Approximately optimum stratification for ratio and regression methods of estimation

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Abstract

This work considers the problem of optimum stratification for two study variables $Y_j$ ($j = 1, 2$) when samples from different strata are collected by simple random sampling with a replacement scheme and the information on the auxiliary variable is used to estimate the population mean using the ratio and regression method of estimation. A cumulative cube root rule for determination of optimum strata boundaries for the ratio and regression method of estimation under compromise allocation has been proposed. A limiting expression for the trace of the variance–covariance matrix has also been suggested. The work concludes with a numerical illustration.

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1. Introduction

Ghosh [4] considered the problem of optimum stratification with two characters under a proportional method of allocation assuming the stratification variable as identical to the estimation variable under consideration. It is unrealistic to assume the distribution of the study variable in advance. Rizvi et al. [3] considered the case of optimum stratification for two study variables in the case of a compromise method of allocation using the auxiliary variable as the stratification variable.

If information on the auxiliary variable, which is highly correlated with the variable under study, is available in that case the population mean can be estimated by using ratio and regression method of estimation as compared to simple random sampling. The information on the auxiliary variable can further be used to increase the precision of the estimator of the population mean by using the technique of optimum stratification.

2. Optimum stratification for ratio and regression methods of estimation

For theoretical development, let us assume that the population under study is divided into $L$ strata. A stratified simple random sample (with replacement) of size $n$ is drawn from it; the sample of $n_h$ units is drawn from the $h$-th stratum such that $\sum_{h=1}^{L} n_h = n$. Let $Y_j$ ($j = 1, 2$) denote the value of the $j$-th variable in the population.
The separate ratio estimator for estimating the population mean in stratified sampling is given by

$$\bar{y}_j = \sum_{h=1}^{L} W_h \bar{y}_h R_j$$  \hspace{1cm} (2.1)

where

$$\bar{y}_h R_j = \frac{\bar{y}_{hj}}{\bar{x}_h} \bar{X}_h,$$

$$W_h = \text{proportion of units in the } h\text{-th stratum},$$

$$\bar{y}_{hj} = \text{sample mean for the } j\text{-th study variable in the } h\text{-th stratum},$$

$$\bar{x}_h = \text{sample mean for the auxiliary variable } X \text{ in the } h\text{-th stratum},$$

$$\bar{X}_h = \text{population mean for the auxiliary variable } X \text{ in the } h\text{-th stratum}.$$  

The variance of the estimator $\bar{y}_j$ under the ratio method of estimation is given by

$$V(\bar{y}_j) = \sum_{h=1}^{L} \frac{W_h^2}{n_h} \left[ \sigma_{h yj}^2 + R_{jh}^2 \sigma_{h x}^2 - 2 R_{jh} \sigma_{h xyj} \right]$$  \hspace{1cm} (2.2)

where $R_{jh} = \frac{\bar{y}_{hj}}{\bar{X}}.$

2.1. Separate regression estimator

It is often found that even when the regression lines of the study variable are linear and the regression line does not pass through the origin, under such conditions ratio estimators are not efficient; hence it seems to more appropriate to use the regression type of estimators.

The separate regression estimator for estimating the population mean for the $j$-th variable is given by

$$\bar{y}_{jR} = \sum_{h=1}^{L} W_h \hat{\mu}_{hRyj}$$  \hspace{1cm} (2.3)

where

$$\hat{\mu}_{hRyj} = \bar{y}_{hj} + \beta_{jh}(\bar{X}_h - \bar{x}_h),$$

$$V(\bar{y}_{jR}) = \sum_{h=1}^{L} \frac{W_h^2}{n_h} \left[ \sigma_{h yj}^2 + \beta_{jh}^2 \sigma_{h x}^2 - 2 \beta_{jh} \sigma_{h xyj} \right].$$  \hspace{1cm} (2.4)

We observe that the variance expressions under the separate ratio and regression method of estimation (2.2) and (2.4) are the same except that they differ in the constants $R_j$ and $\beta_j.$ Hence we will consider only the regression estimator whose variance is given by (2.4).

3. Compromise allocation in stratified sampling

Sukhatme et al. [1] reviewed the problem of allocation with several characteristics as given by several research workers. They have shown numerically that all the compromise allocations, as compared by them, are more efficient than proportional allocation. However the compromise allocation based on the trace of the variance–covariance matrix is the most efficient. Hence we have considered the case of compromise allocation based on minimization of the trace of the variance–covariance matrix. In the $h$-th stratum, the sample sizes $n_h$ are determined in such a way that for a given total sample size (which amounts to the fixed total cost where the cost per unit in each stratum is the same) $\sum_{j=1}^{L} V(\bar{y}_{jR})$ is minimized subject to the condition $\sum_{h=1}^{L} n_h = n.$ The optimum value of $n_h$ is given by

$$n_h = n \frac{W_h \sqrt{R_{hy1}^2 + R_{hy2}^2}}{\sum_{h=1}^{L} W_h \sqrt{R_{hy1}^2 + R_{hy2}^2}}$$  \hspace{1cm} (3.1)

where

$$R_{hyj}^2 = \sigma_{h yj}^2 + \beta_{jh}^2 \sigma_{h x}^2 - 2 \beta_{jh} \sigma_{h xyj}.$$
Using this value of \( n_h \) we shall obtain the variance expression for compromise allocation. Using (3.1) in (2.4), the optimal variance of the estimated population mean of the study variable \( Y_j \) under compromise allocation is given by

\[
V(\bar{y}_{j,R}) = \frac{1}{n} \sum_{h=1}^{L} \left[ \frac{W_h R_{hy_j}^2}{\sqrt{R_{hy_1}^2 + R_{hy_2}^2}} \right].
\]

(3.2)

### 4. Variance under the super-population model

Let us now assume that the population under consideration is a random sample from an infinite super-population with the same characteristics. Further we assume that the study variable \( Y_j \) is linearly related to the auxiliary variable \( X \) so that the regression of \( Y_j \) on \( X \) is given by the linear model

\[
Y_j = c_j(X) + e_j
\]

(4.1)

where \( c_j(X) = \beta_jX \) is a real valued function of \( X \), \( e_j \) is an error component such that \( E(e_j|X) = 0 \), \( E(e_j e_j'|X, X') = 0 \) for \( x \neq x' \) and \( V(e_j|X) = \phi_j > 0 \) for all \( x \in (a, b) \) where \( (b - a) < \infty \).

If the joint density function of \((X, Y_1, Y_2)\) in the super-population is \( f_s(x, y_1, y_2) \) and the marginal density function of \( X \) is \( f(x) \), then under model (4.1) it can be easily seen that

\[
W_h = \int_{x_{h-1}}^{x_h} f(x)dx, \quad \mu_{hY_j} = \mu_{hc_j} = W_h^{-1} \int_{x_{h-1}}^{x_h} c_j(x)f(x)dx
\]

\[
\sigma_{hc_j}^2 = W_h^{-1} \int_{x_{h-1}}^{x_h} c_j^2(x)f(x)dx - \mu_{hc_j}^2, \quad \sigma_{hy_j}^2 = \sigma_{hc_j}^2 + \mu_{h\phi_j}^2
\]

where \((x_{h-1}, x_h)\) are the boundaries of the \( h \)-th stratum, \( \mu_{h\phi_j} \) is the expected value of the function \( \phi_j(x) \) and \( \phi_j(x) \) is the conditional variance of the \( j \)-th study variable.

**Theorem 4.1.** The variance expression for the estimator of the population mean of the regression estimator under super-population model (4.1) is given by

\[
\sigma_j^2 = V(\bar{y}_{j,R}) = \frac{1}{n} \sum_{h=1}^{L} \left[ \frac{W_h \mu_{h\phi_j}^2}{\sqrt{\mu_{h\phi_1} + \mu_{h\phi_2}}} \right] \sum_{h=1}^{L} W_h \sqrt{\mu_{h\phi_1} + \mu_{h\phi_2}} \right] \quad (j = 1, 2).
\]

### 5. Minimal equations

We assume that the stratification variable is continuous with pdf \( f(x) \), \( a \leq x \leq b \) and the points of demarcation forming \( L \) strata are \( x_1, x_2, \ldots, x_L \). Let us denote the optimum points of stratification as \( \{x_h\} \); then corresponding to these strata boundaries the generalized variance \( G \), the determinant of the variance–covariance matrix, which is a function of the point of stratification, is minimum. These \( \{x_h\} \) are the solutions of the minimal equations. Now the generalized variance \( G \) is given by

\[
\text{Det}(G) = \begin{vmatrix} \sigma_1^2 & \sigma_{12} \ \\ \sigma_{12} & \sigma_2^2 \end{vmatrix} = \sigma_1^2 \sigma_2^2 - \sigma_{12}^2.
\]

(5.1)

It is onerous to obtain even an approximate solution to the minimal equations obtained through minimization of \( G \) under the compromise method of allocation; therefore, we have considered the minimization of the trace of the variance–covariance matrix for the purpose of obtaining minimal equations and their solution. The trace of the variance–covariance matrix is given by

\[
\text{tr}(G) = \frac{1}{n} \left[ \sum_{h=1}^{L} W_h \sqrt{\mu_{h\phi_1} + \mu_{h\phi_2}} \right]^2.
\]

(5.2)
Now minimizing the trace of the variance–covariance matrix \( \text{tr}(G) \) w.r.t. \( \{x_h\} \) along the lines of Singh and Sukhatme [2] and on further simplification we have the required minimal equations as given by

\[
\Phi_1(x_h) + \mu_h \Phi_1 + \phi_2(x_h) + \mu_h \phi_2 = \frac{\Phi_1(x_h) + \mu_i \Phi_1 + \phi_2(x_h) + \mu_i \phi_2}{\sqrt{\mu_i \phi_1 + \mu_i \phi_2}} \tag{5.3}
\]

where \( h = 1, 2, \ldots, L - 1 \) and \( i = 1, 2, \ldots, h + 1 \).

Solutions to these minimal equations (5.3) will give the set of optimum points of stratification. This system of equations comprises functions of parameter values, which themselves are functions of points of strata boundaries. Since it is very difficult to obtain exact solutions of minimal equations, we will try to find approximate solutions to these equations.

6. Approximate solutions to the minimal equations

To obtain the approximate solutions to the minimal equations (5.3) we have to expand both sides of the minimal equations about the point \( x_h \), the common boundary point of the \( h \)-th and \( i \)-th strata. The series expansion for \( W_h \) and \( \mu_h \phi_1 \) can be obtained by using Taylor’s theorem about both the upper and lower boundaries of the \( h \)-th stratum along the lines of Singh and Sukhatme [2].

Now using the various expressions in minimal equation (5.3), we get on simplification

\[
k_h^2[1 + A_2 k_h^2 - A_3 k_h^3 + O(k_h^4)] = k_i^2[1 + A_2 k_i^2 + A_3 k_i^3 + O(k_i^4)] \tag{6.1}
\]

where \( k_h = x_h - x_{h-1}, k_i = x_{h+1} - x_h \) are the stratum widths, and

\[
A_2 = \frac{f(\phi_1' + \phi_2')^2}{32 f(\phi_1 + \phi_2)^2},
\]

\[
A_3 = \frac{1}{96 f(\phi_1 + \phi_2)^2} \frac{d}{dx_h} \left[ \frac{f(\phi_1' + \phi_2')^2}{f(\phi_1 + \phi_2)^{3/2}} \right]
\]

where \( \phi_1' \) is the first-order derivative of \( \phi_1 \) and \( \phi_2' \) is the first-order derivative of \( \phi_2 \).

Using these expansions, the system of minimal equations in (5.3) reduces to

\[
k_h^2 \left[ 1 - \frac{k_h}{3} \cdot \frac{P'(t)}{P(t)} + O(k_h^2) \right] = k_i^2 \left[ 1 + \frac{k_i}{3} \cdot \frac{P'(t)}{P(t)} + O(k_i^2) \right] \tag{6.2}
\]

where \( P(t) = \frac{(\phi_1'(t) + \phi_2'(t))^2}{(\phi_1(t) + \phi_2(t))^{3/2}} \).

On further simplification this reduces to

\[
k_h^2 \int_{x_{h-1}}^{x_h} P(t) f(t) dt = Q(x_{h-1}, x_h)[1 + O(k_h^2)]. \tag{6.3}
\]

Then the system of equations (5.3) can be approximately put as

\[
Q(x_{h-1}, x_h) = \text{Constant}, \quad h = 1, 2, \ldots, L. \tag{6.4}
\]

Various methods of finding approximate solutions to the minimal equations can be established through the system of equations (6.4). Singh and Sukhatme [2] developed different forms of the function \( Q(x_{h-1}, x_h) \) corresponding to the univariate case under Neyman allocation. One such function gives the cumulative \( \frac{\sqrt{M(x)}}{M(x)} \) rule according to which the approximately optimum strata boundaries (AOSB) are solutions of the system of equations (5.3). Proceeding along the same lines, one such form of function \( Q(x_{h-1}, x_h) \) can also be obtained as follows:

\[
\int_{x_{h-1}}^{x_h} \frac{\sqrt{M(t)}}{M(t)} dt = \int_a^b \frac{\sqrt{P(t)}}{P(t)} f(t) dt / L. \tag{6.5}
\]

Thus we get the following cumulative cube root rule for finding AOSB on the auxiliary variable for the two estimation variables.
Cumulative $\sqrt[3]{M(x)}$ Rule:

If the function $M(x) = P(x)f(x)$ is bounded and its first two derivatives exist for all $x$ in $(a, b)$ with $(b - a) < \infty$, then for a given value of $L$ taking equal intervals on the cumulative cube root of $M(x)$ will give approximately optimum strata boundaries (AOSB).

7. Limiting form of the trace of the variance–covariance matrix

The limiting expression for the variance–covariance matrix is particularly important in optimum stratification as it gives an insight into the manner in which the variance of the estimator of the mean is reduced as the number of strata increases. For obtaining the limiting expression for the trace of the variance–covariance matrix $\text{tr}(G)$, we give the following lemma for the bivariate case, which can be proved by using the series expansion of the various terms involved in it, exactly as for the univariate case discussed in Singh and Sukhatme [2] and the bivariate case of Rizvi et al. [3].

**Lemma 7.1.** Under regularity conditions for the $h$-th stratum we have

$$W_h\sqrt{\mu_1 \phi_1 + \mu_2 \phi_2} - \int_{x_{h-1}}^{x_h} \sqrt{\phi_1(t) + \phi_2(t)} f(t) dt = \frac{k_h^2}{96} \int_{x_{h-1}}^{x_h} P(t)f(t) dt [1 + O(k_h^2)]$$

where $P(t)$ is defined in (6.2).

Now making use of the Lemma 7.1 in the expression (5.2), we have

$$\text{tr}(G) = \frac{1}{n} \left[ \int_a^b \sqrt{[\phi_1(t) + \phi_2(t)]} f(t) dt + \sum_{h=1}^{L} \frac{k_h^2}{96} \int_{x_{h-1}}^{x_h} P(t)f(t) dt [1 + O(k_h^2)] \right]^2.$$  \hspace{1cm} (7.1)

Now using the result (3.8) of Singh and Sukhatme [2], the Eq. (7.1) can be put as

$$\text{tr}(G) = \frac{1}{n} \left[ \int_a^b \sqrt{[\phi_1(t) + \phi_2(t)]} f(t) dt + \frac{1}{96} \sum_{h=1}^{L} \left\{ \int_{x_{h-1}}^{x_h} \sqrt{P(t)f(t)} dt \right\}^3 \right]^2.$$ \hspace{1cm} (7.2)

Now if the strata boundaries are determined by making use of the proposed cumulative cube root rule then for $h = 1, 2, \ldots, L$ we have

$$\int_{x_{h-1}}^{x_h} \sqrt{P(t)f(t)} dt = \frac{1}{L} \int_a^b \sqrt{P(t)f(t)} dt.$$ \hspace{1cm} (7.3)

Therefore, Eq. (7.3) reduces to

$$\text{tr}(G) = \frac{1}{n} \left[ \lambda + \psi \right]^2$$ \hspace{1cm} (7.4)

where

$$\lambda = \int_a^b \sqrt{[\phi_1(t) + \phi_2(t)]} f(t) dt$$

$$\psi = \frac{1}{96} \left\{ \int_a^b \sqrt{P(t)f(t)} dt \right\}^3.$$

Now taking the limit as $L \to \infty$ on both sides of (7.4) we get

$$\text{Lim} \text{tr}(G) = \frac{\lambda^2}{n}$$ \hspace{1cm} (7.5)

$L \to \infty$. 
From the above relation it may be concluded that with an increase in the number of strata \( L \), the trace of the generalized variance decreases and as the number of strata becomes large enough, \( \text{tr}(G) \) tends to \( \lambda^2/n \). However if the number of strata \( L \) goes to infinity then the sample size \( n \) goes faster to infinity, because we have to select minimum of one unit from each stratum; hence \( \text{tr}(G) \to 0 \).

8. **Empirical study**

To determine approximately optimum strata boundaries (AOSB) by the use of the proposed cumulative cube rule \( \sqrt[3]{M(x)} \) we consider that the stratification variable \( x \) follows uniform, right triangular and exponential distributions with probability density functions given by

- Uniform distribution \( f(x) = 1 \quad 1 \leq x \leq 2 \)
- Right triangular distribution \( f(x) = 2(2-x) \quad 1 \leq x \leq 2 \)
- Exponential distribution \( f(x) = e^{-x+1} \quad 1 \leq x < \infty \).

The ranges of the uniform and right triangular distributions are finite whereas the range of the exponential distribution is infinite. We have considered that the study variables \( Y_j \) are related to the stratification variable \( x \) as \( Y_1 = x + e_1, \ Y_2 = 2x + e_2 \). The conditional variances of the error terms, i.e. \( V(e_1|x) \) and \( V(e_2|x) \), are assumed to be of the forms \( A_1 x^{g_1} \) and \( A_2 x^{g_2} \) respectively where \( A_1, A_2 > 0, g_1 \) and \( g_2 \) being constants. Here we have taken different combinations of \( g_1 \) and \( g_2 \). The values of \( A_1 \) and \( A_2 \) were determined for the values of \( g_1, g_2, \rho_1 \), and \( \rho_2 \) by using the following formulae:

\[
A_1 = \frac{\beta_1^2 \sigma_x^2 (1 - \rho_1^2)}{\rho_1^2 E(x^{g_1})} \quad \text{and} \quad A_2 = \frac{\beta_2^2 \sigma_x^2 (1 - \rho_2^2)}{\rho_2^2 E(x^{g_2})}
\]

where \( \rho_1 \) and \( \rho_2 \) are the coefficients of correlation between the study variables \( Y_1 \) and \( Y_2 \) with stratification variable \( x \). \( \sigma_x^2 \) is the variance of the stratification variable \( x \). For the purpose of numerical illustration we have assumed \( \rho_1^2 = 0.9 \) and \( \rho_2^2 = 0.7 \). For finding the approximately optimum strata boundaries (AOSB), the ranges of uniform, right triangular and exponential distributions were divided into 10 classes of equal width. The function \( P(x) \) was evaluated at the middle points of the class intervals and \( 3\sqrt{M(x)} \) was then found for each of the 10 classes. These cube roots

<table>
<thead>
<tr>
<th>No. of strata ( L )</th>
<th>Strata boundaries</th>
<th>( n \text{tr}(G) )</th>
<th>Per cent relative efficiency</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g_1 = 2 ) and ( g_2 = 1 )</td>
<td>\begin{align*} 1 \quad 1.46127 &amp; \quad 0.29493 &amp; \quad 100.00 \ 2 \quad 1.29838 &amp; \quad 1.63243 &amp; \quad 0.29217 &amp; \quad 100.00 \ 3 \quad 1.22069 &amp; \quad 1.46127 &amp; \quad 1.72117 &amp; \quad 0.29273 &amp; \quad 100.00 \ 4 \quad 1.17506 &amp; \quad 1.36271 &amp; \quad 1.56295 &amp; \quad 1.77550 &amp; \quad 0.29232 &amp; \quad 100.00 \ 5 \quad 1.14516 &amp; \quad 1.29838 &amp; \quad 1.46127 &amp; \quad 1.63243 &amp; \quad 1.81202 &amp; \quad 0.29207 &amp; \quad 100.00 \ 6 \quad &amp; &amp; &amp; &amp; &amp;</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( g_1 = 1 ) and ( g_2 = 2 )</td>
<td>\begin{align*} 1 \quad 1.47259 &amp; \quad 0.15210 &amp; \quad 100.00 \ 2 \quad 1.30855 &amp; \quad 1.64264 &amp; \quad 0.14829 &amp; \quad 102.57 \ 3 \quad 1.22914 &amp; \quad 1.47259 &amp; \quad 1.72984 &amp; \quad 0.14757 &amp; \quad 103.07 \ 4 \quad 1.18210 &amp; \quad 1.37350 &amp; \quad 1.57387 &amp; \quad 1.78282 &amp; \quad 0.14732 &amp; \quad 103.24 \ 5 \quad 1.15126 &amp; \quad 1.30855 &amp; \quad 1.47259 &amp; \quad 1.64264 &amp; \quad 1.81847 &amp; \quad 0.14714 &amp; \quad 103.37 \ 6 \quad &amp; &amp; &amp; &amp; &amp;</td>
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<tr>
<td>( g_1 = 2 ) and ( g_2 = 2 )</td>
<td>\begin{align*} 1 \quad 1.47172 &amp; \quad 0.15213 &amp; \quad 100.00 \ 2 \quad 1.30774 &amp; \quad 1.64188 &amp; \quad 0.14808 &amp; \quad 102.73 \ 3 \quad 1.22846 &amp; \quad 1.47172 &amp; \quad 1.72920 &amp; \quad 0.14732 &amp; \quad 103.27 \ 4 \quad 1.18153 &amp; \quad 1.37266 &amp; \quad 1.57304 &amp; \quad 1.78228 &amp; \quad 0.14705 &amp; \quad 103.46 \ 5 \quad 1.15076 &amp; \quad 1.30774 &amp; \quad 1.47172 &amp; \quad 1.64188 &amp; \quad 1.81800 &amp; \quad 0.14685 &amp; \quad 103.59</td>
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Table 8.2
Per cent relative efficiency of stratification for right triangular distribution

<table>
<thead>
<tr>
<th>No. of strata $L$</th>
<th>Strata boundaries</th>
<th>$n$ tr($G$)</th>
<th>Per cent relative efficiency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$g_1 = 2$ and $g_2 = 1$</td>
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<td>100.00</td>
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<td>2</td>
<td>1.37469</td>
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<td>1.11311 1.23731 1.37469 1.53118 1.71742</td>
<td>0.19512</td>
<td>100.79</td>
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Table 8.3
Per cent relative efficiency of stratification for exponential distribution

<table>
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<tr>
<th>No. of strata $L$</th>
<th>Strata boundaries</th>
<th>$n$ tr($G$)</th>
<th>Per cent relative efficiency</th>
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</thead>
<tbody>
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<td>1</td>
<td>$g_1 = 2$ and $g_2 = 1$</td>
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<td>2.84166</td>
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Table 8.4
Per cent relative efficiency of stratification for exponential distribution

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<th>Strata boundaries</th>
<th>$n$ tr($G$)</th>
<th>Per cent relative efficiency</th>
</tr>
</thead>
<tbody>
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<td>$g_1 = 2$ and $g_2 = 1$</td>
<td>2.54299</td>
<td>100.00</td>
</tr>
<tr>
<td>2</td>
<td>2.30353 3.00931 3.48119</td>
<td>2.38472</td>
<td>106.64</td>
</tr>
<tr>
<td>3</td>
<td>1.77060 2.30353 3.48119 3.82751</td>
<td>2.36346</td>
<td>107.60</td>
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<tr>
<td>4</td>
<td>1.53775 2.30353 3.48119 3.82751</td>
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<td>5</td>
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<td>2.36321</td>
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References