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QUASI-RESIDUAL QUASI-SYMMETRIC DESIGNS

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It is shown that for each $\lambda \geq 3$, there are only finitely many quasi-residual quasi-symmetric (QRQS) designs and that for each pair of intersection numbers (x, y) not equal to $(0, 1)$ or $(1, 2)$, there are only finitely many QRQS designs.

A design is shown to be affine if and only if it is QRQS with $x = 0$. A projective design is defined as a symmetric design which has an affine residual. For a projective design, the block-derived design and the dual of the point-derivate of the residual are multiples of symmetric designs.

1. Quasi-residual quasi-symmetric designs

Let $\mathcal{P}$ be a set of elements (points) and $\mathcal{B}$ a family of subsets (blocks) of $\mathcal{P}$. (The same subset may be repeated, i.e., occur more than once as a block.) $(\mathcal{P}, \mathcal{B})$ is a *block design* with parameters (v, b, r, k, λ) if

(i) $|\mathcal{P}| = v$, $|\mathcal{B}| = b$, $v > k \geq 3$;
(ii) each point lies in r blocks;
(iii) each block contains k points;
(iv) each pair of points occurs in exactly λ blocks.

A design is *symmetric* if $v = b$ or equivalently, $r = k$. It is well-known that symmetric designs are just those designs in which any two blocks intersect in just λ points. A design is *quasi-symmetric* (QS) if the number of points in the intersection of two blocks takes just two values, x and y $(x < y)$. The following known results concerning QS designs can be obtained by counting frequencies of the intersection numbers x and y (see [7] for an alternative proof using linear algebraic methods).

Proposition 1.

$$k(r-1)(x+y) = (b-1)xy + k(k-1)(\lambda-1) + k(r-1)$$

Proposition 2. $y - x \mid k - x$ *and* $y - x \mid r - \lambda$.

A residual design is a design obtained from a symmetric (v, k, λ) design by deleting a block and all of its points. The parameters of the resulting residual design will then be $(v-k, v-1, k, k-\lambda, \lambda)$. A design is called *quasi-residual* (QR) if its parameters are of this form. It is easy to see that a design is quasi-residual if and only if $r = k + \lambda$. We note that a QR design is not necessarily embeddable in a symmetric design, however, Hall and Connor ([8] or [9]) have shown that for $\lambda = 1, 2$, the notions of residual and quasi-residual are equivalent. More generally, Bose, Shrikhande and Singhi [4] have shown that if k is larger than a certain function $g(\lambda)$ of λ, then a QR design is embeddable in a unique symmetric design. We will subsequently utilize the following consequence of Hall and Conner's result:

Theorem 3. *Let D be an arbitrary design with $\lambda = 2$. Then D is residual if and only if D is QS with intersection numbers $(x, y) = (1, 2)$.*

Hereafter, unless otherwise indicated we consider only quasi-residual, quasi-symmetric (QRQS) designs. Specializing Propositions 1 and 2 to these designs we have:

Proposition 4

$$k^2\lambda^2 - k\lambda(k+\lambda-1)(x+y) + (k^2+2k\lambda+\lambda^2-k-2\lambda)xy = 0$$

or equivalently,

$$(k-x)(k-y)\lambda^2 - (k-1)(kx+ky-2xy)\lambda + k(k-1)xy = 0.$$

Proposition 5. $y - x \mid x$ *and* $y - x \mid k$.

We consider two questions with regard to QRQS designs:

(1) Given λ, what are the possible values of (x, y) and the other design parameters?

(2) Given (x, y), what are the possible values of λ and the other design parameters?

The following results are helpful in partially answering these questions.

Lemma 6. *If D is an arbitrary design with $\lambda = 1$, then D is either symmetric or QS with $(x, y) = (0, 1)$.*

Proof. Since two points lie in exactly one block, two blocks intersect in at most one point. If each pair of blocks intersect in the same number of points then the design is symmetric. Otherwise the design must be QS with $x = 0$, $y = 1$. □

The following partial converse results upon substituting $x = 0$, $y = 1$ in Proposition 1.

Lemma 7. *If D is* QS *with* $(x, y) = (0, 1)$, *then* $\lambda = 1$.

Lemma 8. *If* $x = 1$, *then* $y = 2$ *and the design is either the complete design* (5, 10, 6, 3, 3) *or a design with parameters of the form* $(\binom{k+1}{2}, \binom{k+2}{2}, k+2, k, 2)$.

Proof. By Proposition 5, $y - 1 \mid 1$, so $y = 2$. By Proposition 4,

$$(k-1)(k-2)\lambda^2 - (k-1)(3k-4)\lambda + 2k(k-1) = 0.$$

Hence $[\lambda - 2][(k-2)\lambda - k] = 0$, so $\lambda = 2$ or $k/(k-2)$. If $\lambda = k/(k-2) = 1 + 2/(k-2)$, then $k = 3$, $\lambda = 3$ or $k = 4$, $\lambda = 2$. In the case $\lambda = 3$, the parameters are (5, 10, 6, 3, 3). This is a complete design and can easily be shown to have the required intersection numbers. If $\lambda = 2$, then $r = k + 2$ and the parameters are $(\binom{k+1}{2}, \binom{k+2}{2}, k+2, k, 2)$. □

We can now answer questions (1) and (2) above for the smallest parameter values.

(i) For $\lambda = 1$, every QRQS design has $(x, y) = (0, 1)$. The parameters must have the form

$$(k^2, k^2 + k, k + 1, k, 1)$$

which is an affine plane $A(2, k)$.

(ii) For $\lambda = 2$, we have, by Theorem 3, $(x, y) = (1, 2)$ and the parameters must be of the form

$$(\binom{k+1}{2}, \binom{k+2}{2}, k+2, k, 2).$$

(iii) For $(x, y) = (0, 1)$, we have $\lambda = 1$ and parameters as indicated in (i) above.

(iv) For $(x, y) = (1, 2)$, Lemma 8 stipulates that either $\lambda = 2$ with parameters as indicated in (ii) above or $\lambda = 3$ with parameters (5, 10, 6, 3, 3).

In each of the above cases, there are infinitely many possible parameter sets although not all of these may have solutions. We now show that each of the remaining cases yields only finitely many possible parameter sets.

Lemma 9. *In a* (v, b, r, k, λ) QR *design*,

(a) $v = k + k(k-1)/\lambda = (k^2 + \lambda k - k)/\lambda$,

(b) $b = v + k + \lambda - 1 = (k + \lambda)(k + \lambda - 1)/\lambda$.

Proof. (a) follows immediately from the basic relation $(v-1)\lambda = r(k-1)$ and the fact that $r = k + \lambda$ in a QR design.

(b) follows from (a) and the basic relation $bk = vr$. □

For QRQS designs in which $x = 0$ we obtain the following:

Lemma 10. *Let* $x = 0$. *Then for some integer* $n \geq 2$

(a) $ny = k$,
(b) $y = k\lambda/(k+\lambda-1)$,
(c) $y = \lambda-(\lambda-1)/n$, i.e., $\lambda = y+(y-1)/(n-1)$,
(d) $v = nk$,
(e) $y = k^2/v$.

Proof. (a) follows from Proposition 5, and (b) is a consequence of Proposition 4. For (c), note that $k = nk\lambda/(k+\lambda-1)$ by (a) and (b). Hence $k+\lambda-1 = n\lambda$ or (using (a) again) $ny = (n-1)\lambda+1$ from which (c) follows. Using the fact that $k-1 = (n-1)\lambda$ together with Lemma 9 (a), we get $v = k+k(k-1)/\lambda = k+(n-1)k$ and so (d) holds. (e) follows from (a) and (d). □

As direct consequences of Lemma 10, we get,

Corollary 11. *If* $x = 0$, *then*
(a) $n \mid k$, $n \mid \lambda-1$ *and* $n-1 \mid y-1$,
(b) $\lambda \mid k-1$, $k \mid v$ *and* $\lambda-y \mid \lambda-1$,
(c) $v \mid k^2$.

We let a denote the number of blocks meeting a given block B_0 in y points. It is not difficult to show that a, the *frequency* of the intersection number y, is independent of B_0. The following proposition occurs upon specializing a result of [10] to QRQS designs:

Proposition 12

$$(b-a-1)(x-(\lambda-1))(x-\lambda)+a(y-(\lambda-1))(y-\lambda) = \lambda(\lambda-1)(\lambda-2).$$

We derive the following consequence:

Corollary 13. $(x, y) = (\lambda-1, \lambda)$ *if and only if* $\lambda = 1$ *or* 2.

Proof. If $\lambda = 1$ or 2, the right hand side of the equation of Proposition 12 becomes 0. Since there are not integers between $\lambda-1$ and λ, each term on the left hand side is non-negative and so must be zero. The converse is straightforward.

Proposition 14. *If* $y = \lambda \geqslant 3$, *then* $k = (\lambda-2)x/(\lambda-1-x)$.

Proof. From Proposition 4,

$$k[(2\lambda-1)\lambda x-\lambda(\lambda-1)(x+\lambda)]+\lambda(\lambda-2)\lambda x = 0,$$

i.e.,

$$k\lambda(\lambda x-\lambda(\lambda-1))+\lambda(\lambda-2)\lambda x = 0,$$

i.e.,

$$k(\lambda-1-x)=(\lambda-2)x.$$

By Corollary 13, $x\neq\lambda-1$, hence the result. □

Proposition 15. *Suppose* $y\neq\lambda$. *Then* $k=(B\pm D)/2A$, *where*

$$A=(\lambda-x)(\lambda-y),\qquad B=(\lambda-1)(x+y)-(2\lambda-1)xy,$$

$$C=\lambda(\lambda-2)xy,\qquad D^2=B^2-4AC.$$

Proof. If $y\neq\lambda$, then $A\neq0$ and the second equation of Proposition 4, which is $Ak^2-Bk+C=0$ restricts k to the stated values. □

Proposition 16. *Let D be an arbitrary design with a repeated block. Then D is QS if and only if D is a multiple of a symmetric design.*

Proof. Let D be a (v, b, r, k, λ) QS design with a repeated block. Each block has exactly a blocks meeting it in y points. But since D has a repeated block, $y=k$. Therefore every block is repeated a times. Thus D is a multiple of a (v, b', r', k, λ') design with no repeated blocks, where $b/b'=r/r'=\lambda/\lambda'=a+1$. But this means that every pair of blocks in the latter design intersect in x points. Hence this design is symmetric.

The converse is obvious. □

Proposition 17. *Let D be a QS design with parameters* (v, b, r, k, λ). *If* $y=k$, *then* $x=k\lambda/r$.

Proof. We use the notation of the previous proof. It was shown that every pair of blocks in the symmetric (v, b', r', k, λ') design intersect in x points. Hence $x=\lambda'$. But $\lambda'=r'\lambda/r$ and $r'=k$ (by symmetry). Therefore $x=k\lambda/r$. □

Proposition 18. *If* $(x, y)\neq(0, 1)$, *then* $x<\lambda<y^2$.

Proof. If $x=0$, then by Lemma 10 (c),

$$\lambda\leqslant 2y-1=y^2-(y-1)^2<y^2.$$

Suppose now $x\neq0$. Let

$$\theta=\frac{(k-1)(\lambda-1)+(r-1)}{r-1}\quad\text{and}\quad \mu=\frac{k(r-1)}{b-1}$$

(i.e., for a given block B_0, μ is the "mean" of the values $|B_0\cap B|$ as B ranges over the remaining blocks). Then Proposition 1 becomes $x+y=xy/\mu+\theta$. Since $x<\mu$,

$x+y<y+\theta$ and hence $x<\theta$. Since $k<r$,

$$\theta=\frac{(k-1)(\lambda-1)}{r-1}+1<\lambda-1+1=\lambda$$

and so $x<\lambda$. Note, again using the above equation, that since $y>\mu$, $x+y>x+\theta$ and thus $y>\theta$. It follows that $(k-y)(\lambda-y)<y(y-1)$. If $y\neq k$, then $\lambda-y<y^2-y$ or $\lambda<y^2$. If $y=k$, then by Proposition 17,

$$x=\frac{k\lambda}{k+\lambda}=\frac{y\lambda}{y+\lambda}.$$

Therefore $\lambda=xy/(y-x)\leqslant xy<y^2$, so that in either case, $\lambda<y^2$. □

Although we shall utilize Proposition 18 as stated, we should also note the following result of Bose, Shrikhande and Singhi [4]: If $k>2\lambda^3-4\lambda^2+4\lambda-2$ in a QRQS design, then $y\leqslant\lambda$.

We can now complete our answers to questions (1) and (2) above.

Theorem 19. *For a fixed value of $\lambda\geqslant 3$, there are only a finite number of* QRQS *designs.*

Proof. By Proposition 18, $x<\lambda$. If $x=0$, then by Lemma 10(c) and Corollary 11, y can take only finitely many values each fixing the value of n, and hence by Lemma 10(a) for each such value there is at most one value of k and therefore only one possible set of design parameters. If $x=1$, then by Lemma 8, there is only one design if $\lambda=3$ and none if $\lambda>3$.

The remaining case is $2\leqslant x\leqslant\lambda-1$. Since $y-x\mid x$, there are finitely many y for each x. If $y=\lambda$, then by Proposition 14, there is at most one value of k and therefore only one possible parameter set. If $y\neq\lambda$, then by Proposition 15, there is at most one value of k and therefore only one possible parameter set. If $y\neq\lambda$, then by Proposition 15, there are at most two values of k, yielding two parameter sets. □

Theorem 20. *For fixed (x, y) where $x=0$ and $y\geqslant 2$, there are only a finite number of* QRQS *designs.*

Proof. By Lemma 10(c) and Corollary 11, λ can take only finitely many values, each fixing the value of n, and hence by Lemma 10(a) for each such value there is at most one value of k, giving only one parameter set. □

Theorem 21. *For fixed $x\geqslant 2$ there are finitely many possible values of y and for each such pair (x, y) there are only a finite number of* QRQS *designs.*

Proof. Since $y-x\mid x$, there are finitely many y for each x. By Proposition 18,

$x < \lambda < y^2$. Hence there are finitely many values for λ. Also, $\lambda \geq 3$. If $y = \lambda$, then by Proposition 14 there is at most one value of k and therefore only one possible parameter set. If $y \neq \lambda$, then by Proposition 17, there are at most two values of k, yielding two parameter sets. □

It should be noted that the results of this section make it possible to construct an algorithm which will generate a comprehensive listing of possible parameter sets for QRQS designs. We have, in fact, generated such a listing with the aid of a computer. Of course for many of these designs, the question of actual existence remains unanswered.

2. Projective and affine designs

Let $D = (\mathcal{P}, \mathcal{B})$ be an arbitrary design. An equivalence relation $\|$ on $\mathcal{B}$ is a *parallelism* if it satisfies the "Euclidean Axiom":

"For all $p \in \mathcal{P}$ and $B \in \mathcal{B}$, there is a unique $C \in \mathcal{B}$ with $p \in C$ and $B \parallel C$."

A *resolvable* design is a design which admits a parallelism. We note that in a resolvable design, if $B \parallel C$, then $B = C$ or $B \cap C = \emptyset$. An *affine* design is a resolvable design satisfying the following condition:

"There is an integer $y > 0$ such that if $B \nparallel C$, then $|B \cap C| = y$."

The following basic result is due to Bose [1].

Theorem 22. *If D is a resolvable design, then D is quasiresidual if and only if D is affine.*

Our next result characterizes affine designs:

Theorem 23. *A design D is affine if and only if D is* QRQS *with* $x = 0$.

Proof. Note that if D is affine, then D is QS with $x = 0$ and is QR by Theorem 22. For the converse, suppose that D is QRQS with $x = 0$. We define the relation $\|$ on the blocks by:

"$B \parallel C$ if and only if $B = C$ or $B \cap C = \emptyset$"

Clearly the relation $\|$ is reflexive and symmetric. Suppose $A \parallel B$ and $B \parallel C$. If any two of A, B, C are identical, then $A \parallel C$ trivially, so we may suppose they are all distinct. For contradiction, assume that $A \nparallel C$, i.e., A and C intersect in y points. We let a denote the number of blocks intersecting A and d the number of blocks intersecting both A and B. By straightforward counting arguments we have

(i) $ay = k(r-1)$, and
(ii) $dy^2 = k^2\lambda$.

From (i), (ii) and Lemma 10(b), it follows that $a = d$. Hence each block which intersects A also intersects B so that C intersects B, contradicting $B \parallel C$. Thus $A \parallel C$ and $\parallel$ is an equivalence relation.

To see that the Euclidean Axiom is satisfied, let p be a point, B a block and $\bar{B}$ the equivalence class of B. Suppose for all $C \in \bar{B}$, $p \notin C$. Then for all C such that $p \in C$, $|B \cap C| = y$. Counting the number of pairs (q, C) where $q \in B \cap C$ and $p \in C$, we have $k\lambda = ry$. Since $a = d$, we have $ay^2 = k^2\lambda$ from (ii) above and thus $ay^2 = kry$ or $ay = kr$. This contradicts (i) above and hence for some $C \in \bar{B}$ we have $p \in C$. This block C is unique since blocks in the same equivalence class are disjoint. Since $\parallel$ is a parallelism, it follows that D is affine. □

We wish to note that the fact that a QRQS design with $x = 0$ is affine can be determined independently from results of Bose and Shrikhande [3]. For if D is QS with $x = 0$, then the dual of D is a 2 class partially balanced incomplete block design (PBIB) as defined in [3]. Bose and Shrikhande show that each such PBIB is a special PBIB and it can be determined from this (using Lemma 10) that D is affine. In Bose, Bridges and Shrikhande [2] special PBIB's are shown to be equivalent to partial geometric designs satisfying certain conditions.

Let $GF(q)$ be the finite field with $q = p^r$ elements and let V be the vector space of dimension $d+1$ over $GF(q)$. A *projective geometry* $PG(d, q)$ of dimension d over $GF(q)$ is the system of subspaces of V. $P(d, q)$ denotes the block design obtained from $PG(d, q)$ by taking the 0-dimensional subspaces as points and the hyperplanes ($(d-1)$-dimensional subspaces) as blocks. It can be verified that $P(d, q)$ is a symmetric

$$\left(\frac{q^{d+1}-1}{q-1}, \frac{q^d-1}{q-1}, \frac{q^{d-1}-1}{q-1}\right) \text{ design.}$$

$P(2, q)$ has parameters $(q^2+q+1, q+1, 1)$ and is called a *projective plane* or order q. We will allow $P(2, q)$ to denote any design with these parameters, not just those based on $GF(q)$.

An affine (or Euclidean) geometry $AG(d, q)$ is the system of point sets $S-(H \cap S)$ where S ranges over all the subspaces of $PG(d, q)$ and H is a fixed hyperplane. Thus an affine geometry is obtained from a projective geometry by removing a hyperplane and all of its points. $A(d, q)$ denotes the residual of $P(d, q)$ with respect to a block H. Thus $A(d, q)$ is a (q^d, q^{d-1}, q^{d-2}) design. It is known that every $A(d, q)$ is an affine design. $A(2, q)$ is called an *affine plane* of order q.

In [5] Dembowski uses the term "projective design" as a synonym for "symmetric design" because these designs are generalizations of designs derived from projective geometries. However, the relationship between a $P(d, q)$ and an $A(d, q)$ is much closer than that between a symmetric design (Dembowski's "projective design") and an affine design. Every $P(d, q)$ has a residual which is an $A(d, q)$ but a symmetric design does not always have an affine residual. The term "projective design" has fallen into disuse since Dembowski's usage, and this prompts us to revive the term under a different definition. A symmetric (v, k, λ) design is *projective* if there exists a nonnegative integer λ' and a block B such that for every pair of blocks B_1, B_2 different from B and from each other, $B \cap B_1 \cap B_2$ has either λ or λ' points.

If $D = (\mathcal{P}, \mathcal{B})$ is a design and B a block of D, we denote by D^B the residual of D with respect to B as previously defined. We also define the *block-derivate* of D with respect to B to be the structure $D_B = (\mathcal{P}', \mathcal{B}')$ where

$$\mathcal{P}' = B \quad \text{and} \quad \mathcal{B}' = \{B' \cap B \mid B' \in \mathcal{B} \text{ and } B' \neq B\}.$$

If D is a symmetric (v, k, λ) design, then D_B is again a design, with parameters $(k, v-1, k-1, \lambda, \lambda-1)$. If p is a point of D, the *point-derivate* of D with respect to p is the structure $D_p = (\mathcal{P}'', \mathcal{B}'')$ where

$$\mathcal{P}'' = \mathcal{P} - \{p\} \quad \text{and} \quad \mathcal{B}'' = \{B - \{p\} \mid p \in B \text{ and } B \in \mathcal{B}\}.$$

Proposition 24. *Let D be a symmetric design. Then D is projective if and only if D^B is affine for some block B of D.*

Proof. Suppose D is projective. Let C_1, C_2 be two blocks of D^B corresponding to B_1, B_2 in D. If $|B \cap B_1 \cap B_2| = \lambda$, then $|C_1 \cap C_2| = 0$. If $|B \cap B_1 \cap B_2| = \lambda'$, then $|C_1 \cap C_2| = \lambda - \lambda'$. Hence the residual D^B is quasi-symmetric with $x = 0$, $y = \lambda - \lambda'$. Conversely, if D^B is affine, then any triple intersection of B with B_1, B_2 has either λ or $\lambda' = \lambda - y$ points. $\square$

Note that the equation $\lambda' = \lambda - y$ enables us to calculate λ'. Using the relationship between the parameters of D and D^B together with Lemma 10(b) we get $\lambda' = \lambda(\lambda - 1)/(k - 1)$.

In [5], Dembowski shows that the designs $P(d, q)$ are characterized by the fact that we get an affine residual no matter which block is chosen. (This is not true for projective designs in general.)

A t-design is a design in which every set of t points is contained in exactly the same number, λ_t, of blocks. The following result is implicit in Dembowski [5]:

Proposition 25. *Let D be a symmetric (v, k, λ) block design with $v - 1 > k$. Then D is not a 3-design.*

Thus for a nontrivial symmetric design, the number of blocks containing three points assumes at least two values. We now consider the simplest case of the two values, which happens to reduce to the class of designs $P(d, q)$. In a design D, a *line* through two distinct points p, q is the intersection of all blocks containing p and q. D is called *smooth* if any three non-collinear points are contained in the same number of blocks. Note that any three collinear points must be in λ blocks. We call a (v, b, r, k, λ) design a *near-3-design* if the number of blocks containing three distinct points takes exactly two values, one of which is λ.

Proposition 26. *Let D be any design with parameters (v, b, r, k, λ). Then D is smooth if and only if D is a 3-design or a near-3-design.*

Proof. If D is smooth then any 3 collinear points are in λ blocks and any 3 non-collinear points are in λ' blocks. Conversely, if D is a 3-design, D is obviously smooth.

Suppose D is a near-3-design with three distinct points contained in either λ or λ' blocks. Let p_1, p_2, p_3 be three distinct points contained in λ blocks. A fortiori, each of these λ blocks contain p_1, p_2. But in all, there are only λ blocks containing p_1, p_2. Hence every block containing p_1, p_2 also contains p_3. Therefore p_3 is on the line through p_1, p_2. Hence any three non-collinear points must be contained in λ' blocks, i.e., D is smooth. □

An intrinsic characterization of the designs $P(d, q)$ is given by Dembowski and Wagner [6]:

Theorem 27. *Let D be an arbitrary design. Then D is a $P(d, q)$ if and only if D is symmetric and smooth.*

Since $P(d, q)$ is never a 3-design, we can refine the preceding Theorem as follows:

Theorem 28. *D is a $P(d, q)$ if and only if D is a symmetric near-3-design.*

Our final group of theorems develop some results on the structural interrelationships between projective and affine designs. In these theorems, we use D_p^{B*} to denote the dual of the derivate of D^B with respect to the point p of D^B. (The dual of a structure is obtained by interchanging "points" and "blocks". This interchange is known as an "anti-isomorphism".)

Theorem 29. *Let D be a symmetric (v, k, λ) design with $\lambda > 1$ and let B be a block. The following are equivalent:*

(1) D^B is affine.

(2) D_B is a multiple of a symmetric design.

(3) D_p^{B*} is a multiple of a symmetric design for every point p of D^B.

Proof. (1) implies (2) and (3): Since D^B is affine, $n = (v-k)/(k-\lambda) = (k-1)/\lambda$ is an integer ≥ 2. Let $\lambda' = \lambda(\lambda-1)/(k-1)$.

$$D:(v, v, k, k, \lambda)$$

$$D^B:(v-k, v-1, k, k-\lambda, \lambda) \quad x=0,\ y=\lambda-\lambda'$$

$$D_B:(k, v-1, k-1, \lambda, \lambda-1) = (k, nk, n\lambda, \lambda, n\lambda') \quad x=\lambda',\ y=\lambda$$

$$D_p^{B*}:\left(k, \frac{k(k-\lambda-1)}{\lambda}, k-\lambda-1, \lambda, \frac{(k-\lambda-1)(\lambda-1)}{k-1}\right) = (k, (n-1)k, (n-1)\lambda, \lambda, (n-1)\lambda') \quad x=\lambda',\ y=\lambda$$

$$D_2:(k, k, \lambda, \lambda, \lambda') \qquad D_1:(k, k, \lambda, \lambda, \lambda')$$

It can be verified that D^B, D_B and D_p^{B*} are quasi-symmetric designs with parameters and intersection numbers as shown. By Proposition 16, D_B is a multiple of a symmetric (k, λ, λ') design D_1, and D_p^{B*} is a multiple of a symmetric (k, λ, λ') design D_2.

(2) implies (1): D_B has block size λ. Since it is a multiple of a symmetric design it is quasi-symmetric with $x=\mu$ (say) and $y=\lambda$. This means that for any pair of blocks B_1, B_2 of D, $B\cap B_1\cap B_2$ has either λ or μ points. Hence the residual D^B is quasi-symmetric with $x=0$, $y=\lambda-\mu$. Therefore D^B is affine.

(3) implies (1): D_p^{B*} has block size λ. Let C_1, C_2 be two blocks of D_B. Suppose $C_1\cap C_2\neq\emptyset$. Let $p\in C_1\cap C_2$. The anti-isomorphism $D_p^B\to D_p^{B*}$ maps $C_1-\{p\}$, $C_2-\{p\}$ into points p_1, p_2 (say). The pair p_1, p_2 is contained in a fixed number of blocks, say ν (i.e., ν is independent of the choice of C_1, C_2). Hence in D_p^B, $C_1-\{p\}$, $C_2-\{p\}$ have ν points in common. Therefore, in D^B, $|C_1\cap C_2|=\nu+1$.

Since D^B is not symmetric, there must be blocks C_1, C_2 which are disjoint. Hence D^B is quasi-symmetric with $x=0$, $y=\nu+1$. □

Theorem 30. *Let D be a projective (v, k, λ) design with $\lambda>1$. Then $n=(v-k)/(k-\lambda)=(k-1)/\lambda$ is an integer $\geqslant 2$. Moreover, there exist a block B and symmetric (k, λ, λ') designs D_1, D_2 such that $D_B\simeq nD_1$ and $D_p^{B*}\simeq(n-1)D_2$.*

Proof. Since D is projective, D^B is affine for some block B. Thus the proof is the same as the first part of the proof of the previous theorem. □

The next two theorems are specializations of the previous theorem to $P(d, q)$. For $d\geqslant 4$, we have the additional result that D_1 and D_2 are isomorphic, and are in fact the unique $P(d-1, q)$.

Theorem 31. *Let $D=P(d, q)$ and $E=P(d-1, q)$ where $d\geqslant 4$. Let B be any block of D. Then $D^B\simeq A(d, q)$, $D_B\simeq q\cdot E$ and $D_p^{B*}\simeq(q-1)\cdot E$.*

Proof. Let the parameters of D be (v, k, λ). Then

$$v = q^d + q^{d-1} + \cdots + 1, \qquad k = q^{d-1} + \cdots + 1, \qquad \lambda = q^{d-2} + \cdots + 1.$$

Hence

$$n = \frac{v-k}{k-\lambda} = \frac{k-1}{\lambda} = q.$$

We take

$$\lambda' = \frac{\lambda(\lambda-1)}{k-1} = q^{d-3} + \cdots + 1, \qquad \lambda'' = \frac{\lambda'(\lambda'-1)}{\lambda-1} = q^{d-4} + \cdots + 1.$$

It is well known that $D^B \simeq A(d, q)$. By Theorem 29, $D_B \simeq qD_1$ and $D_p^{B*} \simeq (q-1)D_2$ where D_1, D_2 are symmetric (k, λ, λ') designs.

It is easily verified that both D_1 and D_2 have the parameters of $P(d-1, q)$, so it only remains to show that they are isomorphic to $P(d-1, q)$.

To show $D_1 \simeq P(d-1, q)$: Let K, L, M, N be four blocks in D.

$$\dim KL = \dim MN = d-2.$$

$$\dim(KL + MN) = d-2,\ d-1 \quad \text{or} \quad d.$$

$$\begin{aligned}\dim KLMN &= \dim KL + \dim MN - \dim(KL + MN)\\ &= d-4,\ d-3 \quad \text{or} \quad d-2.\end{aligned}$$

Hence $|KLMN| = \lambda''$, λ' or λ.

Since D is self-dual, any 4 points are in λ'', λ' or λ blocks. Therefore any 4 points of D^B are in λ'', λ' or λ blocks. It follows that any 3 points of D_p^B are in λ'', λ' or λ blocks. Hence any 3 blocks of D_p^{B*} intersect in λ'', λ', or λ points. If $q = 2$, $D_p^{B*} = D_1$, so any 3 blocks can intersect only in λ'' or λ' points. If $q > 2$, D_p^{B*} is a multiple of D_1 and again any 3 blocks of D_1 can intersect only in λ'' or λ' points. Hence in any case, any 3 points of D_1^* are in λ'' or λ' blocks. Therefore, D_1^* is smooth and by the Dembowski-Wagner Theorem is the design $P(d-1, q)$. But this design is self-dual, so $D_1 \simeq P(d-1, q)$.

To show $D_2 \simeq P(d-1, q)$: Any 3 points of D are in λ' or λ blocks (since D is self-dual). Hence any 3 points of D_B are in $\lambda'-1$ or $\lambda-1$ blocks. Since $D_B \simeq q \cdot D_2$, any 3 points of D_2 are in $(\lambda'-1)/q$ or $(\lambda-1)/q$ blocks, i.e., λ'' or λ' blocks. Hence D_2 is smooth. Therefore $D_2 \simeq P(d-1, q)$. □

Theorem 32. *Let $D = P(3, q)$. Let B be any block of D. Then $D^B \simeq A(3, q)$, $D_B \simeq q \cdot D_1$ and $D_p^{B*} \simeq (q-1) \cdot D_2$, where D_1, D_2 are projective planes $P(2, q)$.*

Proof. The proof is the same as the first part of the proof of the previous theorem. We cannot conclude $D_1 \simeq D_2$ because there may be more than one $P(2, q)$ for a given q. □

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