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Coupled common fixed point theorems for a pair of commuting mappings in partially ordered complete metric spaces

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ABSTRACT

The purpose of this paper is to establish a coupled coincidence point for a pair of commuting mappings in partially ordered complete metric spaces. We also present a result on the existence and uniqueness of coupled common fixed points. An example is given to support the usability of our results.

1. Introduction and preliminaries

The well-known Banach contraction theorem [1] plays a major role in solving problems in many branches of pure and applied mathematics. A number of generalizations of the Banach contraction theorem were obtained in various directions. Many authors generalized the Banach contraction theorem in ordered metric spaces. The first result in ordered metric spaces was given by Ran and Reurings [2, Theorem 2.1] who presented its applications to the linear and nonlinear metric space. Subsequently, Nieto and Rodŕiguez–López [3] extended the result of Ran and Reurings [2] for nondecreasing mappings and applied to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Agarwal et al. [4] studied generalized contractions in partially ordered metric spaces. Bhaskar and Lakshmikantham [5] introduced the notion of a coupled fixed point and proved some interesting coupled fixed point theorems for mappings satisfying a mixed monotone property. While Lakshmikantham and Ćirić [6] introduced the concept of a mixed g-monotone mapping and proved coupled coincidence and coupled common fixed point theorems that extend theorems due to Bhaskar and Lakshmikantham [5]. Then after, many authors obtained many coupled coincidence and coupled fixed point theorems in ordered metric spaces (see [4,7–15,3,16–22] as examples).

In this paper we establish coupled coincidence points for a pair of commuting mappings in partially ordered complete metric spaces. An example is given to support the useability of our results.

Before presenting the main results of the paper, we start by recalling some definitions introduced in [5].

Recall that if $(X, \preceq)$ is a partially ordered set and $F : X \to X$ is such that for $x, y \in X$, $x \preceq y$ implies $F(x) \preceq F(y)$, then a mapping F is said to be nondecreasing. Similarly, F is defined a nonincreasing mapping.

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Definition 1.1. Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. The mapping F is said to have the mixed monotone property if $F(x, y)$ is monotone nondecreasing in x and is monotone nonincreasing in y, that is, for any

$$x, y \in X,\ x_1, x_2 \in X, \quad x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad y_1 \preceq y_2 \Rightarrow F(x, y_1) \succeq F(x, y_2).$$

This definition coincides with the notion of a mixed monotone function on $\mathbb{R}^2$ and $\leq$ represents the usual total order in $\mathbb{R}$.

Definition 1.2. We call an element $(x, y) \in X \times X$ a coupled fixed point of the mapping $F : X \times X \to X$ if

$$F(x, y) = x \quad \text{and} \quad F(y, x) = y.$$

The concept of the mixed monotone property is generalized in [6].

Definition 1.3 (*[6]*)**.** Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$. The mapping F is said to have the mixed g-monotone property if F is monotone g-nondecreasing in its first argument and is monotone g-nonincreasing in its second argument, that is, for any $x, y \in X$

$$x_1, x_2 \in X, g(x_1) \preceq g(x_2) \Rightarrow F(x_1, y) \preceq F(x_2, y) \tag{1}$$

and

$$y_1, y_2 \in X, g(y_1) \preceq g(y_2) \Rightarrow F(x, y_1) \succeq F(x, y_2). \tag{2}$$

Clearly, if g is the identity mapping, then Definition 1.3 reduces to Definition 1.1.

Definition 1.4. An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if

$$F(x, y) = g(x), \quad \text{and} \quad F(y, x) = g(y).$$

The main theoretical results of Bhaskar and Lakshmikantham in [5] are the following coupled fixed point theorems.

Theorem 1.1 (*[5]*)**.** *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)] \quad \forall x \succeq u \quad \text{and} \quad y \preceq v.$$

If there exist two elements $x_0, y_0 \in X$ with

$$x_0 \preceq F(x_0, y_0) \quad \text{and} \quad y_0 \succeq F(y_0, x_0)$$

then there exist $x, y \in X$ such that

$$x = F(x, y) \quad \text{and} \quad y = F(y, x).$$

Theorem 1.2 (*[5]*)**.** *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Assume that X has the following property:*

(i) *if a nondecreasing sequence $\{x_n\}$ in X converges to $x \in X$, then $x_n \preceq x$ for all n,*
(ii) *if a nonincreasing sequence $\{y_n\}$ in X converges to $y \in X$, then $y_n \succeq y$ for all n.*

Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)] \quad \forall x \succeq u \quad \text{and} \quad y \preceq v.$$

If there exist two elements $x_0, y_0 \in X$ with

$$x_0 \preceq F(x_0, y_0) \quad \text{and} \quad y_0 \succeq F(y_0, x_0)$$

then there exist $x, y \in X$ such that

$$x = F(x, y) \quad \text{and} \quad y = F(y, x).$$

Definition 1.5. Let (X, d) be a metric space and $F : X \times X \to X$ and $g : X \to X$ be mappings. We say F and g commute if

$$F(g(x), g(y)) = g(F(x, y))$$

for all $x, y \in X$.

2. Main theorems

Theorem 2.1. *Let $(X, d, \preceq)$ be an ordered metric space. Let $F : X \times X \to X$ and $g : X \to X$ be mappings such that F has the mixed g-monotone property on X such that there exist two elements $x_0, y_0 \in X$ with $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Suppose there exist non-negative real numbers α, β, L with $\alpha + \beta < 1$ such that*

$$d(F(x, y), F(u, v)) \le \alpha \min\{d(F(x, y), g(x)), d(F(u, v), g(x))\} + \beta \min\{d(F(x, y), g(u)), d(F(u, v), g(u))\} + L \min\{d(F(x, y), g(u)), d(F(u, v), g(x))\} \tag{3}$$

for all $(x, y), (u, v) \in X \times X$ with $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$. Further suppose $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X. Also, suppose that X satisfies the following properties:

(i) *if a nondecreasing sequence $\{x_n\}$ in X converges to $x \in X$, then $x_n \preceq x$ for all n,*
(ii) *if a nonincreasing sequence $\{y_n\}$ in X converges to $y \in X$, then $y_n \succeq y$ for all n.*

Then there exist $x, y \in X$ such that

$$F(x, y) = g(x) \quad \text{and} \quad F(y, x) = g(y),$$

that is, F and g have a coupled coincidence point $(x, y) \in X \times X$.

Proof. Let $x_0, y_0 \in X$ be such that $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$.

In the same way we construct, $g(x_2) = F(x_1, y_1)$ and $g(y_2) = F(y_1, x_1)$.

Continuing in this way we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that,

$$g(x_{n+1}) = F(x_n, y_n) \quad \text{and} \quad g(y_{n+1}) = F(y_n, x_n) \quad \forall n \ge 0. \tag{4}$$

Now we prove that for all $n \ge 0$,

$$g(x_n) \preceq g(x_{n+1}) \tag{5}$$

and

$$g(y_n) \succeq g(y_{n+1}). \tag{6}$$

We shall use the mathematical induction. Let $n = 0$. Since $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$, in view of $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$, we have $g(x_0) \preceq g(x_1)$ and $g(y_0) \succeq g(y_1)$, that is, (5) and (6) hold for $n = 0$. We presume that (5) and (6) hold for some $n > 0$. As F has the mixed g-monotone property and $g(x_n) \preceq g(x_{n+1})$, $g(y_n) \succeq g(y_{n+1})$, from (4), we get

$$g(x_{n+1}) = F(x_n, y_n) \preceq F(x_{n+1}, y_n) \tag{7}$$

and

$$F(y_{n+1}, x_n) \preceq F(y_n, x_n) = g(y_{n+1}). \tag{8}$$

Also for the same reason we have

$$g(x_{n+2}) = F(x_{n+1}, y_{n+1}) \succeq F(x_{n+1}, y_n) \quad \text{and} \quad F(y_{n+1}, x_n) \succeq F(y_{n+1}, x_{n+1}) = g(y_{n+2}).$$

Then from (4) and (5), we obtain

$$g(x_{n+1}) \preceq g(x_{n+2}) \quad \text{and} \quad g(y_{n+1}) \succeq g(y_{n+2}).$$

Thus by the mathematical induction, we conclude that (5) and (6) hold for all $n \ge 0$.

We check easily that

$$g(x_0) \preceq g(x_1) \preceq g(x_2) \preceq \cdots \preceq g(x_{n+1}) \preceq \cdots$$

and

$$g(y_0) \succeq g(y_1) \succeq g(y_2) \succeq \cdots \succeq g(y_{n+1}) \succeq \cdots.$$

Since $g(x_n) \succeq g(x_{n-1})$ and $g(y_n) \preceq g(y_{n-1})$, from (3) and (4), we have

$$\begin{aligned} d(g(x_{n+1}), g(x_n)) &= d(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\ &\le \alpha \min\{d(F(x_n, y_n), g(x_n)), d(F(x_{n-1}, y_{n-1}), g(x_n))\} \\ &\quad + \beta \min\{d(F(x_n, y_n), g(x_{n-1})), d(F(x_{n-1}, y_{n-1}), g(x_{n-1}))\} \\ &\quad + L \min\{d(F(x_n, y_n), g(x_{n-1})), d(F(x_{n-1}, y_{n-1}), g(x_n))\} \end{aligned}$$

or

$$d(g(x_{n+1}), g(x_n)) \le \beta d(g(x_n), g(x_{n-1})). \tag{9}$$

Similarly, since $g(y_{n-1}) \succeq g(y_n)$ and $g(x_{n-1}) \preceq g(x_n)$, from (3) and (4), we have

$$d(g(y_n), g(y_{n+1})) \le \alpha d(g(y_n), g(y_{n-1})). \tag{10}$$

From (9) and (10), we have

$$\begin{aligned} d(g(x_{n+1}), g(x_n)) + d(g(y_n), g(y_{n+1})) &\le \beta d(g(x_n), g(x_{n-1})) + \alpha d(g(y_n), g(y_{n-1})) \\ &\le (\alpha+\beta) d(g(x_n), g(x_{n-1})) + (\alpha+\beta) d(g(y_n), g(y_{n-1})) \\ &= (\alpha+\beta)[d(g(x_n), g(x_{n-1})) + d(g(y_n), g(y_{n-1}))]. \end{aligned}$$

Set $\rho_n = d(g(x_{n+1}), g(x_n)) + d(g(y_{n+1}), g(y_n))$ and $\delta = \alpha + \beta$, then sequence $\{\rho_n\}$ is decreasing as

$$0 \le \rho_n \le \delta\rho_{n-1} \le \delta^2\rho_{n-2} \le \cdots \le \delta^n\rho_0$$

which implies

$$\lim_{n\to\infty} \rho_n = \lim_{n\to\infty} [d(g(x_{n+1}), g(x_n)) + d(g(y_{n+1}), g(y_n))] = 0. \tag{11}$$

Thus,

$$\lim_{n\to\infty} d(g(x_{n+1}), g(x_n)) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(g(y_{n+1}), g(y_n)) = 0.$$

In what follows, we shall prove that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences.

For each $m \ge n$, we have

$$d(g(x_m), g(x_n)) \le d(g(x_m), g(x_{m-1})) + d(g(x_{m-1}), g(x_{m-2})) + \cdots + d(g(x_{n+1}), g(x_n))$$

and

$$d(g(y_m), g(y_n)) \le d(g(y_m), g(y_{m-1})) + d(g(y_{m_1}), g(y_{m-2})) + \cdots + d(g(y_{n+1}), g(y_n)).$$

Therefore

$$\begin{aligned} d(g(x_m), g(x_n)) + d(g(y_m), g(y_n)) &\le \rho_{m-1} + \rho_{m-2} + \cdots + \rho_n \\ &\le (\delta^{m-1} + \delta^{m-2} + \cdots + \delta^n)\rho_0 \\ &\le \frac{\delta^n}{1-\delta}\rho_0 \end{aligned} \tag{12}$$

which implies that

$$\lim_{n,m\to\infty} [d(g(x_m), g(x_n)) + d(g(y_m), g(y_n))] = 0.$$

This imply that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences in $g(X)$. Since $g(X)$ is a complete subspace of X, there exists $(x, y) \in X \times X$ such that $g(x_n) \to g(x)$ and $g(y_n) \to g(y)$. Since $\{g(x_n)\}$ is a nondecreasing sequence and $g(x_n) \to g(x)$ and as $\{g(y_n)\}$ is a nonincreasing sequence and $g(y_n) \to g(y)$, by assumption we have $g(x_n) \preceq g(x)$ and $g(y_n) \succeq g(y)$ for all n. Since

$$\begin{aligned} d(g(x_{n+1}), F(x, y)) &= d(F(x_n, y_n), F(x, y)) \\ &\le \alpha \min\{d(g(x_{n+1}), g(x_n)), d(F(x, y), g(x_n))\} + \beta \min\{d(g(x_{n+1}), g(x)), d(F(x, y), g(x))\} \\ &\quad + L \min\{d(g(x_{n+1}), g(x)), d(F(x, y), g(x_n))\} \end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality, we get $d(g(x), F(x, y)) = 0$. Hence $g(x) = F(x, y)$. Similarly, one can show that $g(y) = F(y, x)$. Thus we proved that F and g have a coupled coincidence point. This concludes the proof. □

Theorem 2.2. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ and $g : X \to X$ be mappings such that F has the mixed g-monotone property on X such that there exist two elements $x_0, y_0 \in X$ with $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Suppose there exist non-negative real numbers α, β, L with $\alpha + \beta < 1$ such that*

$$\begin{aligned} d(F(x, y), F(u, v)) &\le \alpha \min\{d(F(x, y), g(x)), d(F(u, v), g(x))\} + \beta \min\{d(F(x, y), g(u)), d(F(u, v), g(u))\} \\ &\quad + L \min\{d(F(x, y), g(u)), d(F(u, v), g(x))\} \end{aligned} \tag{13}$$

for all $(x, y), (u, v) \in X \times X$ with $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$. Further suppose $F(X \times X) \subseteq g(X)$, g is continuous nondecreasing and commutes with F, and also suppose either

(a) *F is continuous or*
(b) *X has the following property:*
 (i) *if a nondecreasing sequence $\{x_n\}$ in X converges to $x \in X$, then $x_n \preceq x$ for all n,*
 (ii) *if a nonincreasing sequence $\{y_n\}$ in X converges to $y \in X$, then $y_n \succeq y$ for all n.*

Then there exist $x, y \in X$ such that

$$F(x,y) = g(x) \quad \text{and} \quad F(y,x) = g(y),$$

that is, F and g have a coupled coincidence point $(x, y) \in X \times X$.

Proof. Following the proof of Theorem 2.1, we will get two Cauchy sequences (gx_n) and (gy_n) in X such that (gx_n) is a nondecreasing sequence in X and (gy_n) is a nonincreasing sequence in X. Since X is a complete metric space, there is $(x, y) \in X \times X$ such that $gx_n \to x$ and $gy_n \to y$. Since g is continuous, we have $g(gx_n) \to gx$ and $g(gy_n) \to gy$. First, suppose that F is continuous. Then $F(gx_n, gy_n) \to F(x, y)$ and $F(gy_n, gx_n) \to F(y, x)$. On other hand, we have $F(gx_n, gy_n) = gF(x_n, y_n) = g(gx_{n+1}) \to gx$ and $F(gy_n, gx_n) = gF(y_n, x_n) = g(gy_{n+1}) \to gy$. By uniqueness of limit, we get $gx = F(x, y)$ and $gy = F(y, x)$.

Second, suppose that (b) holds. Since $g(x_n)$ is a nondecreasing sequence such that $g(x_n) \to x$, $g(y_n)$ is a nonincreasing sequence such that $g(y_n) \to y$, and g is a nondecreasing function, we get that $g(gx_n) \preceq gx$ and $g(gy_n) \succeq g(y)$ holds for all $n \in \mathbf{N}$. By (13), we have

$$\begin{aligned} d(g(gx_{n+1}), F(x,y)) &= d(F(gx_n, gy_n), F(x,y)) \\ &\le \alpha \min\{d(ggx_{n+1}, ggx_n), d(F(x,y), ggx_n)\} + \beta \min\{d(ggx_{n+1}, gx), d(F(x,y), gx)\} \\ &\quad + L \min\{d(ggx_{n+1}, gx), d(F(x,y), ggx_n)\}. \end{aligned}$$

Letting $n \to +\infty$, we get $d(g(x), F(x, y)) = 0$ and hence $g(x) = F(x, y)$. Similarly, one can show that $g(y) = F(y, x)$. Thus we proved that F and g have a coupled coincidence point. □

Corollary 2.1. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a mapping such that F has the mixed monotone property on X such that there exist two elements $x_0, y_0 \in X$ with $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. Suppose there exist non-negative real numbers α, β and L with $\alpha + \beta < 1$ such that*

$$\begin{aligned} d(F(x,y), F(u,v)) &\le \alpha \min\{d(F(x,y), x), d(F(u,v), x)\} + \beta \min\{d(F(x,y), u), d(F(u,v), u)\} \\ &\quad + L \min\{d(F(x,y), u), d(F(u,v), x)\} \end{aligned} \tag{14}$$

for all $(x, y), (u, v) \in X \times X$ with $x \succeq u$ and $y \preceq v$ and also suppose either

(a) *F is continuous or*
(b) *X has the following property:*
 (i) *if a nondecreasing sequence $\{x_n\}$ in X converges to $x \in X$, then $x_n \preceq x$ for all n,*
 (ii) *if a nonincreasing sequence $\{y_n\}$ in X converges to $y \in X$, then $y_n \succeq y$ for all n,*

then there exist $x, y \in X$ such that

$$F(x,y) = x \quad \text{and} \quad F(y,x) = y,$$

that is, F has a coupled fixed point $(x, y) \in X \times X$.

Proof. In Theorem 2.2, if $g = I$, the identity mapping, then we have the result. □

Corollary 2.2. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ and $g : X \to X$ be mappings such that F has the mixed g-monotone property on X such that there exist two elements $x_0, y_0 \in X$ with $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Suppose that there exist a non-negative real number L such that*

$$d(F(x,y), F(u,v)) \le L \min\{d(F(x,y), g(u)), d(F(u,v), g(x))\} \tag{15}$$

for all $(x, y), (u, v) \in X \times X$ with $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$. Further suppose $F(X \times X) \subseteq g(X)$, g is continuous nondecreasing and commutes with F, and also suppose either

(a) *F is continuous or*
(b) *X has the following property:*
 (i) *if a nondecreasing sequence $\{x_n\}$ in X converges to $x \in X$, then $x_n \preceq x$ for all n,*
 (ii) *if a nonincreasing sequence $\{y_n\}$ in X converges to $y \in X$, then $y_n \succeq y$ for all n,*

then there exist $x, y \in X$ such that

$$F(x, y) = g(x) \quad \text{and} \quad F(y, x) = g(y),$$

that is, F and g have a coupled coincidence point $(x, y) \in X \times X$.

Proof. In Theorem 2.2, if $\alpha = 0 = \beta$, then we have the result. □

Corollary 2.3. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ and $g : X \to X$ be mappings such that F has the mixed g-monotone property on X such that there exist two elements $x_0, y_0 \in X$ with $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Suppose there exist non-negative real numbers α, β and L with $\alpha + \beta < 1$ such that*

$$d(F(x, y), F(u, v)) \le (\alpha + \beta) \min \begin{Bmatrix} d(F(x, y), g(x)), d(F(u, v), g(x)), \\ d(F(x, y), g(u)), d(F(u, v), g(u)) \end{Bmatrix} + L \min\{d(F(x, y), g(u)), d(F(u, v), g(x))\}$$

for all $(x, y), (u, v) \in X \times X$ with $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$. Further suppose $F(X \times X) \subseteq g(X)$, g is continuous nondecreasing and commutes with F, and also suppose either

(a) *F is continuous or*
(b) *X has the following property:*
  (i) *if a nondecreasing sequence $\{x_n\}$ in X converges to $x \in X$, then $x_n \preceq x$ for all n,*
  (ii) *if a nonincreasing sequence $\{y_n\}$ in X converges to $y \in X$, then $y_n \succeq y$ for all n,*

then there exist $x, y \in X$ such that

$$F(x, y) = g(x) \quad \text{and} \quad F(y, x) = g(y),$$

that is, F and g have a coupled coincidence point $(x, y) \in X \times X$.

Proof. Follows from Theorem 2.2, by noting that if α and β are non-negative real numbers, we have

$$(\alpha + \beta) \min \begin{Bmatrix} d(F(x, y), g(x)), d(F(u, v), g(x)), \\ d(F(x, y), g(u)), d(F(u, v), g(u)) \end{Bmatrix} \le \alpha \min\{d(F(x, y), g(x)), d(F(u, v), g(x))\} + \beta \min\{d(F(x, y), g(u)), d(F(u, v), g(u))\}. \quad \square$$

Now we shall prove the existence and uniqueness theorem of a coupled common fixed point. Note that, if $(X, \preceq)$ is a partially ordered set, then we endow the product space $X \times X$ with the following partial order:

$$\text{for } (x, y), (u, v) \in X \times X, \quad (u, v) \preceq (x, y) \Leftrightarrow x \succeq u, \quad y \preceq v.$$

Theorem 2.3. *In addition to the hypotheses of Theorem 2.1, suppose that $L = 0$ and for every $(x, y), (y^*, x^*) \in X \times X$ there exists a $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable to $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$. Then F and g have a unique coupled common fixed point, that is, there exists a unique $(x, y) \in X \times X$ such that*

$$x = g(x) = F(x, y) \quad \text{and} \quad y = g(y) = F(y, x).$$

Proof. From Theorem 2.1, the set of coupled coincidence points of F and g is non-empty. Suppose (x, y) and (x^*, y^*) are coupled coincidence points of F, that is, $g(x) = F(x, y)$, $g(y) = F(y, x)$, $g(x^*) = F(x^*, y^*)$ and $g(y^*) = F(y^*, x^*)$, then

$$g(x) = g(x^*) \quad \text{and} \quad g(y) = g(y^*). \tag{16}$$

By assumption, there exists $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable with $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$. Put $u_0 = u$, $v_0 = v$, and choose $u_1, v_1 \in X$ so that $g(u_1) = F(u_0, v_0)$ and $g(v_1) = F(v_0, u_0)$. Then, similarly as in the proof of Theorem 2.1, we can inductively define sequences $\{g(u_n)\}$, $\{g(v_n)\}$

$$g(u_{n+1}) = F(u_n, v_n) \quad \text{and} \quad g(v_{n+1}) = F(v_n, u_n) \quad \forall n.$$

Further, set $x_0 = x$, $y_0 = y$, $x_0^* = x^*$, $y_0^* = y^*$ and, on the same way, define the sequences $\{g(x_n)\}$, $\{g(y_n)\}$ and $\{g(x_n^*)\}$, $\{g(y_n^*)\}$. Then it is easy to show that

$$g(x_n) \to F(x, y)$$
$$g(y_n) \to F(y, x)$$
$$g(x_n^*) \to F(x^*, y^*),$$

and

$$g(y_n^*) \to F(y^*, x^*)$$

$\forall\, n \geq 1$. Since

$$(F(x,y), F(y,x)) = (g(x_1), g(y_1)) = (g(x), g(y))$$

and

$$(F(u,v), F(v,u)) = (g(u_1), g(v_1))$$

are comparable, then $g(x) \preceq g(u_1)$ and $g(y) \succeq g(v_1)$. It is easy to show that $(g(x), g(y))$ and $(g(u_n), g(v_n))$ are comparable, that is, $g(x) \preceq g(u_n)$ and $g(y) \succeq g(v_n)$ for all $n \geq 1$. Thus from (3), we have

$$\begin{aligned} d(g(x), g(u_{n+1})) &= d(F(x,y), F(u_n, v_n)) \\ &\leq \alpha \min\{d(F(x,y), g(x)), d(F(u_n, v_n), g(x))\} + \beta \min\{d(F(x,y), g(u_n)), d(F(u_n, v_n), g(u_n))\}. \end{aligned}$$

Since $F(x,y) = g(x)$, we have

$$d(g(x), g(u_{n+1})) \leq \beta \min\{d(g(x), g(u_n)), d(F(u_n, v_n), g(u_n))\}.$$

Hence

$$d(g(x), g(u_{n+1})) \leq \beta d(g(x), g(u_n)). \tag{17}$$

Again from (3), we have

$$\begin{aligned} d(g(v_{n+1}), g(y)) &= d(F(v_n, u_n), F(y,x)) \\ &\leq \alpha \min\{d(F(v_n, u_n), g(v_n)), d(F(y,x), g(v_n))\} + \beta \min\{d(F(v_n, u_n), g(y)), d(F(y,x), g(y))\}. \end{aligned}$$

Since $F(y,x) = g(y)$, we have

$$d(g(v_{n+1}), g(y)) \leq \alpha \min\{d(F(v_n, u_n), g(v_n)), d(g(y), g(v_n))\}.$$

Hence

$$d(g(v_{n+1}), g(y)) \leq \alpha d(g(v_n), g(y)). \tag{18}$$

From (17) and (18), we have

$$\begin{aligned} d(g(x), g(u_{n+1})) + d(g(y), g(v_{n+1})) &\leq \beta d(g(x), g(u_n)) + \alpha d(g(v_n), g(y)) \\ &\leq (\alpha + \beta)[d(g(x), g(u_n)) + d(g(y), g(v_n))] \\ &\leq (\alpha + \beta)^2[d(g(x), g(u_{n-1})) + d(g(y), g(v_{n-1}))] \\ &\quad \cdots \\ &\leq (\alpha + \beta)^{n+1}[d(g(x), g(u_0)) + d(g(y), g(v_0))]. \end{aligned}$$

Taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} [d(g(x), g(u_n)) + d(g(y), g(v_n))] = 0.$$

It implies that

$$\lim_{n\to\infty} d(g(x), g(u_n)) = \lim_{n\to\infty} d(g(y), g(v_n)) = 0. \tag{19}$$

Similarly, we show that

$$\lim_{n\to\infty} d(g(x^*), g(u_n)) = \lim_{n\to\infty} d(g(y^*), g(v_n)) = 0. \tag{20}$$

By the triangle inequality, (19) and (20),

$$\begin{aligned} &d(g(x), g(x^*)) \leq d(g(x), g(u_{n+1})) + d(g(x^*), g(u_{n+1})) \to 0 \quad \text{as } n \to \infty, \\ &d(g(y), g(y^*)) \leq d(g(y), g(v_{n+1})) + d(g(y^*), g(v_{n+1})) \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

we have $g(x) = g(x^*)$ and $g(y) = g(y^*)$. Thus we have (16). This implies that $(g(x), g(y)) = (g(x^*), g(y^*))$.

Since $g(x) = F(x,y)$ and $g(y) = F(y,x)$, by commutativity of F and g, we have

$$g(g(x)) = g(F(x,y)) = F(g(x), g(y)) \quad \text{and} \quad g(g(y)) = g(F(y,x)) = F(g(y), g(x)). \tag{21}$$

Denote $g(x) = z$, $g(y) = w$. Then from (21),

$$g(z) = F(z,w) \quad \text{and} \quad g(w) = F(w,z). \tag{22}$$

Thus (z, w) is a coupled coincidence point. Then from (21) with $x^* = z$ and $y^* = w$ it follows $g(z) = g(x)$ and $g(w) = g(y)$, that is,

$$g(z) = z \quad \text{and} \quad g(w) = w. \tag{23}$$

From (22) and (23),

$$z = g(z) = F(z, w) \quad \text{and} \quad w = g(w) = F(w, z).$$

Therefore, (z, w) is a coupled common fixed point of F and g. To prove the uniqueness, assume that (p, q) is another coupled common fixed point. Then by (21) we have $p = g(p) = g(z) = z$ and $q = g(q) = g(w) = w$. □

Corollary 2.4. *In addition to hypotheses of Corollary 2.1, suppose that $L = 0$ and for every $(x, y), (y^*, x^*) \in X \times X$ there exists a $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable to $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$. Then F has a unique coupled fixed point, that is, there exist a unique $(x, y) \in X \times X$ such that*

$$x = F(x, y) \quad \textit{and} \quad y = F(y, x).$$

Proof. In Theorem 2.3, if $g = I$, the identity mapping, then we have the result. □

Theorem 2.4. *In addition to hypotheses of Theorem 2.1, if gx_0 and gy_0 are comparable and $L = 0$, then F and g have a coupled coincidence point (x, y) such that $gx = F(x, y) = F(y, x) = gy$.*

Proof. By Theorem 2.1 we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $gx_n \to gx$ and $gy_n \to gy$, where (x, y) is a coincidence point of F and g. Suppose $gx_0 \preceq gy_0$, then it is an easy matter to show that

$$gx_n \preceq gy_n \quad \text{and } \forall n \in \mathbf{N} \cup \{0\}.$$

Thus, by (3) we have

$$\begin{aligned} d(gx_n, gy_n) &= d(F(x_{n-1}, y_{n-1}), F(y_{n-1}, x_{n-1})) \\ &\leq \alpha \min\{d(F(x_{n-1}, y_{n-1}), gx_{n-1}), d(F(y_{n-1}, x_{n-1}), gx_{n-1})\} \\ &\quad + \beta \min\{d(F(x_{n-1}, y_{n-1}), gy_{n-1}), d(F(y_{n-1}, x_{n-1}), gy_{n-1})\} \\ &= \alpha \min\{d(gx_n, gx_{n-1}), d(gy_n, gx_{n-1})\} + \beta \min\{d(gx_n, gy_{n-1}), d(gy_n, gy_{n-1})\}. \end{aligned}$$

By taking the limit as $n \to +\infty$, we get $d(gx, gy) = 0$. Hence

$$F(x, y) = gx = gy = F(y, x).$$

A similar argument can be used if $gy_0 \preceq gx_0$. □

Corollary 2.5. *In addition to hypotheses of Theorem 2.1, if x_0 and y_0 are comparable and $L = 0$, then F has a coupled fixed point of the form (x, x).*

Proof. In Theorem 2.4, if $g = I$, the identity mapping, then we have the result. □

We demonstrate Theorem 2.1 with the help of the following example.

Example 2.1. Let $X = [0, 1]$. Then $(X, \leq)$ is a partially ordered set with the natural ordering of real numbers. Let

$$d(x, y) = |x - y|$$

for $x, y \in X$. Define $g : X \to X$ by

$$g(x) = x^2$$

and $F : X \times X \to X$ by

$$F(x, y) = \begin{cases} \dfrac{x^2 - y^2}{8}, & x \geq y; \\ 0, & x < y. \end{cases}$$

Then

(1) (X, d) is a complete metric space.
(2) $g(X)$ is complete.
(3) $F(X \times X) \subseteq g(X) = X$.
(4) X satisfies (i) and (ii) of Theorem 2.1.
(5) F has the mixed g-monotone property.

(6) For any $L \in [0, +\infty)$, F and g satisfy

$$d(F(x, y), F(u, v)) \leq \frac{1}{4} \min\{d(F(x, y), g(x)), d(F(u, v), g(x))\} + \frac{1}{4} \min\{d(F(x, y), g(u)), d(F(u, v), g(u))\} + L \min\{d(F(x, y), g(u)), d(F(u, v), g(x))\}$$

for all $gx \preceq gu$ and $gy \succeq gv$.

Thus by Theorem 2.1, F and g have a coupled coincidence point. Moreover $(0, 0)$ is a coupled fixed point of F.

Proof. The proofs of (1)–(5) are clear. The proof of (6) is divided into the following cases:

Case 1. If $x \leq y$ and $u \leq v$, then

$$d(F(x, y), F(u, v)) = d(0, 0) = 0 \leq \frac{1}{4} d(0, x^2) + \frac{1}{4} d(0, u^2) + L \min\{d(0, u^2), d(0, x^2)\}.$$

Case 2. If $x < y$ and $u \geq v$, then

$$\begin{aligned} d(F(x, y), F(u, v)) &= d\left(0, \frac{u^2 - v^2}{8}\right) = \frac{u^2 - v^2}{8} \leq \frac{u^2}{8} \leq \frac{7u^2}{32} \\ &\leq \frac{1}{4}\left(\frac{7u^2 + v^2}{8}\right) = \frac{1}{4} d\left(\frac{u^2 - v^2}{8}, u^2\right) \\ &= \frac{1}{4} \min\left\{d(0, u^2), d\left(\frac{u^2 - v^2}{8}, u^2\right)\right\} \\ &\leq \frac{1}{4} \min\left\{d(0, x^2), d\left(\frac{u^2 - v^2}{8}, x^2\right)\right\} + \frac{1}{4} \min\left\{d(0, u^2), d\left(\frac{u^2 - v^2}{8}, u^2\right)\right\} \\ &\quad + L \min\left\{d(0, u^2), d\left(\frac{u^2 - v^2}{8}, x^2\right)\right\}. \end{aligned}$$

Case 3. If $x \geq y$ and $u < v$, then we have

$$\begin{aligned} d(F(x, y), F(u, v)) &= d\left(\frac{x^2 - y^2}{8}, 0\right) = \frac{x^2 - y^2}{8} \leq \frac{x^2}{8} \leq \frac{7x^2}{32} \\ &\leq \frac{1}{4}\left(\frac{7x^2 + y^2}{8}\right) = \frac{1}{4} d\left(\frac{x^2 - y^2}{8}, x^2\right) \\ &\leq \frac{1}{4} \min\left\{d\left(\frac{x^2 - y^2}{8}, x^2\right), d(0, x^2)\right\} \\ &\leq \frac{1}{4} \min\left\{d\left(\frac{x^2 - y^2}{8}, x^2\right), d(0, x^2)\right\} + \frac{1}{4} \min\left\{d\left(\frac{x^2 - y^2}{8}, u^2\right), d(0, u^2)\right\} \\ &\quad + L \min\left\{d\left(\frac{x^2 - y^2}{8}, u^2\right), d(0, x^2)\right\}. \end{aligned}$$

Case 4. If $x \geq y$ and $u \geq v$, then $v \leq y \leq x \leq u$. Hence

$$\begin{aligned} d(F(x, y), F(u, v)) &= d\left(\frac{x^2 - y^2}{8}, \frac{u^2 - v^2}{8}\right) \\ &= \frac{1}{8} |u^2 - v^2 - x^2 + y^2| \\ &= \frac{1}{8} (u^2 - x^2 + y^2 - v^2) \\ &\leq \frac{1}{8} u^2 \\ &\leq \frac{1}{4} \min\left\{d\left(\frac{x^2 - y^2}{8}, u^2\right), d\left(\frac{u^2 - v^2}{8}, u^2\right)\right\} \\ &\leq \frac{1}{4} \min\left\{d\left(\frac{x^2 - y^2}{8}, x^2\right), d\left(\frac{u^2 - v^2}{8}, x^2\right)\right\} + \frac{1}{4} \min\left\{d\left(\frac{x^2 - y^2}{8}, u^2\right), d\left(\frac{u^2 - v^2}{8}, u^2\right)\right\} \\ &\quad + L \min\left\{d\left(\frac{x^2 - y^2}{8}, u^2\right), d\left(\frac{u^2 - v^2}{8}, x^2\right)\right\}. \end{aligned}$$

In all the above cases, inequality (3) of Theorem 2.1 is satisfied for $\alpha = \beta = \frac{1}{4}$ and any $L \geq 0$. Hence by Theorem 2.1, $(0, 0)$ is a unique coupled coincidence point. Indeed for $x > y$ we have $F(y, x) = 0$ and since $F(y, x) = g(y)$ we have $y = 0$. Then $F(x, 0) = g(x)$ implies $x = 0$. The cases $x = y$ or $x < y$ are similar. □

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References

[1] S. Banach, Sur les opérations dans les ensembles absraites et leurs applications, Fund. Math. 3 (1922) 133–181.
[2] A.C.M. Ran, M.C.B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. 132 (5) (2004) 1435–1443.
[3] J.J. Nieto, R.R. Lopez, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, Order 22 (2005) 223–239.
[4] R.P. Agarwal, M.A. El-Gebeily, D. O'Regan, Generalized contractions in partially ordered metric spaces, Appl. Anal. 87 (2008) 1–8.
[5] T.G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65 (2006) 1379–1393.
[6] V. Lakshmikanthama, Lj.B. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. 70 (2009) 4341–4349.
[7] I. Altun, H. Simsek, Some fixed point theorems on ordered metric spaces and application, Fixed Point Theory Appl. 2010 (2010) Article ID 621492.
[8] A. Amini-Harandi, H. Emami, A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations, Nonlinear Anal. 72 (5) (2010) 2238–2242.
[9] L. Ćirić, N. Cakić, M. Rajović, J.S. Ume, Monotone generalized nonlinear contractions in partially ordered metric spaces, Fixed Point Theory Appl. 2008 (2008) Article ID 131294, 11 pages.
[10] Z. Drici, F.A. Mcrae, J. Vasundhara Devi, Fixed point theorems in partially ordered metric spaces for operators with PPF dependence, Nonlinear Anal. 67 (2) (2007) 641–647.
[11] J. Harjani, K. Sadarangani, Fixed point theorems for weakly contractive mappings in partially ordered sets, Nonlinear Anal. 71 (7–8) (2008) 3403–3410.
[12] J. Harjani, K. Sadarangani, Generalized contractions in partially ordered metric spaces and applications to ordianry differential equations, Nonlinear Anal. 72 (3–4) (2010) 1188–1197.
[13] H.K. Nashine, I. Altun, Fixed point theorems for generalized weakly contractive condition in ordered metric spaces, Fixed point Theory Appl. 2011 (2011) Article ID 132367, 20 pages.
[14] H.K. Nashine, B. Samet, Fixed point results for mappings satisfying (ψ, φ)-weakly contractive condition in partially ordered metric spaces, Nonlinear Anal. 74 (2011) 2201–2209.
[15] H.K. Nashine, B. Samet, C. Vetro, Monotone generalized nonlinear contractions and fixed point theorems in ordered metric spaces, Math. Comput. Modelling 54 (2011) 712–720.
[16] J.J. Nieto, R.R. Lopez, Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations, Acta Math. Sinica, Engl. Ser. 23 (12) (2007) 2205–2212.
[17] D. O'regan, A. Petrutel, Fixed point theorems for generalized contractions in ordered metric spaces, J. Math. Anal. Appl. 341 (2) (2008) 1241–1252.
[18] B. Samet, Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces, Nonlinear Anal. 72 (2010) 4508–4517.
[19] B. Samet, H. Yazidi, Coupled fixed point theorems in partially ordered ε-chainable metric spaces, TJMCS. 1 (3) (2010) 142–151.
[20] W. Shatanawi, Partially ordered cone metric spaces and coupled fixed point results, Computers Math. Appl. 60 (2010) 2508–2515.
[21] Y. Wu, New fixed point theorems and applications of mixed monotone operator, J. Math. Anal. Appl. 341 (2) (2008) 883–893.
[22] Y. Wu, Z. Liang, Existence and uniqueness of fixed points for mixed monotone operators with applications, Nonlinear Anal. 65 (10) (2006) 1913–1924.