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Pure Semisimple Categories and Rings of Finite Representation Type

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PURE SEMISIMPLE CATEGORIES

Let A be a locally finitely presented Grothendieck category and let P.gl.dim A denote the pure global dimension of A (see [12]). A is pure semisimple if P.gl.dim $A = 0$, or, equivalently, if each object in A is a coproduct of Noetherian subobjects (see [11]). A ring R is left pure semisimple if the category R-Mod of all left R-modules is pure semisimple. Recall also that a ring R is said to be of finite representation type if it is left Artinian and up to isomorphism there is only a finite number of indecomposable finitely generated left R-modules. By [5] this property is left–right symmetric.

It is known (see [6]) that a ring R is of finite representation type if and only if R is both left and right pure semisimple. The problem is whether the left pure semisimplicity of R implies that R is of finite representation type. This problem was studied in [3, 7, 13]. A positive solution for Artin algebras is given by Auslander [2]. In this paper a more general categorical problem is solved on the basis of the results in [7, 13]. Our main result is that any pure semisimple category which has only finitely many isomorphism types of simple objects is equivalent to the category of all left modules over a ring of finite representation type.

We have divided the paper in two sections. In Section 1 we recall some definitions and results from [4, 10, 12, 13] which we use in the paper. In Section 2 we show that when a ring R is both left and right perfect then R is left pure semisimple if and only if the functor category fp_R-Mod is perfect, where fp_R denotes the category of all finitely presented right R-modules. This is used in the proof of the main theorem stated above.

1. PRELIMINARIES

We start by recalling some definitions and results from [4, 10, 12]. By an additive category we mean a category together with an Abelian group structure

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on each of its Hom sets such that the composition is bilinear. A category C is skeletally small if the isomorphism classes of objects of C form a set. We say that in the category C idempotents split if each idempotent endomorphism of an object in C has a kernel in C.

Let C be an additive category. A two-sided ideal I in C is a subfunctor of the two-variable functor $\mathrm{Hom}_C : C^{\mathrm{op}} \times C \to Ab$. If I is a two-sided ideal in C one can form a quotient category C/I which has the same objects as C and $\mathrm{Hom}_{C/I}(X, Y) = \mathrm{Hom}_C(X, Y)/I(X, Y)$. An important example of a two-sided ideal in an arbitrary additive category C is the Jacobson radical $J = J(C)$ defined by

$$J(X, Y) = \{f \in \mathrm{Hom}_C(X, Y),\ 1_X - gf \text{ has a two-sided inverse for every } g\}.$$

Observe that $J(X, X)$ is the Jacobson radical of the endomorphism ring End X. Moreover the following lemma holds.

LEMMA 1.1. (a) *Let $C = C_1 \oplus \cdots \oplus C_n$, $C' = C'_1 \oplus \cdots \oplus C'_m$ be objects of an additive category C and let $f = (f_{ij})$: $C \to C'$ be a morphism in C with $f_{ij} : C_j \to C'_i$. Then $f \in J(C, C')$ if and only if $f_{ij} \in J(C_j, C_i')$ for all i and j.*

(b) *Suppose that C is an additive category in which idempotents split and let X, Y be indecomposable objects in C. If* End X *is a local ring then $J(X, Y)$ is the set of all nonisomorphisms from X to Y.*

The proof is left to the reader.

Another important example of a two-sided ideal in an abitrary additive category C is the ideal P defined as follows. For each pair of objects A and B in C we define $P(A, B)$ to be the subset of $\mathrm{Hom}_C(A, B)$ consisting of all morphisms from A to B which factors through projective objects. The quotient category C/P is denoted by $\underline{C}$ and the natural residue functor by $\underline{\ \ } : C \to \underline{C}$.

We now prove the following useful lemma which is essentially due to Auslander [1].

LEMMA 1.2. *Let C be an additive category in which idempotents split. Suppose each object in C is a finite direct sum of indecomposable objects and that each indecomposable projective object in C has a local endomorphism ring. If an object X in C has no nonzero projective summands then the kernel of the natural ring epimorphism* End $X \to$ End $\underline{X}$ *is contained in $J(X, X)$.*

Proof. Let $X = X_1 \oplus \cdots \oplus X_n$ where $X_1, \ldots, X_n$ are nonprojective indecomposable. If $f \in$ End X and $\underline{f} = 0$ then f is a composition $X \to^t P \to^g X$ with P projective. Applying Lemma 1.1 we easily conclude that $g \in J(P, X)$ and hence $f \in J(X, X)$.

As an immediate consequence we have

COROLLARY 1.3. *Let C be as in Lemma* 1.2 *and suppose that A and B are objects in C with no nonzero projective summands. Then*

(a) *a morphism $f: A \to B$ in C is an isomorphism if and only if the morphism $f: \underline{A} \to \underline{B}$ is an isomorphism in $\underline{C}$.*

(b) *A is indecomposable if and only if $\underline{A}$ is indecomposable.*

For any ring R let us denote by fp_R (resp. ${}_R\mathrm{fp}$) the category of all finitely presented right (resp., left) R-modules. If R is semiperfect then the categories ${}_R\mathrm{fp}$ and fp_R satisfy the conditions in Lemma 1.2.

We denote by R-Mod (resp., Mod-R) the category of all left (resp., right) R-modules. The category of all covariant additive functors from a skeletally small additive category C to Abelian groups is denoted by C-Mod. If A is a locally finitely presented Grothendieck category we denote by $\mathrm{fp}(A)$ its full subcategory consisting of all finitely presented objects. A category C-Mod is said to be perfect if each object in C-Mod has a projective cover. A discussion of perfect functor categories and pure semisimple Grothendieck categories can be found in the author's notes [11–13]. In particular, in [12](see also [11]), the following result, which is frequently used in the paper, is proved. Let A be a locally finitely presented Grothendieck category. Then A is pure semisimple if and only if $\mathrm{fp}(A)^{\mathrm{op}}$-Mod is perfect or, equivalently, if $\mathrm{fp}(A)$-Mod is locally Artinian.

2. MAIN RESULTS

We start by giving the following useful result.

PROPOSITION 2.1. (a) *If R is a right perfect ring then* $\mathrm{fp}_R^{\mathrm{op}}$-Mod *is perfect if and only if* $\underline{\mathrm{fp}}_R^{\mathrm{op}}$-Mod *is perfect.*

(b) *If R is a left perfect ring then* fp_R-Mod *is perfect if and only if* $\underline{\mathrm{fp}}_R$-Mod *is perfect.*

Proof. By [12, Theorem 5.4], C^{op}-Mod is perfect if and only if $C/J(C)$ is semisimple and $J(C)$ is right T-nilpotent or, equivalently, if given any sequence

$$C_1 \xrightarrow{f_1} C_2 \longrightarrow \cdots \longrightarrow C_n \xrightarrow{f_n} C_{n+1} \longrightarrow \cdots$$

in C, there are an integer n and a morphism $g: C_{n+1} \to C_n$ such that $f_n \cdots f_1 = g f_{n+1} f_n \cdots f_1$. It follows that if C^{op}-Mod is perfect then $(C/I)^{\mathrm{op}}$-Mod is perfect for any two-sided ideal I in C.

(a). Let us assume that R is a right perfect ring and $\underline{\mathrm{fp}}_R^{\mathrm{op}}$-Mod is perfect. If D is the full subcategory of fp_R consisting of all indecomposable modules then we have an equivalence $\mathrm{fp}_R^{\mathrm{op}}$-Mod $=$ D^{op}-Mod. In view of the discussion above it is sufficient to show that

(i) $J(D)$ is right T-nilpotent and

(ii) $D/J(D)$ is semisimple.

To prove (i) suppose we are given a sequence

$$D_1 \xrightarrow{f_1} D_2 \longrightarrow \cdots \longrightarrow D_n \xrightarrow{f_n} D_{n+1} \longrightarrow \cdots$$

in $J(D)$ and consider the sequence

$$\underline{D}_1 \xrightarrow{\underline{f}_1} \underline{D}_2 \longrightarrow \cdots \longrightarrow \underline{D}_n \xrightarrow{\underline{f}_n} \underline{D}_{n+1} \longrightarrow \cdots$$

in $J(\underline{\mathrm{fp}}_R)$. Since $J(\underline{\mathrm{fp}}_R)$ is right T-nilpotent then there are $1 < n_1 < n_2 < \cdots$ and commutative diagrams

$$\begin{array}{ccccccccccc} D_1 & \xrightarrow{g_0} & D_{n_1} & \xrightarrow{f_{n_1}} & D_{n_1+1} & \xrightarrow{g_1} & D_{n_2} & \xrightarrow{f_{n_2}} & D_{n_2+1} & \xrightarrow{g_2} & D_{n_3} \longrightarrow \cdots \\ & {\scriptstyle s_0}\searrow \nearrow{\scriptstyle t_1} & & & & {\scriptstyle s_1}\searrow \nearrow{\scriptstyle t_2} & & & & {\scriptstyle s_2}\searrow \nearrow{\scriptstyle t_3} & \\ & P_1 & & & & P_2 & & & & P_3 & \end{array}$$

with projective modules $P_1, P_2, P_3, \ldots$ and $g_i = f_{n_{i+1}-1} \cdots f_{n_i+1}$.

Let us denote by pr_R (resp. ${}_R\mathrm{pr}$ the category of all finitely generated projective right (resp. left) R-modules. Since f_{n_i} belongs to $J(\mathrm{fp}_R)$ then $h_i = s_i f_{n_i} t_i$ belongs to $J(\mathrm{pr}_R)$. Since there is an equivalence $\mathrm{pr}_R^{\mathrm{op}}$-Mod $=$ Mod-R and R is right perfect then $J(\mathrm{pr}_R)$ is right T-nilpotent. Hence there is an integer j such that $h_j \cdots h_1 = 0$ and thus we have $f_{n_{j+1}} \cdots f_1 = 0$. This proves part (i).

We now prove (ii). Let X be an object in D. If X is projective then we know that End X is a local ring because R is right perfect. Assume that X is not projective. Then by Lemma 1.2 we know that there is a ring epimorphism End $\underline{X} \to$ End $X/J(X, X)$. Since $\underline{\mathrm{fp}}_R^{\mathrm{op}}$-Mod is perfect it follows from [12, Corollary 5.2](see also [11, Proposition 2.2]) that the ring End $\underline{X}$ is right perfect. Hence End $X/J(X, X)$ is semisimple. But by (i) we know that $J(X, X)$ is right T-nilpotent, then End X must be a local ring because X is indecomposable. Consequently each object in D has a local endomorphism ring. Then by Lemma 1.1 we know that $J(C, C') = \mathrm{Hom}_D(C, C')$ for any pair of nonisomorphic objects C and C' in D. Hence $\mathrm{Hom}_{D/J}(X, Y)$ is a division ring if $X \simeq Y$, and it is zero in the opposite case. It follows that the category $(D/J)^{\mathrm{op}}$-Mod is equivalent to the product of categories Mod-K_X where $K_X =$ End $X/J(X, X)$ is a division ring and X runs through a fixed set of representatives of isomorphy classes of objects in D. This proves that D/J is semisimple, finishing the proof of (a). The proof of (b) proceeds in a similar fashion and is left to the reader (use the duality $\mathrm{pr}_R = {}_R\mathrm{pr}^{\mathrm{op}}$).

COROLLARY 2.2. *Suppose that a ring R is both left and right perfect. Then R is left pure semisimple if and only if the category* fp_R-Mod *is perfect.*

Proof. By [4] we know that there is a duality $\mathrm{fp}_R^{\mathrm{op}} = {}_R\mathrm{fp}$. Then it follows by Proposition 2.1 that fp_R-Mod is perfect if and only if ${}_R\mathrm{fp}^{\mathrm{op}}$-Mod is perfect. On the other hand by [12, Theorem 6.3] we know that R is left pure semisimple if and only if ${}_R\mathrm{fp}^{\mathrm{op}}$-Mod is perfect. Hence the corollary follows.

We are now in a position to establish the main result of the paper.

THEOREM 2.3. *Let A be a locally finitely presented Grothendieck category which has only a finite number of nonisomorphic simple objects. Then A is pure semisimple if and only if A is equivalent to the category of all modules over a ring of finite representation type.*

Proof. Suppose $A =$ Mod-R where R is a ring of finite representation type. Let $X_1, \ldots, X_n$ be a complete set of nonisomorphic indecomposable finitely presented right R-modules. It is clear that there is an equivalence $\mathrm{fp}_R^{\mathrm{op}}$-Mod $=$ Mod-S where S is the endomorphism ring of $X_1 \oplus \cdots \oplus X_n$. Since S is semiprimary then by [12, Theorem 6.3] A is pure semisimple.

Conversely, suppose A is pure semisimple. First we consider the case $A =$ R-Mod where R is a ring. By [8] we know that the category fp_R-Mod is locally Noetherian. Moreover every object in ${}_R\mathrm{fp}$ satisfies the condition $(*)$ in [7] because by [13, Corollary 2.3] we know that the ring R is both left and right Artinian. Then it follows by [7, Corollary] that fp_R-Mod is locally finite. Hence by [2, Theorem 3.1] we know that R is of finite representation type.

Now we consider the general case. Let $S_1, \ldots, S_n$ be a complete set of nonisomorphic simple objects in A and let Λ be the endomorphism ring of the injective envelope of $S_1 \oplus \cdots \oplus S_n$ in A. We know by [13] that Λ is left Artinian, ${}_\Lambda\mathrm{fp}$-Mod is perfect and that there is a duality $\mathrm{fp}(A)^{\mathrm{op}} = {}_\Lambda\mathrm{fp}$. Then according to Corollary 2.2 Λ is right pure semisimple and hence it is of finite representation type. It follows that ${}_\Lambda\mathrm{fp}$ is closed under injective envelopes and thus $\mathrm{fp}(A)$ has a projective generator P. Then there is an equivalence $A =$ Mod-End P and the proof of the theorem is complete.

Recall that an Abelian category C is a length category if each object in C is both Noetherian and Artinian. An Abelian category is of finite representation type if it has only a finite number of nonisomorphic indecomposable objects. In the terminology of Auslander [2] a family of morphisms is called Noetherian in case, for each sequence of nonisomorphisms $M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \rightarrow \cdots$ in the family, there exists an n such that $f_n \cdots f_0 = 0$. The family is co-Noetherian if it satisfies the obvious dual condition.

We now apply Theorem 2.3 to establish the following completion of [2, Theorem 3.1].

THEOREM 2.4. *Let C be a skeletally small Abelian category which has only a* *finite number of nonisomorphic simple objects. Then the following conditions are equivalent*:

(1) C-Mod *is locally finite.*

(2) C-Mod *is locally Artinian.*

(3) C-Mod *is perfect.*

(4) C-Mod *is semi-Artinian and* C *is Noetherian.*

(5) C *is a length category and the family of monomorphisms between indecomposable objects in* C *is Noetherian.*

(6) C *is a length category and the family of epimorphisms between indecomposable objects is co-Noetherian.*

(7) C *is a length category of finite representation type.*

Proof. Let us denote by (n') the statement (n) for the category C^{op}. Moreover we denote by A the category Lex C^{op} consisting of all contravariant left exact functors from C to Abelian groups. Since there is an equivalence $\text{fp}(A) = C$ then it follows from [12, Theorem 6.3] that (3) and (2′) are equivalent, and that A is pure semisimple if and only (3) holds. Then from Theorem 2.3 we conclude that (3) implies (7). Since we know by [2, Theorem 3.1] that the statements (1), (1′), and (7) are equivalent, then all statements (1), (2), (3), (7), (1′), (2′), (3′) are equivalent. The implications (7) $\to$ (5) $\to$ (6′) are obvious and (6′) $\to$ (4) follows from the proof of [2, Theorem 3.1]. Finally, (4) $\to$ (2) is an immediate consequence of [13, Theorem 1.9]. Since similarly we get (7) $\to$ (5′) $\to$ (6) $\to$ (4′) $\to$ (2′) then the theorem is proved.

Remarks. (1) Fuller [6] has proved that a ring R is right pure semisimple if and only if every direct product of projective objects in fp_R^{op}-Mod is projective. Then it follows from [12, Corollaries 2.9, and 2.11] that R is right pure semisimple if and only if every direct product of pure-projective right R-modules is pure-projective.

(2) Since we know that l.P.gl.dim $R = 0$ if and only if R is of finite representation type then the l.P.gl.dim R measures how far R is from being of finite representation type. It was already established in [9] that l.P.gl.dim $R \leqslant n + 1$ whenever $|R| \leqslant \aleph_n$, $n \geqslant 0$. Hence l.P.gl.dim $R = 1$ for any countable ring R which is not of finite representation type.

(3) It would be interesting to know if stably equivalent Artinian rings have the same left pure global dimensions.

Note added in proof. It is possible to give another and easier proof of Theorem 2.3 for $A = R$-mod. Suppose R is left pure semisimple. Then R is Morita dual to some left Artinian ring S, i.e., ${}_R\text{fp} = {}_S\text{fp}^{\text{op}}$. By Corollary 2.2 S is right pure semisimple. In particular it is right Artinian. Then by remarks after Corollary in [8] S is of finite representation type.

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