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Behaviour of distance functions in Hilbert–Finsler geometry

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Abstract

Smooth bounded convex domains equipped with their Hilbert metric provide nice examples of constant negatively curved Finsler manifolds. An important property of these models is that contrary to what happens in Riemannian setting the distance between two points moving at unit speed along intersecting geodesics is not necessarily convex. However we give sharp estimates of the asymptotic behaviour of such functions.

MSC: 53A40; 53C60; 52A50; 52A41

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1. Introduction

Riemannian negatively curved manifolds admit many convexity properties, in particular the distance between two points moving at unit speed along geodesics is a convex function. In this article we show that for Finsler constant negatively curved manifolds such a result is not true any longer and estimate as sharply as possible the asymptotic behavior of this distance function for two intersecting geodesics of a Finsler–Hilbert geometry. In fact we restrict ourselves to Finsler–Hilbert geometry because any simply connected, projectively flat, geodesically complete reversible Finsler manifold of constant negative curvature is isometric to a Hilbert geometry [1,3,8].

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1.1. Hilbert geometry

A *Hilbert geometry* consists of a bounded convex domain C of $\mathbb{R}^n$ equipped with its Hilbert metric h_k (whose definition follows) where k is a real positive number. Any two distinct points x and y of C determine an oriented line joining x to y which intersects the closure of C along an oriented chord $[x^-, x^+]$, the Hilbert distance $h_k(x, y)$ between x and y is then given by

$$h_k(x, y) = \frac{1}{2\sqrt{k}} \ln[x^-, x, y, x^+],$$

where $[x^-, x, y, x^+]$ denotes the cross ratio of x^-, x, y and x^+. If we set $h_k(x, x) = 0$ then h_k is really a metric on C which is moreover complete and defines the same topology as $\mathbb{R}^n$ [2,5,9].

Before explaining when a Hilbert geometry endowed a Finsler structure, let us precise that a (reversible) *Finsler manifold* is a manifold M equipped with a Lagrangian $F : TM \to \mathbb{R}$ such that

- the restriction of F to any tangent space T_xM is a norm,
- the Lagrangian F has class $\mathcal{C}^2$ outside the zero section,
- the matrix of second derivatives $(\frac{\partial^2 F^2}{\partial y_i \partial y_j}(x, y))_{i,j}$ is positive

definite for every nonzero vector y of T_xM.

Under such hypotheses the problem of variations calculus is well defined and F provides a distance on M. A Hilbert geometry (C, h_k) appears as the metric space induced by a Finsler manifold (we will say *Finsler–Hilbert geometry*) if and only if the boundary of C is a $\mathcal{C}^2$ hypersurface of $\mathbb{R}^n$ and for any 2-plane P of $\mathbb{R}^n$ the boundary of the section $C \cap P$ (when nonempty) has a nondegenerated Hessian everywhere except at most along one segment. For such Hilbert geometries, geodesics are exactly traces of lines and Finsler curvature is constant and equal to $-k$ [4,6–8]. Notice that the Hilbert geometry of an open ball of the usual Euclidean space $\mathbb{R}^n$ is the Klein model of constant negatively curved Riemannian manifolds.

1.2. Behaviour of the distance

Let (C, h_k) be a Finsler–Hilbert geometry and β and δ two distinct (oriented) geodesics such that $\beta(0) = \delta(0)$. We want to know how the distance function $d : t \in \mathbb{R} \mapsto h_k(\beta(t), \delta(t))$ behaves near infinity.

These geodesics β and δ define a plane P and d just depends on the affine section $P \cap C$. So from now on we assume C is a bounded convex domain of $\mathbb{R}^2$.

Two points η^+ and η^- can naturally be associated to any oriented geodesic η: the oriented line defined by η intersects the closure of the convex set C along the oriented chord $[\eta^-; \eta^+]$.

Let us $[AB]$ be the chord along which the oriented line $(\beta^+\delta^+)$ intersects the closure of C. The following four different cases can occur (see Fig. 1):

case e: C is not entirely contained in a half plane limited by (AB); so $A = \beta^+$ and $B = \delta^+$;

case f: C is contained in a half plane limited by (AB) with $A = \beta^+$ and $B = \delta^+$;

case g: chords $[AB]$ and $[\beta^+\delta^+]$ have a single end-point in common;

case h: the chord $[\beta^+\delta^+]$ is included in the open chord $]AB[$.

Case e occurs when β and δ do not end on a same segment of the boundary, so in particular when C is strictly convex. But notice that all these cases can occur for general Finsler–Hilbert geometry.

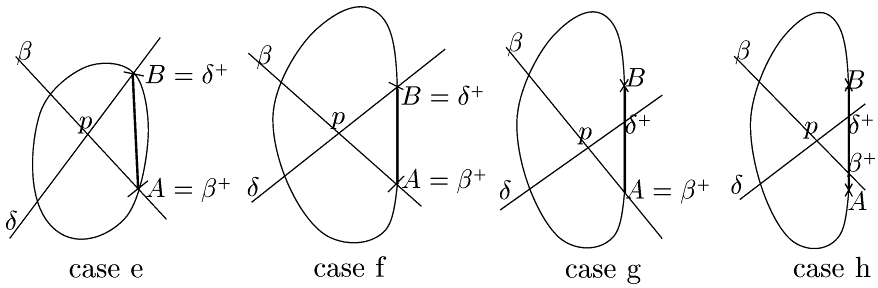


Fig. 1.

Theorem. *Let* (C, h_k) *be a Finsler–Hilbert geometry whose boundary* ∂C *is* $\mathcal{C}^q$ *with* $q \geqslant 2$. *Consider* β *and* δ *two distinct (oriented) geodesics with* $\beta(0) = \delta(0)$ *and parameterized by their arc length. Then the function* $d : t \mapsto h_k(\beta(t), \delta(t))$ *verifies*:

(i) *the function* d *is strictly increasing*;
(ii) *at infinity*:
- *in case e*:

$$d(t) = 2t + \sum_{l=0}^{q-1} K_l \mathrm{e}^{-2l\sqrt{k}\,t} + \mathrm{o}\big(\mathrm{e}^{-2(q-1)\sqrt{k}\,t}\big) \quad \textit{with } K_0 \leqslant 0,$$

- *in case* f *(respectively* g*)*;

$$d(t) \textit{ and } 2t - d(t) \textit{ (respectively } t - d(t)\textit{) goes to } +\infty \textit{ with } t.$$

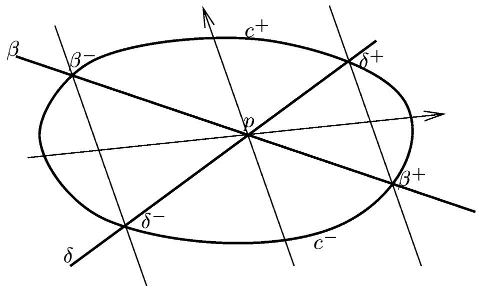
(iii) *for any given positive real number* ε, *the derivative* $\dot{d}$ *satisfies for* t *sufficiently large*:

$$\begin{cases} 2 - \varepsilon < \dot{d}(t) < 2 + \varepsilon & \textit{in case } e, \\ 0 \leqslant \dot{d}(t) \leqslant 2 & \textit{in case } f, \\ 0 \leqslant \dot{d}(t) \leqslant 1 + \varepsilon & \textit{in case } g, \\ 0 \leqslant \dot{d}(t) < \varepsilon & \textit{in case } h; \end{cases}$$

(iv) *in case e the second derivative of* d *vanishes at infinity and has positive values for* t *large enough, i.e.,* d *is convex in an infinite neighbourhood.*

Remarks.

- The sign of K_0 is obviously given by the triangular inequality.
- These results are optimum as it will be shown in Section 4, in particular everything can occur in cases f, g and h for the derivative $\ddot{d}$.
- Some of these results stay true under weaker hypotheses as it will become clear through the proof of the theorem.
- Because Hilbert geometry is not uniform, I was not able to use general results on the variation of length and had to carry out straightforward computations with explicit formulas. It also seems that no uniform control on the behaviour of the geodesic distance can be expected. For strictly convex domains, A.F. Beardon obtained $d(t) = 2t + \mathrm{O}(1)$ under weaker hypothesis (see [2]) but did not control $\mathrm{O}(1)$ uniformly.



$A = (1, c^-(1))$ $\qquad$ $B = (1, c^+(1))$

$\beta^+ = (1, -1)$ $\qquad$ $\beta^- = (-1/r, 1/r)$

$\delta^+ = (1, 1)$ $\qquad$ $\delta^- = (-1/r, -1/r)$

Fig. 2.

2. Preliminaries

Recall that the convex domain C can be assumed of dimension 2. Geodesics β and δ determine two lines $(\beta^+\delta^+)$ and $(\beta^-\delta^-)$. These lines intersect outside C (eventually at infinity) so there exists a projective map which sends C onto a bounded convex set C' and the intersection point between $(\beta^+\delta^+)$ and $(\beta^-\delta^-)$ to infinity. Then this map is an isometry between (C, h_k) and (C', h'_k) which sends β and δ onto two geodesics having the same relative position (case e, f, g or h) as β and δ. Up to this isometry we can then assume that $(\beta^+\delta^+)$ and $(\beta^-\delta^-)$ are parallel. Moreover as β and δ have unit speed, we obtain

Lemma 1. *For all t the line $(\beta(t)\delta(t))$ stays parallel to $(\beta^+\delta^+)$.*

Proof. Let D and D_t be the parallels to $(\beta^+\delta^+)$ through $\beta(0)$ respectively $\beta(t)$. Thus $[\beta^-, \beta(0), \beta(t), \beta^+] = [(\beta^-\delta^-), D, D_t, (\beta^+\delta^+)] = [\delta^-, \delta(0), D_t \cap \delta, \delta^+]$ as $\delta(0) = \beta(0)$. Since $h_k(\delta(0), \delta(t))$ and $h_k(\beta(0), \beta(t))$ are equal, so are $[\delta^-, p, D_t \cap \delta, \delta^+]$ and $[\delta^-, p, \delta(t), \delta^+]$. Finally δ and D_t meet at $\delta(t)$. $\square$

Let us now define the frame $(p, \vec{\imath}, \vec{\jmath})$ we will use to estimate the asymptotic behavior of function d. The origin p is $\beta(0) = \delta(0)$, the vector $2\vec{\jmath}$ is $\overrightarrow{\beta^+\delta^+}$, the line $(p, \vec{\imath})$ is the bisector of β and δ and the first component of β^+ is 1. We fix a scalar product by saying that this frame is orthonormal and denote by r the positive ratio $-\overline{p\beta^+}/\overline{p\beta^-}$.

As C is a $\mathcal{C}^q$ convex domain the intersection between ∂C and the strip $-1/r < x < 1$ can be described as graphs of two $\mathcal{C}^q$ functions. The function c^+ (respectively c^-) giving the upper part of the intersection (respectively the lower one) has to be concave (respectively convex). As $q \geqslant 2$, notice that c^+ and c^- are derivable at 1 in case e and then verify $\dot{c}^+(1) < 1$ and $\dot{c}^-(1) > -1$ by choice of the frame. In the other cases $-\dot{c}^+$ and $\dot{c}^-$ simultaneously tend to $+\infty$ at 1. (See Fig. 2.)

3. Proof of the theorem

The situation is the one described in the previous section. Moreover by definition of Hilbert metrics it's enough to prove the theorem for $2\sqrt{k} = 1$ what we assume now.

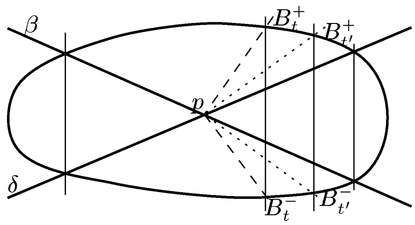


Fig. 3.

3.1. *Why is the function d strictly increasing?*

For each t the oriented line $(\beta(t), \delta(t))$ meets the boundary of C successively at B_t^- and B_t^+. Then $d(t) = \ln[(pB_t^-), \beta, \delta, (pB_t^+)]$. Lemma 1 allows us to draw the illustration (shown in Fig. 3) for $t' > t$ which ends the proof.

3.2. *The first component of $\beta(t)$*

As lines $(\beta^+\delta^+)$ and $(\beta(t)\delta(t))$ stays parallel (Lemma 1), points $\gamma(t)$ and $\delta(t)$ have the same first component denoted by $x(t)$. Thus, since β is parameterized by its Hilbert arc length, we obtain

$$x(t) = \frac{\mathrm{e}^t - 1}{\mathrm{e}^t + r} \quad \text{and} \quad 1 - x(t) = \frac{r+1}{r + \mathrm{e}^t}\cdot \tag{1}$$

We recover that $x(t)$ tends to 1 when t goes to $+\infty$ and obtain the useful identities

$$\dot{x} = (1 - x) - \frac{r}{r+1}(1-x)^2, \tag{2}$$

$$\ddot{x} = -(1 - x) + \frac{3r}{r+1}(1-x)^2 - 2\left(\frac{r}{r+1}\right)^2 (1-x)^3. \tag{3}$$

3.3. *Computation of d*

Making explicit the cross ratio defining function d, it follows that $d = d_+ + d_-$ with d_+ and d_- the functions of class $\mathcal{C}^q$ on $\mathbb{R}$:

$$d_+ = \ln\frac{c^+ \circ x + x}{c^+ \circ x - x} \quad \text{and} \quad d_- = \ln\frac{c^- \circ x - x}{c^- \circ x + x}.$$

As $c^-(1) \leqslant -1 < 1 \leqslant c^+(1)$, the increasing function d goes to infinity with t in all cases except case h in which it converges to a finite limit.

Moreover functions c^+ and c^- can be chosen independently except that their first derivative must be simultaneously finite or infinite at 1 and the study of the following cases gives the behaviour of d_+:

case e': $\dot{c}^+(1)$ is finite and so $c^+(1) = 1$,
case f': $\dot{c}^+(x)$ goes to $-\infty$ when x tends to 1 and $c^+(1) = 1$,
case h': $\dot{c}^+(x)$ goes to $-\infty$ when x tends to 1 and $c^+(1) > 1$.

3.4. *Proof of property (ii)*

- *For case e'.* Using Taylor developments of $c^+ \circ x - x$ and $c^+ \circ x + x$ at 1 we show, as $c^+(1) = 1$ and $\dot{c}^+ < 1$, that there exits two constants k_0 and k_1 such that $d_+ = k_0 - \ln(1-x) + k_1(x-1) + \mathrm{o}(1-x)$. Thus, using expression (1) of $x-1$, we obtain $d_+(t) = t + K_0^+ + K_1^+ \mathrm{e}^{-t} + \mathrm{o}(\mathrm{e}^{-t})$ in a neighbourhood of infinity with K_0^+ and K_1^+ two constants.
  If $q > 2$, this proof immediately shows d_+ expands on the scale e^{-lt} with $l \in \{0, 1, \ldots, q-1\}$.
- *For case f'.* The convexity of C and its regularity gives the following inequality:
  $$-\dot{c}^+ \circ x < \frac{c^+ \circ x - c^+(1)}{1-x}. \tag{4}$$
  Here $\dot{c}^+$ diverges at 1, we have then
  $$1 - x = \mathrm{o}\big(c^+ \circ x - c^+(1)\big) \quad \text{and} \quad c^+ \circ x - x = (c^+ \circ x - 1)\big(1 + \mathrm{o}(1)\big).$$
  It follows $c^+ \circ x + x = 2 + \mathrm{o}(1)$ since $c^+(1) = 1$ and finally $d_+ = -\ln(c^+ \circ x - c^+(1)) + \mathrm{O}(1)$.
- *For case g'.* In this particular case d_+ has a finite limit at infinity.
- *Conclusion.* Similar computation for d_- leads to the looked for expansion of $d = d_+ + d_-$ at infinity by using the independence of c^+ and c^- and the fact that in case f' the function $1 - x$ is negligible with respect to $c^+ \circ x - c^+(1)$ and of the same order as e^{-t}.

3.5. *Proof of property (iii)*

The first derivative d_+ is given by
$$\dot{d}_+ = \frac{\dot{x}(\dot{c}^+ \circ x + 1)}{c^+ \circ x + x} - \frac{\dot{x}(\dot{c}^+ \circ x - 1)}{c^+ \circ x - x} = \frac{2\dot{x}(c^+ \circ x - x\dot{c}^+ \circ x)}{(c^+ \circ x + x)(c^+ \circ x - x)}. \tag{5}$$
As $c^+(1) \geqslant 1$ we deduce from formula (2) that at infinity
$$\dot{d}_+ \sim \frac{2(1-x)(c^+ \circ x - x\dot{c}^+ \circ x)}{(c^+(1) + 1)(c^+ \circ x - x)}.$$

- *For case e'.* Taylor formula leads to $c^+ \circ x - x = (\dot{c}^+(1) - 1)(x - 1) + \mathrm{o}(x - 1)$. As $\dot{c}^+(1) < 1$ and $c^+(1) = 1$ it follows that $\dot{d}_+$ admits 1 as limit at $+\infty$.
- *For case f'.* Here $c^+(1) = 1$, the derivative $\dot{c}^+$ diverges at 1 and $1 - x = \mathrm{o}(c^+ \circ x - c^+(1))$ at infinity, so near infinity
  $$\dot{d}_+ \sim -\frac{(1-x)\dot{c}^+ \circ x}{c^+ \circ x - c^+(1)}.$$
  This equivalent is positive near infinity and bounded by 1 because of inequality (4). Thus $\dot{d}_+$ is positive and bounded by any real number strictly bigger than 1. In fact $\dot{d}_+$ is bounded by 1 near infinity. In order to prove that, let us search the sign of $\dot{d}_+ - 1$ that is the one of $\mathcal{S} = (\dot{d}_+ - 1)(c^+ \circ x + x)(c^+ \circ x - x)$. A direct computation using formulas (5) and (2) leads to $\mathcal{S} = T_1 + T_2 + T_3$ with
  $$T_1 = \frac{2r}{r+1}(1-x)^2(\dot{c}^+ \circ x - 1),$$
  $$T_2 = -\big[c^+ \circ x - c^+(1) + (1-x)\big]\big[c^+ \circ x - c^+(1) - (1-x)\big],$$

$$T_3 = 2\left[c^+ \circ x - c^+(1) + (1-x)\dot{c}^+ \circ x\right]\left[-1 + (1-x) - \frac{r(1-x)^2}{r+1}\right].$$

As $\dot{c}^+$ tends to infinity 1, the concavity of c^+ shows that $y-1 = \mathrm{o}(c^+(y) - c^+(1))$ and $(y-1)\dot{c}^+(y) = \mathrm{O}(c^+(y) - c^+(1))$ for y near 1. We have then near infinity

$$T_1 = \mathrm{o}\big(\big(c^+ \circ x - c^+(1)\big)^2\big),$$
$$T_2 \sim -\big(c^+ \circ x - c^+(1)\big)^2,$$
$$T_3 \sim -2\big(c^+ \circ x - c^+(1) + (1-x)\dot{c}^+ \circ x\big).$$

Equivalents of T_2 and T_3 are negative. Then $\mathcal{S}$ is negative at infinity which ends the proof.

- *For case h'.* Here $c^+(1) > 1$, the derivative $\dot{c}^+$ diverges at 1, so at infinity

$$\dot{d}_+ \sim -\frac{2(1-x)\dot{c}^+ \circ x}{(c^+(1)+1)(c^+(1)-1)}.$$

By inequality (4) the derivative $\dot{d}_+$ vanishes at infinity.

- *Conclusion.* With same methods we prove analogous results for $\dot{d}_-$. As $d = d_+ + d_-$ and c^+ and as c^- are independent, the result immediately follows.

3.6. *Proof of property (iv)*

The derivative of expression (5) of $\dot{d}_+$ leads to

$$\ddot{d}_+(t) = \frac{\ddot{x}(\dot{c}^+ \circ x + 1) + \dot{x}^2\ddot{c}^+ \circ x}{c^+ \circ x + x} - \left(\frac{\dot{x}(\dot{c}^+ \circ x - 1)}{c^+ \circ x + x}\right)^2$$
$$- \frac{\ddot{x}(\dot{c}^+ \circ x - 1) + \dot{x}^2\ddot{c}^+ \circ x}{c^+ \circ x - x} + \left(\frac{\dot{x}(\dot{c}^+ \circ x - 1)}{c^+ \circ x - x}\right)^2.$$

Expressions (2) and (3) of $\dot{x}$ and $\ddot{x}$ show that $\ddot{d}_+$ can be expressed as a compounded function $D \circ x$. And we can expand D on the powers of $(x-1)$ using Taylor formulas for c^+, $\dot{c}^+$ and $\ddot{c}^+$ at 1 (we are in case e). A direct computation leads to

$$D(y) = \left(\frac{r}{r+1} + \frac{\ddot{c}^+(1)}{2(\dot{c}^+(1)-1)} - \frac{\dot{c}^+(1)+1}{2}\right)(1-y) + \mathrm{o}(1-y).$$

So, as going from d^+ to d^- is just exchanging c^+ by $-c^-$, we obtain $\ddot{d}(t) = S(1-x(t)) + \mathrm{o}(1-x(t))$ with

$$S = \frac{2r}{r+1} + \frac{\ddot{c}^+(1)}{2(\dot{c}^+(1)-1)} - \frac{\dot{c}^+(1)+1}{2} + \frac{\ddot{c}^-(1)}{2(\dot{c}^-(1)+1)} + \frac{\dot{c}^-(1)-1}{2}.$$

But in case e, the convexity of the domain C gives the way the tangent line to ∂C at δ^+ and the line $(\beta^-\delta^+)$ (respectively the tangent line at β^+ and $(\delta^-\beta^+)$) are placed: in coordinates $\dot{c}^+(1) \leqslant -\frac{1-1/r}{1+1/r}$ and $\dot{c}^-(1) \geqslant \frac{1-1/r}{1+1/r}$. Finally we have

$$S \geqslant \frac{\ddot{c}^+(1)}{2(\dot{c}^+(1)-1)} + \frac{\ddot{c}^-(1)}{2(\dot{c}^-(1)+1)}$$

which proves S is positive.

4. Examples and counter-examples

The aim of this section is to show by examples that the results we obtained are optimal. We are still in the situation of Section 2 and we just describe the part of the boundary we are interested in. We can do that because there exists a smooth boundary extending the given part. More precisely one has just to pay attention to the given points of the boundary: they have to be out of the interior of the convex hull of the described part.

For all the examples we take $2\sqrt{k}=1$ and so the first component $x(t)$ is $\tanh(t/2)$.

4.1. About property (iv)

Property (iv) of the theorem assures that in case e, function d is convex for t sufficiently large but nothing forces d to be convex on all $\mathbb{R}$ as for negatively curved Riemannian manifolds. In fact there exist convex domains of class $\mathcal{C}^2$ whose boundary has a not degenerated Hessian everywhere and geodesics of this Hilbert geometry for which the distance d is not globally convex.

Otherwise let us consider an open convex bounded domain C of class $\mathcal{C}^2$ whose boundary contains an unique segment $[ab]$ with $a \neq b$ and has nondegenerated Hessian everywhere outside $[ab]$. Now equip C with the Hilbert distance h_1 and chose two distinct points a' and b' on $]ab[$ and two points a'' and b'' of ∂C such that $(a''b'')$ and (ab) are parallel and $(a''b')$ and $(b''a')$ intersect in C. We define δ (respectively β) as the oriented geodesic of speed 1 and of support the oriented line $(a''b')$ (respectively $(b''a')$) such that $\delta(0)$ (respectively $\beta(0)$) is the intersection point $(a''b') \cap (b''a')$. For any positive real number t the line $(\beta(t)\delta(t))$ is parallel to (ab) (Lemma 1). Consider C_t an open convex bounded domain such that the boundary of C_t is of class $\mathcal{C}^2$ with a nondegenerated Hessian everywhere, the parts of C_t and C lying in the half plane limited by $(\beta(t)\delta(t))$ containing a'' coincide and C_t contains points a' and b'. Equip this convex C_t with its Hilbert metric h^t of parameter 1. Maps δ and β are still geodesics of speed one for (C_t, h^t) thus for any s in $[0,t]$ we have $h_1(\delta(s), \beta(s)) = h^t(\delta(s), \beta(s))$. Finally if property (iv) assure d is convex on all $\mathbb{R}^+$, we would obtain that $t \in \mathbb{R}^+ \mapsto h_1(\beta(t), \delta(t))$ is convex which cannot be as it admits a finite limit at infinity.

4.2. What can happen in case f' for the derivative $\dot{d}_+$?

We saw the derivative $\dot{d}_+$ is bounded by 0 and 1 near infinity. In fact it happens this function admits a limit and any limit between 0 and 1 can be obtained, in other hand it occurs function $\dot{d}_+$ is bounded without having any limit at infinity. The following examples illustrate all these situations:

- The derivative $\dot{d}_+$ vanishes at infinity for a boundary described for x in $[1-\mathrm{e}^{-2}, 1[$ by: $c^+ : x \mapsto 1 - 1/\ln(1-x)$.
- For any real number μ in $]0;1[$, a boundary which coincides for x in $[0,1[$ with the graph of $c^+ : x \mapsto 1 + (1-x)^{-}u/\mu$ leads to a function d_+ whose derivative tends to μ at infinity.
- The derivative $\dot{d}_+$ tends to 1 at infinity when C is described on $[1-\mathrm{e}^{-1}, 1]$ by $c^+(x) = 1 - (1-x)\ln(1-x)$.
- For a boundary whose restriction to $[1/2, 1]$ is given by the map $c^+ : x \mapsto 1 + 2[8 + \cos(\ln(1-x))]\sqrt{1-x}$, neither $\dot{d}_+$ nor $\ddot{d}_+$ admit any limit at infinity. However $\ddot{d}_+$ stays bounded and take negative and positive values at any neighborhood of infinity.

4.3. What can happen in cases f' or h' to $\ddot{d}$?

As $\dot{d}$ is bounded, the only possible limit in $\overline{\mathbb{R}}$ for $\ddot{d}$ is zero. Otherwise $\ddot{d}$ can stay bounded without having any limit or be unbounded. In fact any of those situations occurs as following examples illustrate it.

First notice that, as the functions c^+ and c^- can be chosen independently, it is enough to prove all the previously described situations can happen for $\ddot{d}_+$ in cases f' and h'.

Before looking for examples, observe $\ddot{d}_+$ more precisely: using equivalents for $\dot{x}$ and $\ddot{x}$ given by the formulas (2) and (3), inequality (4) and continuity of c^+, we obtain

$$\ddot{d}_+ = -\frac{2\cdot x^2 x \ddot{c}^+ \circ x}{(c^+ \circ x - x)(c^+ \circ x + x)} + \begin{cases} \mathrm{O}(1) & \text{for case } f' \\ \mathrm{o}(1) & \text{for case } h'. \end{cases} \tag{6}$$

- Examples where $\ddot{d}_+$ vanishes at infinity.
  The first three examples of Section 4.2 are such situations of type f'. A boundary whose upper part can be described by the graph of the map $c^+ : x \in [0,1] \mapsto 2 + 2\sqrt{1-x}$ corresponds to case h' and $\ddot{d}_+$ vanishes at infinity.
- Examples for which $\ddot{d}_+$ is bounded without having any limit.
  Such a situation in case f' is given by the last example of Section 4.2. Let us now build an example for case h'.
  First consider the continuous function f defined on $[0,1[$ by -1 except on the intervals $[1-1/n-1/n^4, 1-1/n+1/n^4]$, $n \in \mathbb{N}\backslash\{0,1,2\}$, where $f(x) = -1-n^2+|x-1+1/n|n^6$. In fact f is just a triangle function which satisfies $-1-1/(x-1)^2 \leqslant f \leqslant -1$. Now observe that $g : y \in [0,1[\mapsto \int_0^y f(u)\,\mathrm{d}u$ is a nonincreasing negative function, continuous on $[0,1]$ as $g(1-1/n+1/n^4)$ converges when n goes to infinity. Function $c^+ : x \in [0,1] \mapsto -\int_x^1 g(y)\,\mathrm{d}y + 2\sqrt{1-x} + 2$ is then concave, of class $\mathcal{C}^2$ on $[0,1[$, $\mathcal{C}^1$ on $[0,1]$ and tends to $2 \neq 1$ at 1. Furthermore its first derivative $\dot{c}^+$ diverges to $-\infty$ at 1. Thus it describes a situation of type h' and formula (6) becomes

  $$\ddot{d}_+ = -\frac{2(1+\mathrm{o}(1))\dot{x}^2\ddot{c}^+ \circ x}{3} + \mathrm{o}(1),$$

  that is by formula (2)

  $$\ddot{d}_+ = -\frac{2(1-x)^2}{3} \times \left(f \circ x - \frac{1}{2\sqrt{1-x}^3}\right)\big(1+\mathrm{o}(1)\big) + \mathrm{o}(1).$$

  Thanks to choice of f this last term is bounded but does not have any limit (the sequence defined by $x_n = 1-1/n$ respectively $1-1/n-1/n^4$ tend to $2/3$ respectively zero); so c^+ gives the looked for example.
- Examples where $\ddot{d}_+$ is unbounded.
  Formula (6) shows that if c^+ describes a situation of type h' where $\ddot{d}_+$ is unbounded then $c^+ - c^+(1) + 1$ will give an example of a case f' where $\ddot{d}_+$ is unbounded. So we are just going to give an example for case h'.
  Let f be the continuous triangle function f defined on $[0,1[$ by -1 except on the intervals $[1-1/n-1/n^4, 1-1/n+1/n^4]$, $n \in \mathbb{N}\backslash\{0,1,2\}$, where $f(x) = -1-n^3+|x-1+1/n|n^7$. Now observe that the map $g : y \in [0,1[\mapsto \int_0^y f(u)\,\mathrm{d}u$ is a nonincreasing negative function of class $\mathcal{C}^1$ on $[0,1[$

which tends to $-\infty$ at 1 because $g(1-1/n+1/n^4)$ diverges to infinity with n. A straightforward computation of $\int_{1-1/n}^{1-1/(n+1)} g(u)\,\mathrm{d}u$ shows these terms are the one of a convergent sum, so that the integral $\int_0^1 g(u)\,\mathrm{d}u$ is convergent. Then the function $c^+ : x \in [0,1[\mapsto -\int_x^1 g(y)\,\mathrm{d}y + 2$ is well defined, concave, of class $\mathcal{C}^2$ on $[0,1[$ and tends to $2 \neq 1$ at 1. As $\dot{c}^+$ diverges to $-\infty$ at 1, the described situation is of type h' and here formula (6) leads to $\ddot{d}_+(1-\frac{1}{n}) \sim \frac{2n}{3}$ so $\ddot{d}_+$ is unbounded here.

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