Journal of Pure and Applied Algebra 141 (1999) 249–267

JOURNAL OF PURE AND APPLIED ALGEBRA

www.elsevier.com/locate/jpaa

A-cellular homotopy theories

Alexander Nofech

Department of Mathematics, University of Toronto, Toronto, Canada M5 S3G3

Communicated by J.D. Stasheff; received 31 July 1993; received in revised form 11 November 1997

Abstract

For the categories of pointed spaces, pointed simplicial sets and simplicial groups and for some fixed cofibrant object A there are closed model category structures in which cofibrant objects are built out of "A-cells". The A-cellular structure coincides with the usual structure when $A = S^0$ for spaces and simplicial sets, or $A = Z_{const}$ for simplicial groups. Closed-model category structures are also defined for diagrams in such a way that for diagrams over a contractible category under certain conditions the factorization of a map of diagrams into an A-cofibration followed by an A-trivial fibration commutes with *holim* up to an equivalence.

MSC: 55; 18

0. Introduction

For a given cofibrant object A of a suitable closed simplicial model category $\mathscr{C}$ (see Theorem 2.1), we define a new closed model category structure in which cofibrant objects are built by attaching A-cells of various dimensions (see Corollary 2.1.2 for their characterization). In particular, if $\mathscr{C}$ is the category of pointed spaces, then the cofibrations are relative A-CW complexes in the sense of [8, 9] and their retracts, and weak equivalences are maps that induce isomorphisms on A-homotopy. When $A = S^0$ this specializes to Quillen's closed model category structure [14].

Closed-model category structures are also constructed for diagrams in $\mathscr{C}$ in such a way that if the classifying space of the index category is contractible, with certain additional conditions the *holim* commutes with factorization into an A-cofibration followed by an A-trivial fibration up to an original equivalence.

* *E-mail address:* nofech@math.ualberta.ca (A. Nofech)

PII: S0022-4049(98)00021-8

In [11] closed-model category structures are constructed that correspond to localization with respect to maps of spaces or simplicial sets. If $* \to A$ is a cofibration, a map $f : X \to Y$ is a weak equivalence in the model category of [11] if $P_A f$, the localization of f with respect to A, is an ordinary weak equivalence. In the closed-model category considered here, f is a weak equivalence if $CW_A X$, the colocalization of A with respect to X, is an ordinary weak equivalence. In other words, [11] concentrates on the part of the homotopy theory away from A, while we concentrate on the part of A.

The rest of the introduction is devoted to the small object argument [2] which is used to factor maps into A-cofibrations followed by A-trivial fibrations. We illustrate the small object argument construction in the case of pointed topological spaces and indicate how to extend it to other simplicial categories. There is a short section on notation at the end of the introduction and the axioms of a closed model category that need to be checked are listed at the end of Section 1.

Let A be a fixed pointed cofibrant space (for example, a pointed CW-complex). A map $j : X \to Z$ is called A-cellular if it is built as a (possibly transfinite) composition of pushouts of half-smash products of A with standard inclusions of boundaries into topological simplices

$$\begin{array}{ccc} A \rtimes |\dot{\Delta}[n]| & \longrightarrow & X_i \\ {\scriptstyle Id_A \rtimes |\mathrm{i}[n]|}\downarrow & & \downarrow{\scriptstyle j_i} \\ A \rtimes |\Delta[n]| & \longrightarrow & X_{i+1}, \end{array}$$

so that $X = X_0$, $Z = X_\alpha$ for some ordinal α. The map j_i in the pushout diagram above is the inclusion of the space X_i into the space X_{i+1} obtained by attaching an n-dimensional A-cell. In the same fashion the map $j : X \to Z$ above can be seen as a relative A-CW-complex, or if $X = *$ then Z can be seen as an absolute A-CW complex. Observe that for the choice $A = S^0$ the map $A \rtimes |i[n]|$ becomes simply $|i[n]|$ and so a relative S^0-CW-complex is just a usual relative CW-complex, except that the cells are not required to be attached in the order of dimension.

The weak equivalences in the new closed model category structure are the A-equivalences: namely, maps $f : Z \to Y$ such that they induce a weak homotopy equivalence on the mapping space from A

$$Map(A, f) : Map(A, Z) \xrightarrow{\sim} Map(A, Y).$$

Again if $A = S^0$ an A-equivalence is just a weak homotopy equivalence.

Any map from X to Y can be factored into an A-cofibration followed by a fibration which is an A-equivalence: $X \xrightarrow{Cof^A} Z \xrightarrow{\sim Fib^A} Y$. (we call the second class A-trivial fibrations).

This factoring can be used to construct A-cellular approximations (A-colocalizations) of a space X, or A-localizations of X. Recall from [8, 9] that there exist maps $CW_A X \to X$ and $X \to P_A X$ so that the first is terminal up to homotopy from A-cellular spaces into X and the second is initial up to homotopy from X into spaces whose A-homotopy is trivial.

If the map to be factored is $* \to X$, the result is $CW_A X$, an A-cellular approximation of $X : * \to CW_A X \to X$. If the map is $X \to *$, the result is $P_A X$, an A-localization of X: $X \to P_A X \to *$.

In [1] Bousfield studies the notion of HZ-local space which is analogous in some ways to the notion of A-local space discussed above. He constructs a closed model category structure on the category of spaces with respect to which the map $X \to *$ is a fibration if and only if X is HZ-local. The reader should note that we do something quite different: we construct a closed model category structure on the category of pointed spaces with respect to which the map $X \to *$ is a trivial fibration if and only if X is A-local.

This topological picture can be extended to simplicial objects over an algebraic category as follows: instead of taking the topological half-smash product of a cofibrant space A with the realization of a simplicial set K, one takes the tensor product $A \otimes K$ of a cofibrant object A with a simplicial set K [14]. Hence, all the notions of A-cofibrations, A-equivalences and A-homotopy groups exist as long as the conditions of Theorem 2.1 are satisfied.

The crucial tool in the proof of existence of closed model category structures is the small object argument [2], used to construct the factorizations of the axioms CM5I and CM5II of a closed model category .

Let us illustrate how it works in the case when A is a finite CW-complex and we wish to construct a CM5I factorization of a map $f : X \to Y$.

First, take all possible commutative squares from all maps $A \rtimes |i[n]|$, which we will call generators, and form a map from their wedge to the given map $f : X \to Y$:

$$\begin{array}{ccc} \bigvee_{a,b,n} A \rtimes |\dot{\Delta}[n]| & \xrightarrow{a} & X \\ \downarrow & & \downarrow f \\ \bigvee_{a,b,n} A \rtimes |\Delta[n]| & \xrightarrow{b} & Y \end{array}$$

and then take as the "first approximation to Z" the pushout of the diagram with Y removed. We obtain a space Z_1 and the maps: $X \xrightarrow{j_1} Z_1 \xrightarrow{p_1} Y$.

Then apply the same construction to the map $p_1 : Z_1 \to Y$ and obtain Z_2, etc.

At the step ω take the colimit, which is in this case just the union of all Z_i:

$$X \xrightarrow{j_\omega} Z_\omega \xrightarrow{p_\omega} Y.$$

The map j_ω is a A-cofibration by construction, and p_ω is an A-trivial fibration since it has the RLP, the right lifting property, with respect to all the generators $A \rtimes |i[n]|$ (we use the notation $(A \rtimes |i[n]|) \nearrow p_\omega$).

The right lifting property for the map p_ω with respect to generators $A \rtimes |i[n]|$ follows from the fact that the natural map

$$k : \varinjlim_{i<\omega} Map_*(A \rtimes |i[n]|, p_i) \xrightarrow{\cong} Map_*(A \rtimes |i[n]|, \varinjlim_{i<\omega} p_i).$$

is a bijection by compactness of $A \rtimes |\Delta[n]|$ and the existence of a countable open cover for the pair (Z_ω, X), see Lemma 3.8 for details.

Roughly, the reason for this is that a map, which is actually a commutative square, from the generator into the right side of the bijection above, factors through some $p_n : Z_n \to Y$ and from the construction of $p_{n+1} : Z_{n+1} \to Y$ it follows that the given commutative square is already "glued into" Z_{n+1} and so the lifting exists into the next stage Z_{n+1}.

We say that in this case the small object argument converges at the step ω.

We use the notion of "s-definiteness" defined in [2, 4.2], though not strictly in the sense of "smallness" (since compact spaces and compactly generated spaces are not small) to describe the property of the domains and ranges of the generating maps which causes the small object argument to converge. This property of an object A is the existence of a cardinal γ such that for any γ-sequence Y_α in the category which is continuous at limit ordinals in the sense of colimits, there is a bijection

$$\varinjlim_{\alpha<\gamma} Hom_{\mathscr{C}}(A, Y_\alpha) \xrightarrow{\cong} Hom_{\mathscr{C}}(A, \varinjlim_{\alpha<\gamma} Y_\alpha).$$

See Section 3.1. for additional conditions on the sequence Y_α in the topological case.

We prove in Section 3.1 that the small object argument converges at a step which is a regular cardinal when A a compactly generated space. It would be interesting to know if the least ordinal at which the small object argument construction converges is always a regular cardinal.

By "generators of A-cofibrations" we mean a set of maps that are used together with the "generators of trivial cofibrations" at each step of the small object argument to construct a CM5I factorization of a map into an A-cellular map followed by a map with a right lifting property with respect to the generators. This gives a factoring of map into an A-cofibration followed by an A-trivial fibration if both objects are fibrant. In case the objects of $\mathscr{C}$ are not all fibrant, and the category $\mathscr{C}$ is proper one applies this procedure to the induced map of fibrant models.

The CM5II factorization into a trivial cofibration followed by a fibration remains unchanged and in the case of pointed topological spaces uses the generators:

$$|i[n,k]|_+ : |V[n,k]|_+ \to |\Delta[n]|_+$$

where $V[n,k]$ is the "simplicial horn" obtained by removing the interior and the k-th face of a n-simplex [14].

When the generators form a set, one can use the small object argument with their coproduct as the single generator. Then the construction converges at a cardinal that does not depend on the map that is being factorized. Since the whole construction is functorial, this results in functorial factorizations. In what follows we always assume that the generators form a set.

We use the existence in the original category of adjoint functors $\_ \otimes K$, $(\_)^K$, where K is a not necessarily finite simplicial set, which satisfies the conditions of [14, Part II], to define the generators of cofibrations in Theorem 2.1, and to define an analog of the homotopy inverse limit for an I-diagram $\underline{X}$ over an arbitrary closed simplicial model category : $holim\,\underline{X} = \underset{\longleftarrow}{\lim}\,(X_i)^{I/j}$ where the simplicial diagram $(I/-)$ is defined as in [4].

In the category of pointed spaces examples of localizations and colocalizations with respect to a cofibrant space A include the Quillen plus construction, Postnikov pieces and n-connected covers, but not localization with respect to homology (see [8, 5]).

0.1. Notation

Throughout CW_AX and P_AX stand, respectively, for colocalization and localization of an object X with respect to an object A. A lower bar indicates diagrams or morphisms of diagrams and a lower case "+" stands for a space or a diagram with a disjoint basepoint. As usual LLP and RLP stand for "left lifting property" and "right lifting property" and we also use the shortened "$i \nearrow \varphi$" for "i has the LLP with respect to φ", or equivalently "φ has the RLP with respect to i", applying this to classes of maps as well.

We often specify the kind of a weak equivalence (cofibration, fibration) such that for example $\underline{X} \xrightarrow{\sim A} \underline{Y}$ means a weak A-equivalence of diagrams and $X \xrightarrow{Cof^A} Y$ means an A-cofibration with respect to a cofibrant object A.

If $\mathscr{C}$ is a category and $A \in |\mathscr{C}|$ is a cofibrant object, $\mathscr{C}^A$ will the A-cellular closed model category structure on $\mathscr{C}$.

1. *A*-cellular closed-model category structures

In the following definition $Hom(X, Y)$ denotes the simplicial function complex of [7]. We define weak equivalences as morphisms inducing equivalences on simplicial function complexes from A into fibrant approximations of objects, leave the fibrations unchanged and define the cofibrations by LLP.

Definition 1.0. Let $\mathscr{C}$ be a pointed closed simplicial model category [14, Part II, Section 1] and a A a cofibrant object of $\mathscr{C}$. We call $\mathscr{C}^A$ an A-cellular closed model category structure if the weak equivalences, fibrations and cofibrations in $\mathscr{C}^A$ are defined as follows (the subscript f denotes fibrant approximation, and W_{sSets} is the class of weak equivalences of simplicial sets):

$$W_{\mathscr{C}^A} = \{\varphi \mid \underline{Hom}(A, \varphi_{\mathrm{f}}) \in W_{s\,Sets}\},$$
$$Fib_{\mathscr{C}^A} = Fib_{\mathscr{C}},$$
$$Cof_{\mathscr{C}^A} = \{j \mid j \nearrow (W_{\mathscr{C}^A} \cap Fib_{\mathscr{C}^A})\}.$$

We will call the new cofibrations and the new weak equivalences, respectively A-cofibrations and A-equivalences. It follows from the definitions that weak equivalences in $\mathscr{C}$ are A-equivalences for any cofibrant A.

In the following theorem A-cellular closed-model category structures are constructed for six categories: pointed spaces, pointed simplicial sets, simplicial groups and diagrams of such. These closed model category structures have the property that under suitable assumptions the homotopy limit functor (say from diagrams of spaces to spaces) commutes with a factorization of a map into a cofibration followed by a trivial fibration up to an original equivalence (see Section 5).

Theorem 1.1. *There exist A-cellular closed simplicial model category structures in the following categories*:

(1) *In the category $\mathcal{T}_*$ of pointed spaces with respect to a cofibrant space A*;

(2) *In the category $\mathcal{S}_*$ of pointed simplicial sets with respect to a pointed simplicial set*;

(3) *In the category sGr of simplicial groups with respect to a cofibrant simplicial group*;

(4) *In the category $(\mathcal{T}_*)^{\mathcal{I}}$ of I-diagrams in $\mathcal{T}_*$ with respect to $A \wedge (I/-)_+$, where $(I/-)_+$ is the overcategory with a disjoint basepoint, see* [4].

(5) *In the category $(\mathcal{S}_*)^{\mathcal{I}}$ of diagrams of pointed simplicial sets over I, a small category, with respect to the diagram $A \wedge (I/-)_+$.*

(6) *In the category* s*Gr*^{I} *of I-diagrams of simplicial groups, with respect to $A \otimes (I/-)$, where A is a cofibrant simplicial group.*

For convenience we reproduce the axioms that need to be checked [15]:

(CM1) *The category is closed under finite limits and finite colimits*;
(CM2) *If in a commutative triangle $\gamma = \alpha\cdot\beta$ two of the maps are weak equivalences, then so is the third*;
(CM3) *The three classes of fibrations, cofibrations and weak equivalences are closed under retractions*;
(CM4I) *Cofibrations have LLP with respect to trivial fibrations*;
(CM4II) *Trivial cofibrations have LLP with respect to fibrations*;
(CM5I) *Any map can be factored as a cofibration followed by a trivial fibration*;
(CM5II) *Any map can be factored as a trivial cofibration followed by a fibration.*

2. An existence theorem

We suppose the existence in $\mathscr{C}$ of a set of "generators of trivial cofibrations", i.e. trivial cofibrations $\{t_j\}$ such that a morphism φ is a fibration if and only if $\{t_j\} \nearrow \varphi$, with s-definite domains and codomains. We will also suppose that the original closed-model category $\mathscr{C}$ is *proper*, which means that a pushout of a weak equivalence along a cofibration is a weak equivalence, and a pullback of a weak equivalence along a fibration is also a weak equivalence [3, Definition 1.2 and Appendix A].

Theorem 2.1. *Let $\mathscr{C}$ be a pointed closed simplicial model category with arbitrary colimits and with a set $\{t_j\}$ of generators of trivial cofibrations. Suppose that $\mathscr{C}$ is* *proper* (*or if it is not then all its objects are fibrant*). *Let* A *be a cofibrant*, s-*definite object.*[1] *Then there exists an* A-*cellular closed model category structure* $\mathscr{C}^{\mathscr{A}}$, *as in Definition* 1.0, *which admits functorial factorizations.*

Proof. The proof consists of two steps: first, one proves the existence of the closed model category structure $\mathscr{C}_{\mathrm{f}}^{\mathscr{A}}$ for the subcategory $\mathscr{C}_{\mathrm{f}}$ of fibrant objects, then one uses the characterization of trivial fibrations and of cofibrations and goes through the proof again extending the closed model category structure to all of the category $\mathscr{C}$. At the first reading the fibrant approximation of any object is supposed to be the object itself, and the CM5I factorization is obtained by a straightforward use of the small object argument .

CM1, CM2, CM3 and CM4I are immediate.

CM4II: We need to show the existence of a lifting in a diagram:

$$\begin{array}{ccc} B & \longrightarrow & X \\ {\scriptstyle \sim A}\downarrow{\scriptstyle Cof^A} & & \downarrow{\scriptstyle Fib^A} \\ C & \longrightarrow & Y. \end{array} \tag{1}$$

Since $Fib^A = Fib$, it would be enough to show that a trivial A-cofibration is a trivial cofibration. To do this consider a CM5II factorization of the left vertical map in $\mathscr{C}$

$$\begin{array}{ccc} B & \xrightarrow{\sim Cof} & B' \\ {\scriptstyle \sim A}\downarrow{\scriptstyle Cof^A} & & {\scriptstyle \sim A}\downarrow{\scriptstyle Fib^A} \\ C & \xrightarrow{Id} & C. \end{array} \tag{2}$$

The vertical map on the right-hand side of Eq. (2) is an A-equivalence since the upper horizontal map is an equivalence, hence an A-equivalence. The left vertical map is an A-cofibration, hence a lifting $C \to B'$ exists. This represents the map $B \to C$ as a retract of the map $B \to B'$, which is a trivial cofibration in $\mathscr{C}$. Since trivial cofibrations in $\mathscr{C}$ are closed under retracts, it follows that $B \to C$ is a trivial cofibration in $\mathscr{C}$ and this implies the existence of a lifting $C \to X$ in Eq. (1).

CM5I: Let $i[n] : \dot{\Delta}[n] \to \Delta[n]$ be the usual inclusions, and t_j the generators of trivial cofibrations in $\mathscr{C}$. For an arbitrary map $f : X \to Y$ take a factorization $X_{\mathrm{f}} \xrightarrow{i} Z'' \xrightarrow{\varphi_{\mathrm{f}}} Y_{\mathrm{f}}$ of the induced map of fibrant approximations (see diagram below), obtained by using the set of morphisms $\{A \otimes i[n], t_j\}$ at every successor step of the small object argument. Convergence follows from the s-definiteness of $A \otimes K$ for a finite simplicial set K, which can be seen by applying the functor $Hom_{\mathscr{C}}(K,)$ to the isomorphism in the definition of s-definiteness of A.

The second map φ_{f} is a fibration since it has the RLP with respect to t_j's. To see that φ_{f} is an A-equivalence note that from $A \otimes i[n] \nearrow \varphi_{\mathrm{f}}$ it follows that $i[n] \nearrow Hom_{\mathscr{C}}(A, \varphi_{\mathrm{f}})$, hence this map is a trivial fibration and so $Hom_{\mathscr{C}}(A, \varphi_{\mathrm{f}})$ is an weak equivalence. Hence the pullback $\varphi' : Z' \to Y$ of φ_f is a fibration and an A-equivalence. Now factor the map $X \to Z'$ to obtain a cofibration $j : X \to Z$ followed by a trivial

[1] see [1, 4.2] and the introduction.

fibration. Since $\mathscr{C}$ is proper, the characterization of A-cofibrations below (Part 3) implies that j is an A-cofibration.

$$\begin{array}{ccccccc}
X & \xrightarrow{j} & Z & \xrightarrow[\sim]{Fib} & Z' & \xrightarrow[\sim A]{Fib} & Y \\
{\scriptstyle\sim}\downarrow{\scriptstyle Cof} & & \downarrow & & {\scriptstyle\sim}\downarrow & & {\scriptstyle\sim}\downarrow{\scriptstyle Cof} \\
X_{\mathrm{f}} & \xrightarrow{Cof^A} & Z'' & \xrightarrow{=} & Z'' & \xrightarrow[\sim A]{Fib} & Y_{\mathrm{f}}
\end{array}$$

CM5II – follows from CM5II in $\mathscr{C}$, since the fibrations and the trivial cofibrations are the same. □

Now we give the characterizations of A-trivial fibrations and A-cofibrations in case there are non-fibrant objects in $\mathscr{C}$.

Characterization of A-trivial fibrations: A fibration ϕ is in $Fib_{\mathscr{C}} \cap W_{\mathscr{C}^A}$ if and only if it fits in a diagram

$$\begin{array}{ccc}
X & \xrightarrow[Fib]{\phi} & Y \\
{\scriptstyle i_X}\downarrow{\scriptstyle \sim Cof} & & {\scriptstyle i_Y}\downarrow{\scriptstyle \sim Cof} \\
X_{\mathrm{f}} & \xrightarrow[Fib]{\phi_{\mathrm{f}}} & Y_{\mathrm{f}}
\end{array}$$

where i_X and i_Y are trivial cofibrations and ϕ_{f} is an A-trivial fibration in $\mathscr{C}_{\mathrm{f}}$.

Proof. Factor the fibrant approximation of ϕ into a trivial cofibration followed by a fibration and absorb the first term into i_X. □

Characterization of A-cofibrations: The following are equivalent:

(1) The map $j : X \to Y$ has the LLP with respect to all A-trivial fibrations: $j \nearrow (Fib_{\mathscr{C}} \cap W_{\mathscr{C}^A})$.

(2) The map $j : X \to Y$ is a cofibration in $\mathscr{C}$ and fits in a diagram

$$\begin{array}{ccc}
X & \xrightarrow{j} & Y \\
{\scriptstyle 1_X}\downarrow & & {\scriptstyle \sim Cof}\downarrow{\scriptstyle i_Y} \\
X & \xrightarrow{k} & Y_{\mathrm{f}}
\end{array}$$

where i_Y is a trivial cofibration, Y_{f} is fibrant, and k is a retract of an A-cellular map;

(3) The map $j : X \to Y$ is a cofibration and fits in a diagram above where i_Y is an equivalence, but not necessarily a cofibration.

Proof. (1) ⇒ (2) Let j have the required LLP and let k be the pushout of j along a trivial cofibration $i_X : X \to X_{\mathrm{f}}$ with X_{f} fibrant

$$\begin{array}{ccc}
X & \xrightarrow{j} & Y \\
{\scriptstyle i_X}\downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\
X_{\mathrm{f}} & \xrightarrow{k} & P
\end{array}$$

The right vertical map is an equivalence by properness, since j is clearly a cofibration in $\mathscr{C}$, and k has the same LLP as j. Now consider a factoring of a fibrant approximation $j_{\mathrm{f}}: X_{\mathrm{f}} \to Y_{\mathrm{f}}$ into an A-cellular map i_{f} followed by an A-trivial fibration φ_{f}:

$$X \xrightarrow{i_X} X_{\mathrm{f}} \xrightarrow{i_{\mathrm{f}}} Z_{\mathrm{f}} \xrightarrow{\varphi_{\mathrm{f}}} Y_{\mathrm{f}}.$$

Comparing the two factorings:

$$\begin{array}{ccccc} X_{\mathrm{f}} & \xrightarrow{k} & P & \xrightarrow{\sim} & Y_{\mathrm{f}} \\ {\scriptstyle 1_{X_{\mathrm{f}}}}\downarrow & & \downarrow & & \downarrow{\scriptstyle 1_{Y_{\mathrm{f}}}} \\ X_{\mathrm{f}} & \xrightarrow{i_{\mathrm{f}}} & Z_{\mathrm{f}} & \xrightarrow{\varphi_f} & Y_{\mathrm{f}} \end{array}$$

and using[2] Theorem 4.4 in $\mathscr{C}_{\mathrm{f}}^{A}$, we see that the A-trivial fibration φ_{f} is a trivial fibration, and there exists a lifting in a diagram

$$\begin{array}{ccc} X & \xrightarrow{j_f \circ i_X} & Z_{\mathrm{f}} \\ {\scriptstyle i_Y \circ j}\downarrow & & \downarrow{\scriptstyle \varphi_{\mathrm{f}}} \\ Y_{\mathrm{f}} & \xrightarrow{1_{Y_{\mathrm{f}}}} & Y_{\mathrm{f}} \end{array}$$

implying (2), because $i_Y \circ j = k$ is a retract of $i_{\mathrm{f}} \circ i_X$, an A-cellular map. Note that ordinary trivial cofibrations are retracts of A-cellular maps in a trivial way, since we add their generators to $A \otimes i[n]$, the generators of A-cofibrations.

(2) $\Rightarrow$ (3): obvious.

(3) $\Rightarrow$ (1): Consider a commutative square

$$\begin{array}{ccc} X & \xrightarrow{h} & Z \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \varphi} \\ Y & \xrightarrow{t} & W \end{array}$$

with j as in (3) and φ an A-trivial fibration. We need to show the existence of a lifting in this square. Consider a cubic diagram obtained by attaching the sides:

$$\begin{array}{ccc} X & \xrightarrow{j} & Y \\ {\scriptstyle 1_X}\downarrow & & \downarrow{\scriptstyle \sim Cof\ i_Y} \\ X & \xrightarrow{k} & Y_{\mathrm{f}} \end{array} \qquad \begin{array}{ccc} Z & \xrightarrow{\phi} & W \\ {\scriptstyle i_Z}\downarrow{\scriptstyle \sim Cof} & & \downarrow{\scriptstyle \sim Cof\ i_W} \\ Z_{\mathrm{f}} & \xrightarrow{\phi_{\mathrm{f}}} & W_{\mathrm{f}} \end{array}$$

(The left-hand side is given by (3), and the right follows from the characterization of A-trivial fibrations). Since we do not demand that i_Y be a cofibration, we need to show the existence of a map $t_{\mathrm{f}}: Y_{\mathrm{f}} \to W_{\mathrm{f}}$ so that the cubic diagram commutes. This follows from the existence of a lifting in a commutative square:

$$\begin{array}{ccc} X & \xrightarrow{i_W \circ \varphi \circ h} & W_{\mathrm{f}} \\ {\scriptstyle k}\downarrow & & \downarrow \\ Y_{\mathrm{f}} & \longrightarrow & * \end{array}$$

[2] It is applicable, since $\mathscr{C}_{\mathrm{f}}^{\mathscr{A}}$ exists.

by applying the Lemma 2.1.1 to the category of objects under X, where k is a cofibrant, $i_W \circ \varphi \circ h$ is a fibrant, and the zigzag from the first to the second given by the cubic diagram implies the existence of an actual lifting in the square above by Lemma 2.1.1.

Now, a lifting l in a diagram

$$\begin{array}{ccc} X & \longrightarrow & Z_{\mathrm{f}} \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle \varphi_{\mathrm{f}}} \\ Y_{\mathrm{f}} & \longrightarrow & W_{\mathrm{f}} \end{array}$$

exists, since k is a retract of an A-cellular map and φ_{f} is an A-trivial fibration of fibrant objects. Next, consider the commutative square given in the beginning as two objects under X and over W: a cofibrant object $X \xrightarrow{j} Y \xrightarrow{t} W$ and a fibrant object $X \xrightarrow{h} Z \xrightarrow{\varphi} W$. The lifting above gives a zigzag from the cofibrant to the fibrant in the following way: let P be the pullback obtained by using the lifting l, as seen in the diagram

$$\begin{array}{ccc} P & \xrightarrow{\beta} & Z \\ {\scriptstyle \alpha}\downarrow{\scriptstyle \sim} & & {\scriptstyle \sim}\downarrow{\scriptstyle i_Z} \\ Y & \xrightarrow{l \circ i_Y} & Z_{\mathrm{f}}. \end{array}$$

Now, use the map $X \to P$, the equivalence α and the map β to construct a zigzag in $(X \downarrow \mathscr{C} \downarrow W)$:

$$\begin{array}{ccccc} X & \xrightarrow{1_X} & X & \xleftarrow{1_X} & X \\ {\scriptstyle j}\downarrow & & \downarrow & & \downarrow{\scriptstyle h} \\ Y & \xleftarrow[\sim]{\alpha} & P & \xrightarrow{\beta} & Z \\ {\scriptstyle t}\downarrow & & {\scriptstyle \varphi \circ \beta}\downarrow & & \downarrow{\scriptstyle \varphi} \\ W & \xleftarrow{1_W} & W & \xrightarrow{1_W} & W \end{array}$$

and we use the lemma that follows again to conclude the proof. □

Lemma 2.1.1. *Suppose that in the commutative square j is a cofibration, φ is a fibration, and suppose also that in this square*

$$\begin{array}{ccc} X & \longrightarrow & V \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \varphi} \\ Y & \longrightarrow & W \end{array}$$

considered as two objects in $(X \downarrow \mathscr{C} \downarrow W)$, there exists a zigzag from the cofibrant object $X \to Y \to W$ to the fibrant object $X \to V \to W$ where the maps going left are equivalences. Then there exists a lifting $Y \to V$ so that the diagram commutes. (Note: the given condition is different from the existence of a lifting up to homotopy, and implies the existence of two commutative triangles for every morphism in the zigzag.)

Proof. Call the first object j and the second φ. The category $(X \downarrow \mathscr{C} \downarrow W)$ inherits a closed model category structure from $\mathscr{C}$ by [14, Part I]. By Theorem 4.4 of [7] the simplicial function complex $L^H(j,\varphi)$ is weakly equivalent to $(X \downarrow \mathscr{C} \downarrow W)(j,\varphi_*)$, where φ_* is a fibrant simplicial resolution of φ. Since φ is fibrant, in its fibrant simplicial resolution the term φ_0 can be chosen as φ itself, and so the set $(X \downarrow \mathscr{C} \downarrow W)(j,\varphi)$, as the 0th dimension of a non-empty simplicial set $L^H(j,\varphi)$, is non-empty. □

A possible choice of functorial factorizations in $\mathscr{C}^A$ follows from the fact that the generators of both cofibrations and trivial cofibrations form a set, and the construction of CM5I factorization is functorial.

Corollary 2.1.2. *An object is A-cofibrant if and only if it admits a trivial cofibration into a retract of an A-cellular object.*

Remark 2.2. A-cofibrant objects are cofibrant in $\mathscr{C}$. This is a special case of $Cof_{\mathscr{C}^A} \subseteq Cof_{\mathscr{C}}$ which follows from

$$W_{\mathscr{C}} \cap Fib_{\mathscr{C}} \subseteq W_{\mathscr{C}^A} \cap Fib_{\mathscr{C}^A}.$$

The fibrant objects in $\mathscr{C}$ and $\mathscr{C}^A$ are the same.

Theorem 2.3. *Let I be a small category, and let A be a cofibrant object of a closed simplicial model category* $\mathscr{C}$ *as in Theorem* 2.1, *with arbitrary limits and with one additional condition*: *there exists an s-definite object* $\mathscr{S}^0$ *such that*

$$Hom_{\mathscr{C}}(\mathscr{S}^0, f) \in W_{\mathscr{S}} \Rightarrow f \in W_{\mathscr{C}}.$$

Then there exists a closed model category structure on the category of I-diagrams in $\mathscr{C}$ *in which*

$$W_{(\mathscr{C}^A)^I} = \{\underline{f} | holim \underline{f}) \in W_{\mathscr{C}^A}\},$$
$$Fib_{(\mathscr{C}^A)^I} = \{\underline{\varphi} | \underline{\varphi}_i \in Fib_{\mathscr{C}}\},$$
$$Cof_{(\mathscr{C}^A)^I} = \{\underline{j} | \underline{j} \nearrow (W_{(\mathscr{C}^A)^I} \cap Fib_{(\mathscr{C}^A)^I}\}.$$

Proof. The first step is to construct an underlying closed simplicial model category structure in which the weak equivalences and fibrations are defined objectwise. This is given by Theorem 2.2 of [6] with orbits being $\mathscr{S}^0 \otimes \underline{F}_i$, where $\underline{F}_i$ is the free discrete simplicial diagram corresponding to the object $i \in I$.

The second step is to apply Theorem. 2.1 to this underlying closed simplicial model category structure choosing $A \otimes (I/-)$ as the localizing object. We need to check that $A \otimes (I/-)$ is s-definite and cofibrant in $\mathscr{C}^I$. The first follows from Lemma 3.6, and the second from the adjunction

$$\underline{H}om_{\mathscr{C}^{\mathscr{I}}}(A \otimes (I/-), \underline{f}) \cong holim\, \underline{Hom}_{\mathscr{C}}(A, \underline{f})$$

The generators of trivial cofibrations are $\mathscr{S}^0 \otimes \underline{F}_i \otimes i[n,k]$. □

Example 2.4. In the categories $\mathscr{T}_*$, $\mathscr{S}_*$ the $\mathscr{S}^0$ is just S^0, in the category sGr it is the free simplicial group FS^0 =constant$Z$, in the category of I-diagrams it is $\mathscr{S}^0 \otimes \coprod_i \underline{F}_i$.

Remark 2.5 (*Dwyer and Kan* [7, 4.3]). it follows that a simplicial resolution in $\mathscr{C}$ can serve as one in $\mathscr{C}^A$. Hence, in the subcategory of A-cofibrants the original function complexes and simplicial structure are compatible with the new closed model category structure.

3. The proof of Theorem 1.1

The proof consists of checking the conditions of Theorem 2.1 for five cases: pointed spaces, pointed simplicial sets, simplicial groups, diagrams of pointed spaces and of pointed simplicial sets.

3.1. The category $\mathscr{T}_*$ of pointed topological spaces.

In this section we will use the term "s-definiteness" in a specific restricted sense, since compact spaces are not small. Namely, a pointed space K will be called s-definite if there exists a cardinal β such that

$$\varinjlim_{\alpha<\beta} Hom_{\mathscr{T}_*}(K, Y_\alpha) \xrightarrow{\cong} Hom_{\mathscr{T}_*}(K, \varinjlim_{\alpha<\beta} Y_\alpha). \tag{1}$$

where Y_α is a diagram in $\mathscr{T}_*$ indexed by $Seq[\beta]$, the order category of ordinals less then β, whose values at limit ordinals are colimits of values at smaller ordinals with the additional conditions that are discussed in Lemma 3.8. These conditions are satisfied for transfinite sequences that arise from the small object argument construction when the generating map is as in Lemma 3.8.

Let A be a cofibrant pointed space. We need to show that A is s-definite. First we consider the case of a compact A. Then the small object argument converges in each case for the cardinal $\beta = \omega$, see [14]. This is also a special case of Lemma 3.8 where convergence is proved for an arbitrary limit ordinal.

Now consider the general case. A cofibrant pointed space A is a retract of a CW complex and as such a compactly generated space. Let $A = \varinjlim_{i \in I} K_i$, where K_i is the diagram of compact subsets of A and their inclusions. In this case take β a regular cardinal large enough so that inverse limits of cardinality $|Morph(I)|$ commute with colimits of sequences of length β (see Lemma 3.5.) and use the convergence for each K_i.

To show that the small argument converges for the cardinal β we need the following canonical map to be an isomorphism (as sets):

$$k : \varinjlim_{\alpha<\beta} Hom_{\mathscr{T}_*}(A, X_\alpha) \xrightarrow{\cong} Hom_{\mathscr{T}_*}(A, \varinjlim_{\alpha<\beta} X_\alpha).$$

For the left-hand side we have isomorphisms:

$$\varinjlim_{\alpha<\beta} Hom_{\mathscr{T}_*}(A, X_\alpha) \cong \varinjlim_{\alpha<\beta} Hom_{\mathscr{T}_*} \varinjlim_{i} K_i, X_\alpha) \cong \varinjlim_{\alpha<\beta} \varprojlim_{i} Hom_{\mathscr{T}_*}(K_i, X_\alpha).$$

On the right-hand side we have

$$Hom_{\mathcal{T}_*}(A, \varinjlim_{\alpha<\beta} X_\alpha) \cong Hom_{\mathcal{T}_*} \varinjlim_{i} K_i, \varinjlim_{\alpha<\beta} X_\alpha)$$
$$\cong \varprojlim_{i} Hom_{\mathcal{T}_*}(K_i \varinjlim_{\alpha<\beta} X_\alpha) \cong \varprojlim_{i} \varinjlim_{\alpha<\beta} Hom_{\mathcal{T}_*}(K_i, X_\alpha),$$

where the last one follows from compactness of K_i and Lemma 3.8. Now take β to be a regular cardinal such that β is (a) large enough for the convergence of the small object argument and (b) larger than the cardinality of the category of compact susbsets of A and use Lemma 3.5. on the interchange of small limits and "large and regular" filtered colimits. This lemma is proved for simplicity's sake only for (transfinite) sequences, but can be generalized in a form similar to Theorem 1, p. 211 [12].

The generators of trivial cofibrations are the maps $|i[n,k]| : |V[n,k]| \to |\Delta[n]|$.

3.2. The categories $\mathcal{S}_*$ of pointed simplicial sets and sGr of simplicial groups.

The s-definiteness of any simplicial set was proved in [2]. In this case the factorization in CM5I is constructed using the maps

$$A \rtimes i[n] : A \rtimes \dot{\Delta}[n] \to A \rtimes \Delta[n], \qquad i[n,k] : V[n,k] \to \Delta[n]$$

and the factorization of CM5II uses as usual $i[n,k] : V[n,k] \to \Delta[n]$.

The category sGr of simplicial groups:
All objects of this, category are s-definite by Lemma 3.6, the s-definiteness of Gr was proved in [2]. Let A be a cofibrant simplicial group. Then there exists a closed simplicial model category structure in sGr in which

$$W_A = \{f \mid Hom_{sGr}(A, f) \in W_{\mathcal{S}}\}.$$

3.3. The category $\mathcal{T}_*^I$ of diagrams of pointed spaces over a small category I.

For a cofibrant pointed space A we take as a localizing object the diagram $FA = A \rtimes_{\mathcal{K}} |(I/-)|$, where the subscript $\mathcal{K}$ denotes the compactly generated topology. Note that by adjunction

$$Hom_{\mathcal{T}_*^I}(FA, \underline{X}) \cong Hom_{\mathcal{T}_*}(A, holim\ \underline{X}),$$

where we use the compactly generated version of the ordinary homotopy inverse limit. Hence the equivalences in $(\mathcal{T}_*^I)^A$ are those and only those maps of diagrams that induce an A-equivalence on *holim*.

The s-definiteness of FA (with the same restrictions as in Section 3.1) follows from Lemma 3.6. and the s-definiteness of any cofibrant space in $\mathcal{T}_*$.

The generating maps of trivial cofibrations are the maps of diagrams

$$|(Id_{\underline{F}_i})_+ \wedge i[n,k]_+| : |(\underline{F}_i)_+ \wedge V[n,k]_+| \to |(F_i)_+ \wedge \Delta[n]_+|,$$

where $\underline{F}_i$ are the free orbit diagrams. The initial closed simplicial model category structure is given by

$$Fib_{\mathscr{T}_*^I} = \{\underline{f} \mid (\underline{F}_i)_+ \wedge i[n,k]_+ \nearrow \underline{f}\},$$
$$Fib_{\mathscr{T}_*^I} \cap W_{\mathscr{T}_*^I} = \{\underline{f} \mid (\underline{F}_i)_+ \wedge i[n,k]_+ \mid \nearrow \underline{f}, (\underline{F}_i)_+ \wedge i[n]_+ \nearrow \underline{f}\}.$$

This structure is related to the one described in [4, p. 314] in the same way as the closed simplicial model category structure of pointed spaces is related to that of pointed simplicial sets.

In the new closed simplicial model category structure fibrations are the same and the A-trivial fibrations are

$$Fib_{(\mathscr{T}_*^A)^I} \cap W_{(\mathscr{T}_*^A)^I} = \{\underline{f} \mid A \wedge |(I/-)_+ \wedge i[n]_+| \nearrow \underline{f},\ |(\underline{F}_i)_+ \wedge i[n,k]_+| \nearrow \underline{f}\}.$$

3.4. The category $\mathscr{S}_*^{\mathscr{I}}$ of pointed simplicial diagrams over a small category I.

In this case the initial closed simplicial model category structure is the one of [4, p. 314]. The localizing object is $FA = A \rtimes (I/-)$. Its s-definiteness follows from Lemma 3.6. and from s-definiteness of simplicial sets. The A-equivalences are exactly those maps of diagrams that induce an A-equivalence on $|holim_{-}|$.

Lemma 3.5. *Let β be a regular cardinal. If $seq[\beta]$ is a category with one object for any ordinal $\alpha < \beta$ and one morphism $\alpha \to \gamma$ for $\alpha < \gamma < \beta$ and P is a category with*

$$|Morph\, P| < \beta,$$

then for any bifunctor $F : P \times seq[\beta] \mapsto Sets$ the canonical map

$$k : \varinjlim_{\alpha} \varprojlim_{p} F(p,\alpha) \xrightarrow{\cong} \varprojlim_{p} \varinjlim_{\alpha} F(p,\alpha)$$

is an isomorphism.

Proof. Use the construction of colimits from coproducts

$$\varinjlim_{\alpha} F(p,\alpha) \cong \coprod_{\alpha} F(p,\alpha)/\sim,$$

where the equivalence relation is: $x \sim x'$ for $x \in F(p,\alpha)$, $x' \in F(p,\alpha')$ if and only if $F(p,(\alpha \to \gamma))(x) = F(p,(\alpha' \to \gamma))(x')$ for some ordinal γ such that $\alpha < \gamma$, $\alpha' < \gamma$. The proof for P finite in [12, p. 212] consists of finding a bound for a finite set of p's and morphisms between them, and this is assured in our case by regularity of the cardinal β. □

Lemma 3.6. *For a small category I an I-diagram in $\mathscr{C}$ with s-definite objects is itself s-definite in the category of diagrams $\mathscr{C}^I$.*

Proof. Let W be the subdivision category of I, and let $\underline{A}$, $\underline{X}$ be objects of $\mathscr{C}^I$. Then

$$Hom_{\mathscr{C}^I}(\underline{A},\underline{X}) \cong \varprojlim_{W} Hom_{\mathscr{C}}(A_i,X_j).$$

Let γ_i be a limit ordinal corresponding to A_i as in Section 3.1. Choose β to be a regular cardinal greater then $sup\{\gamma_i, card(W)\}$, and apply Lemma 3.5. □

Corollary 3.7. *For a category $\mathscr{C}$ with s-definite objects, the objects of $s\mathscr{C}$ are also s-definite.*

Lemma 3.8. *Let j be the wedge for all $n \geq 0$ of the inclusions below, where A is a cofibrant, compact pointed space.*

$$A \rtimes |i[n]| : A \rtimes |\dot{\Delta}[n]| \to A \rtimes |\Delta[n]|.$$

For a map $f : X \to Y$ denote as

$$X = Z_0 \xrightarrow{i_\alpha} Z_\alpha \xrightarrow{p_\alpha} Y,$$

the factorization of f at the stage α of the small object argument construction whose generator is j. Then the construction converges at each limit ordinal γ, or in other words $j \nearrow p_\gamma$.

Proof. First note that for each inclusion one can fix open neighborhoods (which we will denote U) so that the inclusion of the domain into the neighborhood is a deformation retract. This is done by choosing appropriate neighborhoods for the simplices and taking their half smash with the identity of A.

Let the stages of the construction be

$$X = Z_0 \longrightarrow Z_1 \longrightarrow \cdots \longrightarrow Z_\omega \longrightarrow \cdots \longrightarrow Z_\gamma \xrightarrow{p_\gamma} Y.$$

We construct inductively open covers for each stage Z_α, indexed by α. They will be denoted U_i^α, $i < \alpha$, and we also denote $Z_\alpha = U_\alpha^\alpha$, so that

$$\cdots \subset U_i^\alpha \subset U_{i+1}^\alpha \subset \cdots \subset U_\alpha^\alpha = Z_\alpha$$

and $Z_i \subset U_i^\alpha$.

For any $U_i^\alpha \subset Z_\alpha$ denote by gU_i^α the open neighborhood of U_i^α in $Z_{\alpha+1}$, obtained by replacing each pushout of j by the pushout of the inclusion of the domain of j in U in a step of the small object argument construction applied to the restriction map $U_i^\alpha \to Y$.

Now, we define U_i^α inductively

$$U_0^0 = Z_0,$$
$$U_i^{\alpha+1} = gU_i^\alpha,$$
$$U_i^\gamma = \bigcup_{\alpha<\gamma} U_i^\alpha \quad \text{if } \lim(\gamma).$$

Since the glueing construction g is functorial, one can show that there is a deformation retraction

$$r_i^\alpha : U_i^\alpha \to Z_i.$$

Now let $j \to p_\gamma$ be a commutative square for γ a limit ordinal. The image of the domain of j in Z_γ is contained in some U_α^γ, and the retraction r_α^γ takes it to Z_α, which defines a map of the range of j into $Z_{\alpha+1}$. Now one can repeat the procedure for the homotopy, etc., exactly as in the case of $\gamma = \omega$. □

4. Relations between the original and the *A*-cellular homotopy categories

We collect some of the relations in the following:

Theorem 4.1. *Let A be a cofibrant object of $\mathscr{C}$ and K a finite simplicial set. Then*:
(1) *An equivalence in $\mathscr{C}$ is an A-equivalence*;
(2) *An A-equivalence is an $A \otimes K$-equivalence*;
(3) *An $A \otimes K$-cofibration is an A-cofibration*;
(4) *An A-cofibration is an cofibration in $\mathscr{C}$*;
(5) *A-trivial A-cofibrations are exactly the trivial cofibrations in $\mathscr{C}$.*

Proof. (2) follows from the adjunction

$$Hom_{\mathscr{C}}(A \otimes K, X) \cong Hom_{\mathscr{C}}(K, Hom_{\mathscr{C}}(A, X)),$$

(3)–(5) follow since cofibrations and trivial cofibrations are defined by the *LLP* with respect to trivial fibrations and fibrations, respectively. □

Of special interest is the closed model category structure on the category of I-diagrams with $A = S^0$:

Corollary 4.2. *There exists a closed model category structure on diagrams of pointed simplicial sets* (*and on diagrams of pointed spaces*) *in which*

$$\underline{Fib}^{S^0} = \{\underline{\varphi} | \ \underline{\varphi}_i \in Fib\},$$

$$\underline{W}^{S^0} = \{\underline{\varphi} | \ holim\, \underline{\varphi_f} \in W\},$$

$$\underline{Cof}^{S^0} = \{\underline{j} | \ \underline{j} \nearrow (\underline{Fib}^{S^0} \cap \underline{W}^{S^0})\}.$$

A particular case is a closed simplicial model category structure on cosimplicial spaces in which the weak equivalences are determined by the Tot functor.

A-trivial fibrations can be characterized in terms of homotopy groups with coefficients in A [8, 9], and there are corresponding long exact sequences. As an example we give a definition in the category of I-diagrams of pointed spaces.

Definition 4.3. If $\underline{\varphi} \in map_*^V(A \wedge (V/-)_+, \underline{X})$ is a point in $Map_*(A, holim\underline{X})$, we denote

$$\pi_k(\underline{X}; A)_{\underline{\varphi}} = \pi_k(holim\,\underline{X}; A)_\varphi = \pi_k(Map_*(A, holim\underline{X}))_\varphi$$

where $\varphi : A \to holim\,\underline{X}$ is a map of pointed spaces adjoint to $\underline{\varphi}$.

We now turn to the relation between the categories $Ho - \mathscr{C}$ and $Ho - \mathscr{C}^A$.

Theorem 4.4. *Let $f : X \to Y$ be an A-equivalence of A-cofibrant, fibrant objects. Then f is an equivalence in $\mathscr{C}$.*

Proof. Since each of X, Y is A-cofibrant and fibrant, there exists a homotopy inverse g and left homotopies

$$h_X : g \circ f \overset{l}{\sim} Id_X, \qquad h_Y : f \circ g \overset{l}{\sim} Id_Y$$

We claim that the cylinder object (for X, for example) can be chosen to be $X \otimes \Delta[1]$. It is enough to show that $X \otimes i[1] : X \otimes \dot{\Delta}[1] \to X \otimes \Delta[1]$ is an A-cofibration, and this follows from Remark 2.5 and [2, Lemma 6.4], since the map $* \to X$ is an A-cofibration. Finally, note that this cylinder object is also a cylinder object in $\mathscr{C}$. □

Corollary 4.5. *Let $\overline{\gamma}$, $\overline{\gamma}^A$ be the localization functors in $\mathscr{C}$ and $\mathscr{C}^A$, and let L, L^A be the respective total left derived functors. Then the functors*

$$L^A\overline{\gamma} : Ho - \mathscr{C}^A \longleftrightarrow Ho - \mathscr{C} : L\overline{\gamma}^A$$

are an adjoint pair, where the functor on the left is a full embedding of the category $Ho - \mathscr{C}^A$ in $Ho - \mathscr{C}$ and the functor on the right is surjective.

Proof. By [14, Part II, Theorem 1] we need to consider the category of cofibrant, fibrant objects in $\mathscr{C}$ and its subcategory of A-cofibrant fibrant objects. By Theorem 4.9 the A-cofibrant approximation of a fibrant object in $\mathscr{C}$ is determined up to original equivalence. The isomorphism of Hom-sets is induced by π_0 from the equivalence of simplicial sets

$$\underline{Hom}(X, CW_AY) \xrightarrow{\sim} \underline{Hom}(X, Y),$$

in the diagram below:

$$\begin{array}{ccc} * & \longrightarrow & CW_AY_{cf} \\ {\scriptstyle Cof^A}\downarrow & & \sim\downarrow{\scriptstyle A} \\ X_{cf}^A & \longrightarrow & Y_{cf} \end{array}$$

where X_{cf}^A is an A-cofibrant, fibrant representative of a homotopy type in $\mathscr{H}0 - \mathscr{C}^A$, and Y_{cf} is a cofibrant, fibrant representative of a homotopy class in $\mathscr{C}$. □

Example 4.6. Let $\mathscr{C} = \mathscr{T}_*$ and $A = S^2$. In this case the embedding of the subcategory of 1-connected spaces is left adjoint to the universal cover functor.

Corollary 4.7. *An A-equivalence of A-cofibrant fibrant diagrams is an ordinary equivalence on each object.*

5. Lemmas on interchange of factorization and *holim* for contractible categories

In this section V denotes a small category whose classifying space (also denoted V) is contractible, and we look at V-diagrams over a closed simplicial model category $\mathscr{C}$, satisfying the conditions of Theorem 2.3 and whose objects are fibrant. In applications we will use the fact that the objects of $\mathscr{T}_*^A$, $(\mathscr{T}_*^A)^V$ are fibrant.

Consider the adjoint functors:

$$FX = X \otimes (V/-),\ F : \mathscr{C} \mapsto \mathscr{C}^V,$$
$$holim- :\ \mathscr{C}^V \mapsto \mathscr{C},$$

where the *holim* is defined as in the introduction.

Lemma 5.1. *The unit of the adjunction* $u_X : X \to holim\, FX$ *is an* $\mathscr{S}^0$*-equivalence for any object* X.

Corollary 5.2. *The counit of the adjunction* (*the evaluation map*) $ev : F\, holim\, \underline{Y} \to \underline{Y}$ *is an* $\mathscr{S}^0$*-equivalence for any V-diagram* $\underline{Y}$.

Corollary 5.3. *For any* $X \in \mathscr{C}$, $\underline{Y} \in \mathscr{C}^V$ *a morphism* $t : X \to holim\, \underline{Y}$ *is an* $\mathscr{S}^0$*-equivalence if and only if its adjoint* $adj\ t : FX \to \underline{Y}$ *is an* $\mathscr{S}^0$ *equivalence.*

Lemma 5.4. *Let* $t : X \to holim\, \underline{Y}$ *be a morphism for an object X and a diagram* $\underline{Y}$ *over a contractible small category V, let Z be its factorization and* $\underline{Z}$ *be the factorization of the morphism* $adj\ t : FX \to \underline{Y}$. *Then the morphism* $FZ \to \underline{Z}$ *is an* $\mathscr{S}^0 \otimes (V/-)$*-equivalence under FX, and the morphism* $Z \to holim\underline{Z}$ *is an* $\mathscr{S}^0$*-equivalence under X.*

Corollary 5.5. *The factorization of the morphism of diagrams* $\underline{*} \to \underline{X}$ *is objectwise* $\mathscr{S}^0$*-equivalent to the constant diagram on* $CW_A(holim\, \underline{X})$.

Lemma 5.6. *The functor holim preserves right homotopies.*

Lemma 5.7. *The functor holim preserves CM5I factorizations up to an* $\mathscr{S}^0$ *-equivalence if the closed model category* $(\mathscr{C}^A)^V$ *is proper.*

6. Examples

(1) $A = S^n$: Equivalences are maps that induce isomorphisms on π_k for $k \geq n$. A-cofibrants: $n-1$-connected spaces.

(2) Let f_α be a self-map of a countable wedge of circles inducing an identity on integral homology, and let $A = \bigvee_\alpha Cf_\alpha$ be a wedge of cones of f_α where the index α belongs to the set of homotopy equivalence classes of all such maps. It follows from [5, 10] that the trivial fibrant approximation in this structure is the plus construction.

(3) Let $V = (a \to b \leftarrow c)$ be the pullback category. Then Theorem 4.4 implies that if a map of two $A \otimes (V/-)$-cofibrant diagrams induces an A-equivalence on *holim*, then it is an equivalence on each object in the original structure. One can also show that if $[A, X] = *$ then the $A \otimes (V/-)$-localization of a diagram $(* \to X \leftarrow *)$ is equivalent to $(* \to P_{\Sigma A}X \leftarrow *)$, recovering the result of [8] in a general setting. Note that $A \otimes (V/-)$-factoring of a morphism of diagrams is not necessarily an A-factoring for each object.

Acknowledgements

I wish to thank Prof. E. Dror Farjoun for his advice and Prof. J. Smith for many conversations.

References

[1] A.K. Bousfield, The localization of spaces with respect to homology, Topology 14 (1975) 133–150.
[2] A.K. Bousfield, Factorization systems in categories, J. Pure Appl. Algeb. 9 (1977) 207–220.
[3] A.K. Bousfield, E.M. Friedlander, Homotopy theory of Γ-spaces, spectra, and bisimplicial sets, in: M.G. Barratt, M.E. Mahowald (Ed.), Geometric Applications of Homotopy Theory II, Lecture Notes in Mathematics, vol. 658, Springer, Berlin, 1978, pp. 80–130.
[4] A.K. Bousfield, D.M. Kan, Homotopy limits, completions and localizations, Lecture Notes in Mathematics, vol. 304, 1972.
[5] C. Casacuberta, Anderson localization from a modern point of view, Contemporary Math. 181 (1995) 35–44; The Čech Centennial Conf., Boston, MA, 1993.
[6] W.G. Dwyer, D.M. Kan, Singular functors and realization functors, Nederl. Acad. Wetensch. Indag. Math. 46 (2) (1984) 147–153.
[7] W.G. Dwyer, D.M. Kan, Function complexes in homotopical algebra, Topology 19 (1980) 427–440.
[8] E. Dror Farjoun, Localization, fibrations and conic structures, preprint.
[9] E. Dror Farjoun, Homotopy localization and v_1-periodic spaces, in: Algebraic Topology San Feliu de Guixols, Lecture Notes in Mathematics, vol. 1509, Springer, Berlin, 1992, pp. 104–113.
[10] E. Dror Farjoun, K. Orr, S. Shelah, Localization with respect to homology and algebraic closure of groups, Israel J. Math. 66 (1989) 143–153.
[11] P.S. Hirschhorn, Localization, of model categories, Preprint.
[12] S. MacLane, Categories for the Working Mathematician, Springer, Berlin, 1971.
[13] A. Nofech, A version of an E^2 closed model category structure, in: C. Broto, C. Casacuberta, G. Mislin (Eds.), Algebraic Topology: New Trends in Localization and Periodicity, Progress in Mathematics vol. 136, Birkhauser, Basel, 1996.
[14] D.G. Quillen, Homotopical Algebra, Lecture Notes in Mathematics vol. 43, Springer, Berlin, 1967.
[15] D.G. Quillen, Rational homotopy theory, Ann. math. 90 (1969), 205–295.