The Conley index along heteroclinic trajectories of reaction–diffusion equations

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ARTICLE INFO

Article history:
Received 20 July 2011
Revised 22 November 2011
Available online 28 January 2012

Keywords:
Conley index
Connecting orbits
Reaction–diffusion equations

ABSTRACT

It is well known that hyperbolic equilibria of reaction–diffusion equations have the homotopy Conley index of a pointed sphere, the dimension of which is the Morse index of the equilibrium. A similar result concerning the homotopy Conley index along heteroclinic solutions of ordinary differential equations under the assumption that the respective stable and unstable manifolds intersect transversally, is due to McCord. This result has recently been generalized by Dancer to some reaction–diffusion equations by using finite-dimensional approximations. We extend McCord’s result to reaction–diffusion equations. Additionally, an error in the original proof is corrected.

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1. Introduction and main result

It is well known that hyperbolic equilibria of reaction–diffusion equations have the homotopy Conley index of a pointed sphere, the dimension of which is the Morse index of the equilibrium. A similar result concerning the homotopy Conley index along heteroclinic solutions of ordinary differential equations under the assumption that the respective stable and unstable manifolds intersect transversally, is due to McCord (see [10, Theorem 3.1]). This result has recently been generalized by Dancer [6] to some reaction–diffusion equations by using finite-dimensional approximations. Roughly speaking, the homotopy Conley index is calculated in $L^2(\Omega)$ under remarkably weak smoothness assumptions on the non-linearity. As Dancer remarks [6, Remark 2.2], his result also covers the Čech cohomology in $L^p(\Omega)$, $1 < p < \infty$.

Unfortunately, the proof of [10, Theorem 3.1] contains an error and, as such, is incomplete. To see this, consider the following ordinary differential equation on $\mathbb{R}^2$:

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\[\begin{align*}
\dot{x} &= 1 - x^2 \\
\dot{y} &= x^2 y.
\end{align*}\]

\((-1, 0)\) and \((1, 0)\) are hyperbolic critical points and there is a solution \((u(t), 0)\) connecting \((-1, 0)\) and \((1, 0)\). It is easy to see that \(\{(x, y) \in \mathbb{R}^2 : x < 1\}\) is the unstable manifold of \((-1, 0)\) and \(\{(x, 0) \in \mathbb{R}^2 : x > -1\}\) is the stable manifold of \((1, 0)\). The tangential spaces of both manifolds intersect transversally in every point \((u(t), 0), t \in \mathbb{R}\). According to the proof of \([10, \text{Theorem 3.1}]\), there is a continuation to

\[\begin{align*}
\dot{x} &= 1 - x^2 \\
\dot{y} &= 0^2 y.
\end{align*}\]

Evidently, \([-1, 1] \times \{0\}\) is not even an isolated invariant set relative to this flow. One might conjecture that this problem could be resolved by an arbitrarily small perturbation. However, there are also examples which show that this is generally not possible. The proof (of \([10, \text{Theorem 3.1}]\)) relies on the assumption that 0 is an isolated rest point with respect to \(\dot{y} = A(x) y\) for every \(x \in [-1, 1]\). Now let \(\varepsilon > 0\) and consider the following perturbation of our original equation

\[\begin{align*}
\dot{x} &= 1 - x^2 \\
\dot{y} &= (x^2 - \varepsilon^2) y =: A(x) y.
\end{align*}\]

This means that 0 is not isolated with respect to \(\dot{y} = A(\pm \varepsilon) y\) and every sufficiently small perturbation will retain these problematic points. Furthermore, the homotopy index of 0 relative to \(\dot{y} = -\varepsilon^2 y\) is not \(\Sigma^1\) as stated but \(\Sigma^0\).

Dancer notes in \([6]\) that “it should be possible to give a more natural direct proof [...] at least in the \(C^1\) case”. In this paper\(^1\) we provide a genuinely infinite-dimensional proof for a theorem which is closely related to Dancer’s result in the \(C^1\) case. It is possible to compute the homotopy Conley index in \(L^p(\Omega)\) (not only the cohomology) directly, provided the solution is sufficiently regular. We face several technical difficulties due to the infinite-dimensional situation, which, fortunately, are all overcome.

We are now in a position to state our main result. Let \(\Omega \subset \mathbb{R}^N\) be a bounded domain and let \(\partial \Omega\) be of class \(C^2\). Let \(2 \leq p < \infty\) and \(f : \Omega \times \mathbb{R} \to \mathbb{R}\). Suppose that for almost all \(x \in \Omega\) there is a partial derivative \(f_u(x, u)\) which is continuous in \(u\) and that \(\text{ess sup}_{x \in \Omega} \sup_{|u| \leq r} |f_u(x, u)| < \infty\) for all \(r \in \mathbb{R}^+\).

Assume further that \(f\) and \((x, u) \mapsto f_u(x, u)\) are Carathéodory functions.

We consider the problem

\[\begin{align*}
u_t(x, t) &= \Delta u(x, t) + f(x, u(x, t)) & t > 0, x \in \Omega \\
u(x, t) &= 0 & t > 0, x \in \partial \Omega. \quad (1.1)
\end{align*}\]

Let \(A_p\) denote the closure of \(-\Delta : \{u \in C^2(\Omega) : u|_{\partial \Omega} = 0\} \to L^p(\Omega) =: X\) in \(W^{2, p}(\Omega)\) and define the Nemitskii (superposition) operator \(\hat{f} \in C^1(C(\bar{\Omega}), L^p(\Omega))\) by

\[(\hat{f}(u))(x) := f(x, u(x)) \quad x \in \Omega\]

so that \((D\hat{f}(\xi))\eta(x) = f_u(x, \xi(x))\eta(x)\) a.e.

\(^1\) The paper is a part of the author’s PhD thesis.
For \( k \) sufficiently large, \( A_p + kI \) is a positive operator having compact resolvent. Letting \( \xi \in X^\alpha \), it follows that all eigenvalues of \( A - D\hat{f}(\xi) \) are real.

Let \( p \geq \max\{2, N\} \), \( A := A_p \), and \( v : \mathbb{R} \to X^\alpha \) be a heteroclinic mild solution of

\[
\dot{x} + Ax = \hat{f}(x)
\]

and suppose that \( v(t) \to e^\pm \) as \( t \to \pm \infty \) in \( X^\alpha \) (resp. \( C(\bar{\Omega}) \)). It follows that \( v \in C^1(\mathbb{R}, L^p(\Omega)) \). Choosing \( 0 < \alpha < 1 \) large enough, we can further assume that there is a continuous inclusion \( X^\alpha \subset C(\bar{\Omega}) \) (see [8, Theorem 1.6.1]).

In the following theorem we will replace transversality by weaker assumptions, which have the advantage of not relying on the existence of global stable manifolds (see also [6]).

**Theorem 1.1.** Let \( u \) be a heteroclinic mild solution of (1.2) with \( u(t) \to e^\pm \) as \( t \to \infty \) in \( X^\alpha \) (resp. \( C(\bar{\Omega}) \)) and suppose that

1. \( e^+, e^- \) are hyperbolic equilibria,
2. the Morse indices satisfy \( m(e^-) = m(e^+) + 1 \),
3. all eigenvalues of \( A - Df(e^\pm) \) are simple,
4. \( e^{\lambda t}(u(t) - e^+) \not\to 0 \) for some \( \lambda \in \mathbb{R} \), and
5. every full bounded (in \( C(\bar{\Omega}) \)) mild solution of

\[
\dot{y} + Ay = D\hat{f}(u(t))y
\]

is a multiple of \( \dot{u} \).

Then the homotopy Conley index \( h(\pi, \bar{u}) \) of \( \bar{u} := \text{cl}\{u(t) : t \in \mathbb{R}\} \) is well defined and trivial, that is, \( h(\pi, \bar{u}) = 0 \), where \( \pi \) denotes the semiflow which is induced by mild solutions of (1.2).

Conditions ensuring that the assumptions of Theorem 1.1 hold for every heteroclinic mild solution of (1.2) are discussed in the following section. In view of the growth condition in Theorem 1.1, it should be noted that in [11] Meshkov gives an example of an equation

\[
\Delta u = q(x, t)u
\]
on the three-dimensional torus which has a non-trivial solution \( u(x, t) \) with \( |u(x, t)| \leq Ce^{-\alpha t^2} \) for some real constants \( c, C \).

Theorem 1.1 is proved by reducing the general problem subsequently to a special case, the homotopy index of which can be calculated.

It follows from Theorem 3.2 that \( u(t) \) satisfies the hypotheses of Proposition 4.4. Therefore, we can apply Theorem 5.12, which is the main result of Section 5 and states that the homotopy index of \( \bar{u} \) relative to \( \pi \) equals the homotopy index of a suitable linear skew product semiflow.

The structure of a certain class of linear skew product semiflows, which are defined on a trivial bundle, is discussed in Section 6. Theorem 6.7 is the main result of this section and completes the proof of Theorem 1.1.

2. Preliminaries

2.1. Notation

Although most of the notation is more or less standard, a couple of symbols should at least be mentioned. \( \mathbb{R}^+ \) (resp. \( \mathbb{R}^- \)) denotes the set of all non-negative (resp. non-negative) real numbers. \( W^u \) and \( W^s \) denote unstable respectively stable manifolds, the precise meaning is given when they are
used. \( \sigma \) is used to designate the spectrum of an operator. The open (resp. closed) ball with radius \( r \) and center \( x \) is denoted by \( B_r(x) \) (resp. \( \bar{B}_r(x) \)).

We will frequently deal with trivial vector bundles. They are considered as continuous families \( U(x), x \in [a, b], \) of vector space homomorphisms. When no confusion can arise, we will identify \( U \) with its image, just like the notation of the topology is usually suppressed. A more detailed exposition of this terminology can be found in Appendix A.

Given normed spaces \( X \) and \( Y, \) and a continuous linear operator \( F \in \mathcal{L}(X, Y), \) \( \|F\|_{X,Y} \) is used sometimes to make the norm unambiguous. The notion of fractional power spaces follows [8]. If \( F \in \mathcal{L}(X^\alpha, X^\beta), \) then \( \|F\|_{\alpha, \beta} \) denotes the operator norm.

Finally, if \( X, Y \) are topological spaces, \( f : X \to Y \) is a homeomorphism, and \( \pi \) is a (local) semiflow on \( X, \) then \( f[\pi] \) is the semiflow on \( Y \) which is obtained by conjugacy, that is, \( u \) is a solution of \( \pi \) if and only if \( f \circ u \) is a solution of \( f[\pi] . \)

2.2. Exponential decay

In addition to the assumptions in the previous section, suppose that for every \( r \in \mathbb{R} \) there exist constants \( \delta > 0 \) and \( C \in \mathbb{R}^+ \) such that

\[
\text{ess sup}_{x \in \Omega} \sup_{|u_1 - u_2| \leq r} |f_u(x, u_1) - f_u(x, u_2)| \leq C|u_1 - u_2|^{\delta}.
\]

Then \( \hat{D}f : C(\hat{\Omega}) \to \mathcal{L}(L^2(\Omega), L^2(\Omega)) \) is locally Hölder continuous.

Let \( u(t) \) be a mild solution of (1.2) defined for all \( t \in \mathbb{R}^+ \) with \( u(0) \neq e^+ \) and \( u(t) \to e^+ \) in \( C(\hat{\Omega}) \) as \( t \to \infty. \) \( u(t) \) has a continuous derivative \( \dot{u} : \mathbb{R}^+ \to X = L^p(\Omega). \) Suppose that \( \lambda(u) := \sup_{\mu \in \mathbb{R}^+} e^{\mu t}\|u(t) - e^+\|_\alpha \to 0 \) as \( t \to \infty \) for all \( \lambda \in \mathbb{R}^+. \)

Define \( B(t) \in C(\mathbb{R}^+, \mathcal{L}(L^2(\Omega), L^2(\Omega))) \) by \( (B(t)y)(x) := f_u(x, u(x))y(x) \) and \( B(\infty) \in \mathcal{L}(L^2(\Omega), L^2(\Omega)) \) by \( (B(\infty)y)(x) := f_u(x, e^+(x))y(x) \). Due to the Hölder-continuity of \( \hat{D}f, \) there is a real constant \( \hat{C} \) with \( \|B(t) - B(\infty)\| \leq \hat{C}e^{-t} \) for all \( t \in \mathbb{R}^+ \).

Now, \( \dot{u}(t) \) is a mild solution of

\[
\dot{y} + A_2y = B(t)y,
\]

where we take \( X := H := L^2(\Omega), \) and \( \alpha = 0. \)

Using the continuity of the inclusion \( L^p(\Omega) \subset L^2(\Omega) \) and Lemma 3.6, it follows that \( e^{\lambda t}\|\dot{u}(t)\|_2 \to 0 \) as \( t \to \infty \) for all \( \lambda \in \mathbb{R}^+. \)

We can apply Proposition 3.12 and obtain an \( \varepsilon > 0 \) such that \( \dot{u}(t) = 0 \) for all \( t \in \mathbb{R}^+ \) with

\[
\sup_{s \geq t} \|B(s) - B(\infty)\| \leq \varepsilon^2.
\]

Let

\[
t_0 := \inf\{ t \in \mathbb{R}^+: \dot{u}(t) = 0 \}
\]

and assume that \( 0 < t_0. \) For all \( t \geq t_0, \) it follows that \( u(t) = e^+ \) and \( B(t) = B(\infty), \) so, by the continuity of \( u(t) \), there is a \( 0 \leq \tilde{t} < t_0 \) with \( \sup_{s \geq \tilde{t}} \|B(s) - B(\infty)\| \leq \varepsilon^2. \) We thus have \( \dot{u}(t) = 0 \) for all \( t \geq t_0, \) a contradiction to the minimality of \( t_0. \)

Lemma 3.6 now implies that \( \lambda(u) = \lambda(\dot{u}) < \infty. \)
2.3. Hyperbolicity, transversality, and simple eigenvalues

It has been shown in [2] that generically (with respect to the non-linearity) all equilibria are hyperbolic, the eigenvalues of their linearizations are simple, and their respective stable and unstable manifolds intersect transversally.

As already noted in [6], it is not necessary to assume the existence of global stable manifolds. Indeed, a sufficient condition can be formulated solely in terms of the linear equation.

To show that the assumptions of Theorem 1.1 hold in the case of transversality, let \( e^+, e^- \in X^\pi \) be hyperbolic equilibria with Morse indices \( m(e^+) = n \) and \( m(e^-) = n + 1 \) for some \( n \in \mathbb{N} \), and let \( u(t) \) be a mild solution of (1.2) with \( u(t) \to e^\pm \) as \( t \to \pm \infty \).

The tangential spaces are characterized in [2, Lemma 4.b.1]. Translated to our notation (see Definition 5.5), we have

\[
T_{u(t)} W^\pi(e^+) = B^+(T \pi, u(t)),
\]

\[
T_{u(t)} W^\pi(e^-) = B^-(T \pi, u(t)).
\]

Since \( \text{codim} T_{u(t)} W^\pi(e^+) = \text{dim} T_{u(t)} W^\pi(e^-) - 1 \) (using the Morse indices of \( e^\pm \)), one has \( \text{dim}(B^+(T \pi, u(t)) \cap B^-(T \pi, u(t))) = 1 \), that is, every full bounded (in \( X^\pi \)) mild solution of

\[
\dot{y} + Ay = D\hat{f}(u(t))y \tag{2.1}
\]

is a multiple of \( \hat{u} \) as stated in the assumptions of Theorem 1.1. Of course, if \( v : \mathbb{R} \to X = L^p(\Omega) \) is a mild solution of (2.1), then \( v(t) \in X^\pi \) for all \( t \in \mathbb{R} \) and \( \sup_{t \in \mathbb{R}} \|v(t)\|_\alpha < \infty \).

2.4. Conley index

The purpose of this section is to give a short overview over the most important concepts of Conley index theory for semiflows on metric spaces. A more detailed exposition can be found in [3] and [12].

Let \( \hat{A}, \hat{B} \) be a topological space and \( A \subset \hat{B} \). Let \((\hat{A}, \hat{B}) := (A, \hat{B})\) if \( \hat{B} \neq \emptyset \) and \((\hat{A}, \hat{B}) := (A \cup \{\ast\}, \{\ast\})\) (endowed with the sum topology) otherwise. Here, we assume that \( \ast \notin \hat{A} \). Now let \( A/B \) denote the set of equivalence classes in \( \hat{A} \) where \( a, \tilde{a} \in \hat{A} \) are related if they are equal or \( \{a, \tilde{a}\} \subset \hat{B} \). A/B is equipped with the quotient topology.

Let \( \pi \) be a local semiflow defined on a metric space \( X \). A subset \( S \subset X \) is called invariant if for every \( x \in S \) there exists a full solution \( u : \mathbb{R} \to S \) of \( \pi \) through \( x \) that is, \( u(0) = x \).

Let \( Y \subset X \), \((x_n)_n\) a sequence in \( Y \), and \((t_n)_n\) a sequence in \( \mathbb{R}^+ \) such that \( t_n \to \infty \) and \( x_0 \pi \mathbb{P}_{[0, t_n]} \subset Y \). \( Y \) is called admissible if the sequence of endpoints \( x_0 \pi t_n \) is precompact for every such pair of sequences. We say that \( \pi \) does not explode in \( Y \) if for every \( x \in X \) either \( x \pi t \) is defined for all \( t \in \mathbb{R}^+ \) or there is a \( t_0 \in \mathbb{R}^+ \) such that \( x \pi [0, t_0] \) is defined and \( x \pi t_0 \notin Y \). \( Y \) is called strongly \( \pi \)-admissible if it is admissible and \( \pi \) does not explode in \( Y \).

Now let \( Z \subset Y \subset X \). \( Z \) is called \( Y \)-positively invariant if it holds that \( x \pi [0, t] \subset Y \) whenever \( x \pi [0, t] \) is defined and \( x \pi t \) is defined in \( Z \).

A subset \( Z \subset X \subset Y \) is called an exit ramp for \( Y \) if for every \( x \in Y \) with \( x \pi [0, t] \) defined and \( x \pi [0, t] \notin Z \), there is a \( t_0 \in [0, t] \) such that \( x \pi [0, t_0] \subset Y \).

**Definition 2.1.** (See Definition 2.4 in [3].) A pair \((N, \pi, S) \) is called an FM-index pair for \((\pi, S) \) if:

1. \( N_1 \) and \( N_2 \) are closed subsets of \( X \) with \( N_2 \subset N_1 \) and \( N_2 \) is \( N_1 \)-positively invariant;
2. \( N_2 \) is an exit ramp for \( N_1 \);
3. \( S \) is closed, \( S \subset \text{int}_X(N_1 \setminus N_2) \) and \( S \) is the largest invariant set in \( \text{cl}_X(N_1 \setminus N_2) \).

Assume that there exists a strongly \( \pi \)-admissible isolating neighborhood \( N \) for \( S \), that is, \( N \subset X \) is a closed and strongly \( \pi \)-admissible neighborhood of \( S \) such that \( S \) is the largest invariant set
in $N$. Then the homotopy Conley index $h(\pi, S)$ is defined to be the homotopy type of $(N_1/N_2, [[N_2]])$ where $(N_1, N_2)$ is an FM-index pair for $(\pi, S)$ such that $cl_X(N_1 \setminus N_2)$ is strongly $\pi$-admissible.

Let $u(t)$ satisfy the assumptions of Theorem 1.1 and let $\pi$ denote the semiflow on $X^\alpha$ induced by mild solutions of (1.2). Then $S := \overline{u}$ is an isolated invariant set admitting a strongly $\pi$-admissible isolating neighborhood. In particular, the homotopy Conley index $h(\pi, \overline{u})$ is well defined under these assumptions.

3. Abstract semilinear parabolic equations

Let $H$ be a real Hilbert space, and let $A_H : D(A_H) \subset H \rightarrow H$ be a sectorial operator such that

(1) $A_H$ has compact resolvent;
(2) $A_H$ is densely defined;
(3) $\operatorname{Re} \lambda > 0$ for all $\lambda \in \sigma(A_H)$.

Let $X$ be a real Banach space with continuous inclusion $X \subset H$, and let $A : D(A) \subset X \rightarrow X$ be a sectorial operator such that

(1) $A$ is densely defined;
(2) $A$ has compact resolvent;
(3) $Ax = A_Hx$ for all $x \in D(A)$.

Fix $\alpha \in [0, 1]$, let $X^\alpha$ denote the $\alpha$-th fractional power space (see [8]), and let $f \in C^1(U, X^0)$ where $U \subset X^\alpha$ is open.

We consider mild solutions of the Cauchy problem

\[ \dot{x}(t) + Ax(t) = f(x(t)) \]
\[ x(0) = x_0, \]

which induce a local semiflow on $X^\alpha$ (see [8, Theorem 3.3.3], [1, Theorem A.3]). This semiflow is denoted by $\pi_f$, respectively $\pi$ whenever the meaning is clear.

Definition 3.1. For $u : [0, \infty[ \rightarrow X^\alpha$ let

\[ \lambda(u) := \sup \{ \gamma \in \mathbb{R} : e^{\gamma t} \| u(t) \|_\alpha \rightarrow 0 \text{ as } t \rightarrow \infty \}. \]

Theorem 3.2. Let $u : \mathbb{R} \rightarrow X^\alpha$ be a heteroclinic solution of (3.1) with

\[ u(t) \rightarrow e^- \quad t \rightarrow -\infty \]
\[ u(t) \rightarrow e^+ \quad t \rightarrow \infty. \]

For each $e \in \{e^-, e^+\}$ assume that $A - Df(e)$ is hyperbolic and that the spectrum $\sigma(A - Df(e))$ consists of isolated simple eigenvalues, all of which are real. Assume further that $\lambda(u - e^+) < \infty$ (Definition 3.1).

Letting $\rho^+(v, t) := \int_t^\infty \| \dot{\nu}(s) \|_\alpha ds$, the following holds:

(1) There is a $0 < \lambda^+ \in \sigma(A - Df(e^+))$ and an associated eigenvector $\eta^+$ such that $\rho^+(u, t)^{-1}u(t) \rightarrow \eta^+$ as $t \rightarrow \infty$. 
and there is another solution $v^+$ of (3.1) defined for all $t \geq 0$ such that

$$(2) \quad \rho^+(v^+, t)^{-1}v^+(t) \to -\eta^+ \text{ in } X^\alpha \text{ as } t \to \infty.$$  

Moreover, with $\rho^-(v, t) := \int_{-\infty}^{t} \|\dot{v}(s)\|_\alpha \, ds$,

$$(3) \text{ there is a } 0 < \lambda^- \in \sigma(A - Df(e^-)) \text{ and an associated eigenvector } \eta^- \text{ such that } \rho^-(u, t)^{-1}u(t) \to \eta^- \text{ in } X^\alpha \text{ as } t \to -\infty$$

and there is another solution $v^-$ of (3.1) defined for all $t \leq 0$ such that

$$(4) \rho^-(v^-, t)^{-1}v^-(t) \to \eta^- \text{ in } X^\alpha \text{ as } t \to -\infty.$$  

**Proof.** Let $L^+ := A - Df(e^+)$, $g^+(x) := f(x) - Df(e^+)x$ and $u^+(t) := u(t) - e^+$. $u^+(t)$ is a solution of

$$\dot{x}(t) + L^+x(t) = g^+(x(t) + e^+) - g^+(e^+).$$  

It follows from Lemma 3.6 that $\lambda(\dot{u}) < \infty$ and from Lemma 3.5 that $\|\dot{u}^+(t)\|_\alpha^{-1}\dot{u}^+(t)$ converges to an eigenvector $\eta$ of $L^+$. Therefore, claim (1) is a consequence of Proposition 3.8; we have $\|v^+(t)\|_\alpha^{-1}v^+(t) \to 0$ as $t \to \infty$. It follows from Lemma 3.5 that there is an eigenvalue $\tilde{\eta}$ of $L^+$ such that $\|\dot{v}^+(t)\|_\alpha^{-1}\dot{v}^+(t) \to \tilde{\eta}$ as $t \to \infty$. Using Lemma 3.7, we conclude $\tilde{\eta} = \eta$, which proves (2).

Analogously, $u^-(t) := u(t) - e^-$, is a solution of

$$\dot{x}(t) + L^-x(t) = g^-(x(t) + e^-) - g^-(e^-)$$

with $L^- := A - Df(e^-)$ and $g^-(x) := f(x) - Df(e^-)x$.

The convergence in (3) now follows from Corollary 3.10 and the existence of $v^-$ follows from Proposition 3.11 and Corollary 3.10. \qed

### 3.1. Estimates

Assume that $f(0) = 0$ and let $u : [0, \infty[ \to X^\alpha$ be a solution of (3.1) with $u(t) \to 0 \in X^\alpha$ as $t \to \infty$. Set $L := A - Df(0)$ and $g(x) := f(x) - Df(0)x, x \in X^\alpha$. Then $L$ is a sectorial operator, $g(0) = 0$, and $u(t)$ is also a solution of

$$\dot{x}(t) + Lx(t) = g(x(t)) \tag{3.2}$$

with $u(t) \to 0$ as $t \to \infty$ and we have

$$\mathcal{L}(X^\alpha, X) \ni Dg(0) = 0. \tag{3.3}$$

Assume that $\sigma(L)$ consists of a sequence of simple eigenvalues $(\lambda_n)_{n \in \mathbb{N}}$ with $\lambda_0 \to \infty$ as $n \to \infty$.

For each $\gamma \in \mathbb{R} \setminus \text{Re } \sigma(L)$ there are linear projections $P^\pm(\gamma) : X \to X$ such that $P^\pm(\gamma)x + P^-\gamma)(x) = x$ for all $x \in X$ and for some real constant $M > 0$ we have

$$\|e^{-Lt}P^-(\gamma)x\|_\alpha \leq Me^{-\gamma t}\|x\|_0 \quad t \leq 0$$

$$\|e^{-Lt}P^+(\gamma)x\|_\alpha \leq Me^{-\gamma t}\|x\|_\alpha \quad t \geq 0$$

$$\|e^{-Lt}P^+(\gamma)x\|_\alpha \leq Me^{-\gamma t}t^{-\alpha}\|x\| \quad t > 0 \tag{3.4}$$

(see [8, Theorem 1.5.3]).
By [8, Lemma 3.3.2], $u$ is differentiable in all $t > 0$ and $\dot{u} : \mathbb{R}^t \xrightarrow{\rightarrow} X^0$ is continuous with $\dot{u}(t) \in X^0$ for all $0 < t \in \mathbb{R}^+$. Let $C_l(\mathbb{R}^+, X^0)$ denote the set of all $f \in C(\mathbb{R}^+, X^0)$ with $\lambda(f) > \mu$, which is equipped with the norm $\|f\|_{C_l} := \sup_{s \in \mathbb{R}^+} \|e^{\mu s} f(s)\|_\alpha$.

Lemma 3.3. Let $\mu \in \mathbb{R}^+ \setminus \Re \sigma(L)$ and

$$P^-(\mu)K_\mu(x_0, f)(t) := -\int_t^\infty e^{-L(t-s)}P^-(\mu)f(s)\, ds$$

$$P^+(\mu)K_\mu(x_0, f)(t) := e^{-Lt}P^+(\mu)x_0 + \int_0^t e^{-L(t-s)}P^+(\mu)f(s)\, ds.$$ 

Then $K_\mu \in L(X^0 \times C_\mu(\mathbb{R}^+, X^0), C_\mu(\mathbb{R}^+, X^0))$.

**Proof.** Let $0 < \delta \in \mathbb{R}^+$ such that $[\mu - \delta, \mu + \delta] \subset \mathbb{R} \setminus \Re \sigma(L)$. We then have $P^-(\mu - \delta) = P^-(\mu)$ and $P^+(\mu + \delta) = P^+(\mu)$, so by (3.4) there is an $M > 0$ such that for all $s, t \in \mathbb{R}^+$

$$\|e^{-L(t-s)}P^-(\mu)f(s)\|_\alpha \leq Me^{-((\mu-\delta)(t-s))}\|f(s)\|_0$$

$t - s \leq 0$

$$\|e^{-L(t-s)}P^+(\mu)f(s)\|_\alpha \leq Me^{-((\mu+\delta)(t-s))}\|f(s)\|_0$$

$t - s > 0$

$$\|e^{-Lt}P^+(\mu)x_0\|_\alpha \leq Me^{-\mu t}\|x_0\|_\alpha.$$ It follows that

$$\|e^{-L(t-s)}P^-(\mu)f(s)\|_\alpha \leq Me^{-\mu t}\delta(t-s)\|f\|_{C_\mu}$$

$t - s \leq 0$

$$\|e^{-L(t-s)}P^+(\mu)f(s)\|_\alpha \leq Me^{-\mu t}(t-s)^{-\alpha}e^{-\delta(t-s)}\|f\|_{C_\mu}$$

$t - s > 0$

showing that $K_\mu$ is well defined and

$$\|K_\mu(x_0, f)\|_{C_\mu} \leq M\|x_0\|_\alpha + \left(M \int_0^\infty e^{-\delta s} ds + M \int_0^\infty s^{-\alpha}e^{-\delta s} ds\right)\|f\|_{C_\mu}. \quad \Box$$

Lemma 3.4. Let $0 \neq x \in X^0$. Then there exists a $\mu \in \mathbb{R} \setminus \Re \sigma(L)$ with $P^-(\mu)x \neq 0$.

**Proof.** Let $(\eta_i)_{i \in \mathbb{N}}$ denote an orthonormal basis for $H$ and $(\lambda_i)_{i \in \mathbb{N}}$ denote the associated eigenvalues. Then there is an eigenvector $\eta_i$ with $\langle x, \eta_i \rangle \neq 0$. Letting $x(t) := e^{-Lt}x$, $t \in \mathbb{R}^+$, and $\mu \in \mathbb{R} \setminus \Re \sigma(L)$ with $\mu > \lambda_i$, it follows that $e^{\mu t}\|x(t)\|_H \to 0$, so by the continuity of the inclusion $X \subset H$, one has $e^{\mu t}\|x(t)\|_{X^0} \to 0$ as $t \to \infty$. We have $x(t) = P^-(\mu)x(t) + P^+(\mu)x(t)$ with $e^{\mu t}\|P^+(\mu)x(t)\|_{X^0} \to 0$ as $t \to \infty$. This shows that $P^-(\mu)x(0) = P^-(\mu)x \neq 0$ whenever $\mu > \lambda_i$. \( \Box \)
3.2. Exponential decay

**Lemma 3.5.** Assume that \( \sigma (L) \subset \mathbb{R} \) and let \( v \in \{ u, \dot{u} \} \) with \( 0 \leq \lambda (v) < \infty \).

Then

1. \( \lambda (v) \in \text{Re} \sigma (L) = \sigma (L) \);
2. there is an eigenvector \( \eta \) of \( L \) which belongs to the eigenvalue \( \lambda (v) \) (that is \( \eta \in \mathcal{D}(L) \) and \( L\eta = \lambda (v)\eta \)) such that

\[
\frac{v(t)}{\| v(t) \|_\alpha} - \eta \frac{\| \eta \|_\alpha}{\| v(t) \|_\alpha} \to 0 \quad \text{as} \quad t \to \infty.
\]

**Proof.** Following [1, A.3.2], let \( B(t) := \int_0^t Dg(su(t)) \, ds \) if \( v = u \) and \( B(t) := Dg(u(t)) \) if \( v = \dot{u} \). In either case we have \( B(t) \to 0 \) in \( \mathcal{L}(X^\alpha, X) \) as \( t \to 0 \), and \( v \) is a mild solution of

\[
\dot{x}(t) + Lx(t) = B(t)x(t).
\]

Now, claim (1) follows from [1, Theorem A.10]. The second claim is a consequence of [1, Corollary A.11] and the assumptions on \( \sigma (L) \).

If \( L \) is hyperbolic, then a particular consequence of Lemma 3.5 is that \( u(t) \in W_{\text{loc}} \) (that is \( \lambda (u) > 0 \)) for all \( t \) large enough, where \( W_{\text{loc}} \) denotes the local stable manifold given by [1, Theorem A.12]. Until further notice, we will assume that \( L \) is hyperbolic.

**Lemma 3.6.** \( \lambda (\dot{u}) = \lambda (u) \) and for all \( t \in \mathbb{R}^+ \)

\[
u(t) = - \int_t^\infty \dot{u}(s) \, ds.
\]

**Proof.** We start with the integral expression. Letting \( t_1, t_2 \in \mathbb{R} \) with \( t_1 < t_2 \), one has

\[
u(t_2) = \nu(t_1) + \int_{t_1}^{t_2} \dot{u}(s) \, ds.
\]

Taking \( t_2 \to \infty \), we obtain all \( t \in \mathbb{R} \)

\[
u(t) = - \int_t^\infty \dot{u}(s) \, ds.
\]  \hspace{1cm} (3.5)

The right side is integrable since by our assumptions there are \( M \in \mathbb{R} \) and \( 0 < \mu \in \mathbb{R} \) such that

\[
\| \dot{u}(t) \|_\alpha \leq Me^{-\mu t} \quad \text{for all} \quad t \geq 0.
\]

It follows from [1, Theorem A.12 d)] that \( \lambda (\dot{u}) \geq \lambda (u) \). Conversely, let \( 0 < \mu \in \mathbb{R} \), such that \( e^{\mu t} \| \dot{u}(t) \|_\alpha \to 0 \) as \( t \to \infty \). Letting \( C := \sup_{s \in \mathbb{R}^+} e^{\mu s} \| \dot{u}(s) \|_\alpha < \infty \), it follows that

\[
\| \nu(t) \|_\alpha \leq \int_t^\infty \| \dot{u}(s) \| \, ds \leq C \int_t^\infty e^{-\mu s} \, ds \leq Ce^{-\mu t} e^{\mu t}
\]

showing that \( \lambda (u) \geq \lambda (\dot{u}) \).
Lemma 3.7. Assume that \( \|\dot{u}(t)\|^{-1} \dot{u}(t) \to -x_0 \) in \( X^\alpha \) as \( t \to \infty \). Then \( \rho(t)^{-1} u(t) \to x_0 \) in \( X^\alpha \) as \( t \to \infty \), where \( \rho(t) := \int_t^\infty \|\dot{u}(s)\|_\alpha \, ds \).

Moreover, \( \rho(t)^{-1} \|u(t)\|_\alpha \to 1 \) as \( t \to \infty \).

Proof. By Lemma 3.6, we have for all \( t \in \mathbb{R}^+ \)

\[
 u(t) = -\int_t^\infty \dot{u}(s) \, ds
\]

and thus

\[
 u(t) = \int_t^\infty \|\dot{u}(s)\|_\alpha x_0 \, ds - \int_t^\infty \|\dot{u}(s)\|_\alpha + x_0 \, ds,
\]

where

\[
 \left\| \int_t^\infty \|\dot{u}(s)\|_\alpha x_0 + \dot{u}(s) \, ds \right\| \leq \sup_{s \geq t} \|\dot{u}(s)\|_\alpha + x_0 \int_t^\infty \|\dot{u}(s)\|_\alpha \, ds
\]

\[
 = \rho(t) \sup_{s \geq t} \|\dot{u}(s)\|_\alpha + x_0,
\]

showing that

\[
 \frac{\|u(t)\|_\alpha - x_0}{\rho(t)} \to 0 \quad \text{as } t \to \infty.
\]

Our assumptions imply that \( \|x_0\|_\alpha = 1 \), so \( \rho(t)^{-1} \|u(t)\|_\alpha \to \|x_0\|_\alpha = 1 \) as \( t \to \infty \), completing the proof. \( \Box \)

3.3. Convergence as \( t \to \infty \)

Let the assumptions on \( f \) at the beginning of Section 3.1 hold, and let \( u : \mathbb{R}^+ \to X^\alpha \) be a mild solution of (3.2) with \( u(t) \to 0 \) as \( t \to \infty \). Assume that the spectrum of \( L = A - Df(0) \) consists of simple, real, and isolated eigenvalues \( (\lambda_i) \in I \) with \( 0 \neq \lambda_i \) for all \( i \in I \).

We have already mentioned that the angle \( \frac{u(t)}{\|u(t)\|_\alpha} \) converges. The inverse question is whether there exists a solution \( v \) which converges to a given eigenvector of \( L \).

The proof primarily refines a part of [1, Theorem A.12]. But we need more control over the constants involved. In the case of ordinary differential equations in finite dimensions and under slightly more restrictive assumptions on the non-linearity, Proposition 3.8 can also be deduced from [5, Theorem 13.4.5].

Proposition 3.8. Let \( 0 < \lambda \) be an eigenvalue of \( L \) and let \( \eta \) denote an associated eigenvector with \( \|\eta\|_\alpha = 1 \). Then there is a solution \( u : [0, \infty[ \to X^\alpha \) of (3.1) with

\[
 \|\|u(t)\|^{-1} u(t) - \eta\|_\alpha \to 0 \quad \text{as } t \to \infty.
\]

(3.6)
Let $B(t) \in C([0, \infty[ , \mathcal{L}(X^\alpha, X^0))$ and consider the following perturbation of (3.2)

$$
\dot{u}(t) + Lu(t) = g(u(t)) + B(t)u(t),
$$

which can also be written as

$$
\dot{u} + Lu = \hat{g}(u) \tag{3.7}
$$

with $\hat{g} : C(\mathbb{R}^+, X^\alpha) \to C(\mathbb{R}^+, X^0), \ \hat{g}(u)(t) := g(u(t)) + B(t)u(t)$. The purpose of introducing $B$ is to cover two variants of the following lemma simultaneously.

**Lemma 3.9.** Let $\mu \in \mathbb{R} \setminus \text{Re} \sigma(L)$ and let $K_\mu$ be given by Lemma 3.3. Let $M = M(\mu) := \max\{2\|K_\mu\|, 1\}$ and $0 < \rho \leq \infty$. Provided that

$$
\kappa(\rho) := \sup_{\|x\|_\alpha \leq \rho, \|y\|_\alpha \leq \rho} \frac{\|g(x) - g(y)\|}{\|x - y\|_\alpha} \leq \frac{1}{2M} \tag{3.9}
$$

and

$$
\sup_{t \in \mathbb{R}^+} \|B(t)\|_{\alpha,0} \leq \frac{1}{2M} \tag{3.10}
$$

do the following hold:

1. If $u : \mathbb{R}^+ \to X^\alpha$ is a solution of (3.8) with $\lambda(u) \geq \mu$, then $\lambda(u) > \mu$ and $u = K_\mu(P^+(\mu)u(0), \hat{g}(u))$.
2. If $u \in C_\mu(\mathbb{R}^+, X^\alpha)$ is a solution of

$$
u = K_\mu(P^+(\mu)u(0), \hat{g}(u)) \tag{3.11}
$$

then $u$ is a mild solution of (3.7).
3. If $u_1, u_2 \in C_\mu(\mathbb{R}^+, X^\alpha)$ are solutions of (3.11) with $\sup_{t \in \mathbb{R}^+} \|u_i(t)\|_{\alpha} \leq \rho$ for $i \in \{1, 2\}$, then

$$
\|u_1 - u_2\|_{\alpha} \leq M\|P^-(\mu)(u_1(0) - u_2(0))\|_{\alpha}.
$$

4. There exists a continuous map $S = S_\mu : B \hat{g} \{0\} \subset X^\alpha \to C_\mu(\mathbb{R}^+, X^\alpha)$ such that for all $x \in \mathcal{D}(S)$ one has $S(x) = K_\mu(x, \hat{g}(S(x))) = K_\mu(P^+(\mu)x, \hat{g}(S(x)))$ and $P^+(\mu)S(x)(0) = P^+(\mu)x$.

**Remark 1.** Since $Dg(0) = 0$ is the Fréchet-derivative, there always exists a $\rho$ such that (3.9) holds.

**Proof.** Letting $C_\mu := C_\mu(\mathbb{R}^+, X^\alpha)$, we have

$$
\left\| K_\mu(x_1, \hat{g}(u)) - K_\mu(x_2, \hat{g}(v)) \right\|_{C_\mu} \leq \frac{M}{2} (\|x_1 - x_2\|_{\alpha} + \kappa(\rho)\|u - v\|_{C_\mu}) + \frac{1}{4}\|u - v\|_{C_\mu}
$$

for all $x_1, x_2 \in X^\alpha$ and for all $u, v \in C(\mathbb{R}^+, X^\alpha)$ with $\|u\|_{C(\mathbb{R}^+, X^\alpha)} \leq \rho$ and $\|v\|_{C(\mathbb{R}^+, X^\alpha)} \leq \rho$.

In view of (3.9)

$$
\left\| K_\mu(x_1, \hat{g}(u)) - K_\mu(x_2, \hat{g}(v)) \right\|_{C_\mu} \leq \frac{M}{2} (\|x_1 - x_2\|_{\alpha} + \frac{1}{2}\|u - v\|_{C_\mu}) \tag{3.12}
$$

for all $x_1, x_2 \in X^\alpha$, and all $u, v \in C_\mu(\mathbb{R}^+, X^\alpha)$ with $\|u\|_{C(\mathbb{R}^+, X^\alpha)} \leq \rho$ and $\|v\|_{C(\mathbb{R}^+, X^\alpha)} \leq \rho$. 

(1) Let \( u \) be a solution of (3.7) with \( \lambda(u) \geq \mu \). By Lemma 3.5, we have \( \lambda(u) > \mu \), so for all \( t \geq r \geq 0 \),

\[
e^{-L(-t)} P^-(\mu) u(t) = e^{Lt} P^-(\mu) u(r) + \int_r^t e^{-L(s-t)} P^-(\mu) \hat{g}(u)(s) \, ds \to 0
\]
as \( t \to \infty \) since for \( (-t) < 0 \) we have \( \|e^{-L(-t)} P^-(\mu) u(t)\|_\alpha \leq M e^{\mu t} \|u(t)\|_\alpha \to 0 \) as \( t \to \infty \). This shows that \( u \) is a solution of (3.11).

(4) Let \( Y := B_\rho[0] \subset C_\mu(\mathbb{R}^+, X^\alpha) \) and let \( x_0 \in P^+(\mu) X^\alpha \) with \( \|x_0\|_\alpha \leq \frac{\rho}{M} \). \( \tilde{K} y := K_\mu(x_0, \hat{g}(y)) \) defines a contraction mapping on \( Y \) since by (3.12)

\[
\|\tilde{K} y\|_{C_\mu} \leq \frac{\rho}{2} + \frac{1}{2} \|y\|_{C_\mu} \leq \rho.
\]

Hence, there is a unique fixed point for every \( x_0 \).

(3) By (3.12), we have

\[
\|u_1 - u_2\|_{C_\mu} = \|K_\mu(x_0, \hat{g}(u_1)) - K_\mu(x_0, \hat{g}(u_2))\|_{C_\mu} \leq \frac{M}{2} \|P^+(\mu)(u_1(0) - u_2(0))\|_\alpha + \frac{1}{2} \|u_1 - u_2\|_{C_\mu},
\]

so

\[
\|u_1 - u_2\|_{C_\mu} \leq M \|P^+(\mu)(u_1(0) - u_2(0))\|_\alpha.
\]

(2) \( u \) is a mild solution of (3.7) since for all \( t_1, t_2 \in \mathbb{R}^+ \) with \( t_1 < t_2 \)

\[
P^-(\mu) u(t_2) - P^-(\mu) e^{-L(t_2-t_1)} u(t_1) = -\int_{t_2}^\infty e^{-L(t_2-s)} P^-(\mu) \hat{g}(u)(s) \, ds
\]

\[+ e^{-L(t_2-t_1)} \int_{t_1}^\infty e^{-L(t_1-s)} P^-(\mu) \hat{g}(u)(s) \, ds\]

\[= \int_{t_1}^{t_2} e^{-L(t_2-s)} P^-(\mu) \hat{g}(u)(s) \, ds. \quad \square\]

**Proof of Proposition 3.8.** Let \( \mu_1 < \lambda < \mu_2 \) be real numbers such that

\[
[\mu_1, \mu_2] \cap \sigma(L) = \{\lambda\},
\]

let \( 1 \leq M(\mu_i), i \in \{1, 2\} \) be given by Lemma 3.9, let \( M := \max\{M(\mu_1), M(\mu_2)\} \), and choose \( \rho > 0 \) small enough that \( \kappa(\rho) < \frac{1}{2M} \).

Let \( 0 < \epsilon \leq \frac{\rho}{M\Gamma} \), and let \( u \) denote the unique solution of

\[
u = K_{\mu_1}(\epsilon \eta, g \circ u).
\]

It follows that \( \sup_{t \in \mathbb{R}^+} \|u(t)\|_\alpha \leq M \|\epsilon \eta\|_\alpha \leq \frac{\rho}{M\Gamma} \).

Suppose that \( \lambda(u) > \lambda \), so by Lemma 3.5 \( \lambda(u) \geq \mu_2 \), which implies that \( u \) is a solution of

\[
    u = K_{\mu_2}(P^+(\mu_2)\varepsilon \eta, g \circ u).
\]

It follows that \( \|u\|_{C_{\mu}} \leq M\|P^+(\mu_2)\varepsilon \eta\|_{\alpha} = 0 \), a contradiction to \( P^+(\mu_1)u(0) = \varepsilon \eta \), implying that \( \lambda(u) = \lambda \).

It is another consequence of Lemma 3.5 that either \( \|u(t)\|_{\alpha}^{-1}u(t) \rightarrow \eta \) or \( \|u(t)\|_{\alpha}^{-1}u(t) \rightarrow -\eta \) as \( t \rightarrow \infty \), so in either case it holds that \( \|u(t)\|_{\alpha}^{-1}P^+(\mu_2)u(t) \rightarrow 0 \) as \( t \rightarrow \infty \).

Suppose that \( \|u(t)\|_{\alpha}^{-1}u(t) \rightarrow -\eta \) as \( t \rightarrow \infty \) and let \( w : \mathbb{R}^+ \rightarrow X^\alpha \) be given by \( w(t) := S_{\mu_2}(P^+(\mu_2)u(t))(0) = S_{\mu_2}(u(t))(0) \). We then have

- \( w(0) = 0 \) since \( P^+(\mu_2)u(0) = 0 \).
- \( \|u(t)\|_{\alpha}^{-1}\|w(t)\|_{\alpha} \leq M\|u(t)\|_{\alpha}^{-1}\|P^+(\mu_2)u(t)\|_{\alpha} \rightarrow 0 \) as \( t \rightarrow \infty \) by the boundedness of \( S \).
- \( P^+(\mu_2)w(t) = P^+(\mu_2)u(t) \) for all \( t \in \mathbb{R}^+ \).

It now follows that \( \|u(t)\|_{\alpha}^{-1}(u(t) - w(t)) \rightarrow -\eta \) as \( t \rightarrow \infty \). By the intermediate value theorem, there exists a \( t_0 \in \mathbb{R}^+ \) such that

\[
    \left( P^+(\mu_1) - P^+(\mu_2) \right)(u(t_0) - w(t_0)) = 0,
\]

and so \( P^+(\mu_1)u(t_0) = P^+(\mu_1)w(t_0) \).

\[
    v := S_{\mu_2}(P^+(\mu_2)w(t_0)) = S_{\mu_2}(w(t_0)) \text{ is a solution of}
\]

\[
    v = K_{\mu_1}(P^+(\mu_1)w(t_0), g \circ v),
\]

and it holds that \( \sup_{t \in \mathbb{R}^+} \|v(t)\|_{\alpha} \leq M\|u(t_0)\|_{\alpha} \leq \rho \). There is another solution of (3.13), namely

\[
    u_{t_0} = K_{\mu_1}(P^+(\mu_1)w(t_0), g \circ u_{t_0}),
\]

where \( u_{t_0}(t) := u(t_0 + t) \), \( t \in \mathbb{R}^+ \), denotes the time-\( t_0 \)-shifted solution.

It follows that \( v = u_{t_0} \), and so \( \lambda(u) = \lambda(v) \geq \mu_2 > \lambda \), a contradiction.

### 3.4. Convergence as \( t \rightarrow -\infty \)

Suppose that \( u : \mathbb{R}^- \rightarrow X^\alpha \) is a solution of (3.1) with \( u(t) \rightarrow 0 \in X^\alpha \) as \( t \rightarrow -\infty \). For large \( t \in \mathbb{R} \), \( u(-t) \) can be described by an ordinary differential equation in finite dimensions (see [14, Theorem 71.1], [8, Theorem 5.1.2], [13, Theorem 3.3]). We can now reverse the time and obtain analogous statements for \( t \rightarrow -\infty \).

**Corollary 3.10.** Let \( \rho^-(t) := \int_{-\infty}^{t} \|\dot{u}(s)\|_{\alpha} \, ds \) (see also Lemma 3.7). There is an eigenvector \( \eta \) of \( L \), which belongs to the eigenvalue \( \lambda \), such that in \( X^\alpha \)

\[
    (\rho^-(t))^{-1}u(t) \rightarrow \eta
\]

and

\[
    \dot{\rho}^-(t)^{-1}\dot{u}(t) = \|\dot{u}(t)\|_{\alpha}^{-1}\dot{u}(t) \rightarrow \eta
\]

as \( t \rightarrow -\infty \).

**Proposition 3.11.** Let \( \lambda < 0 \) be an eigenvalue of \( L \) and let \( \eta \) denote an associated eigenvector with \( \|\eta\|_{\alpha} = 1 \). Then there is a mild solution \( u : [-\infty, 0] \rightarrow X^\alpha \) of (3.1) with

\[
    \|u(t)\|_{\alpha}^{-1}(u(t) - \eta) \rightarrow 0 \quad \text{as} \quad t \rightarrow -\infty.
\]
3.5. A sufficient condition for an exponential decay rate

**Proposition 3.12.** Let $\delta > 0$, $B \in C([0, \infty], \mathcal{L}(H, H))$ symmetric with $e^{2Bt}(B(t) - B(\infty)) \to 0 \in \mathcal{L}(H, H)$ as $t \to \infty$, $A_H$ be symmetric, $(\eta_i)_{i \in \mathbb{N}}$ an orthonormal basis for $H$ which consists of eigenvectors of $L := A_H - B(\infty)$, and let $u : \mathbb{R}_+^+ \to X^\alpha$ be a mild solution of

$$\dot{u}(t) + A_H u(t) = B(t)u(t)$$  \hspace{1cm} (3.17)

with $\lambda(u) = \infty$.

Then there is an $\varepsilon > 0$ such that $u(t) = 0$ for all $t \in \mathbb{R}_+$ with $\sup_{s \geq t} \|B(s) - B(\infty)\|_{H,H} \leq \varepsilon^2$.

**Lemma 3.13.** Let the assumptions of Proposition 3.12 hold and let $K_\mu \in \mathcal{L}(H \times C_{\mu+\delta}(\mathbb{R}_+, H), C_{\mu}(\mathbb{R}_+, H))$ be defined as in Lemma 3.3.

Then $K_\mu$ is well defined and $C_K := \sup_{\mu \in \mathbb{R}_+ \setminus \sigma(L)} \|K_\mu\| < \infty$. Moreover, for all $x \in H$ one has $\|P^+(\mu) x\|_H \to 0$ as $\mu \to \infty$.

$C_\mu$ is defined as before but with respect to $X = H$ and $\alpha = 0$, that is, the norm on $H$ is considered.

**Proof.** For each $i \in \mathbb{N}$, let $\lambda_i$ denote the eigenvalue associated with $\eta_i$. The eigenvalues are (due to the symmetry of $A_H - B(\infty)$) real. We thus have

$$\langle e^{-Lt} x, \eta_i \rangle = e^{-\lambda_i t} \langle x, \eta_i \rangle \quad x \in H, \; i \in \mathbb{N}. \hspace{1cm} (3.18)$$

Every $x \in H$ may now be written as $x = \sum_{i \in \mathbb{N}} \langle x, \eta_i \rangle \eta_i$ and one has

$$\|x\|_H^2 = \sum_{i \in \mathbb{N}} \langle x, \eta_i \rangle^2. \hspace{1cm} (3.19)$$

Since for every $\mu \in \mathbb{R} \setminus \sigma(L)$ the projections $P^-(\mu)$ and $P^+(\mu)$ are the orthogonal projections in $H$, that is,

$$P^-(\mu)x = \sum_{i \in \mathbb{N}: \lambda_i < \mu} \langle x, \eta_i \rangle \eta_i \quad \text{and}$$

$$P^+(\mu)x = \sum_{i \in \mathbb{N}: \lambda_i > \mu} \langle x, \eta_i \rangle \eta_i,$$

it follows that $P^+(\mu)x \to 0$ in $\mathcal{L}(H, H)$ as $\mu \to \infty$.

Furthermore, for every $i \in \mathbb{N}$ we have $e^{-Lt} \eta_i = e^{-\lambda_i t} \eta_i$, which shows that for all $\mu \in \mathbb{R} \setminus \sigma(L)$

$$\|e^{-Lt} P^-(\mu)x\|_H \leq e^{-\mu t} \|x\|_H \quad t \leq 0$$

$$\|e^{-Lt} P^+(\mu)x\|_H \leq e^{-\mu t} \|x\|_H \quad t > 0.$$

It follows that $K_\mu$ is well defined and

$$\|K_\mu(x_0, f)\|_{C_\mu} \leq \|x_0\|_H + 2 \int_0^\infty e^{-\delta s} ds \|f\|_{C_{\mu+\delta}}. \hspace{1cm} \square$$
Proof of Proposition 3.12. Let $C_K$ be given by Lemma 3.13, suppose that
\[ \| B(t) - B(\infty) \|_{H,H} \leq e^{-2\delta t} M , \]
and choose $\varepsilon := \frac{1}{2} C_K$ and $\tau \in \mathbb{R}^+$ such that
\[ \| B(t) - B(\infty) \|_{H,H} \leq \varepsilon \frac{e^{-2\delta t}}{M} \]
for all $t \geq \tau$. We now have
\[ \| B(t) - B(\infty) \|_{H,H}^2 \leq \varepsilon^2 e^{-2\delta t}. \]

Let $0 < \mu \in \mathbb{R}^+ \setminus \sigma(L)$ be arbitrary. $v := u(\tau + t)$ is a mild solution of
\[ \dot{x} + Lx = \tilde{B}(t)x := (B(t + \tau) - B(\infty))x \tag{3.20} \]
with $\lambda(u) = \lambda(v) = \infty$.

It follows from Lemma 3.9 that $v = K_{\mu}(P^+(\mu)v(0), \tilde{B}v)$, where we set $(\tilde{B}u)(t) := \tilde{B}(t)u(t)$. We thus have
\[ \| v \|_{C_{\mu}} \leq C_K \| P^+(\mu)v(0) \|_{H} + \| \tilde{B}v \|_{C_{\mu+\delta}} \]
\[ \leq C_K \| P^+(\mu)v(0) \|_{H} + C_K \varepsilon \| v \|_{C_{\mu}} \]
\[ \leq C_K \| P^+(\mu)v(0) \|_{H} + \frac{1}{2} \| v \|_{C_{\mu}} \]
and consequently
\[ \| v \|_{C_{\mu}} \leq 2C_K \| P^+(\mu)v(0) \|_{H}. \]

This estimate holds for arbitrary $\mu \in \mathbb{R} \setminus \sigma(L)$, that is,
\[ \| u(\tau) \|_{H} \leq 2C_K \| P^+(\mu)u(\tau) \|_{H} \to 0 \quad \text{as} \quad \mu \to \infty, \]
proving that $u(\tau) = 0$. \quad \Box

4. Construction of the diffeomorphism

Recall the assumptions at the beginning of Section 3. We consider the semiflow induced by mild solutions of
\[ \dot{u}(t) + Au(t) = f(u(t)). \tag{4.1} \]

In particular, we assume that $f \in C^1(\mathcal{U}, X)$, where $\mathcal{U}$ is an open set in $X^\alpha$. Fix an eigenvalue $\eta \in X^1$ of $A$, let $F := \text{span}(\eta)$, and let $X = F \oplus E$. For $\alpha \in [0, 1]$, let $E^\alpha := E \cap X^\alpha$ be endowed with $\| \cdot \|_\alpha$.

Using $L := A$, let the projections $P^-$ and $P^+$ be defined as in Section 3.
Suppose that $u$ is a heteroclinic full solution and $\tilde{u} := \text{cl}\{u(t): t \in \mathbb{R}\}$ is an isolated invariant set. In order to calculate its homotopy index it is helpful to assume that $\tilde{u}$ lies entirely in a one-dimensional subspace of the considered phase space $X^\alpha$. Therefore, we construct a diffeomorphism which maps the image of $\tilde{u}$ into a one-dimensional subspace.

There is a simple “prototypical” situation where the construction is obvious, namely, if one assumes that $u$ has a “main direction” that is, there is a one-dimensional subspace and an associated projection such that the image of $\tilde{u}$ under this projection does not vanish for any $t \in \mathbb{R}$. In this case, one could consider a mapping $(t, e) \mapsto u(t) + e$, $e \in E$, where $E$ denotes the complementary subspace. The following theorem is a generalization of this basic idea.

Obviously, the smoothness of such a mapping is – at least in the direction of $t$ – limited by the smoothness of $u$. There are other problems which have not been considered in this informal introduction: the diffeomorphism should be defined in a neighborhood of $\tilde{u}$ and the semiflow obtained by applying the diffeomorphism should still be induced by mild solutions of a semilinear parabolic equation like (4.1).

Theorems of this kind are often referred to as tubular neighborhood theorems, but (as far as known to the author) they are either stated in a finite-dimensional setting or they require more smoothness than $C^1$ and would thus impose additional restrictions on the non-linearity $f$ in (4.1).

**Theorem 4.1.** Let $\gamma \in C^1([0, 1], X^0)$ such that $0 \neq \gamma(t)$ for all $t \in [0, 1]$, $\mathcal{S} \subset [0, 1]$ be finite, and $\xi \in [0, 1]$ with $\gamma(\xi) \notin E$.

Then there exist a neighborhood $U$ of $[0, 1] \times \{0\}$ in $[0, 1] \times E$ and a diffeomorphism $\varphi : U \to \varphi(U) \subset X$ such that

1. there exists $\mu \in \mathbb{R} \setminus \text{Re}(\sigma(A))$ such that $\varphi(x, y) = \gamma(x) + \Phi(x)P^-(\mu)y + P^+(\mu)y$, where $\Phi : [0, 1] \to L(P^-(\mu)E, P^-(\mu)X)$ is continuous;
2. $\Phi(\xi) = \text{id}$;
3. $\Phi(x)$ is locally constant in a neighborhood of $\mathcal{S}$;
4. for all $(x_0, y_0)$ in $U$ there are continuous (Fréchet-)derivatives $D_\gamma \Phi(x_0, y_0)$ and $D_\gamma (D\Phi(x_0, y_0)^{-1})$.

**Lemma 4.2.** Let the assumptions of Theorem 4.1 hold. Then there exists a $\mu \in \mathbb{R} \setminus \sigma(A)$ with $P^-(\mu)\gamma(t) \neq 0$ for all $t \in [0, 1]$.

**Proof.** It follows from Lemma 3.4 that for every $t \in [0, 1]$ there is a $\mu_t \in \mathbb{R} \setminus \sigma(A)$ with $P^-(\mu_t)\gamma(t) \neq 0$. The continuity of $\gamma(t)$ implies that there is an open neighborhood $U_t$ of $t$ such that $P^-(\mu_t)\gamma(s) \neq 0$ for all $s \in U_t$. $\{U_t\}_{t \in [0, 1]}$ is an open covering of $[0, 1]$, hence by compactness, there is a finite subcovering $\{U_{t_k}\}_{k=1}^{[1, \ldots, n]} \in \mathbb{N}$. Let $\mu := \max\{\mu_{t_k}: k \in \{1, \ldots, n\}\}$. We then have for all $k \in \{1, \ldots, n\}$ and all $s \in U_{t_k}$ $P^-(\mu)P^-(\mu)\gamma(s) = P^-(\mu_t)\gamma(s) \neq 0$ so that $P^-(\mu_t)\gamma(s) \neq 0$ for all $s \in [0, 1]$. □

**Lemma 4.3.** Let $k \in \mathbb{N}$, $\mathcal{S} \subset [0, 1]$ finite, $\xi \in \mathcal{S}$, and $\Phi \in C([0, 1], \text{ISO}(\mathbb{R}^k, \mathbb{R}^k)) \cap C^1([0, 1], L(\mathbb{R}^k, \mathbb{R}^k))$. Then there is a sequence $\Phi_n \in C([0, 1], \text{ISO}(\mathbb{R}^k, \mathbb{R}^k)) \cap C^1([0, 1], L(\mathbb{R}^k, \mathbb{R}^k))$ such that

1. $\Phi_n \to \Phi$ in $C([0, 1], \text{ISO}(\mathbb{R}^k, \mathbb{R}^k))$;
2. $\Phi_n$ is locally constant in a neighborhood of $\mathcal{S}$;
3. $\Phi_n(\xi) = \Phi(\xi)$ for all $n \in \mathbb{N}$.

**Proof.** Using the differentiability of $\Phi$, we can write

$$\Phi(x) = \Phi(\xi) + \int_\xi^x F(s) \, ds$$

with $F \in C([0, 1], L(\mathbb{R}^k, \mathbb{R}^k))$. 

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Define $F_n$ in $L^\infty([0, 1], \mathcal{L}(\mathbb{R}^k, \mathbb{R}^k))$ by

$$F_n(x) := \begin{cases} 0 & \text{dist}(x, \mathcal{S}) \leq \frac{1}{2^n} \\ F(x) & \text{otherwise.} \end{cases}$$

It follows that $F_n$ is well defined and that $\|F_n - F\|_\infty \leq \|F\|_\infty < \infty$.

Finally, choose $\tilde{F}_n \in C([0, 1], \text{ISO}(\mathbb{R}^k, \mathbb{R}^k)) \cap C^1([0, 1], \mathcal{L}(\mathbb{R}^k, \mathbb{R}^k))$ with $\|\tilde{F}_n - F_n\|_\infty \leq 1/n$, and let $\Phi_n$ be defined by

$$\Phi_n(x) = \Phi(\xi) + \int_\xi^x F_n(s) \, ds.$$ 

We have

$$\|\Phi_n - \Phi\|_\infty \leq \|F\|_\infty \frac{\# \mathcal{S}}{n} + \frac{1}{n} \to 0 \text{ as } n \to \infty.$$ 

$\Phi_n$ is an isomorphism for all $n$ sufficiently large. □

**Proof of Theorem 4.1.** Let $\mu_0$ be given by Lemma 4.2, $\mu_0 \leq \mu \in \mathbb{R} \setminus \sigma(A)$, $P := P^-(\mu)$, and $E_0 := PX \subset X^1 (\dim E_0 < \infty)$. By choosing $\mu$ large enough, we can assume that $\eta \in E_0$ ($\eta$ is the eigenvector defining $F$).

$P\dot{\gamma} : [0, 1] \to E_0$ induces a monomorphism $U : [0, 1] \times F \to [0, 1] \times E_0$ of bundles in the sense of Appendix A, where $U(t)(\eta) := tP\dot{\gamma}(t)$. By the assumption that $P\eta = \eta$, one has $E_0 = F \oplus PE$. By Corollary A.9, there exists an isomorphism $\Phi_0 = (U \oplus S_0) \in C([0, 1], \mathcal{L}(E_0, E_0))$ such that $S_0(\xi)y = y$ and $\Phi_0(\xi)y = P\dot{\gamma}(t)$ for all $t \in [0, 1]$.

By the Weierstrass approximation theorem, there is another sequence $\Phi_n = (U \oplus S_n)_{n \in \mathbb{N}}$ in $C([0, 1], \mathcal{L}(E_0, E_0))$ such that for each $n \in \mathbb{N}$, $S_n$ is continuously Fréchet-differentiable, $S_n(\xi) = \text{id}$, and $\Phi_n \to \Phi_0$ uniformly in $t$ with respect to the norm in $\mathcal{L}(E_0, E_0)$. Using Lemma 4.3, we can assume that $\Phi_n$ is locally constant in a neighborhood (depending on $n$) of $\mathcal{S}$ for all $n \in \mathbb{N}$.

Let $t \in [0, 1]$ and define $H_{n,t}$ by

$$\Phi_0(t)^{-1}\Phi_n(t) = \Phi_0(t)^{-1}(\Phi_0(t) + (\Phi_n(t) - \Phi_0(t)))$$

$$= 1 + \Phi_0(t)^{-1}(\Phi_n(t) - \Phi_0(t))$$

$$= 1 + H_{n,t}.$$ 

Using the Neumann series, there exists an inverse of $\Phi_0(t)^{-1}\Phi_n(t)$ whenever $\|H_{n,t}\| < 1$. We have

$$\|H_{n,t}\| \leq \|\Phi_0^{-1}(t)\| \|\Phi_n(t) - \Phi_0(t)\| \leq \sup_{t \in [0, 1]} \|\Phi_0^{-1}(t)\| \sup_{t \in [0, 1]} \|\Phi_n(t) - \Phi_0(t)\|$$

for all $t \in [0, 1]$, where $\sup_{t \in [0, 1]} \|\Phi_0^{-1}(t)\| < \infty$ by Corollary A.4 and $\sup_{t \in [0, 1]} \|\Phi_n(t) - \Phi_0(t)\| \to 0$ as $n \to \infty$ by the uniform approximation. Hence, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for all $t \in [0, 1]$ $\Phi_n(t) = (\Phi_0 \circ \Phi_0^{-1} \circ \Phi_n)(t)$ is an isomorphism in $\mathcal{L}(E_0, E_0)$, and particularly a homeomorphism by Corollary A.4.

Let $\Phi := \Phi_{n_0}$ and define $\varphi : [0, 1] \times E \to X$ by

$$\varphi(t, y) := \gamma(t) + \Phi(t)Py + (1 - P)y.$$
Let \((t_0, y_0) \in [0, 1] \times E\), let \((t, y) \in \mathbb{R} \times E\) and let \(h \in \mathbb{R}^+\). We have for \(h\) small enough

\[
\frac{1}{h}(\psi(t_0 + h t, y_0 + h y) - \psi(t_0, y_0))
\]

\[
= \frac{1}{h}(\gamma(t_0 + h t) - \gamma(t_0) + \Phi(t_0 + h t)(P y_0 + h P y) - \Phi(t_0)P y_0) + (1 - P) y
\]

\[
= \frac{1}{h}(\gamma(t_0 + h t) - \gamma(t_0) + \Phi(t_0 + h t)h P y + \Phi(t_0 + h t)P y_0 - \Phi(t_0)P y_0) + (1 - P) y
\]

\[
\rightarrow t \dot{\gamma}(t) + \Phi(t_0)P y + (D\Phi(t_0)t)P y_0 + (1 - P) y \quad \text{as } h \to 0.
\]

In particular, \((t_0, y_0) \mapsto (D\Phi(t_0)1)P y_0\) is continuous, so there is a continuous Fréchet derivative, namely

\[
D\psi(t_0, y_0)(t, y) = (D\gamma(t_0) + (D\Phi(t_0)1)P y_0)t + \Phi(t_0)P y + (1 - P) y.
\]

We have for \((t, y) \in \mathbb{R} \times E\) and \(t_0 \in [0, 1]\)

\[
PD\psi(t_0, 0)(t, y) = P \dot{\gamma}(t_0)t + P \Phi(t_0)P y
\]

\[
= \Phi(t_0)(\eta t + P y),
\]

showing that \(PD\psi(t_0, 0) : \mathbb{R} \times PE \to PX = E_0\) is an isomorphism for all \(t_0 \in [0, 1]\). Therefore, it follows that

\[
D\psi(t_0, 0)(t, y) = P \Phi(t_0)(\eta t + P y) + (1 - P)(\dot{\gamma}(t_0)t + y)
\]

is an isomorphism, the inverse is given by

\[
(t, y_1) = (PD\psi(t_0, 0))^{-1}P y
\]

\[
D\psi(t_0, 0)^{-1}y = (t, y_1 + (1 - P)(y - \dot{\gamma}(t_0)t)).
\]

The inverse mapping theorem now implies that \(\psi\) is a local diffeomorphism.

Suppose that there does not exist an open neighborhood \(U\) of \([0, 1] \times \{0\}\) in \([0, 1] \times E\) such that \(\psi_U\) is injective. Then there are sequences \((t_n, y_n) \to (t_0, 0)\) in \([0, 1] \times E\) and \((\tilde{t}_n, \tilde{y}_n) \to (\tilde{t}_0, 0)\) in \([0, 1] \times E\) (by the compactness of \([0, 1]\)) such that \(t_n \neq \tilde{t}_n\) and \(\psi(t_n, y_n) = \psi(\tilde{t}_n, \tilde{y}_n)\) for all \(n \in \mathbb{N}\). It follows from the continuity of \(\psi\) that \(\gamma(t_0) = \psi(t_0, 0) = \psi(t_0, 0) = \gamma(\tilde{t}_0)\) and since \(\gamma\) is injective, we have \(t_0 = \tilde{t}_0\). This is a contradiction since \(\psi\) is a local homeomorphism. We have shown that there exists an open neighborhood \(U\) of \([0, 1] \times E\) such that \(\psi_U : U \to \varphi(U)\) is a homeomorphism.

Finally, we have \(D_y D\psi(x_0, y_0) = (D\Phi(x_0)1)P y \cdot p_x\), where \(p_x : \mathbb{R} \times E \to \mathbb{R}, p_x(x, y) = x\) for all \((x, y) \in \mathbb{R} \times E\). Hence \(D_y D\psi(x_0, y_0)^{-1}\) exists and is given by

\[
D_y D\psi(x_0, y_0)^{-1}y = -D\psi(x_0, y_0)^{-1} \circ D_y D\psi(x_0, y_0) y \circ D\psi(x_0, y_0)^{-1}.
\]

**Proposition 4.4.** Let \(u\) be a solution of (4.1) with \(u(t) \to 0 \Rightarrow e^+, u(-t) \to e^-, \) and \(\|u(t)\|_{\alpha}^{-1}u(t) \to \eta\) as \(t \to \infty\).

Then there exist an open neighborhood \(U \subset [0, 1] \times E^\alpha\) of \([0, 1] \times \{0\}\), a neighborhood \(V\) of \(K := c[\{u(t) : t \in \mathbb{R}\}], \) and a diffeomorphism \(\varphi : U \to V\) such that

1. \(\varphi(x, y) = \gamma(x) + \Phi(x)P^{-}(\mu) y + P^{+}(\mu) y\), where \(\mu \in \mathbb{R} \setminus \text{Re}(A(\gamma)), \gamma \in C^1([0, 1], X^\alpha), \) and \(\Phi \in C([0, 1], L(P^{-}(\mu)E, P^{+}(\mu)X))\) is locally constant in a neighborhood of \(\varphi^{-1}([e^-, e^+])\);
(2) \( \Phi(\gamma^{-1}(e^+)) = \text{id}_{p-(\mu_1)E} \);
(3) for all \((x_0, y_0)\) in \(U\) there are continuous (Fréchet-)derivatives \(D\gamma D\Phi(x_0, y_0)\) and \(D\gamma(D\Phi(x_0, y_0)^{-1})\);
(4) \(\varphi(\mathbb{R} \times \{0\} \cap U)\) is invariant under the restriction of \(\pi\) to \(V\) and we have \(K \subset \gamma([0, 1])\);
(5) \(x \mapsto A\varphi(x, 0)\) is continuous.

**Lemma 4.5.** Let \(u, v^+: \mathbb{R} \to X^\alpha\) be given by Theorem 3.2. Then there is a closed neighborhood \([a, b]\) of 0 and a homeomorphism \(p^+: [a, b] \to \{u(t): \ t \in [0, \infty]\} \cup \{v^+(t): \ t \in [0, \infty]\} \cup \{e^+\} \subset X^\alpha\) such that

1. \(p^+ \in C^1([a, b], X^\alpha)\);
2. \(\dot{p}^+(t) \neq 0\) for all \(t \in [a, b]\);
3. \((p^+, \dot{p}^+)(a) = (u(0), \|\dot{u}(0)\|^{-1}_\alpha \dot{u}(0)), p^+(0) = e^+, p^+(b) = v^+(0)\);
4. \(Ap^+\) is continuous.

**Proof.** Let \(\lambda^+, \eta^+\) and \(\rho(t) := \rho(u, t)\) be given by Theorem 3.2. Let \(\rho^{-1}(u, .)\) denote the inverse mapping, that is,

\[
\int_{\rho^{-1}(u, t)}^\infty \|\dot{u}(s)\|^{-1}_\alpha ds = t.
\]

Define further

\[
p^+(t) := \begin{cases} u(\rho^{-1}(u, -t)) & t \in [-\rho(u, 0), 0[ \\ e^+ & t = 0 \\ v^+(\rho^{-1}(v^+, t)) & t \in ]0, \rho(v^+, 0)]. \end{cases}
\]

We now have

\[
\dot{p}^+(t) = -\left(\frac{d}{dt} \rho^{-1}(-t)\right)\dot{u}(\rho^{-1}(-t))
\]
\[
= -\frac{1}{\dot{\rho}(\rho^{-1}(-t))} \dot{u}(\rho^{-1}(-t))
\]
\[
= \frac{1}{\|\dot{u}(\rho^{-1}(-t))\|^{-1}_\alpha} \dot{u}(\rho^{-1}(-t))
\]

and substituting \(t = \rho(s)\) one obtains \(\dot{p}^+(\rho(s)) = -\|\dot{u}(s)\|^{-1}_\alpha \dot{u}(s)\) for all \(s \in [0, \infty]\). We have \(\dot{p}^+(t) \to \eta^+\) as \(t \to 0\) and \(Ap^+(\rho(t)) = Au(t) = f(u(t)) - \dot{u}(t)\). The last term is continuous in \(t\) and it holds that \(f(u(t)) - \dot{u}(t) \to f(e^+) = Ap^+(0)\) as \(t \to \infty\).

The second branch of \(p^+\), that is, the case \(t > 0\), can be treated analogously. \(\square\)

There is an equivalent for negative times to the previous lemma, which can be proven analogously.

**Lemma 4.6.** Let \(u, v^-: \mathbb{R} \to X^\alpha\) be given by Theorem 3.2. Then there is a closed neighborhood \([a, b]\) of 0 and a homeomorphism \(p^-: [a, b] \to \{u(t): \ t \in ]-\infty, 0]\} \cup \{v^-(t): \ t \in ]-\infty, 0]\} \cup \{e^-\} \subset X^\alpha\) such that

1. \(p^- \in C^1([a, b], X^\alpha)\);
2. \(\dot{p}^-(t) \neq 0\) for all \(t \in [a, b]\);
3. \((p^-, \dot{p}^-)(a) = (u(0), \|\dot{u}(0)\|^{-1}_\alpha \dot{u}(0)), p^-(0) = e^-, p^-(b) = v^- (0)\);
4. \(Ap^-\) is continuous.
5.1. Linear skew product semiflows

Definition 5.1.

Lemma 4.6.

Let $\pi$ be a Banach space and let $\pi(\xi,\Phi)$ be a linear skew product semiflow on $[a, b] \times F$, where

$$
(x, y)\pi t = (x\xi t, \Phi(x, t)y)
$$

for every $(x, t) \in D(\xi)$ we have $\Phi(x, t) \in \mathcal{L}(F, F)$.

Let $SK([a, b], F)$ denote the set of all linear skew product semiflows on $[a, b], F$ and let $\pi \in SK := SK([a, b], F) \subset SK([a, b], F)$ if there exists an $\epsilon > 0$ and a $\pi \in SK([a - \epsilon, b + \epsilon], F)$ with $(x, y)\pi t = (x, y)\pi t$ whenever the left side is defined.

Proof of Proposition 4.4. Let $p^- : [a_1, b_1] \to X^\alpha$ and $p^+ : [a_2, b_2] \to X^\alpha$ be given by Lemmas 4.5 and 4.6 and let

$$
\tilde{\gamma}(x) := \begin{cases} 
p^-(a_1 - x) & x \in [a_1 - b_1, 0] 
p^+(a_2 + x) & x \in [0, b_2 - a_2].
\end{cases}
$$

In view of Lemma 4.5 and Lemma 4.6, we have particularly $p^-(a_1 - 0) = p^+(a_2 + 0) = u(0)$ and $\hat{p}^-(a_1) = \hat{p}^+(a_2) = \|\hat{u}(0)\|_a^{-1}\hat{u}(0)$. Therefore, $\gamma : [0, 1] \to X^\alpha$.

$$
\gamma(t) := \tilde{\gamma}(t(b_2 - a_2) + (1 - t)(a_1 - b_1))
$$

is well defined and continuously differentiable.

Since $\|u(t)\|_a^{-1}u(t) \to \eta$ as $t \to \infty$, it is clear that $\dot{\gamma}(\gamma^{-1}(e^+) = \eta$. Hence, we can apply Theorem 4.1 to $\gamma$ and obtain a mapping $\phi$ for which (1), (2), and (3) hold.

(4) and (5) follow from the choice of $\gamma$ (see also Fig. 4.1) and the two lemmas: Lemma 4.5 and Lemma 4.6. □

5. Isolation and homotopy equivalence

For a hyperbolic equilibrium (a stationary solution), it is a usual technique to compute its homology index by computing the homology index of its linearization. Given two equilibria and an orbit connecting them, the assumption that the respective stable and unstable manifolds intersect transversally is a substitute for the hyperbolicity assumption in the zero-dimensional case of a single equilibrium. However, it is not immediately clear what linearization shall mean. Simply passing to the tangential space is not possible, since it is a one-dimensional subbundle of the full tangential space, which corresponds to the given orbit, that is, given a heteroclinic solution $u$ of a differentiable semiflow $\pi$, the pair $(u, \lambda\hat{u})$ is a solution of $T\pi$ (using Definition 5.2) for every $\lambda \in \mathbb{R}$. Hence, $K := \text{cl}\{(u(t), 0): t \in \mathbb{R}\}$ is not an isolated invariant set, and $\hat{K} := \text{cl}\{(u(t), \lambda\hat{u}(t))): \lambda, t \in \mathbb{R}\}$ is not compact, which means that there does not exist a $T\pi$-admissible isolating neighborhood of $\hat{K}$.

5.1. Linear skew product semiflows

Sell and You use in [14] the notion of linear skew product semiflows. We will borrow the concept since it is a suitable abstraction for our Conley index calculations.

Definition 5.1. Let $F$ be a Banach space and let $a < b$ be real numbers. A linear skew product semiflow on $(a, b], F)$ is a semiflow $\pi = (\xi, \Phi)$ on $[a, b] \times F$, where

$$
(x, y)\pi t = (x\xi t, \Phi(x, t)y) \quad \forall (t, x, y) \in D(\pi).
$$

Here, $\xi$ is a flow on $[a, b]$ and for every $(x, t) \in D(\xi)$ we have $\Phi(x, t) \in \mathcal{L}(F, F)$.
Given a decomposition $F = F_1 \oplus F_2$ into closed subspaces and semiflows $\pi_1 = (\xi, \Phi_1) \in \text{SK}([a, b], F_1)$, $\pi_2 = (\xi, \Phi_2) \in \text{SK}(\xi, \Phi_2)$, define $\pi_1 \oplus \pi_2 \in \text{SK}([a, b], F)$ by $\pi_1 \oplus \pi_2 = (\xi, \Phi_1 \oplus \Phi_2)$, where $(\Phi_1 \oplus \Phi_2)(t, x)(y_1 \oplus y_2) = \Phi_1(t, x)y_1 \oplus \Phi_2(t, x)y_2$.

Let $M$ be an open subset of a Banach space $F$ and let $\gamma : [0, 1] \rightarrow \Gamma$ be a diffeomorphism. One may regard $TM$ as $M \times F$, and $[0, 1] \times F$ is diffeomorphic to $\Gamma \times X^\alpha$. $U : [0, 1] \times \mathbb{R} \rightarrow TM$, $U(x)y := \gamma'(x) \cdot y$ is a subbundle in the sense of Appendix A. In particular, it follows from Corollary A.9 that $TM/T\Gamma$ is a metric space (the definition according to Appendix A and the definition below coincide).

**Definition 5.2.** Let $M$ be an open subset of a Banach space $F$ and let $\Gamma$ be a $C^1$-submanifold of $M$. For $x \in \Gamma$ define

$$TxM/Tx\Gamma := \{ \{ \eta + \eta' : \eta' \in T_x\Gamma \} : \eta \in TxM \}$$

and

$$TM/T\Gamma := \{ (x, \eta) : x \in \Gamma \text{ and } \eta \in T_xM/Tx\Gamma \}.$$ 

Let $\pi$ be a $C^1$-semiflow on $M$ and let $\Gamma$ be invariant under $\pi$. Then $\pi$ induces a natural semiflow $T\pi$ on $TM$ which is defined by

$$T\pi(t, (x, \eta)) := (x\pi t, D(\pi(t, .))(x)\eta).$$

By the linearized semiflow $\pi'(\Gamma)$ along $\Gamma$ we mean the linear skew product semiflow on $TM/T\Gamma$ which is defined by

$$\pi'(t, (x, \eta)) := p(T\pi(t, (x, \eta))).$$

where $p : \{ (x, \eta) \in TM : x \in \Gamma \} =: TM(\Gamma) \rightarrow TM/T\Gamma$ denotes the canonical projection that is, $p(x, \eta) = (x, [\eta])$.

Let $TM/T\Gamma$ be equipped with the quotient topology and let each fiber be equipped with the norm $\|y\|_{\xi} := \|y\|_{T_xM/T_x\Gamma} := \inf\|y - y'\| : y' \in T\Gamma, \xi \in \Gamma$.

**Lemma 5.3.** Let $M$ and $\Gamma$ satisfy the assumptions of Definition 5.2. Then $\pi' := \pi'(\Gamma)$ is a semiflow.

**Proof.** First, one has to show that $\pi'$ is well defined. Since $T\pi$ is a linear skew product semiflow, it may be decomposed into its components: let $T\pi = (\xi, \Phi)$. Now, let $y_1, y_2 \in F$ with $[y_1]_{F/T_x\Gamma} = [y_2]_{F/T_x\Gamma}$, let $x \in [a, b]$ and let $t \in \mathbb{R}^+$ such that $\Phi(t, x)$ is defined. We then have $y_1 - y_2 \in T\Gamma$ so that $\Phi(t, x)y_1 - \Phi(t, x)y_2 = \Phi(t, x)(y_1 - y_2) \in T\Gamma$ due to the invariance of $T\Gamma$, implying that $[\Phi(t, x)y_1]_{F/T_x\Gamma} = [\Phi(t, x)y_2]_{F/T_x\Gamma}$.

Now, $\pi'$ inherits its properties from $T\pi$. In particular, it is continuous due to the choice of the quotient topology and

$$(x, y)[\pi'(t_1 + t_2) + y] = \left( x[\pi'(t_1 + t_2)], \Phi(x, t_1 + t_2) + y \right)$$

$$= \left( x[\pi'(t_1)]\pi'(t_2), \Phi(x, t_1 + t_2) \right) \in D(T\pi). \quad \square$$
Lemma 5.4. Let \( M \) and \( \Gamma \) satisfy the assumptions of Definition 5.2 and \( \dim \Gamma = n \in \mathbb{N} \).

Further, let \( T \pi = (\xi, \Phi) \) and suppose that \( \Phi(t, x) \neq 0 \) for all \((t, x) \in \mathcal{D}(\Phi) \) and all \( 0 \neq y \in T_x \Gamma \).

Finally, let \([u(t), v(t)]\) be a solution of \( \pi' \) which is defined for all \( t \in [-t_0, 0] \). Then there is a unique solution \((u(t), \tilde{v}(t))\) of \( T \pi \) satisfying \([u(t), v(t)] = [u(t), \tilde{v}(t)] \) and \( v(0) = \tilde{v}(0) \).

**Proof.** We have \([\Phi(t_0, u(-t_0))v(-t_0)] = [v(0)]\), so there is a solution \((u(t), w(t))\) of \( T \pi \) with \( w(t) - v(t) \in T_{u(t)} \Gamma \) for all \( t \in [-t_0, 0] \).

Moreover, the restriction \( \Phi(t_0, u(-t_0)) : T_u(-t_0) \Gamma \to T_{\Phi(t_0)} \Gamma \) is an isomorphism since it is injective, and so there exists a unique \( \eta \in T_{\Phi(t_0)} \Gamma \) with \( \Phi(t_0, u(-t_0))\eta = v(0) - w(0) \).

The linearity of \( \Phi(t_0, u(-t_0)) \) now implies that \( \Phi(t_0, u(-t_0))(v(-t_0) + \eta) = w(0) + \Phi(t_0, u(-t_0))\eta = v(0) \). By the invariance of \( T \Gamma \), we have \([u(t), \tilde{v}(t)] = [u(t), w(t)]\) for all \( t \in [-t_0, 0] \), where we set \( \tilde{v}(t) := w(t) + \Phi(t + t_0)\eta \).

**Definition 5.5.** Let \( M \) and \( \pi \) satisfy the assumptions of Definition 5.2.

For every \( x \in \Gamma \) let \( y \in B^-(T \pi, x) \) iff there is a solution \((u, v) : \mathbb{R} \to TM \) of \( T \pi \) such that \((u(0), v(0)) = (x, y)\) and \( \sup_{t \in \mathbb{R}} \|v(t)\| < \infty \); and let \( y \in B^+(T \pi, x) \) iff there exists a solution \((u, v) : \mathbb{R}^+ \to TM \) of \( T \pi \) with \((u(0), v(0)) = (x, y)\) and \( \sup_{t \in \mathbb{R}^+} \|v(t)\| < \infty \).

The above notion of a bounded solution can be translated to \( TM/T \Gamma' \).

**Definition 5.6.** Let \( M, \Gamma \), and \( \pi \) satisfy the assumptions of Definition 5.2.

For every \( x \in \Gamma \) let \( y \in B^-(\pi', x) \) iff there is a solution \((u, v) : \mathbb{R} \to TM/T \Gamma \) of \( \pi' \) such that \((u(0), v(0)) = (x, y)\) and \( \sup_{t \in \mathbb{R}} \|v(t)\|_{T_{u(t)}M/T_{u(t)}\Gamma'} < \infty \); and let \( y \in B^+(\pi', x) \) iff there exists a solution \((u, v) : \mathbb{R}^+ \to TM/T \Gamma \) of \( \pi' \) with \((u(0), v(0)) = (x, y)\) and \( \sup_{t \in \mathbb{R}^+} \|v(t)\|_{T_{u(t)}M/T_{u(t)}\Gamma'} < \infty \).

The transversal intersection of the respective stable and unstable manifolds (or weaker, of the respective local stable manifold and the unstable manifold) has one implication concerning \( T \pi \) which is crucial (and sufficient) for the following linearization procedure, namely

**Definition 5.7.** Let \( M \) be an open subset of a Banach space \( F \), and let \( \pi \) be a semiflow on \( M \). Let \( e^+, e^- \in M \) be hyperbolic equilibria, and let \( u(t) \) be a heteroclinic solution with \( u(t) \to e^- \) as \( t \to -\infty \) and \( u(t) \to e^+ \) as \( t \to \infty \) (not necessarily \( e^- \neq e^+ \)).

\( u \) is said to be normal if for all \( t \in \mathbb{R} \)

\[
\dim(B^-(T \pi, u(t)) \cap B^+(T \pi, u(t))) = 1. \tag{5.1}
\]

5.2. Isolation

Recall the assumptions we made in Section 3. In particular, the semiflow \( \pi \) is induced by mild solutions of

\[
\dot{u}(t) + Au(t) = f(u(t)), \tag{5.2}
\]

where \( f \in C^1(\mathcal{U}, X) \), \( \mathcal{U} \subset X^\alpha \) is open, and \( A \) has compact resolvent. We will use \( F = X^\alpha \) and \( T \pi \) is the semiflow induced by mild solutions of

\[
\dot{u}(t) + Au(t) = f(u(t)) \quad \text{and} \quad \dot{v}(t) + Av(t) = Df(u(t))v(t).
\]
Let $u(t)$ be a solution of (5.2) such that Proposition 4.4 can be applied, and let $\varphi : U \rightarrow V$ and $E$ be given by that proposition. Then the assumptions in Definition 5.2 are satisfied for $F = X^u$, $M = V$ and $\Gamma = \varphi([0, 1] \times \{0\})$.

If the equilibria $e^-$, $e^+$ are hyperbolic,

$$B^+(T\pi, u(t)) + B^-(T\pi, u(t)) = T_{u(t)}M \quad \text{for all } t \in \mathbb{R}, \quad \text{and}$$

$$\dim B^-(T\pi, u(t)) = \text{codim} B^+(T\pi, u(t)) + 1,$$

then (5.1) holds.

We are now in a position to state the main result of this section.

**Proposition 5.8.** Suppose that $u$ is normal. $K_0 := [K \times \{0\}]_{TM/\Gamma}$ is an isolated invariant set relative to $\pi'$, that is, there exists an isolating neighborhood $N$ of $K_0$ in $TM/\Gamma$.

The proof of Proposition 5.8 relies on

**Lemma 5.9.** The following holds for all $x_0 \in K_0$

$$B^+(\pi', x_0) \subset \left[B^+(T\pi, x_0)\right]$$

$$B^-(\pi', x_0) \subset \left[B^-(T\pi, x_0)\right].$$

**Proof.** We can assume w.l.o.g. that $u(0) = x_0$ or $x_0 \in \{e^-, e^+\}$.

Let $[y] \in B^+(\pi', x_0)$ and let $(u, v) : \mathbb{R}^+ \rightarrow TM$ be a solution of $T\pi$ with $v(0) = y$.

We have $u(t) \rightarrow e \in [e^+, e^-]$ and $\|u(t)\|^{-1} \rightarrow \eta$ as $t \rightarrow \infty$, where $\eta$ is an eigenvector of $L := A - Df(e)$. Let $0 < \lambda$ be the associated eigenvalue, and let $P_\eta$ denote the projection onto the eigenspace spanned by $\eta$ that is, $P_\eta = \lim_{t \rightarrow \infty} P^- (\lambda + \delta) - P^- (\lambda - \delta)$.

By Lemma A.10, there exists a neighborhood $V_0$ of $e$ in $\Gamma$ such that for all $x \in V_0$ the canonical projection $Q(x) : (1 - P_\eta)X^u \rightarrow X^u / T_x\Gamma$ lies in $\text{ISO}((1 - P_\eta)X^u, X^u / T_x\Gamma)$ and there are constants $0 \neq m, M \in \mathbb{R}^+$ such that

$$m\|x\|_\alpha \leq \|x\|_{X^u / T_x\Gamma} \leq M\|x\|_\alpha \quad \forall x \in V_0.$$  \hspace{1cm} (5.5)

Let $t_0 \in \mathbb{R}^+$ such that $u(t) \in V_0$ for all $t \geq t_0$, and set $w(t) := Q^{-1}(u(t))([v(t)])$, $t \geq t_0$. Since $\sup_{t \in \mathbb{R}^+} \|v(t)\|_{X^u / T_{u(t)}\Gamma} < \infty$, (5.5) implies that

$$\sup_{t \in \mathbb{R}^+} \|w(t)\|_\alpha < \infty.$$ \hspace{1cm} (5.6)

Moreover, it holds for all $t \geq t_0$ that $[w(t) - v(t)]_{X^u / T_{u(t)}\Gamma} = 0$ and so $v(t) - w(t) \in T_{u(t)}\Gamma$.

Lemma A.7 implies that there is a neighborhood $V_1 \subset V_0$ of $e$ such that $P(x) := P_\eta \in \text{ISO}(T_{u(t)}\Gamma, P_\eta X^u)$ for all $x \in V_1$. There is a $t_1 \in \mathbb{R}^+$ such that $t_1 \geq t_0$ and $u(t) \in V_1$ for all $t \geq t_1$.

Letting $F(t) := Df(u(t)) - Df(e)$, we have

$$P_\eta F(t)v(t) = P_\eta F(t)(v(t) - w(t)) + P_\eta F(t)w(t)$$

$$= P_\eta F(t)P(u(t))^{-1} P_\eta v(t) \xrightarrow{\text{as } t \rightarrow \infty} P_\eta v(t) = \left(\begin{array}{c} P_\eta (v(t) - w(t)) \rightarrow 0 \text{ as } t \rightarrow \infty \end{array}\right).$$
Thus, \( P_\eta(v(t) - w(t)) = P_\eta v(t), \ t \geq t_1, \) is a solution of an ordinary differential equation (in one dimension)

\[
\dot{x} + P_\eta L x = G(x, t).
\]

We can apply [5, Theorem 13.3.1], which states that \( P_\eta(v(t) - w(t)) \) is governed by the eigenvalue \( 0 < \lambda, \) that is, \( \sup_{t \in \mathbb{R}^+} \| P_\eta(v(t) - w(t)) \|_\alpha < \infty. \) It follows that

\[
\sup_{t \in \mathbb{R}^+} \| v(t) \|_\alpha \leq \sup_{t \in \mathbb{R}^+} \| w(t) \|_\alpha + \sup_{t \in \mathbb{R}^+} \| P(u(t))^{-1} \|_{\alpha, \alpha} \| P_\eta(v(t) - w(t)) \|_\alpha < \infty,
\]

and therefore \( y \in B^+(T \pi, x_0), \) implying that \( B^+(\pi', x_0) \subset [B^+(T \pi, x_0)]. \)

Using Lemma 5.4, one can show analogously that \( B^-(\pi', x_0) \subset [B^-(T \pi, x_0)]. \) The proof is therefore omitted. \( \square \)

**Proof of Proposition 5.8.** Let \( N_0 \) be an isolating neighborhood for \( K \) relative to the restriction of \( \pi \) to \( \Gamma \) and define

\[
N := \{ [x, y] \in TM/T \Gamma: x \in N_0 \text{ and } y \in E^\alpha \text{ with } \| y \|_{X^\alpha/T u(t) F} \leq 1 \}.
\]

Further let \( \tilde{u}, \tilde{v}: \mathbb{R} \to N \) be a full solution of \( \pi'. \) It follows from Lemma 5.8 that there exists a full solution \( (\tilde{u}, v) \) of \( T \Gamma \) such that \( (\tilde{u}, [v]) = (\tilde{u}, \tilde{v}). \) \( v \) is bounded, that is \( \sup_{s \in \mathbb{R}} \| v(s) \|_\alpha < \infty \) by Lemma 5.9.

Now, there are two cases: either \( \tilde{u}(t) \in \{ e^-, e^+ \} \) for all \( t \in \mathbb{R}, \) implying that \( v \equiv 0 \) by the hyperbolicity assumption, or \( \tilde{u}(t) = u(t + \tau) \) for some \( \tau \in \mathbb{R}. \) We may assume w.l.o.g. that \( \tau = 0. \)

In the second case, we have for all \( t \in \mathbb{R} \) \( v(t) \in T \Gamma = B^+(T \pi, u(t)) \cap B^-(T \pi, u(t)), \) which is equivalent to \( v(t) = 0 \) and so \( \dot{v} \equiv 0. \) \( \square \)

5.3. Linearization along a solution

As in the previous section, we are given a linear subspace \( E \subset X. \) It is convenient to assume that \( A | E^1 = A(E \cap D(A)) \subset E. \) Let \( \varphi: U \to V, \) and \( \mu \in \mathbb{R} \) be given by Proposition 4.4, and let \( \varphi(x(t), y(t)) \) be a solution of (5.2) which is defined on \( [0, T]. \) Then for all \( t \in ]0, T[ \) \( \varphi(x(t), y(t)) \in X^1, \) \( (x(t), y(t)) \) is differentiable, and

\[
D\varphi(x(t), y(t))(\dot{x}(t), \dot{y}(t)) + A \varphi(x(t), y(t)) = f \circ \varphi(x(t), y(t)).
\]

Letting \( P := P^- (\mu) \) and \( Q := P^+ (\mu), \) we can split (5.7) into an equation on \( PX \) and another one on \( QX. \) We will omit the notation of \( t \) in order to improve the readability.

On \( PX, \) we have

\[
PD \varphi(x, y)(\dot{x}, \dot{y}) = Pf \circ \varphi(x, y) - PA \varphi(x, y)
\]

\[
=: Pf(\varphi(x, y)),
\]

where the right side is again continuously Fréchet-differentiable since \( PX \subset X^1 \) is finite-dimensional.
On $QX$, one obtains

$$Q D \varphi(x, y)(\dot{x}, \dot{y}) + A \frac{Q y}{Q (\varphi(x, y) - \varphi(x, 0))} = Q f(\varphi(x, y)) - AQ \varphi(x, 0)$$

$$=: Q \tilde{f}(\varphi(x, y)).$$

\(\tilde{f}\) is well defined, continuous, and \(\tilde{f} \circ \varphi\) has a continuous Fréchet-derivative \(D_y \tilde{f}\). Furthermore, \((x(t), y(t))\) is a solution of

$$\dot{x}(t) = g_1(x(t), y(t))$$

$$\dot{y}(t) + \tilde{A} y(t) = g_2(x(t), y(t)),\quad (5.8)$$

where we set

$$g(x, y) := (g_1, g_2)(x, y) := D \varphi(x, y)^{-1} \circ \tilde{f} \circ \varphi(x, y)$$

and \(\tilde{A} := AQ\), which is again a sectorial operator since for all \(y \in X^1\) we have \(Ay - \tilde{A} y = AP y\) with \(AP \in \mathcal{L}(X^\alpha, X^0)\). The sectoriality now follows from [8, Corollary 1.4.5]. Moreover, by [8, Theorem 1.4.6], the norms induced by \(A\) and \(\tilde{A}\) are equivalent.

Using Proposition 4.4, one can show

Lemma 5.10. \(g_2 : U \cap E^\alpha \to E^0\) is continuously Fréchet-differentiable in \(y\) (with \(D_y g_2 \in \mathcal{L}(E^\alpha, E^0)\)).

Let the family of semiflows \((\pi_\lambda)_{\lambda \in [0, 1]}\) on \(\mathbb{R} \times E^\alpha (E^\alpha = E \cap X^\alpha)\) be defined as follows:

Definition 5.11. \((x(t), y(t))\) is a solution of \(\pi_\lambda\) if \(\varphi(x(t), \lambda y(t))\) is a mild solution of (5.2) and \(y(t)\) is a mild solution of

$$\dot{y}(t) + \tilde{A} y(t) = \tilde{g}_\lambda(x(t), y(t)),\quad (5.8)$$

where we set

$$\tilde{g}_\lambda(x, y) := \begin{cases} \lambda^{-1} g_2(x, \lambda y) & \lambda > 0 \\ D_y g_2(x, 0) y & \lambda = 0. \end{cases}$$

Given \(\lambda \in [0, 1]\) and a solution \(\varphi(x(t), \lambda y(t))\) of (5.2), it follows that (5.8) holds, that is, \(y(t)\) is a solution of (5.8).

What follows is the main result of this section.

Theorem 5.12. Let the assumptions at the beginning of Section 5.2 hold, and suppose that \(u\) is normal.

Then

1. \(K := \varphi^{-1}(\text{cl } u(\mathbb{R}))\) is an isolated invariant set relative to \(\pi_\lambda\) for all \(\lambda \in [0, 1]\);
2. \(h(\pi_1, K) = h(\pi_0, K)\).

In order to prove the theorem, we can make the following additional assumptions w.l.o.g.:

1. \(U \cap (\mathbb{R} \times \{0\}) = ]0, 1[ \times \{0\}\);
2. \(\|y\|_\alpha \leq 1\) for all \((x, y) \in U\);
(3) \( U \) is convex in \( y \), that is, for all \( \xi \in [0, 1] \) one has \((x, \xi y_1 + (1 - \xi)y_2) \in U\) whenever \((x, y_1)\) and \((x, y_2) \in U\);
(4) \( \sup_{(x, y) \in U} \|g_2(x, y)\|_\alpha < \infty\); 
(5) \( \sup_{(x, y) \in U} \|D_2g_2(x, y)\|_{\alpha, 0} < \infty\).

**Lemma 5.13.** There exists a constant \( L \in \mathbb{R}^+ \) such that
\[
\left\| \tilde{g}_\lambda(x, y_1) - \tilde{g}_\lambda(x, y_2) \right\|_0 \leq L \|y_1 - y_2\|_\alpha
\]
for all \((x, y_1), (x, y_2) \in U\) and all \( \lambda \in [0, 1] \).

**Proof.** Let \( \lambda \in [0, 1] \) and \((x, y_1), (x, y_2) \in U\). We have for all \( \xi \in [0, 1] \)
\[
\left\| \tilde{g}_\lambda(x, y_1) - \tilde{g}_\lambda(x, y_2) \right\|_0 \leq \sup_{\xi \in [0, 1]} \left\| D_2 \tilde{g}_\lambda(x, \xi y_1 + (1 - \xi)y_2) \right\|_{\alpha, 0} \|y_1 - y_2\|_\alpha
\]
\[
\leq \sup_{(x, y) \in U} \left\| D_2 g_2(x, y) \right\|_{\alpha, 0} \|y_1 - y_2\|_\alpha. \quad \square
\]

**Lemma 5.14.** Let \( \lambda_n \to 0 \) in \([0, 1]\), \( T \in \mathbb{R}^+ \), \( \gamma_n \to \gamma_0 \) in \( C([0, T]), [0, 1]\), and \( h_n(t, y) := \tilde{g}_{\lambda_n}(\gamma_n(t), y) \) for \( n \in \mathbb{N} \cup \{0\} \).

Then \( h_n(t, y) \) is continuous in \((t, y)\) for all \( n \in \mathbb{N} \cup \{0\} \) and for every \( 0 < \rho \in \mathbb{R}^+ \) one has
\[
\sup \left\{ \|h_n(t, y) - h_0(t, y)\|_0 : t \in [0, T], y \in E^\alpha, \|y\|_\alpha \leq \rho \right\} \to 0
\]
as \( n \to \infty \).

**Proof.** We have for all \((x_1, y), (x_2, y) \in U\)
\[
\left\| \tilde{g}_{\lambda_n}(x_1, y) - \tilde{g}_0(x_2, y) \right\|_0 \leq \left\| (\tilde{g}_{\lambda_n}(x_1, y) - \tilde{g}_0(x_1, 0)) - (\tilde{g}_0(x_2, y) - \tilde{g}_0(x_2, 0)) \right\|_0
\]
\[
+ \left\| \tilde{g}_{\lambda_n}(x_1, 0) - \tilde{g}_0(x_2, 0) \right\|_0
\]
\[
\leq \sup_{\xi \in [0, 1]} \left\| D_2 g_2(x_1, \xi \lambda_n y) - D_2 g_2(x_2, \xi \lambda_n y) \right\|_{\alpha, 0} \|y\|_\alpha
\]
\[
+ \left\| g_2(x_1, 0) - g_2(x_2, 0) \right\|_0.
\]

Suppose that our claim is not true for some \( \rho \in \mathbb{R}^+ \). Then there are sequences \( t_n \to t_0 \) in \([0, T]\), \( y_n \in E^\alpha \), \( k(n) \to \infty \) in \( \mathbb{N} \) and an \( \varepsilon > 0 \) such that \( \|h_{k(n)}(t_n, y_n) - h_0(t_n, y_n)\| > \varepsilon \) for all \( n \in \mathbb{N} \). In view of the above calculation, we have for \( x_n := \gamma_{k(n)}(t_n) \) and \( \tilde{x}_n := \gamma_0(t_n) \)
\[
\left\| h_{k(n)}(t_n, y_n) - h_0(t_n, y_n) \right\|_0 \leq \sup_{\xi \in [0, 1]} \left\| D_2 g_2(x_n, \xi \lambda_n y) - D_2 g_2(\tilde{x}_n, \xi \lambda_n y) \right\|_{\alpha, 0} \rho
\]
\[
+ \left\| g_2(x_n, 0) - g_2(\tilde{x}_n, 0) \right\|_0
\]
\[
\to 0 \quad \text{as} \quad n \to \infty,
\]
a contradiction. \( \square \)
Using the previous lemmas, we are now able to prove

**Proposition 5.15.** Let \([a, b] \subset V\) such that \(K := [a, b] \times \{0\}\) is an isolated invariant set relative to \(\pi_0\).

Then \((\pi_\lambda)_{\lambda \in [0, 1]}\) is an \(S\)-continuous family of semiflows in the sense of [12, Definition 12.1], that is, for every \(\lambda \in [0, 1]\), \(K\) is an isolated invariant set relative to \(\pi_\lambda\) and there is a neighborhood \(W\) of \(\lambda\) in \([0, 1]\) and a closed set \(N \subset V\) such that

1. For every \(\lambda \in W, N\) is a strongly \(\pi_\lambda\)-admissible isolating neighborhood of \(K\), relative to \(\pi_\lambda\);
2. Whenever \(\lambda_n \to \lambda_0\) in \([0, 1]\), then \(x_n \pi_{\lambda_n} t_n \to x_0 \pi_{\lambda_0} t_0\) as \(n \to \infty\) for every sequence \((x_n, y_n), t_n \to (x_0, y_0), t_0\) in \(U \times \mathbb{R}^+\), and \(N\) is \((\pi_{\lambda_n})_n\)-admissible.

**Proof.** Let \(\lambda_n \to \lambda_0\) in \([0, 1]\). We have to show that \(\pi_n := \pi_{\lambda_n} \to \pi_{\lambda_0} := \pi_0\). For every \(n \in \mathbb{N}\) let

\[(u_n(t), v_n(t)), 0 \leq t \leq t_0\]

be the solution of \(\pi\) for which \((u_n(0), v_n(0)) = (x_n, y_n)\).

Suppose that \(\lambda_0 \neq 0\). It follows that

\[(x_n, y_n) \to (x_0, \lambda_0 y_0),\]

so by the continuity of \(\pi_1\) there is a solution \((u_0(t), v_0(t))\) of \(\pi\) with \((u_0(0), v_0(0)) = (x_0, y_0)\) which is defined for all \(t \in [0, t_0]\) and we have \((u_n(t_0), v_n(t_0)) \to (u_0(t_0), v_0(t_0))\) as \(n \to \infty\). Therefore, \((x_n, y_n) \pi n t_n = (u_n(t_n), \lambda_n^{-1} v_n(t_n)) \to (u_0(t_0), \lambda_0^{-1} v_0(t_0)) = (x_0, y_0)\) \(\pi 0\) that is, \(\pi_n \to \pi_0\).

Now suppose that \(\lambda_0 = 0\). As the continuity of \(\pi_1\), there is a solution \((u_0(t), 0)\) of \(\pi_1\) defined for all \(t \in [0, t_0]\) with \((u_0(t), 0) = \lim_{n \to \infty} (u_n(t), \lambda_n v_n(t))\) for all \(t \in [0, t_0]\).

For every \(\tau \in [0, t_0]\), we have \(\sup_{t \in [0, \tau]} |u_n(t) - u_0(t)| = 0\) as \(n \to \infty\). It follows from Lemma 5.13 that for all \(n \in \mathbb{N}\)

\[
G_n(t, y) := \tilde{g}_{\lambda_n}(u_n(t), \lambda_n y) \quad t \in [0, \tau]
\]

is Lipschitz continuous in \(y\) and from Lemma 5.14 that

\[
\sup_{\|y\|_\alpha \leq \rho} \|G_n(t, y) - G_0(t, y)\| \to 0
\]

as \(n \to \infty\), provided that \(\rho > 0\) is sufficiently small.

Moreover, for each \(n \in \mathbb{N}\), \(v_n(t)\) is a mild solution of

\[
\dot{y} + \tilde{A} y = G_n(t, y).
\]

Let \(v_0(t)\) denote the maximally defined mild solution of

\[
\dot{y} + \tilde{A} y = G_0(t, y)
\]

with \(v_0(0) = y_0\). [14, Theorem 47.5] implies that \(v_n(t) \to v_0(t)\) uniformly on \([0, \tau]\) provided that \(v_0(t)\) is defined on \([0, \tau]\). Because \(v_n(t) \in U\), we have \(\|v_n(t)\|_\alpha \leq 1\) for all \(t \in [0, t_n]\) and all \(n \in \mathbb{N}\) so it follows from Lemma 5.13 and [14, Lemma 47.4] that \(v_0(t)\) is defined for all \(t \in [0, t_0]\). This shows again that \(\pi_n \to \pi_0\).

In order to verify the strong admissibility, let \(0 < \varepsilon \in \mathbb{R}^+\), let \(N_0 \subset U\) be an isolating neighborhood for \(K\) with respect to \(\pi_0\) and define

\[
N := N(\varepsilon) := \{(x, y) \in [0, 1] \times \mathbb{R}^\alpha : (x, 0) \in N_0\} \text{ and } y \in \mathbb{E}_\alpha \text{ with } \|y\|_\alpha \leq \varepsilon\}.
\]

By choosing \(\varepsilon_0\) small enough, \(N(\varepsilon_0) \subset U\), and Lemma 5.13 and [14, Lemma 47.4] imply that \(\pi_\lambda\) does not explode in \(N(\varepsilon)\) for all \(\varepsilon \in [0, \varepsilon_0]\) and all \(\lambda \in [0, 1]\).

Now let there be given sequences \((x_n, y_n)\) in \(N, \lambda_n \to \lambda_0\) in \([0, 1]\) and \(t_n \to \infty\) in \(\mathbb{R}^+\) such that for every \(n \in \mathbb{N}\) and for all \(s \in [0, t_n]\), \(x_s \pi_{\lambda_n} s \in N\), where we set \(\pi_n := \pi_{\lambda_n}\). We may assume that \(x_n \to x_0\). Let \((u_n(s), v_n(s)) := (x_n, y_n) \pi n s, s \in [0, t_n]\). \(v_n(t)\) is a mild solution of (5.8). Hence, it follows exactly
as in the proof of [12, Theorem 1.4.3] that given \( \beta \in ]0, 1[ \) there is a constant \( b \in \mathbb{R}^+ \) such that \( \|v_n(t_n)\|_\beta \leq b \) for all \( n \in \mathbb{N} \) sufficiently large. By [8, Theorem 1.4.8] (A has compact resolvent), the inclusion \( X^\beta \subset X^\alpha \) is compact, so there exists a convergent (in \( X^\alpha \)) subsequence of \( v_n(t_n) \). This proves the claims concerning the admissibility properties.

Suppose that \( N(\epsilon_0) \) is an isolating neighborhood for \( (K, \pi_0) \) (this can always be achieved by choosing \( \epsilon_0 \) small enough) and that there does not exist an \( \epsilon \in ]0, \epsilon_0[ \) such that for all \( \lambda \in ]0, 1] \), \( N(\epsilon) \) is an isolating neighborhood for \( (K, \pi_\lambda) \). Then there is a sequence \( \lambda_n \in ]0, 1] \) and for every \( n \in \mathbb{N} \) a full solution \( (u_n(t), v_n(t)) \) of \( \pi_n := \pi_{\lambda_n} \) with \( 0 < c_n := \sup_{t \in \mathbb{R}} \|v_n(t)\|_\alpha \to 0 \) as \( n \to \infty \).

It follows that \( (u_n(t), c_n^{-1} v_n(t)) \) is a solution of \( \pi_{\lambda_n c_n} \). We may assume that \( 2\|v_n(0)\|_\alpha \geq c_n \) and by admissibility that \( (u_n(0), v_n(0)) \to (x_0, y_0) \). We have

\[
\frac{\|v_n(0)\|_\alpha}{c_n} \geq \frac{\|v_n(0)\|_\alpha}{2\|v_n(0)\|_\alpha} = \frac{1}{2},
\]

showing that \( y_0 \neq 0 \). By [12, Theorem 1.4.5] and since \( c_n\lambda_n \leq c_n \to 0 \) as \( n \to \infty \), \( (x_0, y_0) \in \text{Inv}_{\pi_0}(N) = K \), a contradiction to \( y_0 \neq 0 \). We have shown that there is an \( \epsilon_0 > 0 \) such that for all \( \epsilon \in ]0, \epsilon_0[ \) and all \( \lambda \in ]0, 1] \) \( N(\epsilon) \) is an isolating neighborhood for \( K \) relative to \( \pi_\lambda \). □

**Lemma 5.16.**

\[
p_2 \circ D\Pi_{1, t}(x_0, 0) = \Phi(x_0, t) \circ p_2 \quad (x_0, 0) \in ]0, 1[ \cap U,
\]

where \( p_2 : \mathbb{R} \times E \to E, p_2(x, y) := y \), denotes the canonical projection, \( \Pi_{1, t}x := x\pi_1 t \) and \( \pi_0 = (\xi, \Phi) \in SK([0, 1], E) \).

**Proof.** According to Definition 5.11 and Proposition 5.15, one has for all \( x_0 \in ]0, 1[ \times \{0\} \cap U \) and \( (x, y) \in \mathbb{R} \times E^\alpha \)

\[
\Phi(x_0, t) \circ p_2(x, y) = \Phi(x_0, t)y
\]

\[
= \lim_{\lambda \to 0^+} p_2 \circ \Pi_{\lambda, t}(x_0, y)
\]

\[
= \lim_{\lambda \to 0^+} p_2(\lambda^{-1}(\Pi_{1, t}(x_0, \lambda y) - \Pi_{1, t}(x_0, 0)))
\]

\[
= p_2 \circ D\Pi_{1, t}(x_0, 0)(0, y)
\]

\[
= p_2 \circ D\Pi_{1, t}(x_0, 0)(x, 0) + p_2 \circ D\Pi_{1, t}(x_0, 0)(0, y)
\]

\[
= p_2 D\Pi_{1, t}(x, 0)(x, y),
\]

where we have used the invariance of \( \Gamma \) under \( \pi \) (resp. \([0, 1[ \cap U \) under \( \pi_1 \)). □

**Proof of Theorem 5.12.** Our claims follow from Proposition 5.15 and [12, Theorem 1.12.2] if we show that \( K = \varphi^{-1}(\text{cl}\{u(t) : t \in \mathbb{R}\}) \) is isolated relative to \( \pi_0 \).

Let \( \tilde{M} = ]0, 1[ \times E^\alpha \) and \( \tilde{\Gamma} = ]0, 1[ \times \{0\} \).

\[
\begin{array}{ccc}
T\tilde{M} \times \mathbb{R}^+ & \xrightarrow{T\pi_1} & T\tilde{M} \\
\downarrow \text{id} \times p_2 \times \text{id} & & \downarrow \text{id} \times p_2 \\
]0, 1[ \times E^\alpha \times \mathbb{R}^+ & \xrightarrow{\pi_0} & ]0, 1[ \times E^\alpha
\end{array}
\]

is commutative by Lemma 5.16 and...
\[ \begin{array}{c}
T \tilde{M} \times \mathbb{R}^+ \\ \downarrow \text{id} \times p_2 \times \text{id} \\
\] \[ ] 0, 1[ \times E^\alpha \times \mathbb{R}^+ \] \[ ] 0, 1[ \times E^\alpha \] \[ \downarrow k \times \text{id} \\
T \tilde{M} / T \Gamma \times \mathbb{R}^+ \] \[ \] \[ \downarrow \pi_1' \]
\end{array} \]

\[ T \tilde{M} / T \Gamma \times \mathbb{R}^+ \] \[ \] \[ \downarrow \pi_1' \]

where we set \( k(x, y) := [x, (0, y)] \), by the definition of \( \pi_1' \). Combining the previous two diagrams (\( p_2 \) is an epimorphism) shows that

\[ \] \[ ] 0, 1[ \times E^\alpha \times \mathbb{R}^+ \] \[ ] 0, 1[ \times E^\alpha \] \[ \downarrow k \times \text{id} \\
T \tilde{M} / T \Gamma \times \mathbb{R}^+ \] \[ \] \[ \downarrow \pi_1' \]

commutes.

By Proposition 5.8, \( [K \times \{0, 0\}] \) is an isolated invariant set relative to \( \pi_1' \). \( k \) : \( ]0, 1[ \times E^\alpha \rightarrow T\tilde{M} / T \Gamma \) is a homeomorphism (a continuous bijection; the continuity of the inverse \( [x, (y_1, y_2)] \mapsto (x, (0, y_2)) \) follows from the choice of the quotient topology on \( T\tilde{M} / T \Gamma \)). Hence, \( K \) is isolated relative to \( \pi_0 \).

6. Homotopy index of linear skew product semiflows

This section is concerned with the homotopy index of linear skew product semiflows obtained in the previous section. We consider linear skew product semiflows which are generated by semilinear parabolic equations and are normalized on the zero-section, that is, the semiflow \( \tilde{\pi} = \tilde{\pi}(A, F) \in \text{SK}([-2, 2], X^\alpha) \) is induced by mild solutions of

\[ \dot{x} = 1 - x^2 \tag{6.1} \]
\[ \dot{y} + Ay = F(x) y. \]

Unfortunately, the right side of the above equation is not necessarily locally Lipschitz continuous if one assumes only that \( F \) is a continuous family of linear operators. Therefore, the term \textit{mild solution} is used as follows: \( (u(t), v(t)) \) is called a mild solution of (6.1) if \( u(t) \) is a solution of the first equation, that is, \( \dot{u}(t) = 1 - u(t)^2 \), and \( v(t) \) is a mild solution of \( \dot{y} + Ay = F(u(t)) y \).

Let \([a, b]\) be an arbitrary interval and let \( \tilde{\pi} \in \text{SK}([a, b], X^\alpha) \) be induced by mild solutions of

\[ \dot{x} = f(x) \]
\[ \dot{y} + Ay = F(x) y. \]

such that there exists a homeomorphism \( \varphi : [a, b] \rightarrow [-2, 2] \) such that \( \varphi \circ u(t) \) is a solution of \( \dot{x} = 1 - x^2 \) whenever \( (u(t), v(t)) \) is a solution of \( \tilde{\pi} \). Then \( (\varphi \circ u(t), v(t)) \) is a mild solution of

\[ \dot{x} = 1 - x^2 \]
\[ \dot{y} + Ay = F(\varphi^{-1}(x)) y \]
and \((\tilde{F}(x) := F(\varphi^{-1}(x)))_{x \in [a, b]}\) is again a continuous family of semiflows. This justifies the restriction to semiflows given by (6.1).

6.1. Existence, continuous dependence of solutions, and admissibility

Suppose that

\begin{itemize}
  \item \(X\) is a Banach space;
  \item \(A\) is sectorial linear operator, which is densely defined on \(X\) and has compact resolvent;
  \item \(X^\alpha\) denotes the \(\alpha\)-th fractional power space (see [8]);
\end{itemize}

and

(1) \(F : [-2, 2] \to \mathcal{L}(X^\alpha, X^0)\) is sufficiently continuous, that is, there are \(-2 = x_0 \leq \cdots \leq x_n = 2 \in [-2, 2]\) such that for every interval \([x_i, x_{i+1}]\), \(i \in \{0, \ldots, n - 1\}\), there is an \(\tilde{F} \in \mathcal{L}([x_i, x_{i+1}], X^\alpha, X^0)\) such that \(F(x) = \tilde{F}(x)\) for every \(x \in [x_i, x_{i+1}]\).

(2) \(-1, 1/\in \{x_0, \ldots, x_n\}\).

Lemma 6.1. Let \(F_n \in L^\infty([0, \tau], \mathcal{L}(X^\alpha, X))\), \(n \in \mathbb{N} \cup \{0\}\), and suppose that \(F_n(t) \to F_0(t)\) a.e. in \([0, \tau]\). Let there further exist an \(M \in \mathbb{R}^+\) with

\[2\|F_n\|_{\infty} \leq M\]

for all \(n \in \mathbb{N} \cup \{0\}\).

Then,

\[K_n v(t) = \int_0^t e^{-\alpha(t-s)} (F_n - F_0)(s) v(s) \, ds \quad t \in [0, \tau]\]

defines a sequence of operators in \(\mathcal{L}(C([0, \tau], X^\alpha), C([0, \tau], X^\alpha))\) with \(\|K_n\| \to 0\) as \(n \to \infty\).

**Proof.** We have

\[K_n v(t) = \int_0^t e^{-\alpha(t-s)} (F_n - F_0)(s) v(s) \, ds \quad t \in [0, \tau].\]

Using standard estimates (see [8]), there exist \(1 \leq \tilde{M}, \bar{M} \in \mathbb{R}^+\) and \(\mu \in \mathbb{R}\) such that Re \(\sigma(A) > \mu\) and

\[\|e^{-At}\|_{\alpha,0} \leq \tilde{M} t^{-\alpha} e^{-\mu t} \leq t^{-\alpha} \bar{M} \quad t \in [0, \tau]\]
\[\|e^{-At}\|_{0,0} \leq \tilde{M} e^{-\mu t} \leq \bar{M} \quad t \in [0, \tau].\]

Let \(\varepsilon > 0\) and \(v \in C([0, \tau], X^\alpha)\). There exists a \(\delta = \delta(\varepsilon) > 0\) with

\[\bar{M} \int_0^t s^{-\alpha} ds < \varepsilon \quad \text{for all } t \in [0, \delta].\]
Consequently, we obtain that

\[
\| K_n v(t) \|_\alpha = \left\| \int_0^t e^{-A(t-s)} (F_n - F_0)(s)v(s) \, ds \right\|_\alpha \leq M \varepsilon \| v \|_{C([0, \tau], X^\alpha)}
\]

for all \( t \in [0, \delta] \).

By Egorov’s theorem (see [7]), there exists a measurable set \( C \subset [\delta, \tau] \) with Lebesgue measure \( \lambda(C) \leq \varepsilon \) and \( \| F_n(t) - F_0(t) \|_{\alpha, 0} \to 0 \) uniformly on \( [\delta, \tau] \setminus C \).

For every \( t \in [\delta, \tau] \), we have

\[
\| K_n v(t) \|_\alpha \leq \left\| \int_{[\delta, \tau] \setminus C} e^{-A(t-s)} (F_n - F_0)(s)v(s) \, ds \right\|_\alpha \\
+ \left\| \int_{C} e^{-A(t-s)} (F_n - F_0)(s)v(s) \, ds \right\|_\alpha \\
+ e^{-A(\tau-\delta)} \int_{0}^{\delta} e^{-A(\delta-s)} (F_n - F_0)(s)v(s) \, ds \\
\leq \delta^{-\alpha} \tilde{M} \sup_{s \in [\delta, \tau] \setminus C} \| F_n(s) - F_0(s) \|_{\alpha, 0} \| v \|_{C([0, \tau], X^\alpha)} \\
+ \varepsilon \tilde{M} M \sup_{s \in C} \| F_n(s) - F_0(s) \|_{\alpha, 0} \| v \|_{C([0, \tau], X^\alpha)} + \varepsilon \tilde{M} M \| v \|_{C([0, \tau], X^\alpha)}.
\]

Let \( N = N(\varepsilon) \in \mathbb{N} \) such that for all \( n \geq N \)

\[
\sup_{s \in [\delta, \tau] \setminus C} \| F_n(s) - F_0(s) \|_{\alpha, 0} \leq \varepsilon \delta^\alpha.
\]

In conjunction with (6.2), we have shown that for all \( t \in [0, \tau] \) and all \( n \geq N(\varepsilon) \),

\[
\| K_n v(t) \|_\alpha \leq \tilde{M} (1 + 2M) \varepsilon \| v \|_{C([0, \tau], X^\alpha)},
\]

where \( \varepsilon > 0 \) was arbitrary. \( \square \)

**Proposition 6.2.** For every \((x_0, y_0) \in ]-2, 2[ \times X^\alpha\), there is a unique, maximally defined mild solution \((u(t), v(t))\) of (6.1), which is defined on \( J \subset \mathbb{R}^+ \) and satisfies \((u(0), v(0)) = (x_0, y_0)\).

Moreover, if \( J \neq \mathbb{R}^+ \), then there is a \( t_0 \in \mathbb{R}^+ \) with \( u(t) \to -2 \) as \( t \to t_0^- \).

**Proof.** Let \( u(t), t \in [0, T[ \) be the maximally defined solution of

\[
\dot{x} = 1 - x^2 \quad x \in ]-2, 2[.
\]

It follows from [14, Theorem 44.1] that there is a unique solution of

\[
\dot{v} + Av = F(u(t))v \quad t \in [0, T[ \]

with \( v(0) = y_0 \). \( \square \)
Proposition 6.3. Let $F_n \to F_0 \in L^\infty([-2, 2], \mathcal{L}(X^\alpha, X))$, $n \in \mathbb{N}$, and suppose that $F_n$, $n \in \mathbb{N} \cup \{0\}$, are sufficiently continuous.

Further, let $(x_n, y_n) \to (x_0, y_0) \in [-2, 2] \times X^\alpha$ be sequences, and $(u_n, v_n) : [0, T_n] \to [-2, 2] \times X^\alpha$, $n \in \mathbb{N} \cup \{0\}$, the maximally defined mild solutions of $\pi(A, F_n)$ with $(u_n(0), v_n(0)) = (x_n, y_n)$. Then $T_n \to T_0$ and $\sup_{s \in [0, t]} \|v_n(s) - v_0(s)\|_{\alpha} \to 0$ as $n \to \infty$ whenever $t \in [0, T_0]$.

Proof. It follows from Proposition 6.2 that $T_n \to T_0$ since the maximal domain of $(u_n, v_n)$ depends only on $u_n$, which is a solution of (6.3).

In order to show the convergence, it is sufficient to consider small times $t$. Assume that

$$\|e^{-At}\|_\alpha \leq M\|x\|_\alpha$$
$$\|e^{-At}\|_\alpha \leq Mt^{-\alpha}\|x\|_0$$

for some $M \in \mathbb{R}^+$ and for all $t \in [0, 1]$. Assume further that $\bar{\tau} \in [0, 1]$ is small enough that

$$M \int_0^t (t-s)^{-\alpha} \left\| F_n(u(s)) \right\|_{\alpha,0} ds \leq \frac{1}{2}$$

for all $t \in [0, \bar{\tau}]$.

Provided that $[0, \tau] \subset [0, T_n] \cap [0, T_0] \cap [0, \bar{\tau}]$, we now have for all $t \in [0, \tau]$

$$v_n(t) - v_0(t) = e^{-At}(v_n(0) - v_0(0)) + \int_0^t e^{-A(t-s)} F_n(u(s))(v_n(s) - v_0(s))$$
$$+ (F_n(u(s)) - F_0(u(s)))v_0(s) ds,$$

and thus

$$\left\| v_n(t) - v_0(t) \right\|_{\alpha} \leq 2M\|v_n(0) - v_0(0)\|_{\alpha} + 2\|K_n v_0\|$$

for some $M \in \mathbb{R}^+$ where $K_n$ is given by Lemma 6.1. Hence, the convergence follows if we show that

$$\left\| F_n(u_n(t)) - F_0(u_0(t)) \right\|_{\alpha,0} \to 0 \quad \text{as} \quad n \to \infty \text{ a.e. on } [0, T_0]. \quad (6.4)$$

For each $n \in \mathbb{N}$, we have either $u_n(t) \in [-1, 1]$ or $u_n(t) \notin [-1, 1]$ for all $t \in \mathbb{R}$. It is thus sufficient to assume that either $u_n(t) \notin [-1, 1]$ for all $n \in \mathbb{N}$ and all $t$ or $u_n(t) \in (-1, 1)$ for all $n \in \mathbb{N}$ and all $t$.

In the first case, let $0 = t_0 \leq t_1 \leq \cdots \leq t_l = T_0$ such that $F_0 \circ u_0$ is continuous on each of the subintervals $]t_k, t_{k+1}[$. For every $k \in \{0, \ldots, l-1\}$, every $n \in \mathbb{N}$ large enough, and almost every $s \in ]t_k, t_{k+1}[$, it holds that

$$\left\| F_n(u_n(s)) - F_0(u_0(s)) \right\|_{\alpha,0} \leq \|F_n(u_n(s)) - F_0(u_n(s))\|_{\alpha,0}$$
$$+ \|F_0(u_n(s)) - F_0(u_0(s))\|_{\alpha,0} \to 0$$

as $n \to \infty$.

In the second case, $x_0 := u_n(0)$ is independent of $n$. Each $F_n$ is continuous in a small neighborhood of $x_0$, so there exists a sequence $x_0' \in [-2, 2]$ with $|x_n' - x_0| \to 0$, $\|F_n(x_0) - F_n(x_n')\|_{\alpha,0} \to 0$, and $\|F_n(x_n') - F_0(x_n')\|_{\alpha,0} \to 0$ as $n \to \infty$. We have
$$\| F_n(x_0) - F_0(x_0) \|_{\alpha,0} \leq \| F_n(x_0) - F_n(x_n') \|_{\alpha,0}$$
$$+ \| F_n(x_n') - F_0(x_n') \|_{\alpha,0} + \| F_0(x_n') - F_0(x_0) \|_{\alpha,0} \to 0$$
as \( n \to \infty \). \( \square \)

**Corollary 6.4.** Let the assumptions of Proposition 6.3 hold. Then

1. \( \pi (A, F_n) \) is a semiflow for all \( n \in \mathbb{N} \cup \{0\} \);
2. \( \pi (A, F_n) \to \pi (A, F) \) and
3. every closed set \( N \subset [-2, 2] \times X^\alpha \) which is bounded with respect to \( \| \cdot \|_{\mathbb{R} \times X^\alpha} \) is strongly \( \pi (A, F_n) \)-admissible.

**Proof.** The first two claims are a restatement of Proposition 6.3. In particular, it follows from Proposition 6.2 that for every \( n \in \mathbb{N} \) \( \pi_n := \pi (A, F_n) \) does not explode in \( N \). Admissibility now follows as in the proof of [12, Theorem 1.4.3] (which is stated only for solutions in the sense of [8]). \( \square \)

### 6.2. The classes \( \text{SK}_i, i \in \{0, 1, 2\} \)

For the rest of this section, let us make the following assumptions in addition to the previous section:

1. \( F : [-2, 2] \to \mathcal{L}(X^\alpha, X^0) \) is sufficiently continuous;
2. \( A \) and \( A - F(1) \) are hyperbolic and have simple eigenvalues, all of which are real; let \( E^\pm (\pi, e) := E^\pm (e) := P^\pm (0)X, e \in \{-1, 1\} \), denote the associated subspaces of \( X \), where \( P^\pm (0) := P^\pm (0) \) is the projection onto the subspaces which belong to the positive respectively negative part of the spectrum of \( \Lambda := A - F(e) \), where \( \pi = \pi (A, F) \) (see Section 3).

(6.1) implies that there are exactly two equilibria, namely \((-1, 0)\) and \((1, 0)\), all of which are hyperbolic.

**Definition 6.5.** Let \( \text{SK}_0 := \text{SK}_0 (X, A) \subset \text{SK}([-2, 2], X^\alpha) \) denote the set of linear skew product semiflows which is given by \( \pi \in \text{SK}_0 \) iff

1. \( \pi \) is induced by mild solutions of (6.1), which satisfies the assumptions above;
2. \( K := [-1, 1] \times \{0\} \) is an isolated invariant set relative to \( \pi \);
3. \( \dim E^-(1) = \dim E^-(1) < \infty \).

**Definition 6.6.** Let \( \pi_0, \pi_1 \in \text{SK}_0 \). Then \( \pi_0 \sim \pi_1 \) iff there exists a homotopy, that is, an \( \mathcal{S} \)-continuous family \( (\pi_\lambda, [-1, 1] \times \{0\})_{\lambda \in [0, 1]} \) such that for all \( \lambda \in [0, 1] \)

1. \( \pi_\lambda \in \text{SK}_0 \), and
2. \( E^-(\pi_\lambda, -1) \) and \( E^-(\pi_\lambda, 1) \) are constant.

The main result of this section is stated in the theorem below. What follows are several normalization steps, either isomorphisms of bundles as defined in Appendix A or equivalences in the sense of Definition 6.6.

**Theorem 6.7.** \( h(\pi, [-1, 1] \times \{0\}) = 0 \) for all \( \pi \in \text{SK}_0 \).

Here, \( h \) denotes the homotopy index as defined in [12].
Proof. Lemma 6.10 and Lemma 6.11 show that the theorem holds if and only if it holds for all \( \pi \in SK_2 \) (which is defined below). The result now follows from Corollary 6.25. \( \square \)

6.3. Local constancy of \( F(x) \)

According to our assumptions in the previous section, we have \( F \in L^\infty([-2, 2], L(X^\alpha, X^0)) \) (in particular, the assumption of sufficient continuity is stronger). Let \( \|F\| := \|F\|_\infty := \|F\|_{L^\infty([-2, 2], L(X^\alpha, X^0))} := \text{ess sup}_{x \in [-2, 2]} \|F(x)\|_{L(X^\alpha, X^0)} \).

Lemma 6.8. Suppose that:

1. \( \pi = \pi(A, F) \) is induced by mild solutions of (6.1), which satisfies the assumptions at the beginning of Section 6.2;
2. \( \dim E^-(1) = \dim E^-(1) < \infty \).

Then \( K := [-1, 1] \times [0] \) is an isolated invariant set relative to \( \pi \) if and only if the following holds:

Whenever \( (x(t), y(t)) \) is a full bounded solution of \( \pi \) with \( x(0) = 0 \), then \( y(t) \equiv 0 \).

For every solution \( (x(t), y(t)), |x(t)| \) is a priori bounded. Hence, a solution \( (x(t), y(t)) \) is bounded if and only if it is bounded in \( y \) that is, \( \sup_t \|y(t)\|_\alpha < \infty \) where the supremum is taken over all \( t \in \mathbb{R} \) for which \( (x(t), y(t)) \) is defined.

Proof. Suppose that every full bounded solution \( (x(t), y(t)) \) with \( x(0) = 0 \) satisfies \( y(t) \equiv 0 \). Let

\[
N := [-3/2, 3/2] \times B_1[0] \subset [-2, 2] \times X^\alpha.
\] (6.5)

\( \lambda \in [0, 1] \) and \( (x(t), y(t)) \) be a full solution with \( (x(t), y(t)) \in N \) for all \( t \in \mathbb{R} \). \( y(t) \) is bounded, that is, \( \sup_{t \in \mathbb{R}} \|y(t)\|_\alpha < \infty \). Since \( x(t) = 1 - x^2(t) \), we have \( x(t) \in [-1, 1] \) for all \( t \in \mathbb{R} \). Either \( x(t) \in [-1, 1] \) for all \( t \in \mathbb{R} \), in which case we have \( y(t) \equiv 0 \) by the assumption above, or \( x(t) \in (-1, 1) \) for all \( t \in \mathbb{R} \), in which case \( y(t) \equiv 0 \) by the hyperbolicity of \( A - F(\pm 1) \). Therefore, we have \( (x(t), y(t)) \) in \( K \) for all \( t \in \mathbb{R} \), showing that \( N \) is an isolating neighborhood for \( (\pi, K) \).

Now, suppose that \( K \) is an isolated invariant set, and let \( N \) be an isolating neighborhood for \( K \). Setting \( \varepsilon := \inf \|y(t)\|_\alpha, x \in [-1, 1] \) and \( (x, y) \in N \), it is clear that \( \varepsilon > 0 \). Let \( (x(t), y(t)) \) be a full bounded solution of \( \pi = (\xi, \Phi) \). Due to the linearity of \( \Phi \), \( (x(t), \mu y(t)) \) is again a solution of \( \pi \). Choosing \( 0 < \mu \) small enough, it holds that \( \|\mu y(t)\|_\alpha \leq \varepsilon \) for all \( t \in \mathbb{R} \) that is, \( (x(t), \mu y(t)) \) in \( N \). It follows that \( \mu y(t) \equiv 0 \) and so \( y(t) \equiv 0 \). \( \square \)

Lemma 6.9. For \( \lambda \in [0, 1] \), let \( \pi_\lambda := (\pi(A, F_\lambda)) \) satisfy the assumptions of Lemma 6.8, and assume that \( \lambda \mapsto F_\lambda \) is continuous.

If it holds for every \( \lambda \in [0, 1] \) and for every full bounded solution \( (x(t), y(t)) \) of \( \pi_\lambda \) with \( x(0) = 0 \) that \( y(t) \equiv 0 \), then \( \pi_0 \sim \pi_1 \).

Proof. We have to show that the family \( \pi_\lambda(K) \) is \( \mathcal{S} \)-continuous. Let \( N \) be given by (6.5). It follows from Lemma 6.8 that \( N \) is an isolating neighborhood for \([-1, 1] \times [0] \) relative to \( \pi_\lambda \) for all \( \lambda \in [0, 1] \).

The continuity and admissibility properties are a consequence of Corollary 6.4. \( \square \)

Let \( SK_1 \subset SK_0 \) denote the subset of all semiflows \( \pi(A, F) \) where \( F \) is locally constant in a neighborhood of \([-1, 1] \), that is, there exists a \( \delta > 0 \) such that for all \( x \in [-1 - \delta, -1 + \delta] \) we have \( F(x) = F(-1) \) and for all \( x \in [1 - \delta, 1 + \delta] \) \( F(x) = F(1) \).
Lemma 6.10. For every $\pi (A, F) \in \text{SK}_0$ there is a $\lambda_0 \in [0, 1]$ such that $\pi (A, F) \sim \pi (A, F_{\lambda}) \in \text{SK}_1$ for all $\lambda \in [0, \lambda_0]$, where we set

$$F_{\lambda}(x) := \begin{cases} F(-1) & x \in [-1 - \lambda, -1 + \lambda] \\ F(1) & x \in [1 - \lambda, 1 + \lambda] \\ F(x) & \text{otherwise.} \end{cases}$$

Proof. We have $\|F_{\lambda} - F\|_\infty \to 0$ because $F$ is continuous in a neighborhood of $(-1, 1)$. Thus it follows from Corollary 6.4 that the assumptions of [12, Theorem I.4.5] hold. Let $\pi_{\lambda} := \pi (A, F_{\lambda})$ and note that $F_{\lambda}(1)$ and $F_{\lambda}(-1)$ are constant in $\lambda$ so that the hyperbolicity at each of the equilibria and the subspaces $E^-(\pm 1)$ are preserved.

Suppose that for every $\delta \in [0, 1]$ there is a $\lambda =: \lambda(\delta) \in [0, \delta]$ and a full bounded solution $(x(t), y(t))$ of $\pi_{\lambda}$ with $x(0) = 0$ and $\|y(0)\|_\alpha = 1$. By [12, Theorem I.4.5] there is a full bounded solution of $\pi_0$ with $x(0) = 0$ and $y(0) \neq 0$, which cannot exist in view of Lemma 6.8 since $K = [-1, 1] \times \{0\}$ is isolated relative to $\pi_0$.

Hence, Lemma 6.9 implies that there exists a $\lambda_0 \in [0, 1]$ such that $\pi_0 \sim \pi_{\lambda}$ for all $\lambda \in [0, 1]$. □

Let $\text{SK}_2 \subset \text{SK}_1$ denote the subset of all those semiflows which satisfy the following stronger restriction (compared to the definition of $\text{SK}_1$): There exists a $\delta > 0$ such that $F(x) = F(-1)$ for all $x \in [-2, -1 + \delta]$ and $F(x) = F(1)$ for all $x \in [1 - \delta, 2]$.

Lemma 6.11. For every $\pi (A, F) \in \text{SK}_1$, it holds that $\pi (A, F) \sim \pi (A, \tilde{F}) \in \text{SK}_2$, where we set

$$\tilde{F}(x) := \begin{cases} F(-1) & -2 \leq x \leq -1 \\ F(1) & 1 \leq x \leq 2. \end{cases}$$

Proof. Let $F_{\lambda}$ be given by

$$F_{\lambda}(x) := \lambda \tilde{F}(x) + (1 - \lambda)F(x).$$

Let $\lambda \in [0, 1]$ and let $(x(t), y(t))$ be a full bounded solution of $\pi_{\lambda} := \pi (A, F_{\lambda})$ with $x(0) = 0$. We have $x(t) \in [-1, 1]$ for all $t \in \mathbb{R}$, showing that $(x(t), y(t))$ is also a solution of $\pi_0$. Therefore, $y(t) \equiv 0$.

Now, the claim follows from Lemma 6.9. □

6.4. Decomposition into “unstable” and “stable” subbundles

Let $\pi_0 = (\xi, \Phi) \in \text{SK}_2$, that is, $\pi_0 = \pi (A, F)$ and there is a $\delta \in [0, 1]$ such that $F(x) = F(-1)$ for all $x \in [-2, -1 + \delta]$ and $F(x) = F(1)$ for all $x \in [1 - \delta, 2]$. The goal of this section is to define a subbundle $U$ in the sense of A.5 such that every solution $(x(t), y(t))$ defined for $t \in \mathbb{R}$ with $\sup_{t \in \mathbb{R}} \|y(t)\|_\alpha < \infty$ satisfies $(x(0), y(0)) \in U$. As a consequence, $\pi$ continues to a direct sum of two linear skew product semiflows, which arise from restrictions of $\pi_0$ to $U$ respectively an appropriate complementary subbundle (later denoted by $S$).

Let $E^- := E^-(\pi_0, -1)$ and define $U(x) \in \mathcal{L}(E^-, X^\alpha)$ by

$$U(x)y := y \quad x \in [-2, -1 + \delta], \quad y \in E^-.$$
We continue along \([-2, 2]\) by following the semiflow, that is,

\[
U(x) := U(-1 + \delta)\Phi(-1 + \delta, tx) \quad x \in [-1 + \delta, 1 - \delta]
\]

where \((-1 + \delta)\xi t x = x\) defines \(t x\).

**Lemma 6.12.** \(U(x)\) is well defined and \(U \in C([-2, 1 - \delta], \mathcal{L}(E^-, X^\alpha))\). Moreover, \(U(x)\) is injective for all \(x \in [-2, 1 - \delta]\).

**Proof.** The linearity of \(U(x)\) follows from the linearity of \(\Phi(-1 + \delta, t)\). Let \(\tau\) be given by \((-1 + \delta)\xi \tau = 1 - \delta\). It follows that \([-1 + \delta, 1 - \delta] = \xi([-1 + \delta] \times [0, \tau])\) and the restriction of \(\xi\) to \([-1 + \delta] \times [0, \tau]\) is a homeomorphism. Hence \(t x\) is well defined for all \(x \in [-1 + \delta, 1 - \delta]\) and we have \(t x \to t_{x_0}\) whenever \(x \to x_0\) and also \(\Phi(-1 + \delta, t_{x})y \to \Phi(-1 + \delta, t_{x_0})y\) for all \(y \in E^-.\) It is clear that \(U(x)\) is bounded for every \(x \in [-2, 1 - \delta]\) since \(\dim E^- < \infty\).

Let \(x \in [-1 + \delta, 1 - \delta]\) and \(y \in E^-\) with \(U(x)y = 0\). Then there is a full solution \((u(t), v(t))\) of \(\pi_0\) with \(u(0) = -1 - \delta, v(0) = y \in E^-\) and \(v(t_{1 - \delta}) = 0\). We have \(\sup_{s \geq 0} \|v(s)\|_{\alpha} < \infty\) since \(v(0) \in E^-\) and \(\sup_{s \geq 0} \|v(s)\|_{\alpha} \leq \sup_{s \in [0, t_{1 - \delta}]} \|v(s)\|_{\alpha} < \infty\) since \(v(t_{1 - \delta}) = 0\). It follows from Lemma 6.8 that \(v(0) = y = 0\). \(\square\)

**Lemma 6.13.** \(P^-_1(0) \circ U(1 - \delta)\) is a bijection.

**Proof.** Let \(y \in E^-(1)\) with \(P^-_1(0) \circ U(1 - \delta)y = 0\). It follows that \(U(1 - \delta)y = \Phi(-1 + \delta, t_{1 - \delta})y \in E^+(1, 0)\), so \(\sup_{s \geq 0} \|\Phi(-1 + \delta, s)y\|_{\alpha} < \infty\). As in the previous proof, it follows from the isolation of \([-1, 1] \times [0]\) that \(y = 0\), showing the injectivity of \(P^-_1(0) \circ U(1 - \delta)\).

Surjectivity holds since \(\dim E^-(-1) = \dim E^-(1)\). \(\square\)

Therefore, given \(y_0 \in E^-(1)\), there is a \(w \in E^-(-1)\) with \(P^-_1(0) \circ U(1 - \delta)w = y_0\). Choose a basis \(\{\eta_i; \ i = 1, \ldots, \dim E^-(1)\}\) for \(E^-(1)\) such that each \(\eta_i\) is an eigenvector of \(L := A - F(1)\).

Further, let \(\lambda_i < 0\) denote the real eigenvalue \(\lambda_i\) which corresponds to \(\eta_i\), that is, \(e^{-\lambda_i t} \eta_i = e^{-\lambda_i t} \eta_i\). For each \(i \in \{1, \ldots, \dim E^-\}\), there is an \(\eta_i^+ \in E^+(1)\) with \(\eta_i + \eta_i^+ \in U(1 - \delta)E^-(-1)\). Let \(y_i \in E^-\) be given by \(U(1 - \delta)y_i = \eta_i + \eta_i^+\) and define

\[
U(x)y_i := \eta_i + e^{-(L - \lambda_i)(t x - t_{1 - \delta})} \eta_i^+ \quad x \in [1 - \delta, 1[, \ i = 1, \ldots, \dim E^-.
\]

Finally, let

\[
U(x)y := \lim_{x \to 1} U(\tilde{x})y \quad x \in [1, 2], \ y \in E^-.
\]

**Remark 2.** Using the construction above, one has \(U(1)E^-(-1) = E^-(-1)\).

Reading \(U\) as a morphism in the sense of Appendix A, we say that \(U\) is \(\pi_0\) invariant if \(\{(x, U(x)y); (x, y) \in [-2, 2] \times E^-\}\) is \(\pi_0\)-invariant.

**Lemma 6.14.** \(U(x) \in C([-2, 2], \mathcal{L}(E^-, X^\alpha))\) is well defined and \(\pi_0\)-invariant.

**Proof.** Let \(x_n\) be a sequence in \([-2, 2]\) with \(x_n \to 1^-\). We have \(t_n := t_{1 - \delta + x_n} - t_{1 - \delta} \to \infty\) as \(n \to \infty\) and thus \(U(x_n)y_i - \eta_i = e^{\lambda_i t_n} e^{-\lambda_i t_{1 - \delta}} \eta_i^+ \to 0\) as \(n \to \infty\) (recall that \(\lambda_i < 0\)) showing that

\[
U(x) \to P^-_1(0) \circ U(1 - \delta) \quad \text{as} \ x \to 1.
\]

(6.6)
Lemma 6.15. There exist morphisms (of bundles) \( S^\beta \in C([-2, 2], \mathcal{L}(E^+ \cap X^\beta, X^\beta)), \beta \in [0, 1], \) such that for all \( \beta \in [0, 1] \)

\[
S^\beta(x)y = S^0(x)y \quad x \in [-2, 2], \ y \in X^\beta
\]

and

\[
U(x) \oplus S^\beta(x) \in \text{ISO}(X^\beta, X^\beta) \quad x \in [-2, 2].
\]

**Proof.** First, we show that there is a \( \mu \in \mathbb{R} \setminus \sigma(A - F(\mu)) \) such that \( P^{-1}(\mu)U(x) \) is injective for all \( x \in [-2, 2] \). Suppose that this is not true. Then there are sequences \( \mu_n \to \infty \) in \( \mathbb{R} \), \( x_n \to x_0 \) in \([-2, 2]\), and \( y_n \to y_0 \neq 0 \) in \( E^- \) such that \( P^{-1}(\mu_n)U(x_n)y_n = 0 \) for all \( n \in \mathbb{N} \). We can assume w.l.o.g. that \( (\mu_n)_n \) is monotone increasing.

Let \( k \in \mathbb{N} \) be arbitrary but fixed. We have

\[
P_{-1}(\mu_k)U(x_0)y_0 = \lim_{n \to \infty} P_{-1}(\mu_k)U(x_n)y_n = \lim_{n \to \infty} 0 = 0
\]

since \( \mu_n \geq \mu_k \) implies that \( P_{-1}(\mu_k)U(x_n)y_n = 0 \).

Now, it follows from Lemma 3.4 that \( U(x_0)y_0 = 0 \), a contradiction to the injectivity of \( U(x_0) \).

Let \( E_0 := P_{-1}(\mu)X \). By Lemma A.8, there is a complementary subbundle \( \tilde{S} \in C([-2, 2], \mathcal{L}(E_0, E_0)) \) for \( P_{-1}(\mu)U \) in \( E_0 \), which is continuous regardless of the norm on \( E_0 \).

We can now define

\[
S^\beta(x)y := \tilde{S}(x)P_{-1}(\mu)y + P_{-1}(\mu)y \quad x \in [-2, 2], \ y \in X^\beta.
\]

One has \( U(x)y^- + S^\beta(x)y^+ = z \) if and only if

\[
P_{-1}(\mu)(U(x)y_1 + \tilde{S}(x)y_2) = P_{-1}(\mu)z
\]

\[
P_{-1}(\mu)(U(x)y_1 + y_3) = P_{-1}(\mu)z.
\]
where \( y_1 + y_2 \in P^-_1(\mu)X \subset X^1 \) and \( y_3 \in P^+_1(\mu)X^\beta \). The first equation has a continuous inverse regardless of the norm considered, and the second equation yields

\[
y_3 = P^-_1(\mu)z - P^+_1(\mu)U(x)y_1,
\]

which is again continuous with respect to \( \| \cdot \|_\beta \). \( \square \)

From Lemma 6.15, we obtain a complementary subbundle \( S^\alpha \) (complementary to \( U \) in \( X^\alpha \)), which is canonically homeomorphic to the quotient bundle \( ([−2, 2] \times X^\alpha)/U \), that is, \( (x, y) \mapsto (x, [S^\alpha(x)y]) \)
defines a homeomorphism \( E^+ \cap X^\alpha \to ([−2, 2] \times X^\alpha)/U \).

Define \( \pi_U := (\xi, \Phi_U) \in \text{SK}([−2, 2] \times E^-) \) by

\[
U(x\xi t)\Phi_U(x, t)y = \Phi(x, t)U(x)y \quad y \in E^-
\]

and \( \pi_S = (\xi, \Phi_S) \in \text{SK}([−2, 2] \times (E^+ \cap X^\alpha)) \) by

\[
\left[ S^\alpha(x\xi t)\Phi_S(x, t)y \right]_{X^\alpha/U(x\xi t)} = \left[ \Phi(x, t)S^\alpha(x)y \right]_{X^\alpha/U(x\xi t)} \quad y \in E^+ \cap X^\alpha.
\]

**Proposition 6.16.** \( U \oplus S^\alpha \) is an isomorphism of bundles and \( (U \oplus S^\alpha)[\pi_U \oplus \pi_S] \sim \pi_0 \) (see Definition 5.1 for the direct sum of the semiflows).

In order to prove Proposition 6.16, we need the following two lemmas.

**Lemma 6.17.** Let \( e \in \{-1, 1\} \). Then there exist a neighborhood \( V \) of \( e \) in \([−2, 2] \), a local isomorphism \( \phi_e \in \mathcal{C}(V, \mathcal{L}(E^+(e) \cap X^\alpha, E^+(e) \cap X^\alpha)) \), and a \( B_e \in \mathcal{L}(E^+(e) \cap X^\alpha, E^+(e) \cap X^0) \) with \( \text{Res} \sigma(A - B_e) > 0 \) such that \( \phi_e(u(t))v(t) \) is a mild solution of

\[
\dot{x} + (A - B_e)x = 0 \quad B_e \in \mathcal{L}(E^+(e) \cap X^\alpha, E^+(e) \cap X^0)
\]

whenever \( (u(t), v(t)), t \in [0, T], \) is a solution of \( \pi_S \) with \( u(t) \in V \) for all \( t \in [0, T] \).

**Proof.** Letting \( B_e = F(e) \), we have \( P^+_e(0)(A - B_e) = (A - B_e)P^+_e(0) \) due to the choice of the projection \( P^+_e(0) \). Now, let \( V \) be given by Lemma A.10 such that the projection \( p : V \times (E^+(e) \cap X^\alpha) \to p(V \times E^+(e) \cap X^\alpha) \subset (V \times X^\alpha)/U \) \( (U(e) = E^-) \) by Remark 2) which is given by \( p(x, y) := (x, [y]_{X^\alpha/U(x)}) \), is a homeomorphism.

By shrinking \( V \) if necessary, we may assume that \( F(x) = B_e \) for all \( x \in V \). Let \( (u(t), v(t)), t \in [0, T], \) be a solution of (6.9) and let \( (u(t), w(t)), t \in [0, T], \) be a solution of \( \pi_S \) with \( [v(0)] = [S(u(0))w(0)] \). Then, by (6.8),

\[
[v(t)] = \Phi(u(0), t)v(0) = [\Phi(u(0), t)S(u(0))w(0)] = [S(u(t))\Phi_S(u(0), t)w(0)] = [S(u(t))w(t)],
\]

so \( (u(t), v(t)) = p^{-1}(u(t), [S(u(t))w(t)]) \), that is, we can choose \( \phi_e(x, y) = p^{-1}([x, S(x)y]) \). \( \square \)

**Lemma 6.18.** Let \( (u(t), v(t)) \) be a bounded solution of \( \pi_S \) which is defined for all \( t \in \mathbb{R}^- \). Then \( v(t) \equiv 0 \).
Proof. There is an \( e \in \{-1, 1\} \) such that \( u(t) \to e \) as \( t \to -\infty \). Let \( \phi_e \) be given by Lemma 6.17 and assume that \( \phi_e(u(t)) \) is defined for all \( t \leq t_0 \).

\( u(t), w(t) := (u(t), \phi_e(u(t))v(t)), t \leq t_0 \), is a mild solution of (6.9), and \( \text{Re} \sigma(A - B_e) > 0 \) implies that \( w(t) \equiv 0 \). This implies that \( v(t) \equiv 0 \) for all \( t \leq t_0 \), showing that \( v(t) \equiv 0 \). \( \square \)

Proof of Proposition 6.16. It is stated in Lemma 6.15 that \( U \oplus S^\alpha \) is an isomorphism of bundles, that is, particularly a homeomorphism.

For every \( \beta \in [0, 1] \), the direct sum \( E^-(1) \oplus (E^+(1) \cap X^\beta) = X^\beta \) defines continuous projections onto each of the components. Applying \( U \oplus S^\beta \), we obtain morphisms of bundles \( P^\beta, Q^\beta \in C([-2, 2], \mathcal{L}(X^\alpha, X^\beta)) \) such that for each \( x \in [-2, 2] \) it holds that

- \( P^\beta(x) \) is a projection onto \( U(x) = U(x)E^\beta \).
- \( Q^\beta(x) \) is a projection onto \( S^\beta(x) = S(x)(E^+ \cap X^\beta) \), and
- \( P^\beta(x) + Q^\beta(x) = \text{id}_{X^\beta} \).

Suppose that \( \pi_0 = \pi(A, F) \), and let \( \pi_A := \pi(A, F_A) \) where we set

\[
F_A(x) = P^0(x)F \left( P^\alpha(x)y + (1 - \lambda) Q^\alpha(x)y \right) + Q^0(x)F(y). \tag{6.10}
\]

Let \( (u(t), v(t)) \) be a full bounded solution of \( \pi_A \). It follows that there is a full bounded solution \( (u(t), w(t)) \) of \( \pi_S \) with

\[
\left[ (u(t), S^\alpha(u(t))w(t)) \right]_{[-2,2] \times X^\alpha/U} = \left[ u(t), v(t) \right]_{[-2,2] \times X^\alpha/U}.
\]

Hence, \( w(t) \equiv 0 \) by Lemma 6.18, showing that \( v(t) \in U(u(t)) \) for all \( t \in \mathbb{R} \). The semiflow on \( U \) is not changed by \( \lambda \) since \( Q^\alpha(x)U(x) = 0 \) for all \( x \in [-2, 2] \), and so it follows that \( v(t) \equiv 0 \). Lemma 6.9 finally implies that \( \pi_0 \sim \pi_1 \).

Moreover, letting \( \pi_1 = (\xi, \Phi_1) \), it follows from (6.10) that \( P^\alpha(x\xi t)\Phi_1(x, t)U(x)y_1 + S^\alpha(x)y_2) = \Phi_1(x, t)U(x)y_1 = \Phi(x, t)U(x)y_1 \) for all \((y_1, y_2) \in E^-(1) \times (E^+(1) \cap X^\alpha). \) We thus have

\[
P^\alpha(x\xi t)\Phi_1(x, t)\left( U(x)y_1 + S^\alpha(x)y_2 \right) = U(x\xi t)\Phi_1(x, t)y_1.
\]

and

\[
Q^\alpha(x\xi t)\Phi_1(x, t)\left( U(x)y_1 + S^\alpha(x)y_2 \right) = S^\alpha(x\xi t)\Phi_5(x, t)y_2
\]

follows immediately from the invariance of \( U \). This shows that \( (U \oplus S^\alpha)[\pi_U \oplus \pi_S] = \pi_1 \). \( \square \)

We continue by discussing \( \pi_U \) and \( \pi_S \) independently of each other. Until further notice, let \( \pi = (\xi, \Phi) \) denote \( (U \oplus S^\alpha)[\pi_U \oplus \pi_S] \), and \( E^\pm = E^\pm(-1) \).

6.4.1. The situation on \( S^\alpha \)

Lemma 6.19. There exists a strongly \( \pi_S \)-admissible isolating neighborhood for \((\pi_S, [-1, 1] \times \{0\}) \).

Proof. Let \( N \subset [-2, 2] \times X^\alpha \) be a strongly \( \pi \)-admissible isolating neighborhood for \([-1, 1] \times \{0\} \).

We have

\[
\Phi(x, t)S^\alpha(x)y = S^\alpha(x\xi t)\Phi_5(x, t)y \quad \forall y \in E^+,
\]

and \( S^\alpha([-2, 2] \times E^+) \cap N \) is an isolating neighborhood for the restriction of \( \pi \) to \( S^\alpha([-2, 2] \times (E^+ \cap X^\alpha)). \) It follows that \( (S^\alpha)^{-1}(N) = \{(x, y) \in [-2, 2] \times (E^+ \cap X^\alpha) \mid (x, S^\alpha(x)y) \in U \} \) is a strongly \( \pi_S \)-admissible isolating neighborhood for \([-1, 1] \times \{0\} \). \( \square \)
Lemma 6.20. There exist an isolating neighborhood $N_0 = [a, b] \subset \mathbb{R}$ for $[-1, 1]$ relative to $\xi$ and a constant $M \in \mathbb{R}^+$ such that $\|\Phi_S(x, t)y\|_\alpha \leq M \|y\|_\alpha$ whenever $y \in E^+ \cap X^\alpha$, $x \in [0, t]$ is defined and $x \in [0, t] \subset N_0$.

Proof. Let $N$ be given by Lemma 6.19 and choose $N_0$ small enough that $N_0 \times \{0\} \subset N$. Then every closed set $\bar{N} \subset N_0 \times (E^+ \cap X^\alpha)$ with $[-1, 1] \times \{0\} \subset \text{int} \bar{N}$ and $\sup_{(x, y) \in \bar{N}} \|y\|_\alpha < \infty$ is a strongly admissible isolating neighborhood for $\pi_S$ since we can choose $\varepsilon > 0$ small enough that $\{(x, \varepsilon y): (x, y) \in N\} \subset N$.

Suppose that the lemma is not true. Then there are sequences $x_n \to x_0$ in $N_0$ and $y_n \in E^+ \cap X^\alpha$ with $\|y_n\|_\alpha = 1$ and $\bar{t}_n$ in $\mathbb{R}^+$ such that

$$q_n := \sup_{s \in [0, t_n]} \|\Phi_S(x_n, s)y_n\|_\alpha \to \infty.$$

For every $n \in \mathbb{N}$, there exists a $t_n \in [0, \bar{t}_n]$ with $\|\Phi_S(x_n, t_n)y_n\| = q_n$.

Assume that $t_n \to \infty$, that is, by choosing subsequences we may assume that $t_n \to t_0$, implying that $1 = \|\Phi_S(x_n, t_n)q_n^{-1}y_n\| \to \|\Phi_S(x_0, t_0)0\| = 0$, a contradiction, showing that $t_n \to \infty$.

By admissibility, we may further assume that $(x_n, q_n^{-1}y_n)\pi_St_n \to (x_0, y_0) \in [-1, 1] \times (E^+ \cap X^\alpha)$ with $0 \neq y_0$ and $(x_0, y_0) \in \text{Inv}^{-}(N)$. Lemma 6.18 now implies that $y_0 = 0$, a contradiction.  

6.4.2. The situation on $U$

In this section, we will simplify the semiflow on $U$ by constructing a suitable isomorphism.

Lemma 6.21. Let $e \in [-1, 1]$. Then there exist a neighborhood $V$ of $e$ in $[-2, 2]$, a local isomorphism of bundles $\phi_e \in C(V, L(E^-(e), E^-(e)))$, and a $B_e \in L(E^-(e), E^-(e))$ with $\text{Re} \sigma(A - B_e) < 0$ such that $\phi_e(u(t))v(t)$ is a solution of (the ordinary differential equation in finite dimensions)

$$\dot{x} + P_e - (A - B_e)x = 0 \quad x \in E^-(e)$$

(6.11)

whenever $(u(t), v(t))$, $t \in [0, T]$, is a solution of $\pi_U$ with $u(t) \in V$ for all $t \in [0, T]$.

Proof. Let $B_e := F(e)$ and let $P := P_e^{-1}(0)$ be given by the spectral decomposition of $A - B_e$ (see also Section 3). By Lemma A.7, there exists a neighborhood $V$ of $e$ by possibly shrinking $V$ we may assume that $F(x) = B_e$ for all $e \in V$ such that $p : U(V) \to V \times E^-(e)$, $p(x, y) := (x, Py)$, is a homeomorphism.

Let $(u(t), w(t))$, $t \in [0, T]$, be a solution of (6.11) and let $(u(t), v(t))$, $t \in [0, T]$, be a solution of $\pi_U$ with $(u(0), w(0)) = p(u(0), U(u(0))v(0))$. Then by (6.7)

$$(u(t), w(t)) = P(u(t), \Phi(u(0), t)w(0))$$

$$= (u(t), PU(u(t))\Phi_U(u(0), t)v(0))$$

$$= (u(t), PU(u(t))v(t)),$$

so $w(t) = PU(u(t))v(t)$.

Therefore, we can choose $\phi_e(x) := PU(x)$, $x \in V$.  

Proposition 6.22. There exists a strongly $\pi_U$-admissible isolating neighborhood for $(\pi_U, [-1, 1] \times \{0\})$.

Proof. Let $N \subset \mathbb{R} \times X^\alpha$ be a strongly $\pi$-admissible isolating neighborhood for $[-1, 1] \times \{0\}$. We have

$$\Phi(x, t)U(x)y = U(x\xi t)\Phi_U(x, t)y \quad \forall y \in E^-,$$
and \(U(-2, 2] \times E^-) \cap N\) is an isolating neighborhood for the restriction of \(\pi\) to \(U(-2, 2] \times E^-)\). It follows that \(U^{-1}(N) = \{(x, y) \in ]-2, 2[ \times E^- : (x, U(x)y) \in N\}\) is a strongly \(\pi^{-1}\)-admissible isolating neighborhood for \([-1, 1] \times \{0\}\).

Recall that \(F(x)\) is constant on each of the intervals \([e - \delta, e + \delta], e \in \{-1, 1\}\), and let \(a_e < b_e\) such that \([a_e, b_e] \subseteq [e - \delta, e + \delta]\). Further, let \(\tau \in \mathbb{R}^+\) such that \(b_{-1}\) \(\tau = a_1\), and define \(V_1 : [-2, a_1] \times E^-(-1) \to U([-2, a_1])\) by

\[
V_1(x) := \Phi_U(x \xi(-\tau), \tau) \phi_{-1}(x \xi(-\tau))^{-1} y
\]

and \(V_2 : [a_1, 2] \times E^-(1) \to U([a_1, 2])\) by

\[
V_2(x) := \phi_1(x)^{-1} y,
\]

where \(\phi_e, e \in \{-1, 1\}\), is given by Lemma 6.21.

We can now define \(V \in \mathcal{C}([-2, 2], \mathcal{L}(E^-, E^-))\) (note that \(E^- = E^-(-1)\) by definition) by

\[
V(x) := \begin{cases} 
V_1(x) & x \in [-2, a_1] \\
V_2(x)V_2(a_1)^{-1}V_1(a_1)y & x \in [a_1, 2]. 
\end{cases}
\]

Note that \(\text{im} V(x)E^- \subset \text{im} U(x)E^-\) for every \(x \in [-2, 2]\).

For every \(t \in \mathbb{R}^+\) with \(x\xi[0, t] \subset [-2, a_1]\) and every \(y \in E^-,\) we have

\[
\Phi_U(x, t) V(x) y = \Phi_U(x, t) V_1(x) y \\
= \Phi_U(x, t) \Phi_U(x \xi(-\tau), \tau) \phi_{-1}(x \xi(-\tau))^{-1} y \\
= \Phi_U(x \xi(-\tau), t + \tau) \phi_{-1}(x \xi(-\tau))^{-1} y \\
= \Phi_U(x \xi(t - \tau), \tau) \Phi_U(x \xi(-\tau), t) \phi_{-1}(x \xi(-\tau))^{-1} y \\
= V_1(x \xi(t) \phi_{-1}(x \xi(-\tau + t)) \Phi_U(x \xi(-\tau), t) \phi_{-1}(x \xi(-\tau))^{-1} y
\]

and for \(x \in [a_1, 2]\), one obtains

\[
\Phi_U(x, t) V(x) V_1(a_1)^{-1} V_2(a_1)y = \Phi_U(x, t) V_2(x) y \\
= \Phi_U(x, t) \phi_1(x)^{-1} y \\
= \phi_1(x)^{-1} \phi_1(x) \Phi_U(x, t) \phi_1(x)^{-1} y \\
= V_2(x) V_1(a_1)^{-1} V_2(a_1) \phi_1(x) \Phi_U(x, t) \phi_1(x)^{-1} y.
\]

Consider the following system of ordinary differential equation on \(]2, 2[ \times E^-\)

\[
\begin{align*}
\dot{x} &= 1 - x^2 \\
\dot{y} &= \begin{cases} 
G(-1)y := P_{-1}(0)(-A + F(-1))y & x \leq a_1 \\
G(1)y := V_1(a_1)^{-1}V_2(a_1) P_{-1}(0)(-A + F(1)) V_2(a_1)^{-1} V_1(a_1)y & a_1 < x.
\end{cases}
\]

(6.14)
Let \((u(t), v(t))\) be a solution of (6.14) which is defined on \([0, T]\). If \(u(t) \in ]−2, a_1]\) for all \(t \in [0, T]\), then \(u(t)\xi(−τ) \in ]−2, b_{−1}\) and

\[ v(t) = φ_{−1}(u(t)\xi(−τ))Φ_U(u(0)\xi(−τ), t)φ_{−1}(u(0)\xi(−τ))^{−1}. \]

In conjunction with (6.12), we obtain

\[ \Phi_U(u(0), t)V(u(0))v(0) = V(u(t))v(t), \]

that is, \((u(t), V(u(t))v(t))\) is a solution of \(V[π]\). Now, suppose that \(u(t) \in [a_1, 2]\) for all \(t \in [0, T]\). We have

\[ V_2(a_1)^{−1}V_1(a_1)v(t) = φ_1(u(t))Φ_U(u(0), t)φ_{−1}(u(0))^{−1}V_2(a_1)^{−1}V_1(a_1)v(t). \]

Using (6.13), we can conclude that

\[ \Phi_U(u(0), t)V(u(0))v(0) = V(u(t))v(t), \]

which shows again that \((u(t), V(u(t))v(t))\) is a solution of \(V[π]\) lying entirely in \(U\).

Therefore, \(V^{−1}[π_U]\) is induced by mild solutions of (6.14).

**Proposition 6.23.** \((ξ, π_n) \sim V^{−1}[π_U]\), where \(π_n, n := \dim E^−\), denotes the flow on \(E^−\) which is induced by solutions of \(y = y\).

**Proof.** All eigenvalues \(λ_i, i \in \{1, ..., n\}\), of \(G(1)\) and \(G(−1)\) are positive real numbers, so there are \(Te \in \text{ISO}(\mathbb{R}^n, E^−), \epsilon \in [−1, 1]\), such that \(G(ε)\) is a diagonal matrix, namely

\[ G(ε) = Te \begin{pmatrix} λ_1 & & 0 \\ & \ddots & \\ 0 & & λ_n \end{pmatrix} T_{e}^{−1}. \]

Let \(G^v\) be defined by

\[ G^v(ε) = Te \begin{pmatrix} λ_1^v & & 0 \\ & \ddots & \\ 0 & & λ_n^v \end{pmatrix} T_{e}^{−1}. \]

\([-1, 1] \times \{0\}\) is an isolated invariant set relative to \(χ_v\) for all \(v \in [−1, 1]\), where \(χ_v\) is induced by mild solutions of

\[ \begin{cases} x = 1 − x^2 \\ ˙y = \begin{cases} G(−1)^vy & x < a_1 \\ G(1)^vy & a_1 \leq x. \end{cases} \]

It follows that \(V^{−1}[π_U] = χ_1 \sim χ_0 = (ξ, π_n). \)
6.5. Calculation of the homotopy index

Proposition 6.24. Let $F$ be a Banach space, let $\pi = (\xi, \Phi) \in SK([-2, 2], F)$ such that

1. \([a, b], [b]) is an isolating block for \((\xi, [-1, 1])\);
2. there exists a constant $1 \leq M \in \mathbb{R}^+$ such that $\|\Phi(x, t)y\| \leq M\|y\|$ whenever $x\xi[0, t]$ is defined with $x\xi[0, t] \subset [a, b]$;
3. there is a strongly $\pi$-admissible isolating neighborhood $\tilde{N}$ for $K := [-1, 1] \times \{0\}$ relative to $\pi$ with $[a, b] \times \{0\} \subset \tilde{N}$.

Then $h(\pi, K) = 0$.

Proof. Let

$$N_1 := \{(x, y) \in [a, b] \times F : \|\Phi(x, t)y\| \leq 1 \text{ for all } t \geq 0 \text{ with } x\xi t \leq b\}$$

$$N_2 := \{(x, y) \in N_1 : x = b\}.$$

Suppose that $N_1$ is not closed in $]-2, 2[ \times F$. Then there is a sequence $(x_n, y_n) \to (x_0, y_0)$ in $[a, b] \times F$ such that $(x_n, y_n) \in N_1$ for all $n \in \mathbb{N}$ and $(x_0, y_0) \notin N_1$. We thus have $\|\Phi(x_0, t_0)0\| > 1$ for some $t_0 \in \mathbb{R}^+$ with $x_0\xi t_0 < b$. It follows that $x_n\xi t_0 \leq b$ for all $n \in \mathbb{N}$ sufficiently large. Consequently, we have $\|\Phi(x_n, t_0)y_n\| > 1$ for all $n$ large enough, a contradiction to $(x_n, y_n) \in N_1$. $N_2$ is closed in $N_1$ and hence also in $]-2, 2[ \times F$.

Let $(x, y) \in [a, b] \times F$ with $\|y\| \leq \frac{1}{2M}$ and let $t \in \mathbb{R}^+$ with $x\xi[0, t] \subset [a, b]$. It follows that $\|\Phi(x, t)y\| \leq \frac{1}{2}$ and thus $(x, y) \in N_1$. Hence $[-1, 1] \times \{0\} \subset \text{Int} N_1 \setminus N_2$.

Let $(x, y) \in N_2$ that is, $(x, y) = (b, y)$. Then $x\xi t \notin [a, b]$ for all $t \in \mathbb{R}^+$ with $x\xi t$ defined, showing that $N_2$ is $N_1$-positively invariant.

Let $(x, y) \in N_1$ and $t \in [0, \infty]$ such that $(x, y)\pi t$ is defined and $(x, y)\pi t \notin N_1$. It follows that $x\xi t > b$, so there is an $s \in [0, t]$ with $x\xi s = b$, showing that $N_2$ is an exit ramp for $N_1$.

Furthermore, there exists an $\varepsilon > 0$ such that

$$N_1 \subset \tilde{N}_\varepsilon := \{(x, \varepsilon^{-1}y) \in ]-2, 2[ \times F : (x, y) \in \tilde{N}\}$$

since $\sup_{(x, y) \in N_1} \|y\| \leq 1$. $\tilde{N}$ is a strongly admissible isolating neighborhood and so is $\tilde{N}_\varepsilon$. This implies that the closed subsets $N_1$ and $\text{cl}(N_1 \setminus N_2)$ are strongly admissible isolating neighborhoods for $(\pi, K)$. Hence, $(N_1, N_2)$ is a strongly admissible FM-index pair for $(\pi, K)$.

Define a homotopy $H(x, y, \lambda) : (N_1, N_2) \times [0, 1] \to (N_1, N_2)$ by

$$H(x, y, \lambda) := (x, \lambda y).$$

Let $(x, t) \in [a, b] \times \mathbb{R}^+$ such that $x\xi[0, t] \subset [a, b]$. It follows from the linearity of $\Phi(x, t)$ that given $(x, y) \in N_1$ and $\lambda \in [0, 1]$ we also have $(x, \lambda y) \in N_1$. Thus, $H$ is well defined and

$$\left\{(N_1/N_2, \{[N_2]\})\right\} = \{(\{a, b\}, \{[b]\})\} = \tilde{N}_\varepsilon. \quad \square$$

Corollary 6.25. $h(\pi, [-1, 1] \times \{0\}) = 0$ for all $\pi \in SK_2$.

Proof. We have $\pi \sim (U \oplus S^\nu)([\pi_U \oplus \pi_S])$ (see Lemma 6.20) and $\pi_U \oplus \pi_S = V(\xi, \pi_n) \oplus \pi_S$ (see Lemma 6.23). Recall that $\pi_n$ is induced by the differential equation $\dot{y} = y$ on $E^-$ where we set $n := \dim E^-$, so $V[\pi_n] \oplus \pi_S$ can be considered as the product of $\pi_S$ with $\pi_n$. Moreover, $h([-1, 1] \times \{0\}, \pi_S) = 0$ has been proved in Proposition 6.24.
It is well known (see [4]) that in the case of product semiflows the homotopy index equals the smash product of the indices of its factors, that is,

$$h([-1, 1] \times \{0\}, (\pi_a, \pi_\Sigma)) = \Sigma^n \wedge h([-1, 1] \times \{0\}, \pi_\Sigma)$$

$$= \Sigma^n \wedge \emptyset = \emptyset. \quad \square$$

Appendix A. Trivial vector bundles

Although one could certainly use the notion of a vector bundle as defined in [9], this would create a large overhead due to formalism since the structure of the vector bundles used here is relatively simple. Therefore, definitions restricted to the use case will be given.

Let $[a, b] \subset \mathbb{R}$ be fixed and let $E, F$ denote arbitrary Banach spaces. We will write $E = E_1 \oplus E_2$ iff $E_1$ and $E_2$ are closed linear subspaces of $E$ with $E = E_1 + E_2$ and $E_1 \cap E_2 = \{0\}$. Given a linear subspace $E_1 \subset E$, another linear subspace $E_2$ is called a topological complement iff $E = E_1 \oplus E_2$. In particular, such a complement exists if either $\dim E_1 < \infty$ or $\dim E_1 < \infty$.

Definition A.1. A (trivial) bundle is the Cartesian product $[a, b] \times E$ equipped with the product metric.

Taking (trivial) bundles as objects of a category $\mathcal{B} = \mathcal{B}([a, b])$, one needs to define morphisms:

Definition A.2. A morphism in $\mathcal{B}$ is a continuous mapping $G : [a, b] \to \mathcal{L}(E, F)$. $G$ is called a splitting if for every $x \in [a, b]$, $G(x)E$ has a topological complement in $F$.

Given bundles $[a, b] \times E$ and $[a, b] \times \tilde{E}$ and a morphism $F$ between them, $F$ can be applied to $[a, b] \times E$ in the following way: $F(x, \eta) := (x, F(x)\eta)$.

If $F_1, F_2$ are morphisms, then $(F_1 \circ F_2)(x) := F_1(x) \circ F_2(x)$ is again a morphism. In particular, a morphism $F$ is an isomorphism iff for every $x \in [a, b]$ $F(x) \in \mathcal{L}(E, F)$ is an isomorphism and iff the induced mapping $\tilde{F}$ is a homeomorphism.

Lemma A.3. Let $G \in C([a, b], \mathcal{L}(E, F))$ and suppose that $G(x_0)$ is an isomorphism in $\mathcal{L}(E, F)$. Then there is a neighborhood $U$ of $x_0$ in $[a, b]$ such that $G(x)$ is an isomorphism for all $x \in U$. Moreover, $G(x)^{-1}$ is continuous in $x$ for all $x \in U$.

Corollary A.4. $G \in C([a, b], \mathcal{L}(E, F))$ is an isomorphism if and only if for every $x \in [a, b]$ $G(x)$ is an isomorphism in $\mathcal{L}(E, F)$.

Definition A.5. A subset $U \subset [a, b] \times F$ is called a subbundle if there exist another bundle $[a, b] \times E$ and a splitting monomorphism $G : [a, b] \times E \to [a, b] \times F$ such that $U = \tilde{G}([a, b] \times E)$.

Lemma A.6. $\tilde{G} : [a, b] \times E \to U$ is a homeomorphism and the norms on the fibers are equivalent, that is, there are constants $m, M \in \mathbb{R}^+$ such that $0 \neq m$ and $m \|\eta\|_E \leq \|G(x)\eta\|_F \leq M \|\eta\|_E$ for all $(x, \eta) \in [a, b] \times E$.

Given a splitting monomorphism $U : [a, b] \times E \to [a, b] \times F$, one can speak of a subbundle, that is, identifying $U$ with its image $\tilde{U}(a, b] \times E)$. Then the fibers are given by $U(x) := U(x)E$ for $x \in [a, b]$. If $V \subset [a, b]$, then we write $U(V) := \bigcup_{x \in V} \{x\} \times U(x)$.

Lemma A.7. Let $U : [a, b] \times E \to [a, b] \times F$ be a subbundle, let $x_0 \in [a, b]$ and let $P : F \to U(x_0)$ be a continuous projection onto $U(x_0)$. Then there exists a neighborhood $V$ of $x_0$ in $[a, b]$ such that $p : U(V) \to V \times U(x_0)$, $p(x, y) = (x, Py)$ is a homeomorphism and the norms on the fibers are equivalent, that is, there are constants $m, M \in \mathbb{R}^+$ such that $0 \neq m$ and $m \|\eta\| \leq \|P\eta\| \leq M \|\eta\|$ for all $(x, \eta) \in U(V)$. 
As before, let $U : [a, b] \times E \to B$ be a subbundle, where $B := [a, b] \times F$. Define the quotient bundle $B / U$ to be the disjoint union of the quotients on the fibers, that is,

$$B / U := \bigcup_{x \in [a, b]} \{ x \} \times (F / U(x)).$$

It is natural to endow $B / U$ with the quotient topology and to assign to each fiber the norm

$$\| y \|_{F / U(x)} := \inf \{ \| y - z \|_F : z \in U(x) \} \quad y \in F.$$

**Lemma A.8.** Let $E = E_1 \oplus E_2$ and let $U : [a, b] \times E_1 \to B = [a, b] \times E$ be a subbundle. Then there exists another subbundle $S : [a, b] \times E_2 \to [a, b] \times E$ such that $U \oplus S : [a, b] \times (E_1 \oplus E_2) \to E$, which is defined by $(U \oplus S)(x)(y_1 \oplus y_2) = U(x)y_1 + S(x)y_2$, is an isomorphism.

Furthermore, if $E = U(\xi) \oplus E_2$ for some $\xi \in [a, b]$, then we can assume that $S(x) = \text{id}_{E_2}$ for all $x$ in a sufficiently small neighborhood of $\xi$ in $[a, b]$.

A consequence of the previous lemma is, that $B / U$ is again a metric space, which allows for example to consider the Conley index on $B / U$.

**Corollary A.9.** Let the assumptions of Lemma A.8 hold, let $U$ and $S$ be given by that lemma and let the canonical projection $p : S \to B / U$ be defined by $p(x, y) := (x, [y])$.

Then $p \circ S : [a, b] \times E_2 \to B / U$ is a homeomorphism and the norms on the fibers are equivalent, that is, there are constants $m, M \in \mathbb{R}^+$ such that $0 \neq m$ and $m\| \eta \|_E \leq \| [S(x)\eta] \|_{E / U(x)} \leq M\| \eta \|_E$ for all $(x, \eta) \in [a, b] \times E_2$.

**Lemma A.10.** Let $U : [a, b] \times E_1 \oplus E_2 \to [a, b] \times E$ be a subbundle, let $x_0 \in [a, b]$ and let

$$E = U(x_0) \oplus E_2.$$

Then there exists a neighborhood $V$ of $x_0$ in $[a, b]$ such that $p : V \times E_2 \to p(V \times E_2) \subset B / U$, $p(x, y) = (x, [y])$ is a homeomorphism and the norms on the fibers are equivalent, that is, there are constants $m, M \in \mathbb{R}^+$ such that $0 \neq m$ and $m\| \eta \|_E \leq \| [\eta] \|_{E / U(x)} \leq M\| \eta \|_E$ for all $(x, \eta) \in V \times E_2$.

**References**


