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Biharmonic ideal hypersurfaces in Euclidean spaces

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ABSTRACT

Let $x: M \to \mathbb{E}^m$ be an isometric immersion from a Riemannian n-manifold into a Euclidean m-space. Denote by Δ and $\vec{x}$ the Laplace operator and the position vector of M, respectively. Then M is called biharmonic if $\Delta^2\vec{x} = 0$. The following Chen's Biharmonic Conjecture made in 1991 is well-known and stays open: *The only biharmonic submanifolds of Euclidean spaces are the minimal ones*. In this paper we prove that the biharmonic conjecture is true for $\delta(2)$-ideal and $\delta(3)$-ideal hypersurfaces of a Euclidean space of arbitrary dimension.

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1. Introduction

Let $x: M \to \mathbb{E}^m$ be an isometric immersion from a Riemannian n-manifold into a Euclidean m-space. Denote by Δ, $\vec{x}$ and $\vec{H}$ the Laplace operator, the position vector and the mean curvature vector of M, respectively. Then M is called a biharmonic submanifold if $\Delta^2\vec{x} = 0$. Due to the well-known Beltrami's formula, $\Delta\vec{x} = -n\vec{H}$, it is obvious that every minimal submanifold of $\mathbb{E}^m$ is a biharmonic submanifold.

The study of biharmonic submanifolds was initiated by B.-Y. Chen in the middle of 1980s (cf. [9,11,14–16,21–23]). He proved in 1985 that biharmonic surfaces in $\mathbb{E}^3$ are minimal. This result was the starting point of I. Dimitrić's work on his doctoral thesis [22]. In particular, Dimitrić extended Chen's result on biharmonic surfaces in $\mathbb{E}^3$ to that if M is a biharmonic hypersurface of $\mathbb{E}^m$ with at most two distinct principal curvatures, then M is minimal [22,23]. Since conformally flat hypersurfaces of $\mathbb{E}^m$ with $m \geqslant 5$ have at most two distinct principal curvatures, Dimitrić's result implies that biharmonic conformally flat hypersurfaces of $\mathbb{E}^m$ with $m \geqslant 5$ are minimal. Dimitrić also proved that every biharmonic curve in $\mathbb{E}^n$ is an open part of a straight line and each biharmonic submanifold of finite type in $\mathbb{E}^m$ is minimal. Another extension of Chen's result was given by T. Hasanis and T. Vlachos in [24] (see also [20]). They proved that biharmonic hypersurfaces of $\mathbb{E}^4$ are minimal.

In 1991, B.-Y. Chen [9] made the following.

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Biharmonic Conjecture. *The only biharmonic submanifolds of Euclidean spaces are the minimal ones.*

In the same spirit of Chen's result, R. Caddeo, S. Montaldo and C. Oniciuc [6] proved that any biharmonic surface in the hyperbolic 3-space $\mathbb{H}^3(-1)$ is minimal. They also proved that biharmonic hypersurfaces of $\mathbb{H}^n(-1)$ with at most two distinct principal curvatures are minimal [5]. Based on these, they made the following.

The generalized Chen's conjecture. *Any biharmonic submanifold of a Riemannian manifold with non-positive sectional curvature is minimal.*

The study of biharmonic submanifolds is nowadays a very active research subject. In particular, there exist many results on the generalized Chen's conjecture (see, for instance, [1–4,26–30]). Very recently, N. Nakauchi and H. Urakawa [27] proved that the generalized Chen's conjecture is true for every complete biharmonic submanifold M with finite total mean curvature, i.e. $\int_M |\overrightarrow{H}|^2 * 1 < \infty$. On the other hand, it was proved recently by Y.-L. Ou and L. Tang [30] that the generalized Chen's conjecture is false in general by constructing foliations of proper biharmonic hyperplanes in a 5-dimensional conformally flat space with negative sectional curvature. In contrast, the original Chen's biharmonic conjecture made in 1991 stays open in general.

A submanifold of a Euclidean space is called *k-harmonic* if its mean curvature vector satisfies $\Delta^{k-1}\overrightarrow{H} = 0$. It follows from Hopf's lemma that such submanifolds are always non-compact. Some relationships between k-harmonic and harmonic maps of Riemannian manifolds into Euclidean m-space $\mathbb{E}^m$ have been obtained by Chen in [13]. Very recently, S. Maeta [25] found some relations between k-harmonic and harmonic maps of Riemannian manifolds into non-flat real space forms.

From [13, Proposition 3.1] it was known that every k-harmonic submanifold of $\mathbb{E}^m$ is either minimal or of infinite type (in the sense of [8]). On the other hand, it was shown in 1991 that k-harmonic curves in $\mathbb{E}^m$ are of finite type [17, Proposition 4.1]. Consequently, it was known that every k-harmonic curve in $\mathbb{E}^m$ is an open portion of line. This known fact was recently rediscovered by Maeta in [25, Theorem 5.5]. Based on this known fact, Maeta [25] made another generalized Chen's conjecture; namely,

"The only k-harmonic submanifolds of a Euclidean space are the minimal ones."

Now, let us recall the notion of δ-invariants of Riemannian manifolds. Denote by $K(\pi)$ the sectional curvature of a given Riemannian n-manifold M associated with a plane section $\pi \subset T_pM$, $p \in M$. For any orthonormal basis $e_1, \dots, e_n$ of the tangent space T_pM, the scalar curvature τ at p is defined to be

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j). \tag{1.1}$$

Let L be a subspace of T_pM of dimension $r \geqslant 2$ and $\{e_1, \dots, e_r\}$ an orthonormal basis of L. The scalar curvature $\tau(L)$ of L is defined by

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta), \quad 1 \leqslant \alpha, \beta \leqslant r. \tag{1.2}$$

For an integer $r \in [2, n-1]$, the δ*-invariant* $\delta(r)$ of M is defined by (cf. [12,14])

$$\delta(r)(p) = \tau(p) - \inf\{\tau(L)\}, \tag{1.3}$$

where L runs over all r-dimensional linear subspaces of T_pM.

For any n-dimensional submanifold M in $\mathbb{E}^m$ and any integer $r \in [2, n-1]$, Chen proved the following general sharp inequality (cf. [12,14]):

$$\delta(r) \leqslant \frac{n^2(n-r)}{2(n-r+1)} H^2, \tag{1.4}$$

where $H^2 = \langle \overrightarrow{H}, \overrightarrow{H} \rangle$ is the squared mean curvature.

A submanifold in $\mathbb{E}^m$ is called $\delta(r)$*-ideal* if it satisfies the equality case of (1.4) identically. Roughly speaking, ideal submanifolds are submanifolds which receive the least possible tension from its ambient space. Ideal submanifolds have many interesting properties and were studied by many geometers (see [14] for details).

In this paper we prove that Chen's original biharmonic conjecture is true for $\delta(2)$-ideal and $\delta(3)$-ideal hypersurfaces of a Euclidean space of arbitrary dimension.

2. Preliminaries

Let M be a hypersurface of a Euclidean $(n+1)$-space $\mathbb{E}^{n+1}$. Denote by ∇ the Levi-Civita connection on M and by $\mathring{\nabla}$ the canonical flat connection on $\mathbb{E}^{n+1}$.

Recall the formulas of Gauss and Weingarten (cf. [7,14])

$$(\mathbf{G})\quad \mathring{\nabla}_X Y = \nabla_X Y + h(X,Y)\xi, \qquad (\mathbf{W})\quad \mathring{\nabla}_X \xi = -SX,$$

where X, Y are tangent to M, ξ is a unit vector normal to M, h is the scalar second fundamental form, and S is the shape operator associated to ξ. We know that h and S are related by $h(X,Y) = \langle SX, Y\rangle$.

The mean curvature vector field $\vec{H}$ can be expressed as $\vec{H} = H\xi$ with

$$H = \frac{1}{n}\operatorname{trace} S, \tag{2.1}$$

where ξ is a unit normal vector field.

We recall, for later use, the Gauss and Codazzi equations:

$$(\mathbf{EG})\quad R_{XY}Z = \langle SY, Z\rangle SX - \langle SX, Z\rangle SY,$$

$$(\mathbf{EC})\quad (\nabla_X S)Y = (\nabla_Y S)X,$$

for all X, Y, Z tangent to M. All over this paper, the curvature R is given by $R_{XY} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.

If we consider a local orthonormal frame $\{e_1, \dots, e_n\}$ on M, then the Laplacian Δ acting on $\vec{H}$ is given by

$$\Delta\vec{H} = \sum_{i=1}^{n}\big[\mathring{\nabla}_{\nabla_{e_i}e_i}\vec{H} - \mathring{\nabla}_{e_i}\mathring{\nabla}_{e_i}\vec{H}\big]. \tag{2.2}$$

Since $\vec{H} = H\xi$, by identifying the tangent and the normal parts in (2.2), we obtain a necessary and sufficient condition for M to be biharmonic in $\mathbb{E}^{n+1}$, namely

$$S(\nabla H) = -\frac{n}{2}H\nabla H, \tag{2.3}$$

$$\Delta H + H\operatorname{trace} S^2 = 0, \tag{2.4}$$

where ∇H is the gradient of the mean curvature H. Recall that the Laplacian acts on functions on M in the following way

$$\Delta H = \sum_{i=1}^{n}[\nabla_{e_i}e_i H - e_i e_i H].$$

A hypersurface in $\mathbb{E}^{n+1}$ is called an H-*hypersurface* if it satisfies (2.3) (cf. [24]). Clearly, every hypersurface with constant mean curvature in a Euclidean space is an H-hypersurface.

3. Biharmonic $\delta(2)$-ideal hypersurfaces in $\mathbb{E}^{n+1}$

In this section we classify $\delta(2)$-ideal biharmonic hypersurfaces and $\delta(2)$-ideal H-hypersurfaces in $\mathbb{E}^{n+1}$.

By using (1.3) and (1.4) (or Lemma 3.2 in [10]), we have

$$\inf K \geqslant \tau - \frac{n^2(n-2)}{2(n-1)}H^2. \tag{3.1}$$

As M being a hypersurface, equality in (3.1) holds if and only if, with respect to a suitable orthonormal frame $\{e_1, \dots, e_n\}$, the shape operator takes the form:

$$S = \begin{pmatrix} a & 0 & 0 & \dots & 0\\ 0 & b & 0 & \dots & 0\\ 0 & 0 & a+b & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & a+b \end{pmatrix} \tag{3.2}$$

for some functions a and b on M. If this happens, M is $\delta(2)$-ideal (see, e.g. [14]).

With Chen's biharmonic conjecture in mind, we are asking whether there exist non-minimal biharmonic $\delta(2)$-ideal hypersurfaces in $\mathbb{E}^{n+1}$.

Without loss of the generality we may assume that H is non-constant. Otherwise, if H would be a constant, it should be zero by virtue of (2.4), and hence M would be minimal.

Let us choose an orthonormal frame $\{e_1, \dots, e_n\}$ on M such that S is given as in (3.2). We give the following result.

Lemma 3.1. *Let M be a $\delta(2)$-ideal H-hypersurface in $\mathbb{E}^{n+1}$, $n \geqslant 3$, with non-constant mean curvature H. If the shape operator S is given by (3.2), then we have*

$$\{a, b\} = \left\{-\frac{n}{2}H, \frac{n(n+1)}{2(n-1)}H\right\}. \tag{3.3}$$

Proof. Since $e_1, \dots, e_n$ are eigenvectors of S, there exist functions $\lambda_1, \dots, \lambda_n$ on M such that $\nabla H = \sum_{i=1}^n \lambda_i e_i$. We have

$$\begin{aligned} S(\nabla H) &= \sum_{i=1}^n \lambda_i S e_i \\ &= \lambda_1 a e_1 + \lambda_2 b e_2 + \sum_{i \geqslant 3} \lambda_i (a+b) e_i \\ &= (a+b)\nabla H - \lambda_1 b e_1 - \lambda_2 a e_2. \end{aligned}$$

Since M is an H-hypersurface, Eq. (2.3) is fulfilled and it yields

$$\left(a + b + \frac{n}{2}H\right)\nabla H = \lambda_1 b e_1 + \lambda_2 a e_2.$$

Hence $\lambda_3 = \dots = \lambda_n = 0$ and

$$\lambda_1\left(a + \frac{n}{2}H\right) = 0 \quad \text{and} \quad \lambda_2\left(b + \frac{n}{2}H\right) = 0. \tag{3.4}$$

Since H is not constant, ∇H is different from 0. Therefore, at least one of λ_1 and λ_2 does not vanish.

If both $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, then $a = b = -\frac{n}{2}H$. Hence we get

$$nH = \operatorname{trace} S = (n-1)(a+b) = -(n-1)nH$$

which implies $H = 0$. This is a contradiction. Consequently, we obtain either

(i) $\lambda_1 \neq 0$ and $\lambda_2 = 0$, or
(ii) $\lambda_1 = 0$ and $\lambda_2 \neq 0$.

In case (i) it follows $a = -\frac{n}{2}H$ and since $a + b = \frac{n}{n-1}H$ one gets $b = \frac{n(n+1)}{2(n-1)}H$. Case (ii) can be discussed in a similar way. This completes the proof. □

From this lemma it turns out that we can take e_1 in the direction of ∇H and the shape operator may be expressed as

$$S = \begin{pmatrix} c_1 H & 0 & 0 & \dots & 0 \\ 0 & c_2 H & 0 & \dots & 0 \\ 0 & 0 & c_3 H & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & c_n H \end{pmatrix} \tag{3.5}$$

with $c_1 = -\frac{n}{2}$, $c_2 = \frac{n(n+1)}{2(n-1)}$ and $c_k = c_1 + c_2$ for $k \geqslant 3$. Moreover, we also have

$$e_1 H \neq 0, \qquad e_k H = 0, \quad \forall k > 1. \tag{3.6}$$

Let $\omega_{ij}^k \in C^\infty(M)$ be defined by $\nabla_{e_i} e_j = \sum_{k=1}^n \omega_{ij}^k e_k$.

Theorem 3.2. *Every $\delta(2)$-ideal biharmonic hypersurface of $\mathbb{E}^{n+1}$ with $n \geqslant 3$ is minimal.*

Proof. Since the case $n = 3$ was already studied in general in [19,24], from now on we will consider $n \geqslant 4$ only. Let us assume that H is non-constant.

By definition we have

$$(\nabla_X S)Y = \nabla_X(SY) - S\nabla_X Y.$$

Using the equation of Codazzi **(EC)** for $X = e_i$ and $Y = e_j$ we obtain

$$(\nabla_{e_i} S)e_j = c_j(e_i H)e_j + H\sum_k (c_j - c_k)\omega_{ij}^k e_k.$$

Then, we continue with special choices of i and j.

For $i=1$, $j=2$ we get

$$c_2(e_1H)e_2+H\sum_k(c_2-c_k)\omega_{12}^k e_k=H\sum_k(c_1-c_k)\omega_{21}^k e_k.$$

Identifying the coefficients corresponding to $\{e_1,\ldots,e_n\}$ we find

$$\omega_{12}^1=0, \tag{3.7}$$

$$e_1H+\left(1-\frac{c_1}{c_2}\right)H\omega_{21}^2=0, \tag{3.8}$$

$$c_1\omega_{12}^k=c_2\omega_{21}^k, \quad k\geqslant 3. \tag{3.9}$$

For $i=1$, $j\geqslant 3$ we get

$$c_j(e_1H)e_j+H\sum_k(c_j-c_k)\omega_{1j}^k e_k=H\sum_k(c_1-c_k)\omega_{j1}^k e_k.$$

Identifying the coefficients as above we obtain

$$\omega_{1j}^1=0, \quad j\geqslant 3, \tag{3.10}$$

$$\omega_{1j}^2=\left(1-\frac{c_2}{c_1}\right)\omega_{j1}^2, \quad j\geqslant 3, \tag{3.11}$$

$$c_j(e_1H)\delta_{jk}+c_2H\omega_{j1}^k=0, \quad j,k\geqslant 3. \tag{3.12}$$

For $i=2$, $j\geqslant 3$ we get

$$H\sum_k(c_j-c_k)\omega_{2j}^k e_k=H\sum_k(c_2-c_k)\omega_{j2}^k e_k.$$

Identifying the coefficients we discover that

$$\omega_{2j}^1=\left(1-\frac{c_1}{c_2}\right)\omega_{j2}^1, \quad j\geqslant 3, \tag{3.13}$$

$$\omega_{2j}^2=0, \quad j\geqslant 3, \tag{3.14}$$

$$\omega_{j2}^k=0, \quad j,k\geqslant 3. \tag{3.15}$$

From (3.6) we know $[e_2,e_j](H)=0$. So we have $\sum_k(\omega_{2j}^k-\omega_{j2}^k)e_kH=0$. Taking into account (3.6) again, we get $\omega_{2j}^1=\omega_{j2}^1$, for $j\geqslant 3$. Combining with (3.13) gives

$$\omega_{2j}^1=\omega_{j2}^1=0. \tag{3.16}$$

Since $\{e_k\}_{k=1}^n$ is an orthonormal basis, we have successively:

(**a**) $0=e_i\langle e_j,e_j\rangle=2\langle\nabla_{e_i}e_j,e_j\rangle=2\omega_{ij}^j$, $\forall i,j=1,\ldots,n$. Hence,

$$\omega_{11}^1=0, \qquad \omega_{12}^2=0, \qquad \omega_{1j}^j=0, \quad j\geqslant 3, \tag{3.17a}$$

$$\omega_{21}^1=0, \qquad \omega_{22}^2=0, \qquad \omega_{2j}^j=0, \quad j\geqslant 3, \tag{3.17b}$$

$$\omega_{k1}^1=0, \qquad \omega_{k2}^2=0, \qquad \omega_{kj}^j=0, \quad j,k\geqslant 3. \tag{3.17c}$$

(**b**) $0=e_i\langle e_1,e_2\rangle=\langle\nabla_{e_i}e_1,e_2\rangle+\langle e_1,\nabla_{e_i}e_2\rangle=\omega_{i1}^2+\omega_{i2}^1$, $\forall i=1,\ldots,n$. Combining (**b**) with (3.7), (3.8) and (3.16) we derive that

$$\omega_{11}^2=0, \qquad \omega_{22}^1=\frac{c_2e_1H}{(c_2-c_1)H}, \qquad \omega_{j1}^2=0, \quad j\geqslant 3. \tag{3.18}$$

Moreover, from (3.11) and (3.18) we find

$$\omega_{1j}^2=0, \quad j\geqslant 3. \tag{3.19}$$

(**c**) $0=e_1\langle e_2,e_j\rangle=\langle\nabla_{e_1}e_2,e_j\rangle+\langle e_2,\nabla_{e_1}e_j\rangle=\omega_{12}^j+\omega_{1j}^2$, $j\geqslant 3$. By using (**c**) and (3.19), and then combining with (3.9) we get

$$\omega_{12}^j=0, \qquad \omega_{21}^j=0, \quad j\geqslant 3. \tag{3.20}$$

(**d**) In the same way we find $\omega_{11}^j + \omega_{1j}^1 = 0$, $\omega_{22}^j + \omega_{2j}^2 = 0$, and $\omega_{jj}^1 + \omega_{j1}^j = 0$, $j \geqslant 3$. Thus it follows that

$$\omega_{11}^j = 0, \qquad \omega_{22}^j = 0, \qquad \omega_{jj}^1 = \frac{(c_1 + c_2)e_1 H}{c_2 H}, \quad j \geqslant 3. \tag{3.21}$$

Now the Codazzi equations $(\nabla_{e_i} S)e_j = (\nabla_{e_j} S)e_i$, for $i, j \geqslant 3$, yield

$$H \sum_k (c_j - c_k)\omega_{ij}^k e_k = H \sum_k (c_i - c_k)\omega_{ji}^k e_k.$$

Subsequently, we find

$$\omega_{ij}^1 = \omega_{ji}^1, \qquad \omega_{ij}^2 = \omega_{ji}^2, \quad i, j \geqslant 3. \tag{3.22}$$

Using (3.6), (3.7) and (3.17b) we have

$$[e_1, e_2](H) = \sum_k (\omega_{12}^k - \omega_{21}^k) e_k H = 0.$$

Therefore we get

$$e_2 e_1 H = 0. \tag{3.23}$$

In the same way, it follows from (3.6), (3.10) and (3.17c) that

$$e_j e_1 H = 0, \quad j \geqslant 3. \tag{3.24}$$

At this point we have all needed coefficients ω_{ij}^k in order to apply Gauss' equation (EG). Write it for some X, Y, Z and pick up the coefficient of a convenient vector (call it W). We respectively obtain:

(1) $X = e_1$, $Y = e_2$ and $Z = e_1$ ($W = e_2$)

$$e_1\left(\frac{e_1 H}{H}\right) + \frac{c_2}{c_1 - c_2}\left(\frac{e_1 H}{H}\right)^2 + c_1(c_1 - c_2)H^2 = 0, \tag{3.25}$$

(2) $X = e_1$, $Y = e_j$ and $Z = e_1$ ($W = e_j$)

$$e_1\left(\frac{e_1 H}{H}\right) - \frac{c_1 + c_2}{c_2}\left(\frac{e_1 H}{H}\right)^2 - c_1 c_2 H^2 = 0, \tag{3.26}$$

(3) $X = e_2$, $Y = e_j$ and $Z = e_2$ ($W = e_j$)

$$\left(\frac{e_1 H}{H}\right)^2 - c_2(c_1 - c_2)H^2 = 0. \tag{3.27}$$

Taking into account (3.17a), (3.18) and (3.21), Eq. (2.4) becomes

$$\frac{e_1 e_1 H}{H} - \left[\frac{c_2}{c_2 - c_1} + (n-2)\frac{c_1 + c_2}{c_2}\right]\left(\frac{e_1 H}{H}\right)^2 - \left[c_1^2 + c_2^2 + (n-2)(c_1 + c_2)^2\right]H^2 = 0. \tag{3.28}$$

From (3.25), (3.26) and (3.28) we immediately obtain that $H = 0$, which is a contradiction. Consequently, H should be constant and due to (2.4), M has to be minimal. $\square$

For $\delta(2)$-ideal H-hypersurfaces we have the following result which generalizes Theorem 3.2.

Theorem 3.3. *Every $\delta(2)$-ideal H-hypersurface of a Euclidean $(n+1)$-space is either minimal or an open portion of a spherical hypercylinder $\mathbb{R} \times \mathbb{S}^{n-1}(r)$.*

Proof. According to Lemma 3.1 we can consider the orthonormal basis just as in Theorem 3.2. Thus, by using the same technique as in the proof of Theorem 3.2 we conclude that the mean curvature H should be constant. Therefore, after applying Theorem 1 of [18] or Theorem 20.13 in [14, page 423], we obtain the theorem. $\square$

4. Biharmonic $\delta(3)$-ideal hypersurfaces in $\mathbb{E}^{n+1}$

In this section we study biharmonic hypersurfaces in $\mathbb{E}^{n+1}$ which are $\delta(3)$-ideal.

If M is a Riemannian n-manifold, we have (see [14])

$$\delta(3)(p) = \tau(p) - \inf_L \tau(L), \quad p \in M, \tag{4.1}$$

where L runs over 3-dimensional subspaces of T_pM. If L is spanned by orthonormal vectors e_1, e_2, e_3, then the scalar curvature $\tau(L)$ is defined by

$$\tau(L) = \sum_{1 \leqslant \alpha < \beta \leqslant 3} K(e_\alpha \wedge e_\beta).$$

We recall the following sharp result from [14, Theorem 13.7].

Proposition 4.1. *Let M be a hypersurface in the Euclidean space $\mathbb{E}^{n+1}$. Then*

$$\delta(3) \leqslant \frac{n^2(n-3)}{2(n-2)} H^2. \tag{4.2}$$

The equality case holds at p if and only if there is an orthonormal basis $e_1, \ldots, e_n$ at p such that the shape operator at p satisfies

$$S = \begin{pmatrix} a & 0 & 0 & 0 & \ldots & 0 \\ 0 & b & 0 & 0 & \ldots & 0 \\ 0 & 0 & c & 0 & \ldots & 0 \\ 0 & 0 & 0 & a+b+c & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & a+b+c \end{pmatrix}, \tag{4.3}$$

where a, b, c are functions on M. If this happens at every point, M is a $\delta(3)$-ideal hypersurface.

We ask the same question as in the previous section:

"Do there exist non-minimal biharmonic $\delta(3)$-ideal hypersurfaces in $\mathbb{E}^{n+1}$?"

It follows from (2.4) that every biharmonic hypersurface with constant mean curvature in $\mathbb{E}^{n+1}$ is minimal. Thus from now on we make the following.

Assumption. The hypersurface M has non-constant mean curvature.

Let us choose an orthonormal frame $e_1, \ldots, e_n$ such that the shape operator S is given by (4.3) with respect to $e_1, \ldots, e_n$. We need the following.

Lemma 4.2. *Let M be a $\delta(3)$-ideal H-hypersurface in $\mathbb{E}^{n+1}$, $n \geqslant 4$, with non-constant mean curvature H. If the shape operator of M satisfies (4.3), then, up to reordering of a, b and c, we have either*

(i) *$a = b = -\frac{n}{2}H$ and $c = \frac{n(n-1)}{n-2}H$, or*
(ii) *$a = -\frac{n}{2}H$ and $c = \frac{n^2}{2(n-2)}H - b$.*

Proof. We proceed in the same way as in the proof of Lemma 3.1. Since $e_1, \ldots, e_n$ are eigenvectors of the shape operator S, we may write $\nabla H = \sum_{i=1}^n \lambda_i e_i$ for some functions $\lambda_1, \ldots, \lambda_n$ on M. Since (2.3) is satisfied, after applying (4.3), we obtain that $\nabla H \in \operatorname{span}\{e_1, e_2, e_3\}$. Hence $\lambda_4 = \cdots = \lambda_n = 0$. Thus we find

$$\left(a + b + c + \frac{n}{2}H\right)\nabla H = \lambda_1(b+c)e_1 + \lambda_2(c+a)e_2 + \lambda_3(a+b)e_3.$$

Consequently, we find

$$\left(a + \frac{n}{2}H\right)\lambda_1 = 0, \qquad \left(b + \frac{n}{2}H\right)\lambda_2 = 0, \qquad \left(c + \frac{n}{2}H\right)\lambda_3 = 0,$$

which lead to the following:

(1) If all λ_1, λ_2 and λ_3 are different from 0, then $a=b=c=-\frac{n}{2}H$. This contradicts the fact that $a+b+c=\frac{n}{n-2}H$ and $H\neq 0$.
(2) If two of $\lambda_1,\lambda_2,\lambda_3$ are different from 0, say $\lambda_1,\lambda_2\neq 0$, then $a=b=-\frac{n}{2}H$. So, we find $c=\frac{n(n-1)}{n-2}H$.
(3) If only one of $\lambda_1,\lambda_2,\lambda_3$ is different from 0, say $\lambda_1\neq 0$, then $a=-\frac{n}{2}H$ and $b+c=\frac{n^2}{2(n-2)}H$.
(4) Since H is non-constant, λ_1, λ_2 and λ_3 cannot vanish simultaneously.

This completes the proof. $\square$

Now, let us focus our attention to the case (ii) of Lemma 4.2. Thus the shape operator takes the form:

$$S=\begin{pmatrix} c_1H & 0 & 0 & 0 & \dots & 0\\ 0 & \varphi & 0 & 0 & \dots & 0\\ 0 & 0 & c_2H-\varphi & 0 & \dots & 0\\ 0 & 0 & 0 & (c_1+c_2)H & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \dots & (c_1+c_2)H \end{pmatrix}, \tag{4.4}$$

where $c_1=-\frac{n}{2}$, $c_2=\frac{n^2}{2(n-2)}$ and φ is a function on M.

Moreover, we exclude the following cases which will be discussed later, together with the case (i) of the previous lemma: $\varphi=\pm c_1H$, $\varphi=\frac{c_2H}{2}$, $\varphi=(c_2\pm c_1)H$.

We have that ∇H is collinear to e_1 and then,

$$e_1H\neq 0,\qquad e_2H=\cdots=e_nH=0. \tag{4.5}$$

We define $\omega_{ij}^k\in C^\infty(M)$ by $\nabla_{e_i}e_j=\sum_{k=1}^n\omega_{ij}^k e_k$.

After applying the equation of Codazzi for $X=e_i$, $Y=e_j$, $i,j=1,\dots,n$, we derive the following:
For $i=1,\ j=2$

$$\omega_{12}^1=0,\qquad \omega_{21}^2=\frac{e_1\varphi}{c_1H-\varphi}, \tag{4.6a}$$

$$(2\varphi-c_2H)\omega_{12}^3=(\varphi+c_1H-c_2H)\omega_{21}^3, \tag{4.6b}$$

$$\big[\varphi-(c_1+c_2)H\big]\omega_{12}^k=-c_2H\omega_{21}^k,\quad k\geqslant 4. \tag{4.6c}$$

For $i=1,\ j=3$

$$\omega_{13}^1=0,\qquad \omega_{31}^3=\frac{c_2e_1H-e_1\varphi}{\varphi+c_1H-c_2H}, \tag{4.7a}$$

$$(2\varphi-c_2H)\omega_{13}^2=(\varphi-c_1H)\omega_{31}^2, \tag{4.7b}$$

$$(\varphi+c_1H)\omega_{13}^k=c_2H\omega_{31}^k,\quad k\geqslant 4. \tag{4.7c}$$

For $i=1,\ j\geqslant 4$

$$\omega_{1j}^1=0,\qquad \omega_{j1}^k=-\frac{(c_1+c_2)e_1H}{c_2H}\delta_{jk},\quad k\geqslant 4, \tag{4.8a}$$

$$\big[\varphi-(c_1+c_2)H\big]\omega_{1j}^2=(\varphi-c_1H)\omega_{j1}^2, \tag{4.8b}$$

$$(\varphi+c_1H)\omega_{1j}^3=(\varphi+c_1H-c_2H)\omega_{j1}^3. \tag{4.8c}$$

For $i=2,\ j=3$

$$(\varphi+c_1H-c_2H)\omega_{23}^1=(c_1H-\varphi)\omega_{32}^1, \tag{4.9a}$$

$$\omega_{23}^2=\frac{e_3\varphi}{c_2H-2\varphi},\qquad \omega_{32}^3=\frac{e_2\varphi}{c_2H-2\varphi}, \tag{4.9b}$$

$$(\varphi+c_1H)\omega_{23}^k=\big[(c_1+c_2)H-\varphi\big]\omega_{32}^k,\quad k\geqslant 4. \tag{4.9c}$$

For $i=2,\ j\geqslant 4$

$$c_2H\omega_{2j}^1=(\varphi-c_1H)\omega_{j2}^1, \tag{4.10a}$$

$$\omega_{2j}^2=\frac{e_j\varphi}{(c_1+c_2)H-\varphi},\qquad \omega_{j2}^k=0,\quad k\geqslant 4, \tag{4.10b}$$

$$(\varphi+c_1H)\omega_{2j}^3=(2\varphi-c_2H)\omega_{j2}^3. \tag{4.10c}$$

For $i=3,\ j\geqslant 4$

$$-c_2H\omega_{3j}^1=(\varphi+c_1H-c_2H)\omega_{j3}^1, \tag{4.11a}$$

$$\big[\varphi-(c_1+c_2)H\big]\omega_{3j}^2=(2\varphi-c_2H)\omega_{j3}^2, \tag{4.11b}$$

$$\omega_{3j}^3=-\frac{e_j\varphi}{c_1H+\varphi},\qquad \omega_{j3}^k=0,\quad k\geqslant 4. \tag{4.11c}$$

Further on we write

$$0=[e_2,e_3](H)=\sum_{k=1}^n\big(\omega_{23}^k-\omega_{32}^k\big)e_kH=\big(\omega_{23}^1-\omega_{32}^1\big)e_1H$$

and using (4.9a) we get

$$\omega_{23}^1=\omega_{32}^1=0. \tag{4.12}$$

In the same way we prove

$$\omega_{2j}^1=\omega_{j2}^1=0,\qquad \omega_{3j}^1=\omega_{j3}^1=0,\quad j\geqslant 4. \tag{4.13}$$

We also have $e_i\langle e_k,e_l\rangle=0$ which implies $\omega_{ik}^l+\omega_{il}^k=0$, for all $i,k,l=1,\dots,n$. Thus we successively obtain:

- $l=1,\ k=2$ using (4.6a) and (4.12)

$$\omega_{11}^2=0,\qquad \omega_{22}^1=\frac{e_1\varphi}{\varphi-c_1H},\qquad \omega_{31}^2=0,\qquad \omega_{j1}^2+\omega_{j2}^1=0,\quad j\geqslant 4; \tag{4.14}$$

- $l=1,\ k=3$ using (4.7a) and (4.12)

$$\omega_{11}^3=0,\qquad \omega_{33}^1=\frac{e_1\varphi-c_2e_1H}{\varphi+c_1H-c_2H},$$

$$\omega_{21}^3=0,\qquad \omega_{j1}^3+\omega_{j3}^1=0,\quad j\geqslant 4; \tag{4.15}$$

- $l=2,\ k=3$ using (4.9b), (4.6b) and (4.7b)

$$\omega_{22}^3=\frac{e_3\varphi}{2\varphi-c_2H},\qquad \omega_{33}^2=\frac{e_2\varphi}{2\varphi-c_2H},$$

$$\omega_{12}^3=\omega_{13}^2=0,\qquad \omega_{j2}^3+\omega_{j3}^2=0,\quad j\geqslant 4; \tag{4.16}$$

- $l=1,\ k\geqslant 4$ using (4.8a) and (4.13)

$$\omega_{11}^k=0,\qquad \omega_{21}^k=0,\qquad \omega_{31}^k=0,\qquad \omega_{j1}^k+\omega_{jk}^1=0,\quad j\geqslant 4, \tag{4.17}$$

  which combining with (4.6c), (4.7c) and (4.8a) yield also

$$\omega_{12}^k=0,\qquad \omega_{13}^k=0,\qquad \omega_{jk}^1=\frac{(c_1+c_2)e_1H}{c_2H}\delta_{jk},\quad j\geqslant 4; \tag{4.18}$$

- $l=2,\ k\geqslant 4$ using (4.10b) and (4.18)

$$\omega_{1k}^2=0,\qquad \omega_{22}^k=\frac{e_k\varphi}{\varphi-(c_1+c_2)H},\qquad \omega_{jk}^2=0,\quad j\geqslant 4, \tag{4.19a}$$

$$\omega_{32}^k+\omega_{3k}^2=0, \tag{4.19b}$$

  which combining with (4.8b) yield

$$\omega_{k1}^2=0; \tag{4.20}$$

- $l=3,\ k\geqslant 4$ using (4.11c) and (4.18)

$$\omega_{1k}^3=0,\qquad \omega_{33}^k=\frac{e_k\varphi}{\varphi+c_1H},\qquad \omega_{jk}^3=0,\quad j\geqslant 4, \tag{4.21a}$$

$$\omega_{23}^k+\omega_{2k}^3=0, \tag{4.21b}$$

  which combined with (4.8c) yield

$$\omega_{k1}^3=0; \tag{4.22}$$

- $k=l$

$$\omega_{ik}^k = 0, \quad i,k = 1,\dots,n. \tag{4.23}$$

By using (4.9c), (4.10c), (4.16), (4.19b) and (4.21b) we get

$$\begin{aligned}
&\omega_{23}^k = -\omega_{2k}^3 = \frac{y_k}{\varphi + c_1 H},\\
&\omega_{k3}^2 = -\omega_{k2}^3 = \frac{y_k}{2\varphi - c_2 H},\\
&\omega_{3k}^2 = -\omega_{32}^k = \frac{y_k}{\varphi - (c_1+c_2)H},
\end{aligned} \tag{4.24}$$

with $y_k = (\varphi + c_1 H)\omega_{23}^k$.

At this point we may write down the expression of the Levi-Civita connection on M. We have

$$\nabla_{e_1} e_1 = \nabla_{e_1} e_2 = \nabla_{e_1} e_3 = 0, \qquad \nabla_{e_1} e_j = \sum_{k \geqslant 4} \omega_{1j}^k e_k, \tag{4.25a}$$

$$\nabla_{e_2} e_1 = -A e_2, \qquad \nabla_{e_2} e_2 = A e_1 + B e_3 + \sum_{k \geqslant 4} P_k e_k, \tag{4.25b}$$

$$\nabla_{e_2} e_3 = -B e_2 + \sum_{k \geqslant 4} S_k e_k, \qquad \nabla_{e_2} e_j = -P_j e_2 - S_j e_3 + \sum_{k \geqslant 4} \omega_{2j}^k e_k,$$

$$\nabla_{e_3} e_1 = F e_3, \qquad \nabla_{e_3} e_3 = -F e_1 + T e_2 + \sum_{k \geqslant 4} Q_k e_k, \tag{4.25c}$$

$$\nabla_{e_3} e_2 = -T e_3 - \sum_{k \geqslant 4} U_k e_k, \qquad \nabla_{e_3} e_j = U_j e_2 - Q_j e_3 + \sum_{k \geqslant 4} \omega_{3j}^k e_k,$$

$$\begin{aligned}
&\nabla_{e_j} e_1 = -L e_j, \qquad \nabla_{e_j} e_2 = -V_j e_3, \qquad \nabla_{e_j} e_3 = V_j e_2,\\
&\nabla_{e_j} e_l = L\delta_{jl} e_1 + \sum_{k \geqslant 4} \omega_{jl}^k e_k, \quad j,l \geqslant 4,
\end{aligned} \tag{4.25d}$$

where, for the sake of simplicity, we put

$$\begin{aligned}
&A = \frac{e_1\varphi}{\varphi - c_1 H}, \qquad B = \frac{e_3\varphi}{2\varphi - c_2 H}, \qquad F = \frac{c_2 e_1 H - e_1\varphi}{\varphi + c_1 H - c_2 H},\\
&T = \frac{e_2\varphi}{2\varphi - c_2 H}, \qquad L = \frac{(c_1+c_2)e_1 H}{c_2 H},\\
&P_j = \frac{e_j\varphi}{\varphi - (c_1+c_2)H}, \qquad Q_j = \frac{e_j\varphi}{\varphi + c_1 H},\\
&S_j = \frac{y_j}{\varphi + c_1 H}, \qquad U_j = \frac{y_j}{\varphi - (c_1+c_2)H} \qquad V_j = \frac{y_j}{2\varphi - c_2 H}, \quad j \geqslant 4.
\end{aligned} \tag{4.26}$$

Now it is easy to compute the curvature tensor R and to apply Gauss' equation (EG) for different values of X, Y and Z. Identifying the coefficients with respect to the orthonormal basis $e_1,\dots,e_n$ we obtain:

- $X = e_1$, $Y = e_2$, $Z = e_1$

$$e_1 A - A^2 = c_1 H\varphi; \tag{4.27}$$

- $X = e_1$, $Y = e_2$, $Z = e_3$

$$e_1 B = AB, \qquad e_1 S_j = A S_j + \sum_{k \geqslant 4} \omega_{1j}^k S_k, \quad j \geqslant 4; \tag{4.28}$$

- $X = e_1$, $Y = e_3$, $Z = e_1$

$$e_1 F + F^2 = c_1 H(\varphi - c_2 H); \tag{4.29}$$

- $X = e_1$, $Y = e_3$, $Z = e_2$

$$e_1 T + FT = 0, \qquad e_1 U_j = -F U_j + \sum_{k \geqslant 4} \omega_{1j}^k U_k, \quad j \geqslant 4; \tag{4.30}$$

- $X = e_1$, $Y = e_j$, $Z = e_1$

$$e_1 L - L^2 = c_1(c_1 + c_2)H^2; \tag{4.31}$$

- $X = e_2$, $Y = e_3$, $Z = e_1$

$$e_3 A = B(A + F), \qquad e_2 F = T(A + F),$$
$$(F + L)S_j = (A - L)U_j, \quad j \geqslant 4; \tag{4.32}$$

- $X = e_2$, $Y = e_j$, $Z = e_1$

$$e_j A = P_j(A - L), \qquad e_2 L = 0,$$
$$(F + L)S_j = (A + F)V_j, \quad j \geqslant 4; \tag{4.33}$$

- $X = e_3$, $Y = e_j$, $Z = e_1$

$$e_j F = Q_j(F + L), \qquad e_3 L = 0; \tag{4.34}$$

- $X = e_j$, $Y = e_k$, $Z = e_1$, $j, k \geqslant 4$

$$e_j L = 0. \tag{4.35}$$

Let us develop the equation $\Delta H + H \operatorname{trace} S^2 = 0$ from (2.4). We have

$$\Delta H = -e_1 e_1 H + \big[A - F + (n-3)L\big] e_1 H,$$

and

$$\operatorname{trace} S^2 = cH^2 - 2c_2 H\varphi + 2\varphi^2,$$

with $c = c_1^2 + c_2^2 + (n-3)(c_1 + c_2)^2$. Hence

$$e_1 e_1 H - \big[A - F + (n-3)L\big] e_1 H - H\big(cH^2 - 2c_2 H\varphi + 2\varphi^2\big) = 0. \tag{4.36}$$

Moreover, by computing $[e_1, e_i](H)$, $i = 2, \dots, n$, we get

$$e_2 e_1 H = 0, \qquad e_3 e_1 H = 0, \qquad e_j e_1 H = 0, \quad j \geqslant 4, \tag{4.37}$$

which implies

$$e_2 e_1 e_1 H = 0, \qquad e_3 e_1 e_1 H = 0, \qquad e_j e_1 e_1 H = 0, \quad j \geqslant 4. \tag{4.38}$$

After applying e_j, $j \geqslant 4$, to (4.36), by using (4.38), one gets

$$\big[e_j A - e_j F + (n-3) e_j L\big] e_1 H + 2H(2\varphi - c_2 H) e_j \varphi = 0.$$

By using (4.33), (4.34) and (4.35) we derive that

$$\big[Q_j(F + L) - P_j(A - L)\big] e_1 H = 2H(2\varphi - c_2 H) e_j \varphi, \quad j \geqslant 4.$$

Replacing P_j and Q_j from (4.26) it follows

$$\left[\frac{F + L}{\varphi + c_1 H} - \frac{A - L}{\varphi - (c_1 + c_2)H}\right] e_1 H e_j \varphi = 2H(2\varphi - c_2 H) e_j \varphi.$$

We claim that

$$e_j \varphi = 0, \quad j \geqslant 4. \tag{4.39}$$

Indeed, if $e_j \varphi \neq 0$ for a certain $j \geqslant 4$, we could write

$$\left[\frac{F + L}{\varphi + c_1 H} - \frac{A - L}{\varphi - (c_1 + c_2)H}\right] e_1 H = 2H(2\varphi - c_2 H).$$

Applying e_j, and since

$$e_j\left(\frac{F + L}{\varphi + c_1 H}\right) = 0, \qquad e_j\left(\frac{A - L}{\varphi - (c_1 + c_2)H}\right) = 0,$$

we get $0 = 4H e_j \varphi$, which is a contradiction. Hence the claim is proved.

It follows that

$$P_j = 0, \qquad Q_j = 0, \quad j \geqslant 4. \tag{4.40}$$

Return to (4.36) and apply e_2 to it. We write

$$\big[e_2A - e_2F + (n-3)e_2L\big]e_1H + 2H(2\varphi - c_2H)e_2\varphi = 0.$$

By using (4.32) and (4.33) we get

$$e_2A = T(A+F) - \frac{2HT}{e_1H}(2\varphi - c_2H)^2. \tag{4.41}$$

In the same way, by applying e_3 to (4.36), and combining then with (4.32) and (4.34) we obtain

$$e_3F = B(A+F) + \frac{2HB}{e_1H}(2\varphi - c_2H)^2. \tag{4.42}$$

Compute now $[e_1,e_2](A)$. On one hand we have $[e_1,e_2](A) = Ae_2A$ and on the other hand $[e_1,e_2](A) = e_1e_2A - e_2e_1A$. Thus after using (4.27), (4.29) and (4.30) we get

$$\begin{aligned}
&-FT\left[A + F - \frac{2H}{e_1H}(2\varphi - c_2H)^2\right] + T\big[A^2 + c_1H\varphi - F^2 + c_1H(\varphi - c_2H)\big]\\
&\quad - 2Te_1\left(\frac{H}{e_1H}\right)(2\varphi - c_2H)^2 - \frac{4HT}{e_1H}(2\varphi - c_2H)e_1(2\varphi - c_2H)\\
&\quad - 3AT\left[A + F - \frac{2H}{e_1H}(2\varphi - c_2H)^2\right] - c_1HT(2\varphi - c_2H) = 0.
\end{aligned}$$

Next, we claim that $T = 0$. Otherwise, if $T \neq 0$, we divide the previous equality by T and after some computations we get

$$e_1\left(\frac{H}{e_1H}\right) = \frac{H}{e_1H}\left[F + 3A - 2\frac{e_1(2\varphi - c_2H)}{2\varphi - c_2H}\right] - \frac{(A+F)^2}{(2\varphi - c_2H)^2}.$$

Acting with e_2 we get

$$0 = \frac{H}{e_1H}\left[e_2F + 3e_2A - 2e_2\left(\frac{e_1(2\varphi - c_2H)}{2\varphi - c_2H}\right)\right] - \frac{2(A+F)}{2\varphi - c_2H}e_2\left(\frac{A+F}{2\varphi - c_2H}\right). \tag{4.43}$$

Straightforward computations yield

$$e_2\left(\frac{e_1(2\varphi - c_2H)}{2\varphi - c_2H}\right) = -2T(A+F) \tag{4.44}$$

and

$$e_2\left(\frac{A+F}{2\varphi - c_2H}\right) = -\frac{2HT}{e_1H}(2\varphi - c_2H). \tag{4.45}$$

After substituting (4.44) and (4.45) into (4.43) and taking into account (4.32) and (4.41) we obtain

$$2(A+F) = \frac{H}{e_1H}(2\varphi - c_2H)^2. \tag{4.46}$$

Acting again with e_2 and since T is supposed to be different from 0, we get

$$A + F = \frac{2H}{e_1H}(2\varphi - c_2H)^2. \tag{4.47}$$

In our hypothesis (4.46) and (4.47) run into a contradiction. So, we have proved our claim:

$$T = 0. \tag{4.48}$$

In a similar way, starting by computing $[e_1,e_3](F)$, we can prove that

$$B = 0. \tag{4.49}$$

Once we have $B = 0$, $T = 0$, $P_j = 0$ and $Q_j = 0$ for $j \geqslant 4$, let us continue with Gauss' equation (EG). We have

- $X = e_2$, $Y = e_3$, $Z = e_2$ (resp. $Z = e_3$)

$$\begin{gathered}
\varphi(\varphi - c_2H) + AF + 2\sum_{k\geqslant 4} S_kU_k = 0,\\
e_2U_j = \sum_{k\geqslant 4}\omega_{2j}^kU_k, \qquad e_3S_j = \sum_{k\geqslant 4}\omega_{3j}^kS_k,
\end{gathered} \tag{4.50}$$

- $X = e_2$, $Y = e_j$, $Z = e_2$ (resp. $Z = e_3$), $j \geqslant 4$

$$\begin{aligned}&\big[AL + (c_1 + c_2)H\varphi\big]\delta_{jk} = 2V_j S_k,\\&e_2 V_j = \sum_{k\geqslant 4}\omega_{2j}^k V_k, \qquad e_j S_k = \sum_{l\geqslant 4}\omega_{jk}^l S_l,\end{aligned} \tag{4.51}$$

- $X = e_3$, $Y = e_j$, $Z = e_2$ (resp. $Z = e_3$), $j \geqslant 4$

$$\begin{aligned}&\big[FL + (c_1 + c_2)(\varphi - c_2 H)H\big]\delta_{jk} = -2V_j U_k,\\&e_3 V_j = \sum_{k\geqslant 4}\omega_{3j}^k V_k, \qquad e_j U_k = \sum_{l\geqslant 4}\omega_{jk}^l U_l,\end{aligned} \tag{4.52}$$

- $X = e_1$, $Y = e_j$, $Z = e_2$ (resp. $Z = e_3$), $j \geqslant 4$

$$e_1 V_j = L V_j + \sum_{k\geqslant 4}\omega_{1j}^k V_k. \tag{4.53}$$

So, for $n \geqslant 5$ we can take $j \neq k$ and we get, from the first equation in (4.52), that $y_j = 0$ for all $j \geqslant 4$. Therefore, it follows that

$$\begin{aligned}&\varphi(\varphi - c_2 H) + AF = 0,\\&(c_1 + c_2)H\varphi + AL = 0,\\&(c_1 + c_2)(\varphi - c_2 H)H + FL = 0.\end{aligned} \tag{4.54}$$

Hence we obtain

$$\big[(c_1 + c_2)^2 H^2 + L^2\big]AF = 0.$$

In our hypothesis, this is a contradiction.

When $n = 4$, we have $c_1 = -2$ and $c_2 = 4$. Moreover, we have $S_4 = U_4 = 2V_4$. The third equations in (4.32) and (4.33) lead to either

(a) $y_4 = 0$ and hence $S_4 = U_4 = V_4 = 0$, or
(b) $A + F = 0$ when $y_4 \neq 0$.

In case (a), we may proceed as above (see (4.54)) and obtain $AF(L^2 + 4H^2) = 0$, which is a contradiction.

In case (b), we get $A = L$, namely

$$2He_1\varphi = (e_1 H)(\varphi + 2H). \tag{4.55}$$

The first equations in (4.50), (4.51) and (4.52) become

$$\varphi(\varphi - 4H) + AF + 2S^2 = 0, \tag{4.56a}$$
$$AL + 2\varphi H - S^2 = 0, \tag{4.56b}$$
$$FL + 2(\varphi - 4H)H + S^2 = 0, \tag{4.56c}$$

where we put S for S_4.

Adding (4.56a) with (4.56b) and subtracting (4.56c) give

$$AF + \varphi(\varphi - 4H) + L(A - F) + 8H^2 = 0. \tag{4.57}$$

By replacing the expression of A from (4.26) in (4.27) we find

$$e_1 e_1 \varphi = 2A^2(\varphi + 2H) - 2\varphi H(\varphi + 2H) + 2Ae_1 H. \tag{4.58}$$

Similarly, by replacing F from (4.26) in (4.29) we get

$$4e_1 e_1 H - e_1 e_1 \varphi = -2F^2(\varphi - 6H) - 2H(\varphi - 4H)(\varphi - 6H) - 2Fe_1 H. \tag{4.59}$$

Adding (4.58) and (4.59) yields

$$e_1 e_1 H = -H\big(\varphi^2 - 4\varphi H + 12H^2\big) + \frac{1}{2}(A - F)e_1 H + \frac{1}{2}A^2(\varphi + 2H) - \frac{1}{2}F^2(\varphi - 6H). \tag{4.60}$$

We may compute using also that $F+L=0$:

$$A^2(\varphi+2H)-F^2(\varphi-6H)=8AFH+4(A-F)e_1H. \tag{4.61}$$

On the other hand, from (4.57) we find

$$AFH=-\varphi H(\varphi-4H)-\frac{1}{2}(A-F)e_1H-8H^3. \tag{4.62}$$

By combining (4.61) and (4.62) and replacing in (4.60) we derive that

$$e_1e_1H=\frac{1}{2}(A-F)e_1H-H\bigl(5\varphi^2-20\varphi H+44H^2\bigr). \tag{4.63}$$

Now, by considering (4.36) for $n=4$, we find

$$e_1e_1H=(A-F)e_1H+\frac{(e_1H)^2}{2H}+2H\bigl(\varphi^2-4\varphi H-12H^2\bigr). \tag{4.64}$$

Then (4.31) yields

$$e_1e_1H-\frac{3(e_1H)^2}{2H}=-8H^3. \tag{4.65}$$

From (4.63)–(4.65) we get

$$e_1e_1H=-H\bigl(9\varphi^2-36\varphi H+86H^2\bigr), \tag{4.66}$$

$$(A-F)e_1H=-4H\bigl(2\varphi^2-8\varphi H+21H^2\bigr). \tag{4.67}$$

By acting with e_1 on both sides of (4.67), and using (4.27) and (4.29) we have

$$\bigl(A^2+F^2\bigr)e_1H+(A-F)e_1e_1H=-16H(e_1\varphi)(\varphi-2H)-4(e_1H)\bigl(2\varphi^2-16\varphi H+61H^2\bigr).$$

Using now (4.55), we obtain

$$\bigl(A^2+F^2\bigr)e_1H+(A-F)e_1e_1H=-4(e_1H)\bigl(4\varphi^2-16\varphi H+53H^2\bigr). \tag{4.68}$$

On the other hand, from (4.57) and combining with (4.67) we get

$$AF=3\varphi^2-12\varphi H+34H^2. \tag{4.69}$$

By subtracting $2AFe_1H$ from both sides of (4.68) and using (4.69), we find

$$(A-F)^2e_1H+(A-F)e_1e_1H=-2(e_1H)\bigl(11\varphi^2-44\varphi H+140H^2\bigr).$$

At this point use (4.66) and (4.67), we compute the following expression

$$17(A-F)H\bigl(\varphi^2-4\varphi H+10H^2\bigr)=2(e_1H)\bigl(11\varphi^2-44\varphi H+140H^2\bigr). \tag{4.70}$$

If we multiply (4.70) by e_1H and then by using (4.65), (4.66) and (4.67), we have

$$\alpha^2-29\alpha+120=0$$

with the solution $\alpha_1=5$ and $\alpha_2=24$, where we put

$$\alpha=\frac{\varphi^2-4\varphi H+10H^2}{H^2}.$$

By taking the derivative with respect to e_1 and using (4.55) we get

$$e_1\alpha=-\frac{e_1H}{H}(\alpha-6). \tag{4.71}$$

Now, by applying e_1 to (4.70) and using (4.27), (4.29), $A+F=0$, (4.66) and (4.71), we derive that

$$\begin{aligned}&-17\alpha H^3(3\alpha+8)+51(A-F)e_1H\\&\quad=-2H^3(9\alpha-4)(11\alpha+30)-11(\alpha-6)\frac{2(e_1H)^2}{H}.\end{aligned}$$

Finally, by applying (4.65), (4.66), and (4.67), we get $9\alpha^2-83\alpha+426=0$. Since neither α_1 nor α_2 are solutions of this equation, we obtain a contradiction.

Finally, let us study the remained particular situations as well as the case (i) of the lemma. For all of them, the shape operator S can be written in the form:

$$S=\begin{pmatrix} c_1H & 0 & 0 & 0 & \dots & 0\\ 0 & c_2H & 0 & 0 & \dots & 0\\ 0 & 0 & c_3H & 0 & \dots & 0\\ 0 & 0 & 0 & (c_1+c_2+c_3)H & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \dots & (c_1+c_2+c_3)H \end{pmatrix} \tag{4.72}$$

where c_1, c_2 and c_3 are real constants.

We have to analyze the following five situations: (i) $c_1=c_2$; (ii) $c_1=-c_2$; (iii) $c_1=c_3$; (iv) $c_1=-c_3$; (v) $c_2=c_3$.

Notice that (iii) is similar to (i); and (iv) is similar to (ii). Thus they will be omitted. Now, we consider the remaining cases.

Case (i): $c_1=c_2=-\frac{n}{2}$, $c_3=\frac{n(n-1)}{n-2}$. In this case, by writing the equation of Codazzi for $X=e_1$ and $Y=e_2$ we get as coefficient of e_2 that $c_2e_1H=0$, which is a contradiction since e_1H is nonzero.

Case (ii): $c_1=-\frac{n}{2}$, $c_2=\frac{n}{2}$, $c_3=\frac{n}{n-2}$. This situation is very similar to that we had for the $\delta(2)$-ideals. After we obtain equations analogue to (3.25), (3.26), and (3.27), the contradiction follows immediately.

Case (v): $c_1=-\frac{n}{2}$, $c_2=c_3=\frac{n^2}{4(n-2)}$. We may use the same strategy as in the general case. By applying in a convenient way the equations of Gauss and Codazzi, we have

$$e_1\left(\frac{e_1H}{H}\right)+\frac{c_2}{c_1-c_2}\left(\frac{e_1H}{H}\right)^2+c_1(c_1-c_2)H^2=0.$$

After doing the computations, we obtain

$$e_1e_1H-\frac{4(n-1)}{3n-4}\frac{(e_1H)^2}{H}+\frac{n^2(3n-4)}{8(n-2)}H^3=0. \tag{4.73}$$

Moreover, we may compute

$$\Delta H=-e_1e_1H+\frac{2n}{3n-4}\frac{(e_1H)^2}{H}.$$

Now, using (2.4), we find

$$-e_1e_1H+\frac{2n}{3n-4}\frac{(e_1H)^2}{H}+\frac{n^2(3n^2-16)}{8(n-2)^2}H^3=0. \tag{4.74}$$

Combining (4.73) and (4.74) yields again a contradiction.

Therefore, we have proved that the assumption "H is non-constant" implies that $H=0$. This leads to a contradiction. Consequently, we have proved the following.

Theorem 4.3. *Every $\delta(3)$-ideal biharmonic hypersurface of $\mathbb{E}^{n+1}$ with $n\geqslant 4$ is minimal.*

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