n$), and let $\operatorname{Re} P(\xi)$ is bounded above and r-coercive for some $r \in (0, m]$. If there exist $j \in \mathbf{N}_0$ and $\beta \in [0, 1)$ such that $f \in C^{j+\beta}([0, T], Y_\gamma)$, where $\gamma > (\frac{n}{2} + 1)(m - r) - r\beta$, then for every $x \in Y_{\alpha(m-r)}$, $\alpha > \frac{n}{2}$, (1) (with $a_\mu(t) \equiv a_\mu$) has a unique solution $u \in C^{j+1}((0, T], X)$ satisfying*

$$\left\|u(t)\right\| \leqslant M\left(\|x\|_{\alpha(m-r)} + \sup_{0 \leqslant s \leqslant t} \left\|f(s)\right\|_{\gamma - m + r\beta}\right) \quad \text{for } t \in [0, T]. \tag{37}$$

In particular, $f \in C^\infty([0,T], Y_\delta)$ for some $\delta > (\frac{n}{2}+1)(m-r)-r$ implies $u \in C^\infty((0,T], X)$.

Proof. By [14, Theorem 3.1], $\overline{P(A)}$ generates a C-regularized semigroup $(T(t))_{t\geqslant 0}$ with $T(\cdot) \in C^\infty((0,\infty), B(A^\infty))$, where $C = (1+|A|^2)^{-\alpha(m-r)/2}$. Then (cf. [14]) $w := T(\cdot)C^{-1}x$ is a solution of the Cauchy problem $w'(t) = P(A)w(t)$ $(t > 0)$, $w(0) = x$. Moreover, we have $w \in C^\infty((0,T], X)$.

We now define $v(t) = \int_0^t v_{t-s}(A)C_1^{-1} f(s)\, ds$ for $t \in [0,T]$, where $C_1 = (1+|A|^2)^{-(\gamma-m+r\beta)/2}$ and $v_t = (1+|\cdot|^2)^{-(\gamma-m+r\beta)/2} e^{tP}$. Then, from the proof of Theorem 2 one has that $u := w + v$ is a solution of (1) (with $a_\mu(t) \equiv a_\mu$) and satisfies (37).

Since $w \in C^\infty((0,T], X)$, it remains to show $v \in C^{j+1}((0,T], X)$. Indeed, as seen in (17), an induction on j leads to

$$v^{(j+1)}(t) = C_1^{-1} f^{(j)}(t) + \sum_{k=1}^{j} \left(\frac{d}{dt}\right)^k v_t(A) C_1^{-1} f^{(j-k)}(0) + \int_0^t P(A) v_{t-s}(A) C_1^{-1} f^{(j)}(s)\, ds \quad \text{for } t \in (0,T].$$

Because $f^{(j)}$ satisfies the same condition as f in Theorem 2, it follows from the proof of Theorem 2 that $v^{(j+1)}(t)$ $(t \in (0,T])$ exists and is in $C((0,T], X)$. $\square$

We now summarize the above results (with $B = P(D)$ and $A = D$), as well as Corollary 1(c)–(d) (with $a_\mu(t) \equiv a_\mu$) in the following corollary.

Corollary 3. *Let $P(\xi) = \sum_{|\mu|\leqslant m} a_\mu \xi^\mu$ $(\xi \in \mathbf{R}^n)$.*

(a) *If $P(\xi)$ is strongly elliptic, and if $f \in C^{j+\beta}([0,T], X)$ for some $j \in \mathbf{N}_0$ and $\beta \in (0,1)$, then for every $x \in X$,* (31) *(with $a_\mu(t) \equiv a_\mu$) has a unique solution $u \in C^{j+1+\beta}([\delta,T], X)$ for $\delta \in (0,T)$ such that* (32) *holds. Moreover, $f \in H(\Sigma)$ implies $u \in H(\Sigma', X)$.*

(b) *If* $\operatorname{Re} P(\xi)$ *is bounded above and r-coercive for some $r \in (0,m]$, and if there exist $j \in \mathbf{N}_0$ and $\beta \in [0,1)$ such that $f \in C^{j+\beta}([0,T], W^{\gamma,X}(\mathbf{R}^n))$, where $\gamma > (n_X+1)(m-r) - r\beta$, then for every $u_0 \in W^{n_X(m-r),X}(\mathbf{R}^n)$,* (31) *(with $a_\mu(t) \equiv a_\mu$) has a unique solution $u \in C^{j+1}((0,T], X)$ satisfying* (33). *In particular, $f \in C^\infty([0,T], W^{\delta,X}(\mathbf{R}^n))$ for some $\delta > (n_X+1)(m-r)-r$ implies $u \in C^\infty((0,T], X)$.*

(c) *If* $\operatorname{Re} P(\xi)$ *is bounded above, and if $f \in C([0,T], W^{\gamma,X}(\mathbf{R}^n))$ where $\gamma > m(n_X+1)$, then for every $u_0 \in W^{m(n_X+1),X}(\mathbf{R}^n)$,* (31) *(with $a_\mu(t) \equiv a_\mu$) has a unique solution u satisfying* (34) *and* (35).

(d) *If* $f \in C([0,T], C_c^\infty(\mathbf{R}^n))$ *then there exists a dense subspace* $\mathcal{D}$*, which contains* $C_c^\infty(\mathbf{R}^n)$*, such that for every* $u_0 \in \mathcal{D}$*,* (31) (*with* $a_\mu(t) \equiv a_\mu$) *has a unique solution* $u \in C^1([0,T], X)$*. In particular,* $u_0 \in C_c^\infty(\mathbf{R}^n)$ *implies* $u \in C^1([0,T], C^\infty(\mathbf{R}^n))$*. Moreover, if in addition* f *is entire then so is the solution* u*.*

We conclude this paper with several examples.

Example 1. We first consider the following equation with space-dependent coefficients and Dirichlet boundary condition

$$\begin{cases} \dfrac{\partial}{\partial t} u(t,x) = P\big(t, q(x)D\big) u(t,x) + f(t,x) \\ \quad \text{for } (t,x) \in (0,T] \times \mathrm{I}^n, \\ u(t,x) = 0 \quad \text{for } t \in (0,T] \text{ and } x \in \partial \mathrm{I}^n, \\ u(0,x) = u_0(x) \quad \text{for } x \in \mathrm{I}^n, \end{cases} \tag{38}$$

on $C_0(\mathrm{I}^n) := \{f \in C(\mathrm{I}^n);\ f|_{\partial \mathrm{I}^n} = 0\}$, where $\mathrm{I} = [0,1]$, $\partial \mathrm{I}^n$ denotes the boundary of I^n, $q(x)D = (q(x_1)D_1, \ldots, q(x_n)D_n)$ and $q(x_j) = x_j^\alpha (1 - x_j)^\alpha$ for some $\alpha \geqslant 1$.

By [1, Proposition 3] we know that $q(x_j) i D_j$ $(1 \leqslant j \leqslant n)$ are the generators of commuting bounded C_0-groups on $C_0(\mathrm{I}^n)$, if $P(t,\xi)$ is strongly elliptic for every $t \in [0,T]$, and if there exists $\beta \in (0,1)$ such that $a_\mu \in C^\beta[0,T]$ $(|\mu| \leqslant m)$ and $f \in C^\beta([0,T], C_0(\mathrm{I}^n))$, then for every $u_0 \in C_0(\mathrm{I}^n)$, Corollary 1(a) implies that (38) has a unique solution $u \in C([0,T], C_0(\mathrm{I}^n)) \cap C^{1+\gamma}([\delta,T], C_0(\mathrm{I}^n))$, where $\delta \in (0,T)$ and $\gamma \in (0,\beta)$. Moreover,

$$\big\| u(t,\cdot) \big\|_0 \leqslant M \Big(\|u_0\|_0 + \sup_{0 \leqslant s \leqslant t} \big\| f(s,\cdot) \big\|_0 \Big) \quad \text{for } t \in [0,T],$$

where $\|\cdot\|_0$ denotes the sup-norm of $C_0(\mathrm{I}^n)$.

Example 2. Next, we consider the n-dimensional linearized KdV-Burgers equation

$$\begin{cases} \dfrac{\partial}{\partial t} u(t,x) = \displaystyle\sum_{1 \leqslant |\mu| \leqslant 3} a_\mu (iD)^\mu u(t,x) + f(t,x) \\ \quad \text{for } (t,x) \in (0,T] \times \mathbf{R}^n, \\ u(0,x) = u_0(x) \quad \text{for } x \in \mathbf{R}^n, \end{cases} \tag{39}$$

on $L^p(\mathbf{R}^n)$ $(1 \leqslant p < \infty)$, where $a_\mu \in \mathbf{R}$ $(|\mu| = 1,2,3)$. We note that, except in the case $p = 2$ (cf. [9, Section 8.5]), (39) cannot be treated by C_0-semigroups.

If $\sum_{|\mu|=2} a_\mu \xi^\mu > 0$ for $\xi \neq 0$, and if $f \in C^{j+\beta}([0,T], W^{\gamma,p}(\mathbf{R}^n))$ for some $j \in \mathbf{N}_0$, $\beta \in [0,1)$ and $\gamma > n_p - 2\beta + 1$ then by Corollary 3(b), for every $u_0 \in W^{n_p,p}(\mathbf{R}^n)$, (39) has a unique solution $u \in C([0,T], L^p(\mathbf{R}^n)) \cap C^{j+1}((0,T], L^p(\mathbf{R}^n))$ such that

$$\|u(t,\cdot)\|_{L^p} \leqslant M\Bigl(\|u_0\|_{n_p,L^p} + \sup_{0\leqslant s\leqslant t}\|f(s,\cdot)\|_{\gamma+2\beta-3,L^p}\Bigr) \quad \text{for } t \in [0,T].$$

Moreover, if in addition $f \in C^\infty([0,T], W^{\delta,p}(\mathbf{R}^n))$ for some $\delta > n_p - 1$ then $u \in C^\infty((0,T], X)$.

If $a_\mu = 0$ ($|\mu| = 2$) then (39) is the n-dimensional linearized KdV equation. In this case we assume that $f \in C([0,T], W^{\gamma,p}(\mathbf{R}^n))$ where $\gamma > 3(n_p+1)$. Then by Corollary 3(c), for every $u_0 \in W^{3(n_p+1),p}(\mathbf{R}^n)$, (39) (with $a_\mu = 0$ for $|\mu| = 2$) has a unique solution $u \in C([0,T], W^{3,p}(\mathbf{R}^n)) \cap C^1([0,T], L^p(\mathbf{R}^n))$ such that

$$\|u(t,\cdot)\|_{3,L^p} \leqslant M\Bigl(\|u_0\|_{3(n_p+1),L^p} + \sup_{0\leqslant s\leqslant t}\|f(s,\cdot)\|_{\gamma,L^p}\Bigr) \quad \text{for } t \in [0,T].$$

Example 3. Finally, consider the first order equation

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = \displaystyle\sum_{j=1}^n a_j(t)\frac{\partial}{\partial x_j}u(t,x) + a_0(t)u(t,x) + f(t,x) \\ \quad \text{for } (t,x) \in (0,T] \times \mathbf{R}^n, \\ u(0,x) = u_0(x) \quad \text{for } x \in \mathbf{R}^n, \end{cases} \tag{40}$$

on X, where $a_j \in C[0,T]$ $(0 \leqslant j \leqslant n)$.

If a_j $(1 \leqslant j \leqslant n)$ are real valued, and $f \in C([0,T], W^{\alpha,X}(\mathbf{R}^n))$, where $\alpha = 1$ for $X = L^p(\mathbf{R}^n)$ $(1 < p < \infty)$ and $\alpha > 1$ otherwise, then by Theorem 4 and Miklin's multiplier theorem [11], for every $u_0 \in W^{\alpha,X}(\mathbf{R}^n)$, (40) has a unique solution $u \in C([0,T], W^{\alpha,X}(\mathbf{R}^n)) \cap C^1([0,T], X)$ such that

$$\|u(t,\cdot)\|_{\alpha,X} \leqslant M\Bigl(\|u_0\|_{\alpha,X} + \sup_{0\leqslant s\leqslant t}\|f(s)\|_{\alpha,X}\Bigr) \quad \text{for } t \in [0,T].$$

In the case $X = L^1(\mathbf{R}^n)$, if $W^{1,1}(\mathbf{R}^n)$ is understood as the usual Sobolev space then the conclusion (with $\alpha = 1$) still holds.

If a_j $(1 \leqslant j \leqslant n)$ are purely imaginary valued and $f \in C([0,T], C_c^\infty(\mathbf{R}^n))$ then, by Corollary 1(d), there exists a dense subspace $\mathcal{D}$, which contains $C_c^\infty(\mathbf{R}^n)$, such that for every $u_0 \in \mathcal{D}$, (40) has a unique solution $u \in C^1([0,T], X)$. In particular, $u_0 \in C_c^\infty(\mathbf{R}^n)$ implies $u \in C^1([0,T], C^\infty(\mathbf{R}^n))$. Moreover, if in addition a_j $(0 \leqslant j \leqslant n)$ and f are all entire then so is the solution u.

In the case when a_j $(1 \leqslant j \leqslant n)$ are real valued, the statement "(40), even with constant coefficients, cannot be treated by integrated semigroups" in [7, p. 817] is not right. Indeed, (40) can be treated by evolution families, even by C_0-groups directly (see the proof of Theorem 4). Meanwhile, our result improves [7, Corollary 3.3], in which $\alpha > 1 + n_X$ is required.

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