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Orthogonal polynomial solutions of linear ordinary differential equations

W.N. Everitt[a,*], K.H. Kwon[b], L.L. Littlejohn[c,1], R. Wellman[c]

[a] *School of Mathematics and Statistics, University of Birmingham, Birmingham B15 2TT, UK*
[b] *Department of Mathematics, Kaist, 373-1 Kusong-Dong Yusong-Ku, Taejon 305-701, South Korea*
[c] *Department of Mathematics, Utah State University, Logan, UT, 84322-3900, USA*

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Dedicated to Professor T.S. Chihara

Abstract

This paper surveys the latest known results concerning the classification of differential equations of the form

$$\sum_{k=1}^{N} a_k(x)y^{(k)}(x) = \lambda y(x)$$

having a sequence of polynomial eigenfunctions $\{p_n(x)\}_{n=0}^{\infty}$ that are orthogonal with respect to some real bilinear form. Since the publishing of the Erice Report in 1990, several new significant results and generalizations have been discovered.

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1. Introduction

In 1991, Everitt and Littlejohn published a paper [30] in the Proceedings of the *Third International Symposium on Orthogonal Polynomials and their Applications* held in Erice, Sicily in

* Corresponding author.
E-mail addresses: w.n.everitt@bham.ac.uk (W.N. Everitt), khkwon@jacobi.kaist.ac.kr (K.H. Kwon), lance@math.usu.edu (L.L. Littlejohn), wellman@math.usu.edu (R. Wellman).
[1] Current address: Departamento de Matemáticas, Universidad Carlos III de Madrid, Avda de la Universidad 30, 28911 Leganés (Madrid), Spain.

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June 1990. This paper surveyed the known results at the time on various algebraic, analytic and functional–analytic problems associated with the classification of orthogonal polynomial eigenfunctions $\{p_n(x)\}_{n=0}^{\infty}$ of ordinary, linear differential equations of the form

$$L_N[y](x) = \sum_{r=1}^{N} a_r(x) y^{(r)}(x) = \lambda_n y(x). \tag{1.1}$$

The origins of this classification problem were borne in the important papers of Bochner [18] in 1929 and Krall [61,62] in the years 1938–1940.

The paper [30], known as the *Erice Report*, dealt specifically with various problems associated with the classification of those polynomial eigenfunctions $\{p_n(x)\}_{n=0}^{\infty}$ of $L_N[\cdot]$ that are orthogonal with respect to bilinear forms of the type

$$\int_{\mathbb{R}} pq \, \mathrm{d}\mu, \tag{1.2}$$

where μ is a (possibly signed) real, finite Borel measure. Indeed, at the time, *all* of the *known* examples involved only this type of orthogonality. However, since the publishing of [30], different types of orthogonality, specifically Sobolev orthogonality, of the polynomial eigenfunctions have been discovered. This surprising development has led to some important, and unexpected, results and a greater appreciation of a more general classification problem.

This paper, together with [30], deals exclusively with *finite-order* equations of the type (1.1). We remark that there is interest in infinite-order differential equations; for example, see [63], where it is shown that any sequence of polynomials satisfies (uncountably) many infinite-order differential equations. Moreover, every sequence $\{p_n(x)\}_{n=0}^{\infty}$ of polynomials that is orthogonal with respect to (1.2), and are solutions to (1.1), satisfies a second-order differential equation of the form

$$A_2(x,n)y'' + A_1(x,n)y' + A_0(x,n)y = 0 \quad (n \in \mathbb{N}_0). \tag{1.3}$$

Equations of the form (1.3) having orthogonal polynomial solutions $\{p_n\}_{n=0}^{\infty}$ are very important in the general theory of semi-classical orthogonal polynomials (see [80] and references therein). More recently, such equations have played an important role in the electrostatic interpretation of the zeros of $\{p_n\}_{n=0}^{\infty}$ (see Section 6). However, it is difficult to study the spectral properties of (1.3) which is our principal motivation for studying the above classification problem.

The decade 1990–1999 has witnessed an enormous effort and a large growth in popularity in problems related to the classification of orthogonal polynomial eigenfunctions of ordinary differential equations. Indeed, of the 100 references listed in this bibliography, 78 have been published since 1990. Moreover, as a consequence of these efforts, there is renewed and growing interest in similar classification problems involving difference equations and partial differential equations (for example, see [47,66]).

The contents of this paper are as follows. In Section 2, we review the necessary terminology and definitions, together with some basic informational facts that are pertinent for subsequent discussions. Section 3 deals specifically with the Erice Report and new results (since 1990) pertaining to this report. Section 4 deals with other results concerning problems discussed in [30]. In Section 5, we discuss various extensions and generalizations of the original problems from the Erice Report; in particular, to the burgeoning area of Sobolev orthogonality. Some applications concerning the polynomials and their associated differential equations are considered in Section 6. Lastly, in Section 7, and in the same spirit and format of the original Erice Report, we include some open problems that we hope will entice even more mathematicians into this important intersection of orthogonal polynomials, differential equations, distribution theory, and functional analysis.

2. Notation, terminology, and basic facts

The fields of real numbers and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. The natural numbers $\{1,2,3,\ldots\}$ and the non-negative integers $\{0,1,2,3,\ldots\}$ are written, respectively, as $\mathbb{N}$ and $\mathbb{N}_0$. The set of all real-valued polynomials in the real variable x is denoted by $\mathscr{P}$.

A polynomial system (PS) is a sequence $\{p_n\}_{n=0}^{\infty}\subset\mathscr{P}$ with $\deg(p_n)=n$ for each $n\in\mathbb{N}_0$. The kth derivative of $p\in\mathscr{P}$ is written as $D^k p$ or $p^{(k)}$. For $n,m\in\mathbb{N}_0$ with $n\geqslant m$, we write $P(n,m)=n(n-1)\ldots(n-m+1)$ and $(n)_m=n(n+1)\ldots(n+m-1)$ with the convention $P(n,0)=(n)_0=1$ for $n\in\mathbb{N}$.

An open interval of the real line will be written as (a,b), where $-\infty\leqslant a<b\leqslant\infty$, while a closed interval is an interval of the form $[a,b]$; in the latter case, we identify, for example, the intervals $[-\infty,b]$ and $(-\infty,b]$ if $b\in\mathbb{R}$. If $w>0$ (a.e.) on an arbitrary interval $I\subset\mathbb{R}$, $L^2(I;w)$ denotes the Hilbert space of all Lebesgue measurable functions $f:I\to\mathbb{C}$ satisfying $\|f\|_w^2:=\int_I|f|^2w\,\mathrm{d}x<\infty$.

The textbooks by Chihara [21] and Szegö [94] are the sources that we recommend for the theory, both basic and advanced, of polynomials orthogonal with respect to bilinear forms of the type (1.2).

A *moment functional* σ is a linear mapping $\sigma:\mathscr{P}\to\mathbb{R}$. For $p\in\mathscr{P}$, we write $\langle\sigma,p\rangle$ instead of $\sigma(p)$. For each $n\in\mathbb{N}_0$, the real number $\sigma_n:=\langle\sigma,x^n\rangle$ is called the nth *moment* of σ. It is well known from Boas' moment theorem (see [21, p. 74]) that σ has a representation of the form

$$\langle\sigma,p\rangle=\int_{\mathbb{R}}p\,\mathrm{d}\mu\quad(p\in\mathscr{P}),$$

where μ is a finite, (possibly signed) Borel measure generated from a function $\hat{\mu}:\mathbb{R}\to\mathbb{R}$ of local bounded variation. A more recent result of Duran [25] yields a different representation

$$\langle\sigma,p\rangle=\int_{\mathbb{R}}pw\,\mathrm{d}x\quad(p\in\mathscr{P}),$$

where w is a function in the Schwartz class $\mathscr{S}(\mathbb{R})$. Since the publishing of [30], much progress has been made on the calculus of moment functionals, a study initiated by Maroni [80] and further advanced by the Korean school under the leadership of K.H. Kwon. For example, the derivative σ' of σ and multiplication of σ by a polynomial π are defined to be moment functionals through the formulas

$$\langle\sigma',p\rangle:=-\langle\sigma,p'\rangle\quad\langle\pi\sigma,p\rangle:=\langle\sigma,\pi p\rangle\quad(p\in\mathscr{P}).\tag{2.1}$$

With these definitions in (2.1), it is possible to discuss *moment functional differential equations* of the form

$$\sum_{j=0}^{m}p_j\sigma^{(j)}=0,\tag{2.2}$$

where each $p_j\in\mathscr{P}$. Although (2.2) is actually equivalent to an infinite system of recurrence relations for the moments $\{\sigma_n\}_{n=0}^{\infty}$ of σ, it is convenient, in practice, to seek solutions w to (2.2) in the space $\mathscr{D}'$ of distributions, subject to the requirement that w act on $\mathscr{P}$. We remark that this latter interpretation of equations of the form (2.2) has been effectively used to construct orthogonalizing weight distributions for certain PSs; in particular, Kwon et al. [42] used this method to solve the long outstanding Bessel moment problem in 1991 (see also [26]); see Section 3.5 for further details.

3. The Erice report

We now report on new results — obtained since 1990 — pertaining to Sections 4–8 of the Erice Report [30]; the subsections below have identical names to those sections from [30].

3.1. Connections

In [30], this section was concerned with relationships that existed among the various PSs defined in [30]. We refer the reader to [30, Section 3] for the definitions of the PS classes named: (OPS on $[a,b]$) (briefly, these are the positive-definite orthogonal polynomials), TPS, DPS, and (SDPS on (a,b)). For subsequent discussion, we remind the reader that if $\{p_n\}_{n=0}^{\infty} \in \mathrm{DPS}(N)$, there exists a differential equation of the form (1.1) such that $L_N[p_n] = \lambda_n p_n$ for all $n \in \mathbb{N}_0$.

It is well known that if a differential expression with real, sufficiently differentiable coefficients is Lagrangian symmetrizable, then it is necessarily of even order. In [76], the authors determine necessary and sufficient conditions for a differentiable expression $L[\,\cdot\,]$ (with sufficiently smooth complex coefficients) to be symmetrizable; in the case that

$$L[y](x) = \sum_{j=0}^{2r} a_j(x) y^{(j)}(x)$$

has real coefficients, with $a_{2r}(x) \neq 0$ on some interval of $\mathbb{R}$, these conditions reduce to the system

$$\sum_{i=0}^{2r-2k-1} (-1)^i \binom{i+k}{k} (a_{2k+i+1}(x)y(x))^{(i)} = 0 \quad (k = 0, 1, \ldots, r-1) \tag{3.1}$$

of r homogeneous differential equations — called the *symmetry equations of* $L[\,\cdot\,]$ — having a simultaneous nontrivial solution $w(x)$; such a w is called a *symmetry factor* for $L[\cdot]$ and is necessarily given by

$$w(x) = K \exp\left(\frac{1}{r}\int^{x} \frac{a_{2r-1}(t)}{a_{2r}(t)}\,\mathrm{d}t\right) \quad (K \neq 0), \tag{3.2}$$

we remark that system (3.1) plays an important and fundamental role in the classification of polynomial solutions to differential equations of the form (1.1) that are orthogonal to bilinear forms of the type (1.2); indeed, we shall see this connection in Theorem 3.1 below.

In [73], Kwon and Yoon prove a remarkable result that further connects the classes ($\mathrm{DPS}(N) \cap$ TPS) and $\mathrm{SDPS}(N)$. Specifically, they show that if $\{p_n\}_{n=0}^{\infty}$ is a TPS with $y = p_n(x)$ $(n \in \mathbb{N}_0)$ satisfying the differential equation

$$L_{2r}[y](x) = \sum_{j=1}^{2r} a_j(x) y^{(j)}(x) = \lambda_n y(x),$$

then $L_{2r}[\cdot]$ is Lagrangian symmetrizable; that is to say, the system of symmetry equations in (3.1) has w, defined in (3.2), as a common solution. Consequently, there exists functions $\{q_j\}_{j=1}^r$ such that, on any interval I not containing a root of $a_{2r}(\cdot)$, each $q_j \in C^\infty(I)$ and $L_{2r}[\cdot]$ can be written as

$$L_{2r}[y](x) = \frac{1}{w(x)} \sum_{j=1}^{r} (-1)^j (q_j(x) y^{(j)}(x))^{(j)} \quad (x \in I). \tag{3.3}$$

The upshot of the Kwon–Yoon result is that the classical Glazman–Krein–Naimark (GKN) theory can, in some form, be applied to any member of the (OPS on $[a,b]$)$\cap$DPS(N) class for any $N \in \mathbb{N}$.

The only new class of PSs that is not specifically mentioned in the Erice Report is a *Bochner–Krall class of order N* (or BKS(N)):

Definition 3.1. A PS $\{p_n\}_{n=0}^\infty$ is called a Bochner–Krall OPS of order N if

$$\{p_n\}_{n=0}^\infty \in \mathrm{DPS}(N) \cap \mathrm{TPS} := \mathrm{BKS}(N).$$

Evidently, $\{p_n\}_{n=0}^\infty \in \mathrm{BKS}(N)$ if and only if

(i) there exists a (possibly signed) real Borel measure v on the Borel subsets of the real line such that

$$\int_{\mathbb{R}} p_n p_m \, \mathrm{d}v = k_n \delta_{n,m} \quad (n,m \in \mathbb{N}_0)$$

for some sequence $\{k_n\}_{n=0}^\infty$ of nonzero real numbers, and

(ii) there exists a sequence of real numbers $\{\lambda_n\}_{n=0}^\infty$ and a differential equation of the form (1.1) such that

$$\sum_{k=1}^{N} a_k(x) p_n^{(k)}(x) = \lambda_n p_n(x).$$

The BKS classification problem seeks to determine the contents of BKS(N) for each $N \in \mathbb{N}$.

It is well known that each $a_k(x)$ is necessarily a polynomial of degree $\leqslant k$; furthermore, if we write $a_k(x) = \sum_{j=0}^{k} \ell_{k,j} x^j$ for $k = 1,2,\ldots,N$, then $\lambda_n = \sum_{k=1}^{N} P(n,k)\ell_{k,k}$.

Both Bochner and Lesky (see [18,74]) determined the contents of BKS(2) under a complex linear change of variable. Under this constraint, the contents of BKS(2) are

$$\begin{array}{ll} (1) & \text{the Jacobi PS } \{P_n^{(\alpha,\beta)}\}_{n=0}^\infty, \text{ where } -\alpha, -\beta, -(\alpha+\beta+1) \notin \mathbb{N}, \\ (2) & \text{the Laguerre PS } \{L_n^\alpha\}_{n=0}^\infty, \text{ where } -\alpha \notin \mathbb{N}, \\ (3) & \text{the Hermite PS } \{H_n\}_{n=0}^\infty, \\ (4) & \text{the Bessel PS } \{y_n^a\}_{n=0}^\infty \text{ where } -(a+1) \notin \mathbb{N}. \end{array} \tag{3.4}$$

In [61], H.L. Krall proved that BKS($2N+1$) $= \emptyset$ for all $N \in \mathbb{N}_0$ using his powerful “classification theorem”. We now state this result together with a more recent interpretation (see also [96]) in terms of moment functional differential equations that was obtained by Littlejohn [75] and

Kwon et al. [70, Theorem 2.4]; to date, this continues to be the main structural theorem for the BKS classification problem.

Theorem 3.1. *Suppose* $\{p_n\}_{n=0}^{\infty}$ *is a* TPS *with associated moment sequence* $\{\mu_n\}_{n=0}^{\infty}$. *Then* $\{p_n\}_{n=0}^{\infty} \in \mathrm{BKS}(N)$ *with* $y = p_n(x)$ *satisfying*

$$L_N[y](x) := \sum_{k=1}^{N} \sum_{j=0}^{k} \ell_{k,j} x^j y^{(k)}(x) = \lambda_n y(x) \quad (n \in \mathbb{N}_0)$$

if and only if N *is even and either one of the following equivalent conditions holds*:

(i) [61] *the moments* $\{\mu_n\}_{n=0}^{\infty}$ *satisfy the* $r = [(N+1)/2]$ *recurrence relations*

$$S_k(m) = \sum_{i=2k+1}^{N} \sum_{j=0}^{i} \binom{i-k-1}{k} P(m-2k-1, i-2k-1)\ell_{i,i-j}\mu_{m-j} = 0$$

for $k = 0, 1, \ldots, r-1$ *and all integers* $m \geqslant 2k+1$;

(ii) [75,70] σ *satisfies the* $r = [(N+1)/2]$ *moment functional differential equations*

$$R_k(\sigma) = \sum_{i=0}^{N-2k-1} (-1)^i \binom{i+k}{k} (a_{2k+i+1}\sigma)^{(i)} = 0 \tag{3.5}$$

for $k = 0, 1, \ldots, r-1$.

Observe that the symmetry equations in (3.1) are identical to the moment functional differential equations in (3.5). Krall used this theorem to classify BKS(4) (see [62]) under a complex linear change of variable. In this case, the contents of BKS(4) are the four PSs listed in (3.4) together with:

$$\begin{aligned}
&(5)\ \text{the Legendre-type PS } \{P_n^{(0,0,M,M)}\}_{n=0}^{\infty}, \text{ where } M \neq -2/(n(n+1)) \text{ for } n \in \mathbb{N},\\
&(6)\ \text{the Laguerre-type PS } \{L_n^{0,A}\}_{n=0}^{\infty}, \text{ where } A \neq -1/n \text{ for } n \in \mathbb{N},\\
&(7)\ \text{the Jacobi-type PSs } \{P_n^{(\alpha,0,M,0)}\}_{n=0}^{\infty} \text{ and } \{P_n^{(0,\beta,0,N)}\}_{n=0}^{\infty},\\
&\qquad \text{where } -\alpha, -\beta \in \mathbb{N},\ n^2 + \alpha n + 2^{\alpha}/M \neq 0, \text{ and } n^2 + \beta n + 2^{\beta}/N \neq 0 \text{ for } n \in \mathbb{N}_0.
\end{aligned} \tag{3.6}$$

Koornwinder first studied the Jacobi-type $\{P_n^{(\alpha,\beta,M,N)}\}_{n=0}^{\infty}$ and the Laguerre-type polynomials $\{L_n^{\alpha,A}\}_{n=0}^{\infty}$ in [59]; these PSs are orthogonal with respect to, respectively, the weight distributions

$$w_{\alpha,\beta,M,N}(x) = (1-x)^{\alpha}(1+x)^{\beta} + M\delta(x+1) + N\delta(x-1) \quad (M, N \geqslant 0; \alpha, \beta > -1) \tag{3.7}$$

on $[-1, 1]$ and

$$w_{\alpha,A}(x) = x^{\alpha}\exp(-x) + A\delta(x) \quad (A \geqslant 0; \alpha > -1) \tag{3.8}$$

on $[0, \infty)$. Here, δ refers to the Dirac-delta distribution. We also note that A.M. Krall [60] was the first to examine, in detail, the polynomials listed in (3.6) as well as the spectral theory of the associated fourth-order differential equations; see also [79] where the spectral analysis of the five Legendre- and Legendre-type differential equations are studied.

A complete determination of BKS(6) is still unknown.

3.2. *The class* (OPS *on* $[a,b]$) $\cap$ (SDPS *on* (a,b))

As discussed in the Erice Report, this is the most interesting class of orthogonal polynomial eigenfunctions of ordinary differential equations from the perspective of spectral theory of self-adjoint differential operators.

In a surprising — in fact, completely unexpected — development, the authors in [50] discover a PS in the DPS(4) \ BKS(4) class that is an orthogonal sequence but the associated fourth-order differential equation $\ell_4[\cdot]$ is *not* Lagrangian symmetrizable. Indeed, the fourth-order differential equation

$$\ell_4[y] = (x^2-1)^2 y^{(4)} + 4x(x^2-1)y^{(3)} + 2(x-1)((1+2A)x+2A+3)y'' = \lambda_n y, \tag{3.9}$$

where $\lambda_n = n(n-1)(n^2-n+4A)$ has a sequence of polynomial solutions $\{p_n\}_{n=0}^{\infty}$ which are orthogonal with respect to the positive-definite inner product

$$(p,q)_H = p(1)\bar{q}(1) + \frac{1}{A}p'(-1)\bar{q}'(-1) + \int_{-1}^{+1} p'\bar{q}'\,\mathrm{d}x \quad (p,q\in\mathscr{P}). \tag{3.10}$$

This inner product is not of the classical form (1.2) so $\{p_n\}_{n=0}^{\infty} \notin$ BKS(4). Moreover, and this was equally unexpected, the *non-Lagrangian symmetrizable* expression $\ell_4[\cdot]$ in (3.9) does generate a *self-adjoint* operator, with the PS $\{p_n\}_{n=0}^{\infty}$ as eigenfunctions, in some Hilbert–Sobolev function space H with inner product $(\cdot,\cdot)_H$ (see [28]). Although $\ell_4[\cdot]$ cannot be put into the form (3.3), it is important to note that

$$(\ell_4[p],q)_H = (p,\ell_4[q])_H \quad (p,q\in\mathscr{P}); \tag{3.11}$$

in other words, $\ell_4[\cdot]$ is symmetric in the inner product from H. The moral of this example is that we should only expect a formally Lagrangian symmetric representation (3.3) of a differential equation having a sequence of orthogonal polynomial solutions when this PS belongs to BKS(N) for some $N\in\mathbb{N}$.

A significant step in solving the BKS classification problem was achieved by Kwon et al. [72]. In this contribution, the authors show that if a PS $\{p_n\}_{n=0}^{\infty}$ is orthogonal with respect to the classic bilinear form (1.2) and satisfies (1.1), with the leading coefficient $a_N(x)$ a nonzero constant, then $\{p_n\}_{n=0}^{\infty}$ is, up to a real linear change of variables, either the Hermite or twisted Hermite polynomials (see Section 4).

Another surprising example, with important ramifications on the BKS classification problem, is due to Grünbaum et al. [40]. In this paper, the authors show that the PS $\{p_n\}_{n=0}^{\infty}$, orthogonal with respect to the weight distribution

$$w(x) = \exp(-x) + A\delta(x) - B\delta'(x)$$

on $[0,\infty)$, satisfies a 10th-order differential equation of the form (1.1); that is, $\{p_n\}_{n=0}^{\infty} \in$ BKS(10). An interesting feature of this equation is that the associated spectral analysis must likely be conducted in a Pontryagin space if $B>0$.

In all of the known examples from the (OPS on $[a,b]$)$\cap$(SDPS on (a,b)) class, there is a positive, self-adjoint representation T of the associated differential equation, in some Hilbert space, having the polynomial solutions as eigenfunctions (see [30] and the references therein for specific literature on the spectral theory for these examples). Consequently, as a result of a general theory developed by Littlejohn and Wellman [77], there is a continuum of unique Hilbert spaces $\{H_r\}_{r>0}$ — the space H_r is called the rth *left-definite space* — and, for each $r > 0$, there is a unique self-adjoint representation T_r — called the rth *left-definite operator* — of T in H_r (see Conjecture 5.2 in the Erice Report). In terms of positive-definite orthogonal polynomials in the BKS class, the work of Littlejohn and Wellman show that there is a continuum of nonisometric Hilbert spaces where these polynomials are orthogonal.

A result of significant importance that has recently been obtained is the contribution of Koekoek and Koekoek [54] (see also [12]). In this paper, they determine the minimal order of the differential equation of the form (1.1) having the Jacobi-type polynomials $\{P_n^{(\alpha,\beta,M,N)}(x)\}_{n=0}^{\infty}$ as eigenfunctions. In [57] (see also [52]) and [54], the authors explicitly find these differential equations and show that their orders are:

Conditions on $w_{\alpha,\beta,M,N}$	Order of DE
$M=N=0;\alpha,\ \beta>-1$	2
$M>0,N=0;\ \alpha>-1,\ \beta\in\mathbb{N}_0$	$2\beta+4$
$M=0,N>0;\ \alpha\in\mathbb{N}_0,\beta>-1$	$2\alpha+4$
$M=N>0;\ \alpha=\beta:=n\in\mathbb{N}_0$	$2n+4$
$0<M\neq N>0;\ \alpha,\beta\in\mathbb{N}_0$	$2\alpha+2\beta+6$
Otherwise	∞

(3.12)

Their result also corrects a conjecture in the Erice Report (see [30, p. 38]). In deriving these explicit equations, the authors use a very important technique — called the *inversion formula* — due to Bavinck and Koekoek [16] (see also [7,53]) which has interesting, and important, consequences on its own right. Indeed, a key result in finding the differential equations for the Jacobi-type PSs is the following new identity for the Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$ when $\alpha,\beta>-1$:

$$\sum_{k=j}^{i}\frac{\alpha+\beta+2k+1}{(\alpha+\beta+k+j+1)_{i-j+1}}P_{i-k}^{(-\alpha-i-1,-\beta-i-1)}(x)P_{k-j}^{(\alpha+j,\beta+j)}(x)=\delta_{i,j}\quad (j\leqslant i;\ i,j\in\mathbb{N}_0). \tag{3.13}$$

It is remarkable how this inversion method — initially developed to find differential equations for *nonclassical* orthogonal polynomials — gives new results about *classical* orthogonal polynomials.

The authors (see [51]) had earlier determined, and this was reported in [30], with help from the symbolic program MAPLE, the orders of the differential equations for the Laguerre-type polynomials $\{L_n^{\alpha,A}\}_{n=0}^{\infty}$ when $A\geqslant 0$; the table below lists these orders:

Conditions on $w_{\alpha,A}$	Order of DE
$A=0;\ \alpha>-1$	2
$A>0;\ \alpha\in\mathbb{N}_0$	$2\alpha+4$
$A>0;\ \alpha>-1$ but $\alpha\notin\mathbb{N}_0$	∞

(3.14)

3.3. The DPS classification problem

In view of new developments and a significant generalization of the BKS(N) problem (that we further discuss in the next section), the most important class of PSs is, arguably, the DPS class.

Indeed, as discussed in Section 3.2, there are examples $\{p_n\}_{n=0}^{\infty} \in \mathrm{DPS}(N)\backslash\mathrm{BKS}(N)$ and yet $\{p_n\}_{n=0}^{\infty}$ forms an orthogonal set with respect to some nonclassical-type bilinear form. Consequently, more structural theorems like Theorem 3.1 are needed to determine if a DPS is orthogonal with respect to some specific bilinear form; we offer one such extension in Theorem 5.1 below.

There are seven DPSs, up to a *real* linear change of variables, satisfying second-order equations of the form

$$a_2(x)y'' + a_1(x)y' = \lambda y.$$

These DPSs are determined by the leading, real coefficient $a_2(x)$ of which there are five possibilities:

$a_2(x)$	Number (and name) of DPSs
1	2 (Hermite and twisted Hermite)
x	1 (Laguerre)
x^2	2 (Bessel and $\{x^n\}_{n=0}^{\infty}$)
$1-x^2$	1(Jacobi)
$1+x^2$	1 (twisted Jacobi)

To date, no complete classification is available for any other DPS class; in particular, the contents of DPS(4) is not completely known. Recent work of Jung et al. [50] and Yoon [98] have turned up two new fourth-order differential equations with orthogonal polynomial solutions. Indeed, as previously discussed in Section 3.2, the authors in [50] found a PS in the DPS(4) class (see (3.9)) which is an orthogonal PS but lies outside of the BKS(4) class. It is possible that a closer examination of Krall's classic 1940 paper [62] may reveal further information on this DPS(4) classification. Afterall, Krall determined the contents of the BKS(4) class (up to a complex linear change of variable) by an exhaustive method involving over forty cases. However, Krall was only looking for PSs in the BKS(4) class and dismissed any polynomial sequences that were outside of this set. In [98], Yoon reconsidered the BKS(4) classification problem from the standpoint of a real linear change of variable. He discovered that there is one new PS in BKS(4) — the *twisted Legendre-type polynomials* — emerging from this point of view. This PS is a TPS but is not in the class (OPS on $[a,b]$) for any closed interval $[a,b]$.

3.4. The Koornwinder–Laguerre OPS on $[0,\infty)$

The Laguerre-type polynomials $\{L_n^{\alpha,A}\}_{n=0}^{\infty}$ are orthogonal on $[0,\infty)$ with respect to the weight distribution $w_{\alpha,A}$ given in (3.4). The order of the differential equation $L_\alpha[y]=\lambda y$ that these polynomials satisfy is given in (3.14). A problem was posed in [30, Problem 7] to determine:

(i) if the finite-order Laguerre-type differential equations $L_\alpha[y]=\lambda y$ are Lagrangian symmetrizable and, if so,
(ii) to explicitly determine these formally symmetric equations.

Of course, the result of Kwon and Yoon [73] gives an affirmative answer to (i). Moreover, in [31,78], the authors determine the Lagrangian symmetric form of each of these finite-order equations. Using these symmetric forms, Wellman [97] develops both the right-definite and (first) left-definite theory for each of these symmetric Laguerre-type differential equations. More specifically, in the right-definite setting, for each $\alpha \in \mathbb{N}_0$, he constructs a self-adjoint operator T_α, generated from $L_\alpha[\cdot]$, in the Hilbert space $L^2([0,\infty); w_{\alpha,A})$ having the Laguerre-type polynomials $\{L_n^{\alpha,A}\}_{n=0}^\infty$ as a complete orthogonal set of eigenfunctions. Remarkably, the right-definite operators $\{T_\alpha\}_{\alpha=0}^\infty$ were constructed without the aid of the GKN theory. In the (first) left-definite spectral setting, Wellman first constructs, for each $\alpha \in \mathbb{N}_0$, a Hilbert–Sobolev space $H_\alpha[0,\infty)$, with inner product depending on both the (positive) coefficients of the Lagrangian symmetric form of $L_\alpha[\cdot]$ and the inner product inherited from $L^2([0,\infty); w_{\alpha,A})$. He then constructs a self-adjoint operator S_α (the first left-definite operator) in $H_\alpha[0,\infty)$, generated by $L_\alpha[\cdot]$, having the Laguerre-type polynomials as a complete set of eigenfunctions.

3.5. The spectral theory of the Bessel polynomials

Kwon et al. [42] and Duran [26] independently solved, using different methods, the long-standing Bessel moment problem of constructing a real-valued weight function for the simple Bessel polynomials. This is, unarguably, one of the most important new results obtained in the interplay between orthogonal polynomials, the theory of moments, and differential equations. The (necessarily signed) weight function $w: \mathbb{R} \to \mathbb{R}$, given by Kwon et al. is given explicitly by

$$w(x) = \begin{cases} 0 & \text{if } x < 0, \\ -\exp(-2/x) \displaystyle\int_x^\infty \frac{\exp(2/t)\exp(-t^{1/4})\sin(t^{1/4})}{t^2}\,\mathrm{d}t & \text{if } x \geqslant 0. \end{cases} \tag{3.15}$$

Maroni [81] generalizes the method used in [42] and obtains a weight function for the Bessel polynomials $\{y_n^a(x)\}_{n=0}^\infty$ for all $a \geqslant 12(2/\pi)^4 - 2$. Duran's construction in [26] is more general and gives explicit weight functions in the Schwartz space $\mathscr{S}(\mathbb{R})$ for the general Bessel polynomials $\{y_n^a(x)\}_{n=0}^\infty$ $(-(a+1) \in \mathbb{N})$. To date, a classic spectral analysis — using both "hard" analytic and operator theoretic techniques — of the associated general Bessel differential expression, defined by

$$\ell_B^a[y](x) = x^2 y'' + ((a+2)x + 2)y' \quad (x \in \mathbb{R}; -(a+1) \in \mathbb{N}), \tag{3.16}$$

in the appropriate Krein space, using any of these constructed weight functions has not been attempted. As discussed in the Erice Report, Han and Kwon [43], using a hyperfunctional weight, developed the spectral theory of the simple ($a=0$) Bessel polynomials $\{y_n\}_{n=0}^\infty$ in the appropriate Krein space. In this setting, they obtain a self-adjoint operator, generated from the second-order Bessel polynomial differential expression $\ell_B^0[\cdot]$, having the simple Bessel polynomials as eigenfunctions.

4. Further results on the BKS classification problem

In [68], the authors revisit the original Bochner second-order problem and show that, up to a *real* linear change of variable, the contents of BKS(2) are the four PSs in (3.4), together with:

$$\begin{array}{l} (5) \text{ the twisted Jacobi PS } \{\check{P}_n^{(\alpha,\beta)}\}_{n=0}^\infty \text{ where } -(\alpha+\beta+1) \notin \mathbb{N} \text{ and } \bar{\alpha} = \beta, \\ (6) \text{ the twisted Hermite PS } \{\check{H}_n\}_{n=0}^\infty. \end{array} \tag{4.1}$$

The second-order twisted Jacobi equation is given by

$$(1+x^2)y'' + ((\alpha+\beta+2)x + i(\alpha-\beta))y' = \lambda y \quad (-(\alpha+\beta+1) \notin \mathbb{N} \text{ and } \bar{\alpha} = \beta), \tag{4.2}$$

moreover, when $\lambda = \lambda_n = n(n+\alpha+\beta+1)$, $y = \check{P}_n^{(\alpha,\beta)}(x) := i^n P_n^{(\alpha,\beta)}(-ix)$ $(n \in \mathbb{N}_0)$ is a real polynomial solution of (4.2). The twisted Hermite equation is defined as

$$y'' + 2xy' = \lambda y, \tag{4.3}$$

for each $n \in \mathbb{N}_0$, $y = \check{H}_n(x) := i^n H_n(-ix)$ is a real polynomial solution of (4.3) when $\lambda = \lambda_n = 2n$. Under a complex linear change of variable, Eqs. (4.2) and (4.3) become, respectively, the classical Jacobi and Hermite differential equations. The two PSs in (4.1) belong to the class TPS but are not positive-definite orthogonal polynomials. No spectral analysis on the associated twisted Jacobi or twisted Hermite expressions has been accomplished at the time of this writing.

Another important development in the general BKS classification problem can be seen in the extensive work of Duistermaat and Grünbaum [23], and Grünbaum and Haine (see [37–39]). In these papers, the authors use the Darboux transformation to obtain important and general structural results concerning the BKS classification problem. In particular, they re-obtain the BKS(2) and BKS(4) classifications by applying the Darboux factorization method on the Jacobi matrix associated with the three-term recurrence relation for the orthogonal polynomials.

Although a complete characterization of each BKS(N) is still unknown, there has been significant progress made recently. The following result, due to Kwon et al. [71], characterizes *all* PSs in each BKS(N) class ($N \in \mathbb{N}$) which are orthogonal relative to $\tau = \sigma + v$, where σ is a classical moment functional (i.e. Jacobi $\sigma_J^{(\alpha,\beta)}$, twisted Jacobi $\sigma_{\check{J}}^{(\alpha,\beta)}$, Laguerre σ_L^{α}, Hermite σ_H, twisted Hermite $\sigma_{\check{H}}$, or Bessel $\sigma_B^{(a)}$) and v is a distribution of order 0 with finite support. The theorem below also gives necessary and sufficient conditions for when the Jacobi-type $\{P_n^{(\alpha,\beta,M,N)}\}_{n=0}^{\infty}$ polynomials and Laguerre-type polynomials $\{L_n^{\alpha,A}\}_{n=0}^{\infty}$ satisfy finite-order differential equations; see also [100], where Zhedanov obtains necessary conditions for when the Jacobi-type polynomials $\{P_n^{(\alpha,\beta,M,N)}\}_{n=0}^{\infty}$ satisfy a finite-order differential equation.

Theorem 4.1. *Let*

$$\tau = \sigma + M\delta(x-a) + N\delta(x-b)$$

be a real quasi-definite moment functional with corresponding TPS $\{p_n\}_{n=0}^{\infty}$, *where* $a, b \in \mathbb{C}$ *are such that* $|a| + |b| \neq 0$, *and where* σ *is a classical moment functional. Suppose* $\{p_n\}_{n=0}^{\infty} \in$ BKS(N) *for some (even)* $N \in \mathbb{N}$. *Then* σ *is necessarily either the Jacobi* $\sigma_J^{(\alpha,\beta)}$, *twisted Jacobi* $\sigma_{\check{J}}^{(\alpha,\beta)}$, *or Laguerre* σ_L^{α} *moment functional. More specifically*,

(i) *if* $\sigma = \sigma_J^{(\alpha,\beta)}$, *then* $\tau = \sigma_J^{(\alpha,\beta)} + M\delta(x-1) + N\delta(x+1)$. *Furthermore, if* $M \neq 0$, *then* $\alpha \in \mathbb{N}_0$ *and if* $N \neq 0$, *then* $\beta \in \mathbb{N}_0$.
(ii) *if* $\sigma = \sigma_L^{\alpha}$, *then* $\tau = \sigma_L^{\alpha} + M\delta(x)$. *Furthermore, if* $M \neq 0$, *then* $\alpha \in \mathbb{N}_0$.
(iii) *if* $\sigma = \sigma_{\check{J}}^{(\alpha,\beta)}$, *then* $\tau = \sigma_{\check{J}}^{(\alpha,\beta)} + M\delta(x-i) + \bar{M}\delta(x+i)$. *Furthermore, if* $M \neq 0$, *then* $\alpha = \beta \in \mathbb{N}_0$.

This theorem answers several questions that have been raised in the past. For example, Hendriksen, in [44], studied properties of the simple Bessel-type polynomials $\{y_n^{(0,\lambda)}\}_{n=0}^{\infty}$ which are orthogonal with respect to $\tau = \sigma_{\mathrm{B}}^{(0)} + \lambda\delta(x)$ $(\lambda \neq 0)$, where $\sigma_{\mathrm{B}}^{(0)}$ is the moment functional for the simple Bessel polynomials. Efforts have been made in the past to determine if this Bessel-type PS lies in some BKS(N) class. This theorem says, in fact, that $\{y_n^{(0,\lambda)}\}_{n=0}^{\infty} \notin$ BKS(N) for any $N \in \mathbb{N}$. Moreover, this result explains why certain Jacobi- and Laguerre-type polynomials must satisfy infinite-order differential equations. We remark that the tables in (3.12) and (3.14) give the appropriate order of the BKS class for the polynomials listed in (i) and (ii) above. In [71], the authors show that the twisted Jacobi-type polynomials in (iii) are in the BKS($2\alpha+4$) class when $M \neq 0$ and $\alpha=\beta$. As a further application, they find a new PS in the class BKS(6): the *twisted Krall-type polynomials* $\{\check{K}_n\}_{n=0}^{\infty}$ defined, for each $n \in \mathbb{N}_0$, by $\check{K}_n(x) = i^{-n} P_n^{(0,0,(A+iB)^{-1},(A-iB)^{-1})}(ix)$; this PS is orthogonal with respect to the bilinear form

$$\langle p,q\rangle_{\check{K}} = \int_{-1}^{+1} p\bar{q}\,\mathrm{d}x + \frac{1}{A+iB} p(i)\bar{q}(i) + \frac{1}{A-iB} p(-i)\bar{q}(-i) \quad (A^2+B^2 \neq 0;\ p,q \in \mathscr{P}).$$

5. New developments: Sobolev orthogonal polynomials and a generalization of the BKS problem

In 1993, Koekoek [56] produced a PS $\{p_n\}_{n=0}^{\infty}$ which are eigenfunctions of an eighth-order differential equation of the form (1.1) and are orthogonal with respect to the Sobolev bilinear form

$$\phi(p,q) = \int_0^{\infty} p(x)\bar{q}(x)\exp(-x)\,\mathrm{d}x + A p'(0)\bar{q}'(0) \quad (A>0;\ p,q \in \mathscr{P}). \tag{5.1}$$

Observe that there is no moment functional σ satisfying $\phi(p,q) = \langle \sigma, pq\rangle$ for all $p,q \in \mathscr{P}$; indeed, this bilinear form is the sum of two moment functionals:

$$\phi(p,q) = \langle \sigma, pq\rangle + \langle \tau, p'q'\rangle \quad (p,q \in \mathscr{P}).$$

This example is important in that it opens up new classification and related research (for example, new types of spectral) problems. Moreover, no longer does *orthogonality* necessarily mean with respect to classical Lebesgue-type bilinear forms as defined in (1.2), which was the focus of the Erice Report. Since Koekoek's example first appeared, several new examples have emerged which we summarize below. In view of these discoveries, we make the following definition which generalizes the classic BKS(N) problem.

Definition 5.1. Let $N \in \mathbb{N}$ and $M \in \mathbb{N}_0$. Suppose there exists a PS $\{p_n\}_{n=0}^{\infty}$, a real sequence $\{\lambda_n\}_{n=0}^{\infty}$, and a real, linear differential equation $L_N[\,\cdot\,]$ of order N of the form (1.1) such that

(i) $L_N[p_n](x) = \lambda_n p_n(x)$ ($n \in \mathbb{N}_0$) and
(ii) there exists $M+1$ moment functionals $\sigma_0, \sigma_1, \ldots, \sigma_M$, with $\sigma_M \neq 0$, such that $\{p_n\}_{n=0}^{\infty}$ is orthogonal with respect to the Sobolev bilinear form

$$\phi_M(p,q) := \sum_{r=0}^{M} \langle \sigma_r, p^{(r)}q^{(r)}\rangle \quad (p,q \in \mathscr{P}), \tag{5.2}$$

that is to say, there exists constants $\{K_n\}_{n=0}^{\infty} \subset \mathbb{R}\setminus\{0\}$ such that

$$\phi_M(p_n, p_m) = K_n \delta_{n,m} \quad (n,m \in \mathbb{N}_0).$$

Then we write $\{p_n\}_{n=0}^{\infty} \in$ BKS(N,M) and call $\{p_n\}_{n=0}^{\infty}$ a Bochner–Krall PS of order (N,M).

In particular, observe that $\mathrm{BKS}(N,0)=\mathrm{BKS}(N)$. Koekoek's example, described above, lies in the $\mathrm{BKS}(8,1)$ class. We note that, by Boas' and Duran's theorems, the bilinear form in (5.2) has the representations

$$\phi_M(p,q)=\sum_{r=0}^{M}\int_{\mathbb{R}} p^{(r)}\bar{q}^{(r)}\,\mathrm{d}\mu_r=\sum_{r=0}^{M}\int_{\mathbb{R}} p^{(r)}\bar{q}^{(r)}w_r\,\mathrm{d}x,$$

where each μ_r is a finite, real Borel measure and each $w_r\in\mathscr{S}(\mathbb{R})$, the space of Schwartz functions.

For more than 10 years, the theory of Sobolev orthogonal polynomials has been the subject of an intense study. Certainly, part of the interest concerning this study lies in the fact that many of its features are different from the theory of polynomials orthogonal with respect to (1.2). For example, polynomials orthogonal with respect to (5.2) need not satisfy a three-term recurrence relation, unlike their classical counterparts (see, for example, [27]). For a general discussion of Sobolev orthogonal polynomials and various historical connections, see the contributions [2,82,83,90].

One of the first general results concerning this $\mathrm{BKS}(N,M)$ class can be found in [70]. In this paper, the authors prove that

$$\mathrm{BKS}(2N)\subset\mathrm{BKS}(2N,N)\quad(N\in\mathbb{N}). \tag{5.3}$$

In the special case $N=1$, this result is essentially the Hahn–Sonine characterization (see [41,93]) of the classical orthogonal polynomials. In terms of operator and spectral theory, the inclusion in (5.3) has an important left-definite operator-theoretic interpretation (see [77]).

In [69], Kwon and Littlejohn determined the contents of the $\mathrm{BKS}(2,1)$ class, under both a real and a linear change of variable. The contents, under a real linear change of variable, are the six PSs listed in (3.4) and (4.1), together with

(7) the Jacobi polynomials $\{P_n^{(-1,-1)}\}_{n=0}^{\infty}$,	(5.4)
(8) the Jacobi polynomials $\{P_n^{(-1,\beta)}\}_{n=0}^{\infty}$ $(-\beta\notin\mathbb{N})$,	
(9) the Jacobi polynomials $\{P_n^{(\alpha,-1)}\}_{n=0}^{\infty}$ $(-\alpha\notin\mathbb{N})$,	
(10) the Laguerre polynomials $\{L_n^{-1}\}_{n=0}^{\infty}$,	
(11) the twisted Jacobi polynomials $\{\check{P}_n^{(-1,-1)}\}_{n=0}^{\infty}$.	

A key structural result for the $\mathrm{BKS}(N,1)$ class (see [48,49,65,69]) is the following theorem.

Theorem 5.1. *Consider the differential equation*

$$L_N[y](x)=\sum_{k=1}^{N}\sum_{j=0}^{k}\ell_{k,j}x^j y^{(k)}(x)=\lambda y(x), \tag{5.5}$$

and the quasi-definite bilinear form

$$\phi(p,q)=\langle\sigma,pq\rangle+\langle\tau,p'q'\rangle\quad(p,q\in\mathscr{P}), \tag{5.6}$$

where σ and τ are moment functionals. Let $\{p_n\}_{n=0}^{\infty}$ be a PS. *Then, for each $n\in\mathbb{N}_0$, $y=p_n(x)$ satisfies* (5.5) *and $\{p_n\}_{n=0}^{\infty}$ is orthogonal with respect to* (5.6) *if and only if $N=2r$ is even and σ and τ satisfy the following system of $r+1$ moment functional differential equations*

$$R_{2k+2}(\sigma,\tau)=0\quad(k=0,1,\ldots,r),$$

where

$$R_{2k+2}(\sigma,\tau)=\sum_{j=0}^{N-2k+1}(-1)^{j+1}\binom{j+2k-1}{2k-1}(a_{j+2k-1}\tau)^{(j)}-a_{2k-1}\tau-2(a_{2k}\tau)'-(a_{2k+1}\tau)''$$
$$-\sum_{j=0}^{N-2k}(-1)^{j+1}\binom{j+2k}{2k}(a_{j+2k}\tau')^{(j)}+a_{2k}\tau'+(a_{2k+1}\tau)'$$
$$-\sum_{j=0}^{N-2k-1}(-1)^{j+1}\binom{j+2k+1}{2k+1}(a_{j+2k+1}\sigma)^{(j)}+a_{2k+j}\sigma. \tag{5.7}$$

We remark that the differential equations in (5.7), in general, cannot be uncoupled in σ and τ unless $N=2$. This technical difficulty, when $N>2$, makes classifying even the BKS(4,1) class formidable at the present time; better techniques are currently being sought. We also remark that, when $\tau=0$, the above theorem reduces to Krall's 1938 classification result [61] (see Theorem 3.1).

In a subsequent paper [65], the authors determine the contents of BKS(2,2) using a further generalization of Theorem 5.2. Up to a real linear change of variable, the contents of this class are the eleven sets from BKS(2,1) as well as the PSs listed in the following table:

(12) the Jacobi polynomials $\{P_n^{(-2,-1)}\}_{n=0}^{\infty}$,	
(13) the Jacobi polynomials $\{P_n^{(-1,-2)}\}_{n=0}^{\infty}$,	
(14) the Jacobi polynomials $\{P_n^{(-2,-2)}\}_{n=0}^{\infty}$,	
(15) the Jacobi polynomials $\{P_n^{(-2,\beta)}\}_{n=0}^{\infty}$ $(-(\beta+2)\notin\mathbb{N})$,	(5.8)
(16) the Jacobi polynomials $\{P_n^{(\alpha,-2)}\}_{n=0}^{\infty}$ $(-(\alpha+2)\notin\mathbb{N})$,	
(17) the Laguerre polynomials $\{L_n^{-2}\}_{n=0}^{\infty}$,	
(18) the twisted Jacobi polynomials $\{\check{P}_n^{(-2,-2)}\}_{n=0}^{\infty}$.	

Observe that, in both (5.4) and (5.8), the classifications yield, essentially, the classical orthogonal polynomials with the notable exception that the parameters for the Jacobi, Laguerre, and twisted Jacobi PSs are appropriately relaxed.

Kwon and Littlejohn [67] (see also [88,89] subsequently showed that the Laguerre polynomials $\{L_n^{-k}(x)\}_{n=0}^{\infty}$, for each $k\in\mathbb{N}$, belong to the class BKS$(2,k)$. Of course, Favard's theorem (see [21, p. 75]) shows that $\{L_n^{-k}(x)\}_{n=0}^{\infty}$ cannot be orthogonal with respect to a bilinear form of the type (1.2) when $k\in\mathbb{N}$. In [29], the authors develop the spectral theory associated with the Laguerre differential equation

$$xy''+(1-k-x)y'=\lambda y \tag{5.9}$$

for $k=1$ and 2. A subsequent paper [32] is in preparation that handles the general case $k\in\mathbb{N}$ using methods developed in [77].

The authors in [3] generalize the work of Kwon and Littlejohn and study PSs that are orthogonal with respect to Sobolev bilinear forms of the type

$$\mathscr{B}_S^{(N)}(p,q)=(p(c),p'(c),\ldots,p^{(N-1)}(c))\boldsymbol{A}(q(c),q'(c),\ldots,q^{(N-1)}(c))^{\mathrm{T}}$$
$$+\langle\sigma,p^{(N)}q^{(N)}\rangle\quad(p,q\in\mathscr{P}), \tag{5.10}$$

where $(q(c),\ldots,q^{(N-1)}(c))^{\mathrm{T}}$ denotes the transpose of the vector $(q(c),\ldots,q^{(N-1)}(c))$, $\boldsymbol{A}$ is an $N\times N$ symmetric matrix with nonsingular principle submatrices, and σ is a moment functional. As an application, they show that $\{P_n^{(-N,\beta)}\}_{n=0}^{\infty}\in \mathrm{BKS}(2,N)$ when $-(\beta+N)\notin\mathbb{N}$. To date, no spectral analysis of the associated second-order Jacobi differential equations has been made.

In [4], the authors study the Gegenbauer polynomials $\{C_n^{(-N+1/2)}\}_{n=0}^{\infty}$ for $N\in\mathbb{N}$; this PS cannot be orthogonal with respect to a bilinear form of the type (1.2). They show that $\{C_n^{(-N+1/2)}\}_{n=0}^{\infty}\in \mathrm{BKS}(2,2N)$. Indeed, they prove that these Gegenbauer polynomials are orthogonal with respect to a Sobolev inner product of the form

$$\begin{aligned}
&\mathscr{G}_S^{(N)}(p,q)\\
&\quad=(p(1)\ldots p^{(N-1)}(1),p(-1)\ldots p^{(N-1)}(-1))\boldsymbol{B}(q(1)\ldots q^{(N-1)}(1),q(-1)\ldots q^{(N-1)}(-1))^{\mathrm{T}}\\
&\qquad+\int_{-1}^{+1}p^{(2N)}(x)\bar{q}^{(2N)}(x)(1-x^2)^N\,\mathrm{d}x\quad(p,q\in\mathscr{P}),
\end{aligned}\tag{5.11}$$

where $\boldsymbol{B}$ is a certain symmetric matrix. As of this writing, the spectral theory of the second-order Gegenbauer differential equation

$$(1-x^2)y''+(2N-2)xy'=\lambda y\quad(N\in\mathbb{N}),$$

having the polynomials $\{C_n^{(-N+1/2)}(x)\}_{n=0}^{\infty}$ as eigenfunctions, has not been completed.

The authors in [17] study the generalized Laguerre-type polynomials $\{L_n^{\alpha,M_0,M_1}\}_{n=0}^{\infty}$, where $\alpha>-1$ and $M_0,M_1\geqslant 0$, which are orthogonal with respect to the inner product

$$(p,q)_{\alpha,M_0,M_1}=\frac{1}{\Gamma(\alpha+1)}\int_0^{\infty}p(x)\bar{q}(x)x^{\alpha}\exp(-x)\mathrm{d}x+M_0p(0)\bar{q}(0)+M_1p'(0)\bar{q}'(0)\ \ (p,q\in\mathscr{P}).$$

Information that they obtained on the $\mathrm{BKS}(N,M)$ classification of these polynomials is given in the following table:

Conditions on α,M_0,M_1	Order of DE	$\mathrm{BKS}(N,M)$ class
$M_0=M_1=0$	2	BKS(2)
$M_0>0,M_1=0;\alpha\in\mathbb{N}_0$	$2\alpha+4$	$\mathrm{BKS}(2\alpha+4)$
$M_0=0,M_1>0;\alpha\in\mathbb{N}_0$	$2\alpha+8$	$\mathrm{BKS}(2\alpha+8,1)$
$M_0>0,M_1>0;\alpha\in\mathbb{N}_0$	$4\alpha+10$	$\mathrm{BKS}(4\alpha+10,1)$
$M_0^2+M_1^2\neq 0;\alpha\notin\mathbb{N}_0$	∞	Not applicable (NA)

(5.12)

Moreover, in all cases, the authors explicitly compute the corresponding differential equations.

In [8], Bavinck shows that the polynomials $\{L_n^{\alpha,0,0,\ldots,M_k}\}_{n=0}^{\infty}$, orthogonal with respect to

$$(p,q)_{\alpha,0,0,\ldots,M_k}=\frac{1}{\Gamma(\alpha+1)}\int_0^{\infty}p(x)\bar{q}(x)x^{\alpha}\exp(-x)\,\mathrm{d}x+M_kp^{(k)}(0)\bar{q}^{(k)}(0)\quad(p,q\in\mathscr{P}),$$

are in $\mathrm{BKS}(2\alpha+4k+4,k)$ if $M_k\neq 0$ and $\alpha\in\mathbb{N}_0$; moreover, if $\alpha\notin\mathbb{N}_0$, these polynomials satisfy an infinite-order differential equation. In both cases, Bavinck constructs the differential equations. For an excellent survey on Sobolev orthogonality for generalized Laguerre polynomials, see [11].

In [55] and [58], Koekoek and Meijer study the generalized Laguerre-Sobolev type polynomials $\{L_n^{\alpha,M_1(\ell_1),\ldots,M_k(\ell_k)}\}_{n=0}^{\infty}$ which are orthogonal with respect to the bilinear form

$$(p,q)_{\alpha,M_1(\ell_1),\ldots,M_k(\ell_k)} = \frac{1}{\Gamma(\alpha+1)}\int_0^{\infty} p(x)\bar{q}(x)x^{\alpha}\exp(-x)\,\mathrm{d}x + \sum_{r=1}^{k} M_r p^{(\ell_r)}(0)\bar{q}^{(\ell_r)}(0) \quad (p,q\in\mathscr{P}),$$

where $\{\ell_r\}_{r=1}^{k}$ are nonnegative integers with $\ell_1<\ell_2<\cdots<\ell_k$. At this time of writing, it is not clear what the value of $N=N(\alpha,k,\ell_1,\ldots,\ell_k)\in\mathbb{N}$ is so that $\{L_n^{\alpha,M_1(\ell_1),\ldots,M_k(\ell_k)}\}_{n=0}^{\infty}\in \mathrm{BKS}(N,\ell_k)$ when $\alpha\in\mathbb{N}_0$. However, Bavinck makes the following

Conjecture 5.1 (Bavinck [14]). *Suppose* $\alpha\in\mathbb{N}_0$ *and* $M_r>0$ $(r=1,\ldots,k)$. *Then the generalized Laguerre type polynomials*

$$\{L_n^{\alpha,M_1(\ell_1),\ldots,M_k(\ell_k)}\}_{n=0}^{\infty}\in \mathrm{BKS}\left(2k(\alpha+1)+2+4\sum_{r=1}^{k}\ell_r,\ell_k\right).$$

When $k=2$, Bavinck proves this conjecture in [14] and further shows that $\{L_n^{\alpha,M_1(\ell_1),M_2(\ell_2)}\}_{n=0}^{\infty}$ satisfies an infinite-order differential equation when $\alpha>-1$ but $\alpha\notin\mathbb{N}_0$. Bavinck uses another remarkable technique, which he developed, called the *perturbation method* for proving the existence of these differential equations (see [9,10,15]).

Some progress has been recently made on the appropriate $\mathrm{BKS}(N,M)$ classes and the explicit differential equations for the Jacobi–Sobolev-type polynomials $\{P_n^{(\alpha,\beta,M_0,\ldots,M_r;N_0,\ldots,N_r)}\}_{n=0}^{\infty}$ (see [5]), which are orthogonal with respect to the bilinear form defined by

$$\psi_{JS}(p,q)=\int_{-1}^{+1} p(x)\bar{q}(x)(1-x)^{\alpha}(1+x)^{\beta}\,\mathrm{d}x + \sum_{j=0}^{r} M_j p^{(j)}(-1)\bar{q}^{(j)}(-1)+\sum_{j=0}^{r} N_j p^{(j)}(1)\bar{q}^{(j)}(1) \quad (p,q\in\mathscr{P}).$$

Bavinck, in [13], shows that the Gegenbauer–Sobolev-type polynomials $\{P_n^{(\alpha,\alpha,M_0,M_1;M_0,M_1)}\}_{n=0}^{\infty}$, where $\alpha>-1$ and $M_0,M_1\geqslant 0$, belong to the $\mathrm{BKS}(N,M)$ classes listed in the following table:

Conditions on α,M_0,M_1	Order of DE	$\mathrm{BKS}(N,M)$ class
$M_0=M_1=0$	2	BKS(2)
$M_0>0,M_1=0;\alpha\in\mathbb{N}_0$	$2\alpha+4$	$\mathrm{BKS}(2\alpha+4)$
$M_0=0,M_1>0;\alpha\in\mathbb{N}_0$	$2\alpha+8$	$\mathrm{BKS}(2\alpha+8,1)$
$M_0>0,M_1>0;\alpha\in\mathbb{N}_0$	$4\alpha+10$	$\mathrm{BKS}(4\alpha+10,1)$
$M_0^2+M_1^2\neq 0,\alpha\notin\mathbb{N}_0$	∞	NA

(5.13)

6. Applications

New applications of orthogonal polynomials from the higher-order BKS classes and their connections to other areas of mathematics, physics, and engineering (including signal analysis) have emerged during the past few years.

As documented in the Erice Report, there have been many key contributions linking the theory of orthogonal polynomials to the spectral theory of differential operators. Titchmarsh [95] pioneered this connection with his analytic study (involving the Titchmarsh–Weyl m-coefficient) of the self-adjoint operator in $L^2(-1,1)$ generated by the classic Legendre equation

$$\ell_L[y](x) = -((1-x^2)y'(x))' = \lambda y(x) \quad (x \in (-1,1)). \tag{6.1}$$

In the 1950s the Russian school led by M.A. Naimark, N.I. Akhiezer, M.G. Krein, and I.M. Glazman, applied, and subsequently refined, the von Neumann theory of self-adjoint extensions of Lagrangian formally symmetric differential expressions; see [1,24,84]. Some of the most important applications of their theory involves the spectral analysis of the classical second-order differential equations of Jacobi, Laguerre, and Hermite; see [1, Appendix II] (also see [87]). The first mathematician to study examples from the BKS(N) classes for $N > 2$ was Krall [60] who developed the spectral theory for the differential equations associated with the PSs listed in (3.6). During the past 20 years, an intense study of these higher-order differential equations has been undertaken; see [30] for a list of references to these spectral problems. Since 1993, there has been a concentrated effort to develop methods to deal with spectral problems for the BKS(N,M) classes when $M > 0$. The contributions [28,29,32] specifically address such problems from the BKS$(2,1)$ and BKS$(4,1)$ classes. At this date, there is no general theory available, in a Sobolev space setting, that addresses self-adjoint extensions of differential equations from the BKS(N,M) classes in a way similar to how the GKN theory deals with such extensions for equations from the BKS(N) classes.

A number of important papers have been written that have studied the connections between sampling theory, interpolation theory, and orthogonal polynomials from the BKS(2) class; for example, see [19,20,33,85,99]. Applications of sampling and interpolation theory abound in engineering and physics, especially in the areas of signal processing and communication; these ideas and applications came to the forefront of mathematical engineering with the publishing of Kramer's seminal paper [64]. The importance of the theory lies in an underlying engineering principle that all of the information contained in a signal F_f is, in fact, contained in the sample values $F_f(\lambda_n)$ (see [99]). The recent thesis of Nasri–Roudsari [85] is an impressive account of the general sampling theorem of Kramer, together with extensions of the theory and numerous applications. One of the richest sources of Kramer kernels is the subject of self-adjoint boundary value problems generated from Lagrangian symmetric ordinary differential equations; see [19,20,33,85] and the references cited therein. In particular, the self-adjoint operators generated from the second-order differential equations having classical orthogonal polynomial solutions provide us with excellent examples to illustrate both sampling and interpolation theory. We now briefly outline this connection between orthogonal polynomials, sampling theory, and interpolation theory.

For an open interval I, a *Kramer kernel* is a Lebesgue measurable function $K : I \times \mathbb{C} \to C$ such that $K(\cdot,\lambda) \in L^2(I;w)$ for each $\lambda \in \mathbb{R}$, where w is a positive (a.e.) function, $K(x,\cdot)$ is an entire function on $\mathbb{C}$ for each $x \in I$, and for which there exists a strictly increasing sequence $\{\lambda_n\}_{n=0}^{\infty} \subset \mathbb{R}$ such that $\{K(\cdot,\lambda_n)\}_{n=0}^{\infty}$ is a complete orthogonal set in $L^2(I;w)$.

Define, for each $n \in \mathbb{N}_0$, the *interpolation functions*

$$S_n(\lambda) = \|K(\cdot,\lambda_n)\|_w^{-2} \int_I K(x,\lambda)\overline{K(x,\lambda_n)}w(x)\,\mathrm{d}x,$$

and, for each $f \in L^2(I; w)$, the *signal* function

$$F_f(\lambda) = \int_I K(x, \lambda) f(x) w(x) \, \mathrm{d}x.$$

From the original work of Kramer [64], we have

$$F_f(\lambda) = \sum_{n=0}^{\infty} F_f(\lambda_n) S_n(\lambda),$$

where this series, called a *Whittaker–Shannon sampling series*, converges absolutely in $\mathbb{C}$ for each $\lambda \in \mathbb{R}$. Moreover, if $\{c_n\}_{n=0}^{\infty} \subset \mathbb{C}$ is such that

$$\sum_{n=0}^{\infty} \frac{|c_n|^2}{\|K(\cdot, \lambda_n)\|_w^2} < \infty,$$

then there exists a unique $f \in L^2(I; w)$ such that $F_f(\lambda_n) = c_n$. Lastly, if there exists an entire function $G : \mathbb{C} \to C$ with simple roots $\{\lambda_n \mid n \in \mathbb{N}_0\}$ such that $S_n(\lambda) = G(\lambda)/((\lambda - \lambda_n) G'(\lambda_n))$, we call G an *interpolation function* for K.

For example, the Kramer kernel associated with the self-adjoint operator in $L^2(-1, 1)$, generated from the classical Legendre differential expression $\ell_L[\cdot]$ in (6.1), having the Legendre polynomials $\{P_n(x)\}_{n=0}^{\infty}$ as eigenfunctions, is given by

$$K(x, \lambda) = {}_2F_1(-\sqrt{\lambda} + \tfrac{1}{2}, \sqrt{\lambda} + \tfrac{1}{2}; 1, \tfrac{1}{2}(1 - x)) \quad (x \in (-1, 1), \lambda \in \mathbb{C});$$

in this case, $K(x, (n + \frac{1}{2})^2) = P_n(x)$. Furthermore, for every $f \in L^2(-1, 1)$, we have the Whittaker–Shannon sampling series

$$F_f(\lambda) = \sum_{n=0}^{\infty} F_f((n + \tfrac{1}{2})^2) \frac{(2n + 1) \sin(\pi(\sqrt{\lambda} - n - \frac{1}{2}))}{\pi(\lambda - (n + \frac{1}{2})^2)} \quad (\lambda \in \mathbb{C}),$$

and the interpolation function for the Legendre–Kramer kernel $K(x, \lambda)$ is

$$G(\lambda) = \sin(\pi(\sqrt{\lambda} - \tfrac{1}{2})) \quad (\lambda \in \mathbb{C}).$$

In 1885, Stieltjes (see [94, pp. 140–142]) gave an electrostatic interpretation of the zeros of the Jacobi polynomials in terms of a logarithmic potential; see also [34] and the references therein. In a remarkable series of papers and recent lectures by Arvesú and Marcellán [6], Dimitrov and van Assche [22], Grünbaum [35,36], and Ismail [45,46], the authors give an electrostatic interpretation, involving a differential equation of the form (1.3), of the zeros of a large class of orthogonal polynomials satisfying differential equations, including the Jacobi-type $\{P_n^{(\alpha,\beta,M,N)}\}_{n=0}^{\infty}$ and Laguerre-type $\{L_n^{\alpha,A}\}_{n=0}^{\infty}$ polynomials.

There is also significant progress made on various linearization and connection problems involving the Jacobi–Sobolev- and Laguerre–Sobolev-type polynomials (see [91]).

Recently, Nualart and Schoutens (see [86,92]) have shown that the simple Laguerre-type polynomials $\{L_n^{1,A}\}_{n=0}^{\infty}$ arise naturally in their stochastic model that generalizes the important Lévy process; such models are important, for example, in many areas of finance and stock market analysis.

7. Open problems

1. Prove or disprove the following conjecture:

Conjecture 7.1. *Suppose* $\{p_n\}_{n=0}^{\infty} \in \mathrm{BKS}(2r)$ *for some* $r \in \mathbb{N}$. *Then* $\{p_n\}_{n=0}^{\infty} \in$ (OPS *on* $[a,b]$) *if and only if the positive, classical solution* $w \in C^{\infty}(a,b)$ *of the symmetry equations* (3.1) *satisfies the additional conditions that* $x^n w \in L^1(a,b)$ $(n \in \mathbb{N}_0)$ *and the orthogonalizing moment functional* σ *for* $\{p_n\}_{n=0}^{\infty}$ *has the distributional representation*

$$W = (H_a - H_b)w + M\delta_a + N\delta_b, \tag{7.1}$$

where H_a *and* H_b *are the Heaviside functions with unit jumps at the endpoints* a *and* b, *respectively*, δ_a *and* δ_b *are the Dirac distributions with supports at* a *and* b, *respectively, and* M *and* N *are nonnegative real numbers. Furthermore, the terms in* H_a *and* δ_a (*respectively*, H_b *and* δ_b) *do not appear if* $a = -\infty$ (*respectively*, $b = \infty$).

We remark that this conjecture is closely related to Conjecture 4.3 in the Erice Report [30]. This conjecture is also connected with Conjecture 5.3 in [30] which states, essentially, that the only PSs simultaneously belonging to the classes (OPS on $[a,b]$) and BKS are the Jacobi, Laguerre and Hermite polynomials, subject to the restriction that the parameters satisfy the conditions in (3.12) and (3.14).

From Theorem 3.1, the moment functional σ is a solution of the symmetry equations (3.1); consequently W, given in (7.1), will be a distributional solution of these symmetry equations.

As reported in Section 4, a large step towards proving this conjecture was accomplished in [71] (see Theorem 4.1). On the other hand, in [40], the authors produce an example in the BKS(10) class (that is not, however, in any (OPS on $[a,b]$) class) where the weight distribution involves the first derivative of the Dirac distribution. Is it possible that other such weight distributions (which include derivatives of Dirac delta distributions) could give rise to *positive-definite* inner products and, consequently, disprove Conjecture 7.1?

2. Prove or disprove Conjecture 5.1 in [30]:

Conjecture 7.2. *Suppose* $\{p_n\}_{n=0}^{\infty} \in$ (OPS *on* $[a,b]$) *with associated orthogonalizing measure* μ, *determined by the monotonic increasing function* $\hat{\mu} : \mathbb{R} \to R$. *In addition, suppose* $\{p_n\}_{n=0}^{\infty} \in$ (SDPS *on* (a,b)) *with differential equation*

$$L_{2r}[y] = \lambda y,$$

where $L_{2r}[\,\cdot\,]$ *is defined in* (3.3). *Then*

(i) $q_k,\ w \in L(a,b)$ $(k = 0, 1, \ldots, r)$,

(ii) $q_k(x) \geqslant 0$ *for* $x \in (a,b)$ $(k = 0, 1, \ldots, r)$,

(iii) $\hat{\mu}(x) - \hat{\mu}(c) = \int_c^x w(t)\,\mathrm{d}t$ *for all* $x, c \in (a,b)$.

Kwon and Yoon [73] showed that each differential equation (1.1) associated with a PS in the BKS(N) class is Lagrangian symmetrizable. It remains to show that conditions (i)–(iii) are satisfied.

The nonnegativity of the coefficients q_k would guarantee that a left-definite spectral analysis is possible for $L_{2r}[\cdot]$; in this case, the general left-definite theory developed by Littlejohn and Wellman [77] applies (see [30, Conjecture 5.2]).

3. For each $k \in \mathbb{N}$, determine, up to both a real and a complex linear change of variable, each DPS$(2k)$ class. This was also listed as an open problem in [30]. As discussed earlier, it is known (see [49,61]) that the differential equations associated with the classes BKS(N) and BKS$(N,1)$ are necessarily of even order. Is this also the case for the general BKS(N,M) class for each $M \in \mathbb{N}_0$? If not, then it is necessary to classify DPS$(2k+1)$ $(k \in \mathbb{N})$ as well.
4. Determine the contents of BKS$(4,1)$ and BKS(6) under both a real and a complex linear change of variable.
5. Determine the Lagrangian symmetric form of the finite-order differential equations (see (3.12)) having the Jacobi-type PS $\{P_n^{(\alpha,\beta,M,N)}\}_{n=0}^{\infty}$ as eigenfunctions. Develop the right- and left-definite spectral theory for these symmetric equations. The approach taken by Wellman in [97] may prove useful in the spectral study of these equations.
6. Develop a "classical" spectral analysis (see Section 3.5) of the Bessel polynomial differential expression defined in (3.16) using the weight functions found through the methods in [26] and/or [42]. Similarly, develop a spectral analysis for the twisted Jacobi and twisted Hermite polynomials.
7. A key conjecture in solving the BKS(N) classification problem — or, perhaps more generally, the BKS(N,M) problem — may be the following:

Conjecture 7.3. *Suppose* $\{p_n\}_{n=0}^{\infty} \in$ BKS(N,M) *and* $a_N(x)$ *is the leading coefficient of the corresponding differential equation* (1.1) *having* $\{p_n(x)\}_{n=0}^{\infty}$ *as eigenfunctions. Then* N *is even and*

$$a_N(x) = b^j(x),$$

where $b(x) \in \{1, x, x^2, 1-x^2, 1+x^2\}$ *and* $j \in \{1, 2, \dots, N/2\}$.

We remark that the five polynomials listed above are precisely the leading coefficients for, respectively, the Hermite and twisted Hermite, Laguerre, Bessel, Jacobi, and twisted Jacobi differential equations (see Section 3.3).

8. Develop a Kramer sampling and interpolation theory for the differential equations and associated orthogonal polynomials in the BKS(4) and BKS$(2,1)$ classes. A deep understanding of the properties of the solutions of these equations is essential in order to construct and analyze the kernels associated with these equations. We remark that Butzer and Schöttler [20] develop sampling and interpolation results for certain fourth-order equations; however, these equations are not from the BKS(4) class.
9. As discussed in Section 6, there is now an electrostatic interpretation of the roots of, say, the Legendre type polynomials $\{P_n^{(0,0,M,M)}\}_{n=0}^{\infty}$, which satisfy the fourth-order differential equation

$$(1-x^2)^2 y^{(4)} + 8x(x^2-1)y^{(3)} + (4A+12)(x^2-1)y'' + 8Axy' = \lambda y. \tag{7.2}$$

This interpretation is accomplished through the second-order differential equation of the form (1.3) that these polynomials also satisfy. Is there an electrostatic interpretation of the roots of these polynomials that reproduces (7.2)?

10. The GKN theory provides a recipe, in theory, for determining all self-adjoint extensions in the Hilbert space $L^2(I;w)$ of formally symmetric differential expressions of the form (3.3) on some open interval $I=(a,b)$; we assume here that $w>0$ and each coefficient q_k is sufficiently differentiable on I. This theory works well in developing the spectral theory for the second-order classical differential equations of Jacobi, Laguerre, and Hermite. However, for nonclassical symmetric differential equations (3.3) in the $(\mathrm{OPS}\cap[a,b])\cap\mathrm{BKS}(N)$ classes, $N>2$, the appropriate right-definite spectral setting is $L^2(\bar{I};\mathrm{d}\mu)$, where $\bar{I}$ is the closure of I and where the orthogonalizing measure μ has the form

$$\mathrm{d}\mu = w + \sum_{j=1}^{p}(\alpha_j\delta_a^{(j)} + \beta_j\delta_b^{(j)}) \quad (\alpha_j,\beta_j\geqslant 0;\ p\in\mathbb{N}). \tag{7.3}$$

Here each $\alpha_j=0$ (respectively, $\beta_j=0$) if $a=-\infty$ (respectively, $b=\infty$). Develop a general GKN-type theory for this setting; in particular, provide a "recipe" for determining the self-adjoint operator that has the PS as eigenfunctions.

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