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Stability of regime-switching diffusions[☆]

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Abstract

This work is devoted to stability of regime-switching diffusion processes. After presenting the formulation of regime-switching diffusions, the notion of stability is recalled, and necessary conditions for p-stability are obtained. Then main results on stability and instability for systems arising in approximation are presented. Easily verifiable conditions are established. An example is examined as a demonstration. A remark on linear systems is also provided.

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1. Introduction

Owing to the increasing demands on regime-switching diffusions from emerging applications in financial engineering and wireless communications, much attention has been drawn to switching diffusion processes. A salient feature of such systems is that the systems include both continuous dynamics and discrete events. For example, one of the early efforts of using such hybrid models for financial applications can be traced back to [1], in which both the appreciation rate and the volatility of a stock depend on a continuous-time Markov chain. The introduction of such models makes it possible to describe stochastic volatility in a relatively simple manner. To illustrate, in the simplest case, a stock market may be considered to have two "modes" or

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"regimes", up and down, resulting from the state of the underlying economy, the general mood of investors in the market, and so on. The rationale is that in the different modes or regimes, the volatility and return rates are very different. Another example is from a wireless communication network. Consider the performance analysis of an adaptive linear multiuser detector in a cellular direct-sequence code-division multiple-access wireless network with changing user activity due to an admission or access controller at the base station. Under certain conditions, an optimization problem for the aforementioned application leads to certain optimization problems, which in turn, lead to switching diffusion limits; see [15].

Motivated by the arising applications, we study asymptotic properties of switching diffusion systems. The model under consideration is a two-component Markov process, a continuous component and a discrete-event component. A problem of great interest is concerned with stability of such systems. Continuing on our effort of studying positive recurrence and ergodicity of switching diffusion processes in [16], this paper focuses on stability of such systems. For some of the recent progress, we refer the reader to [4,9,11,14] and references therein. Consider the following important problem. If a linear system is stable, what can we say about the associated nonlinear systems? This paper provides a systematic approach for treating such problems. We develop new methods for solving these problems using Lyapunov function methods.

The rest of the paper is arranged as follows. Section 2 begins with the formulation of the regime-switching diffusions together with an auxiliary result, which is used in our asymptotic analysis. Section 3 recalls the notion of stability, and presents results for p-stability and exponential p-stability. Easily verifiable conditions for stability and instability of the systems are provided in Section 4. To demonstrate our results, we provide an example. Further remarks are made in Section 5 to conclude the paper.

2. Formulation and auxiliary results

Throughout the paper, we use z' to denote the transpose of $z \in \mathbb{R}^{\ell_1 \times \ell_2}$ with $\ell_i \geq 1$, whereas $\mathbb{R}^{\ell \times 1}$ is simply written as $\mathbb{R}^{\ell}$; $\mathbb{1} = (1, 1, \ldots, 1)' \in \mathbb{R}^m$ is a column vector with all entries being 1; the Euclidean norm for a row or column vector x is denoted by $|x|$. As usual, I denotes the identity matrix. For a matrix A, its trace norm is denoted by $|A| = \sqrt{\operatorname{tr}(A'A)}$. If B is a set, its indicator function is denoted by $I_B(\cdot)$.

Suppose that $(X(t), \gamma(t))$ is a two-component Markov process such that $X(\cdot)$ is a continuous component taking values in $\mathbb{R}^n$ and $\gamma(\cdot)$ is a jump process taking values in a finite set $\mathcal{M} = \{1, 2, \ldots, m\}$. The process $(X(t), \gamma(t))$ has a generator $\mathcal{L}$ given as follows. For each $i \in \mathcal{M}$, and for any twice continuously differentiable function $g(\cdot, i)$,

$$\begin{aligned}\mathcal{L}g(x,\cdot)(i) &= \frac{1}{2}\sum_{j,k=1}^{n} a_{jk}(x,i)\frac{\partial^2 g(x,i)}{\partial x_j \partial x_k} + \sum_{j=1}^{n} b_j(x,i)\frac{\partial g(x,i)}{\partial x_j} + Q(x)g(x,\cdot)(i)\\ &= \frac{1}{2}\operatorname{tr}(a(x,i)\nabla^2 g(x,i)) + b'(x,i)\nabla g(x,i) + Q(x)g(x,\cdot)(i), \quad i \in \mathcal{M}, \quad (2.1)\end{aligned}$$

where $x \in \mathbb{R}^n$, and $Q(x) = (q_{ij}(x))$ is an $m \times m$ matrix depending on x satisfying $q_{ij}(x) \geq 0$ for $i \neq j$ and $\sum_{j \in \mathcal{M}} q_{ij}(x) = 0$,

$$Q(x)g(x,\cdot)(i) = \sum_{j\in\mathcal{M}} q_{ij}(x)g(x,j) = \sum_{j\neq i,\, j\in\mathcal{M}} q_{ij}(x)(g(x,j) - g(x,i)), \quad i \in \mathcal{M},$$

and $\nabla g(\cdot, i)$ and $\nabla^2 g(\cdot, i)$ denote the gradient and Hessian of $g(\cdot, i)$, respectively.

Remark 2.1. In what follows, we often write $\mathcal{L}g(x,\cdot)(i)$ as $\mathcal{L}g(x,i)$ whenever it is more convenient.

The process $(X(t),\gamma(t))$ can be described by

$$\mathrm{d}X(t) = b(X(t),\gamma(t))\mathrm{d}t + \sigma(X(t),\gamma(t))\mathrm{d}w(t), \quad X(0) = x,\ \gamma(0) = \gamma, \tag{2.2}$$

and

$$\mathbf{P}\{\gamma(t+\Delta t) = j|\gamma(t) = i, (X(s),\gamma(s)), s \le t\} = q_{ij}(X(t))\Delta t + o(\Delta t), \quad i \ne j, \tag{2.3}$$

where $w(t)$ is a d-dimensional standard Brownian motion, $b(\cdot,\cdot) : \mathbb{R}^n \times \mathcal{M} \mapsto \mathbb{R}^n$, and $\sigma(\cdot,\cdot) : \mathbb{R}^n \times \mathcal{M} \mapsto \mathbb{R}^{n\times d}$ satisfies $\sigma(x,i)\sigma'(x,i) = a(x,i)$.

To proceed, we need conditions regarding the smoothness and growth of the functions involved, and the condition that 0 is the only equilibrium point of the random dynamic system. Hence we assume that the following conditions are valid throughout the paper.

(A1) The matrix-valued function $Q(\cdot)$ is bounded and continuous.
(A2) $b(0,\gamma) = 0$, and $\sigma(0,\gamma) = 0$ for each $\gamma \in \mathcal{M}$. Moreover, assume that $\sigma(x,\gamma)$ vanishes only at $x = 0$ for each $\gamma \in \mathcal{M}$.
(A3) There exists a constant $K_0 > 0$ such that for each $\gamma \in \mathcal{M}$,

$$|b(x,\gamma) - b(y,\gamma)| + |\sigma(x,\gamma) - \sigma(y,\gamma)| \le K_0|x-y|, \quad \text{for any } x, y \in \mathbb{R}^n. \tag{2.4}$$

It is well known that under these conditions, system (2.2) and (2.3) has a unique solution; see [13] for details. In what follows, a process starting from (x,γ) will be denoted by $(X^{x,\gamma}(t),\gamma^{x,\gamma}(t))$ if the emphasis on initial condition is needed. If the context is clear, we simply write $(X(t),\gamma(t))$.

To study stability of the equilibrium point $x = 0$, it will be useful to recall the following "nonzero" property ([9, Lemma 2.1]), which asserts that almost all the sample paths of any solution of (2.2) and (2.3) starting from a nonzero state will never reach the origin:

$$\mathbf{P}\{X^{x,\gamma}(t) \ne 0, t \ge 0\} = 1, \quad \text{for any } x \ne 0,\ \gamma \in \mathcal{M}. \tag{2.5}$$

In view of (2.5), we can work with functions $V(\cdot,i), i \in \mathcal{M}$ which are twice continuously differentiable in the deleted neighborhood of 0 in the sequel.

To proceed, we present an auxiliary result, namely, the solvability of a system of deterministic equations.

(A0) Suppose that Q, an $m \times m$ constant matrix, is the generator of a continuous-time Markov chain $r(t)$ and that Q is irreducible.

Remark 2.2. In the above, by the irreducibility, we mean that the system of equations

$$\nu Q = 0, \qquad \nu\mathbb{1} = 1$$

has a unique solution such that $\nu = (\nu_1,\ldots,\nu_m)$ satisfies $\nu_i > 0$. It should be mentioned that many applications in science and engineering require using invariant distributions of diffusion processes. When dealing with diffusions having fast and slow components, suitable limiting behaviors of the slow component were found by using an invariant measure of the fast component in [12]. In this paper, we treat regime-switching diffusions, where in addition to the continuous component, there are discrete events. Our interest is in the large-time behavior.

In view of (A0), since Q is irreducible, the rank of Q is $m-1$. Denote by $\mathcal{R}(Q)$ and $\mathcal{N}(Q)$ the right range and the null space of Q, respectively. It follows that $\mathcal{N}(Q)$ is one dimensional, spanned by $\mathbb{1}$ (i.e., $\mathcal{N}(Q) = \text{span}\{\mathbb{1}\}$). As a consequence, the Markov chain $r(t)$ is ergodic; see, for example, [3]. In what follows, denote the associated stationary distribution by

$$\nu = (\nu_1, \nu_2, \dots, \nu_m) \in \mathbb{R}^{1\times m}. \tag{2.6}$$

We are interested in solving a linear system of equations

$$Qc = \eta, \tag{2.7}$$

where c and $\eta \in \mathbb{R}^m$. Note that (2.7) is a Poisson equation. The assertion of the following lemma seems to be well known. The proof is thus omitted.

Lemma 2.3. *Under* (A0)*, Eq.* (2.7) *has a solution if and only if* $\nu\eta = 0$*. Moreover, suppose that* c_1 *and* c_2 *are two solutions of* (2.7)*. Then* $c_1 - c_2 = \alpha_0\mathbb{1}$ *for some* $\alpha_0 \in \mathbb{R}$.

3. *p*-Stability

This section is concerned with stability of the equilibrium point $x = 0$ for system (2.2) and (2.3). We first present definitions of stability, p-stability, and exponential p-stability using the terminologies of [6]. Then general results in terms of Lyapunov function are provided.

Definition 3.1. The equilibrium point $x = 0$ of system (2.2) and (2.3) is said to be

(i) *stable in probability*, if for any $\varepsilon > 0$ and any $\gamma \in \mathcal{M}$,

$$\lim_{x\to 0} \mathbf{P}\{\sup_{t\ge 0} |X^{x,\gamma}(t)| > \varepsilon\} = 0;$$

and $x = 0$ is said to be *unstable in probability* if it is not stable in probability;

(ii) *asymptotically stable in probability*, if it is stable in probability and satisfies

$$\lim_{x\to 0} \mathbf{P}\{\lim_{t\to\infty} X^{x,\gamma}(t) = 0\} = 1, \quad \text{for any } \gamma \in \mathcal{M};$$

(iii) *p-stable (for $p > 0$)*, if

$$\lim_{\delta\to 0} \sup_{|x|\le\delta,\gamma\in\mathcal{M},t\ge 0} \mathbf{E}|X^{x,\gamma}(t)|^p = 0;$$

(iv) *asymptotically p-stable*, if it is p-stable and satisfies $\mathbf{E}|X^{x,\gamma}(t)|^p \to 0$ as $t \to \infty$ for any $\gamma \in \mathcal{M}$;

(v) *exponentially p-stable*, if for some positive constants K and k

$$\mathbf{E}|X^{x,\gamma}(t)|^p \le K|x|^p \exp(-kt), \quad \text{for any } \gamma \in \mathcal{M}.$$

Using almost the same arguments as those for [6, Theorems 5.3.1, 5.4.1, and 5.4.2], we can establish the following three lemmas. The statements are given and the detailed proofs are omitted.

Lemma 3.2. *Let* $D \subset \mathbb{R}^n$ *be an open neighborhood of* 0*. Suppose that for each* $i \in \mathcal{M}$*, there exists a nonnegative function* $V(\cdot, i) : D \mapsto \mathbb{R}$ *such that*

(i) $V(\cdot, i)$ *is continuous in* D *and vanishes only at* $x = 0$*;*

(ii) $V(\cdot, i)$ *is twice continuously differentiable in* $D - \{0\}$ *and satisfies*

$$\mathcal{L}V(x, i) \le 0, \quad \textit{for all } x \in D - \{0\}.$$

Then the equilibrium point $x = 0$ *is stable in probability.*

Introduce the notation

$$\tau_{\varepsilon,r}^{x,\gamma} := \inf\{t \ge 0 : |X^{x,\gamma}(t)| = \varepsilon \text{ or } |X^{x,\gamma}(t)| = r\}, \tag{3.1}$$

for any $0 < \varepsilon < r$ and any $(x, \gamma) \in \mathbb{R}^n \times \mathcal{M}$ with $\varepsilon < |x| < r$.

Lemma 3.3. *Assume the conditions of Lemma 3.2. If for any sufficiently small* $0 < \varepsilon < r$ *and any* $(x, \gamma) \in \mathbb{R}^n \times \mathcal{M}$ *with* $\varepsilon < |x| < r$, *we have*

$$\mathbf{P}\{\tau_{\varepsilon,r}^{x,\gamma} < \infty\} = 1, \tag{3.2}$$

then the equilibrium point $x = 0$ *is asymptotically stable in probability.*

Lemma 3.4. *Let* $D \subset \mathbb{R}^n$ *be an open neighborhood of* 0. *Assume that the conditions of Lemma 3.3 hold and that for each* $i \in \mathcal{M}$, *there exists a nonnegative function* $V(\cdot, i) : D \mapsto \mathbb{R}$ *such that* $V(\cdot, i)$ *is twice continuously differentiable in every deleted neighborhood of* 0, *and*

$$\mathcal{L}V(x, i) \le 0 \quad \textit{for all } x \in D - \{0\}; \tag{3.3}$$

$$\lim_{|x|\to 0} V(x, i) = \infty, \quad \textit{for each } i \in \mathcal{M}. \tag{3.4}$$

Then the equilibrium point $x = 0$ *is unstable in probability if (3.2) holds.*

Remark 3.5. Note that (3.2) is an essential assumption in Lemmas 3.3 and 3.4. Here we present two sufficient conditions for which (3.2) holds.

(i) Let $N \subset \mathbb{R}^n$ be an open neighborhood of 0. Assume that for each $i \in \mathcal{M}$, there exists a nonnegative function $V(\cdot, i) : N \mapsto \mathbb{R}$ such that $V(\cdot, i)$ is twice continuously differentiable in every deleted neighborhood of 0, and that for any sufficiently small $0 < \varepsilon < r$ there is a positive constant $\kappa = \kappa(\varepsilon)$ such that

$$\mathcal{L}V(x, i) \le -\kappa, \quad \text{for all } x \in N \text{with } \varepsilon < |x| < r. \tag{3.5}$$

Then (3.2) holds.

(ii) If for any sufficiently small $0 < \varepsilon < r$, there exist some $\iota = 1, 2, \ldots, n$ and some constant $\kappa = \kappa(\varepsilon) > 0$ such that

$$a_{\iota\iota}(x, i) \ge \kappa, \quad \text{for all } (x, i) \in (\{x : \varepsilon < |x| < r\}) \times \mathcal{M}, \tag{3.6}$$

then (3.2) holds.

Assertion (i) can be established using almost the same proof as that for [6, Theorem 3.7.1]. While (ii) follows by observing that if (3.6) is satisfied, then we can construct some Lyapunov function $V(\cdot, \cdot)$ satisfying (3.5); see also a similar argument in the proof of [6, Corollary 3.7.2]. We shall omit the details here for brevity.

Concerning the exponential p-stability of the equilibrium point $x = 0$ of system (2.2) and (2.3), sufficient conditions in terms of the existence of certain Lyapunov functions being homogeneous of degree p were obtained in [9]. In what follows, we first provide a lemma as a preparation of the subsequent study, and then we present a necessary condition for the exponential p-stability.

Lemma 3.6. *Assume that the coefficients b and σ have continuous derivatives with respect to the variable x up to the second order. Then for a fixed $T > 0$ and for any $p > 0$, the function $u(t,x,i) : [0,T] \times \mathbb{R}^n \times \mathcal{M} \mapsto \mathbb{R}$ defined by*

$$u(t,x,i) := \mathbf{E}\left|X^{x,i}(t)\right|^p \tag{3.7}$$

is continuously differentiable with respect to the variable t, twice continuously differentiable with respect to the variable x except possibly at $x = 0$, and satisfies the system of parabolic equations together with the initial data

$$\begin{cases} \dfrac{\partial u(t,x,i)}{\partial t} = \mathcal{L}u(t,x,i), \quad (t,x,i) \in (0,T) \times \mathbb{R}^n \times \mathcal{M}, \\ u(0,x,i) = |x|^p, \quad (x,i) \in \mathbb{R}^n \times \mathcal{M}, \end{cases} \tag{3.8}$$

where $\mathcal{L}$ is defined in (2.1).

Remark 3.7. Note that (3.8) is a system of coupled parabolic equations. For convenience, we have adopted the use of $\mathcal{L}u(t,x,i)$, which should be interpreted as $\mathcal{L}u(t,x,\cdot)(i)$ as mentioned in Remark 2.1. Likewise, a similar remark applies to $\mathcal{L}u^\delta(t,x,i)$ in (3.9).

Proof of Lemma 3.6. The proof is divided into three steps. In the first step, we quote a well-known result to show that a unique solution exists for a "regularized" parabolic system in a subdomain of interest. In the second step, we use Dynkin's formula to obtain a stochastic representation of the solution of the regularized system. In the third step, using interior estimates, we show that the sequence of solutions of the regularized system converges to the desired limit.

Step 1: Let $\delta > 0$ be given and $N_\delta = \{x : |x| \le \delta\}$ with boundary $\partial N_\delta = \{x : |x| = \delta\}$. Consider the parabolic system together with initial and boundary data

$$\begin{cases} \dfrac{\partial u^\delta(t,x,i)}{\partial t} = \mathcal{L}u^\delta(t,x,i), \quad (t,x,i) \in (0,T) \times (\mathbb{R}^n - N_\delta) \times \mathcal{M}, \\ u^\delta(0,x,i) = |x|^p - \delta^p, \quad (x,i) \in \mathbb{R}^n \times \mathcal{M}, \\ u^\delta(t,x,i) = 0, \quad (t,x,i) \in [0,T] \times \partial N_\delta \times \mathcal{M}. \end{cases} \tag{3.9}$$

Then by [8, Theorem 7.1, p. 596], the system (3.9) has a unique solution that is continuously differentiable with respect to t and twice continuously differentiable with respect to x.

Step 2: For each pair $(x,i) \in \mathbb{R}^n \times \mathcal{M}$ with $x \ne 0$ and $\delta < |x|$, define

$$\tau_\delta := \inf\left\{0 \le t \le T : \left|X^{x,i}(t)\right| \le \delta\right\}.$$

Define also $f(x,i) = |x|^p I_{\{\delta \le |x|\}}$ for all $(x,i) \in \mathbb{R}^n \times \mathcal{M}$. Note that $f(X^{x,i}(t), \gamma^{x,i}(t)) = \left|X^{x,i}(t)\right|^p I_{\{t \le \tau_\delta\}}$. Then it follows from Dynkin's formula that

$$\begin{aligned} \mathbf{E}\left|X^{x,i}(t) I_{\{t\le\tau_\delta\}}\right|^p - |x|^p &= \mathbf{E}\int_0^t \mathcal{L}f(X^{x,i}(s), \gamma^{x,i}(s))\mathrm{d}s \\ &= \mathbf{E}\int_0^t \left[\mathcal{L}\left|X^{x,i}(s)\right|^p\right] I_{\{s\le\tau_\delta\}}\mathrm{d}s. \end{aligned} \tag{3.10}$$

Now, it is easily seen that

$$u^\delta(t,x,i) = \mathbf{E}\left|X^{x,i}(t) I_{\{t\le\tau_\delta\}}\right|^p - \delta^p \tag{3.11}$$

satisfies the system of equations in (3.9). In addition, it satisfies both the boundary conditions and initial condition in (3.9).

Step 3: Note that for each $i \in \mathcal{M}$, we have constructed a monotone sequence of bounded and smooth functions $\{u^\delta(\cdot,\cdot,i)\}$ as $\delta \downarrow 0$ in Step 2. The monotonicity can be seen as follows. Noting that for fixed (t,x,i) with $|x| > \delta_1 > \delta_2 > 0$, we have $\tau_{\delta_1} \le \tau_{\delta_2}$ and hence $\{t \le \tau_{\delta_1}\} \subset \{t \le \tau_{\delta_2}\}$. It thus follows from the expression for $u^\delta(t,x,i)$ given in (3.11) that $u^{\delta_1}(t,x,i) \le u^{\delta_2}(t,x,i)$. Therefore $\{u^\delta(\cdot,\cdot,i)\}$ is a monotone sequence as $\delta \downarrow 0$. The interior estimates in [7, Theorem 7.13] enable us to take the limit as $\delta \downarrow 0$ and to conclude that the limit function is continuously differentiable with respect to t and twice continuously differentiable with respect to x for $x \ne 0$. Thus it suffices to obtain the limit representation.

Direct computation leads to

$$\mathcal{L}|x|^p = \left\langle p|x|^{p-2}x, b(x,i)\right\rangle + \frac{1}{2}\mathrm{tr}\left(a(x,i)\left[p|x|^{p-2}I + p(p-2)|x|^{p-4}xx'\right]\right).$$

Then it follows from conditions (A2) and (A3) that

$$\left|\mathcal{L}|x|^p\right| \le p\,|x|^{p-1}\,|b(x,i)| + \frac{1}{2}\,|a(x,i)|\left(p|x|^{p-2} + p|p-2||x|^{p-2}\right) \le K|x|^p,$$

where K is a positive constant depending only on p and the Lipschitz constant K_0 in (2.4). In what follows, K will be used as a generic constant independent of δ, x, and t, whose exact value may change in different appearances. Consequently, we have

$$\begin{aligned}\mathbf{E}\left|\int_0^t \mathcal{L}\left|X^{x,i}(s)\right|^p I_{\{s\le\tau_\delta\}}\mathrm{d}s\right| &\le \mathbf{E}\int_0^t \left|\mathcal{L}\left|X^{x,i}(s)\right|^p\right| I_{\{s\le\tau_\delta\}}\mathrm{d}s\\ &\le K\mathbf{E}\int_0^t \left|X^{x,i}(s)\right|^p \mathrm{d}s \le KT.\end{aligned} \tag{3.12}$$

Note that in the last step, we have used the fact that $\sup_{0\le s\le T}\mathbf{E}\left|X^{x,i}(s)\right|^p \le K$. Note also that (2.5) implies that $\mathbf{P}(\tau_\delta \to \infty \text{ as } \delta \to 0) = 1$, and hence

$$t \wedge \tau_\delta \to t \quad \text{or} \quad I_{\{t\le\tau_\delta\}} \to 1 \quad \text{a.s. as } \delta \to 0. \tag{3.13}$$

Then it follows that

$$\int_0^t I_{\{s\le\tau_\delta\}}\mathcal{L}\left|X^{x,i}(s)\right|^p \mathrm{d}s \to \int_0^t \mathcal{L}\left|X^{x,i}(s)\right|^p \mathrm{d}s \quad \text{a.s. as } \delta \to 0. \tag{3.14}$$

By virtue of the Dominated Convergence Theorem, (3.12) and (3.14) imply that as $\delta \to 0$,

$$\mathbf{E}\int_0^t \mathcal{L}f(X^{x,i}(s), \gamma^{x,i}(s))\mathrm{d}s \to \mathbf{E}\int_0^t \mathcal{L}\left|X^{x,i}(s)\right|^p \mathrm{d}s. \tag{3.15}$$

On the other hand, (3.13) and the Dominated Convergence Theorem imply that

$$\mathbf{E}\left|X^{x,i}(t)\right|^p I_{\{t\le\tau_\delta\}} - \delta^p \to \mathbf{E}\left|X^{x,i}(t)\right|^p \quad \text{as } \delta \to 0. \tag{3.16}$$

Hence (3.10), (3.15) and (3.16) yield

$$\mathbf{E}\left|X^{x,i}(t)\right|^p - |x|^p = \mathbf{E}\int_0^t \mathcal{L}\left|X^{x,i}(s)\right|^p \mathrm{d}s. \tag{3.17}$$

Then it follows that $u(t,x,i)$ defined by (3.7) satisfies (3.8). This completes the proof of the lemma. □

Theorem 3.8. *Suppose that the equilibrium point $x = 0$ is exponentially p-stable. Assume moreover that the coefficients b and σ have continuous bounded derivatives with respect to the variable x up to the second order. Then for each $i \in \mathcal{M}$, there exists a function $V(\cdot, i) : \mathbb{R}^n \mapsto \mathbb{R}$ such that*

$$k_1|x|^p \le V(x,i) \le k_2|x|^p, \quad x \in N, \tag{3.18}$$

$$\mathcal{L}V(x,i) \le -k_3|x|^p \quad \text{for all } x \in N - \{0\}, \tag{3.19}$$

$$\left|\frac{\partial V}{\partial x_j}(x,i)\right| < k_4|x|^{p-1}, \qquad \left|\frac{\partial^2 V}{\partial x_j \partial x_k}(x,i)\right| < k_4|x|^{p-2},$$
$$\text{for all } 1 \le j,k \le n \text{ and } x \in N - \{0\}, \tag{3.20}$$

for some positive constants k_1, k_2, k_3, and k_4, where N is some open neighborhood of 0.

Proof. For each $i \in \mathcal{M}$, consider the function

$$V(x,i) = \int_0^T \mathbf{E}|X^{x,i}(u)|^p \mathrm{d}u,$$

where T is a positive constant such that $\mathbf{E}|X^{x,i}(T)|^p < \frac{1}{2}|x|^p$. Note that the existence of such T follows from the definition of exponential p-stability. Also it follows from Lemma 3.6 that the functions $V(x,i)$, $i \in \mathcal{M}$ are twice continuously differentiable with respect to x except possibly at $x = 0$. Then using an argument similar to that for [6, Theorem 5.7.2], we can verify that the function $V(\cdot, i)$ defined above satisfies all the conditions of the theorem. The details are omitted. □

We end this section with the following results on linear systems. More specifically, we assume that the evolution (2.2) is replaced by

$$\mathrm{d}X(t) = b(\gamma(t))X(t)\mathrm{d}t + \sum_{j=1}^{d} \sigma_j(\gamma(t))X(t)\mathrm{d}w_j(t), \tag{3.21}$$

where $b(i), \sigma_j(i)$ are $n \times n$ constant matrices and $w_j(t)$ are independent one-dimensional standard Brownian motions for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, d$. Then we have the following two theorems.

Theorem 3.9. *The equilibrium point $x = 0$ of system (3.21)–(2.3) is exponentially p-stable if and only if for each $i \in \mathcal{M}$, there is a function $V(\cdot, i) : \mathbb{R}^n \mapsto \mathbb{R}$ satisfying equations (3.18)–(3.20) for some constants $k_i > 0$, $i = 1, \ldots, 4$.*

Proof. The proof of sufficiency was contained in [9], while the necessity follows from Theorem 3.8 since the coefficients of (3.21) and (2.3) satisfy the conditions of Theorem 3.8. We omit the details here. □

Theorem 3.10. *Let $Q(x) \equiv Q$, a constant matrix. Assume that the Markov chain $\gamma(t)$ is independent of the Brownian motion $w(t) = (w_1(t), w_2(t), \ldots, w_d(t))'$ (or equivalently, $\gamma(0)$ is independent of the Brownian motion $w(\cdot)$). If the equilibrium point $x = 0$ of system given by (3.21) and (2.3) is stable in probability, then it is p-stable for sufficiently small $p > 0$.*

Proof. The proof follows from a crucial observation. In this case, since (3.21) is linear in $X(t)$, $X^{\lambda x,\gamma}(t) = \lambda X^{x,\gamma}(t)$. Using an argument similar to that for [6, Lemma 6.4.1], we can conclude the proof; a few details are omitted. □

4. Main result

4.1. Criteria for stability and instability

Assumption (A4) will be used in what follows.

(A4) For each $i \in \mathcal{M}$, there exist $b(i), \sigma_j(i) \in \mathbb{R}^{n\times n}$, $j = 1, 2, \ldots, d$, and a generator of a continuous-time Markov chain $\widehat{Q} = (\widehat{q}_{ij})$ such that as $x \to 0$,

$$\begin{aligned} &b(x, i) = b(i)x + o(|x|),\\ &\sigma(x, i) = (\sigma_1(i)x, \sigma_2(i)x, \ldots, \sigma_d(i)x) + o(|x|),\\ &Q(x) = \widehat{Q} + o(1). \end{aligned} \tag{4.1}$$

Moreover, $\widehat{Q}$ is irreducible and $\widehat{\gamma}(t)$ is a Markov chain with generator $\widehat{Q}$.

Remark 4.1. Note that condition (A4) is rather natural. It is equivalent to $Q(x)$ being continuous at $x = 0$, and $b(x, i)$ and $\sigma(x, i)$ being continuously differentiable at $x = 0$.

It follows from (A4) that $\widehat{\gamma}(t)$ is an ergodic Markov chain. Denote the stationary distribution of $\widehat{\gamma}(t)$ by $\pi = (\pi_1, \pi_2, \ldots, \pi_m) \in \mathbb{R}^{1\times m}$.

Remark 4.2. For any square matrix $A \in \mathbb{R}^{n\times n}$, A can be decomposed as the sum of a symmetric matrix A_1 and an antisymmetric matrix A_2. In fact, $A_1 = (A + A')/2$ and $A_2 = (A - A')/2$. Moreover, the quadratic form satisfies

$$x'Ax = x'\frac{A + A'}{2}x. \tag{4.2}$$

This observation will be used in what follows. To proceed, for any $A \in \mathbb{R}^{n\times n}$, we introduce the notation

$$\begin{aligned} &\Lambda_A = \text{ the maximum eigenvalue of the matrix } \frac{1}{2}(A + A'),\\ &\lambda_A = \text{ the minimum eigenvalue of the matrix } \frac{1}{2}(A + A'),\\ &\rho_A = \text{ the spectral radius of the matrix } \frac{1}{2}(A + A') = \max\{|\lambda_A|, |\Lambda_A|\}. \end{aligned} \tag{4.3}$$

We will take $A = b(i), \sigma_j(i)$, and $a_j(i) = \sigma_j(i)\sigma_j'(i)$ for $i \in \mathcal{M}$, $j = 1, 2, \ldots, d$ in what follows.

Theorem 4.3. *Assume condition* (A4). *Then the equilibrium point $x = 0$ of the system given by* (2.2) *and* (2.3) *is asymptotically stable in probability if*

$$\sum_{i=1}^{m} \pi_i \left(\Lambda_{b(i)} + \frac{1}{2}\sum_{j=1}^{d} \Lambda_{a_j(i)} \right) < 0, \tag{4.4}$$

and is unstable in probability if

$$\sum_{i=1}^{m} \pi_i \left(\lambda_{b(i)} + \frac{1}{2}\sum_{j=1}^{d} [\lambda_{a_j(i)} - 2(\rho_{\sigma_j(i)})^2] \right) > 0. \tag{4.5}$$

Proof. (a) We first prove that the equilibrium point $x = 0$ of system (2.2) and (2.3) is asymptotically stable in probability if (4.4) holds. For notational simplicity, define the column vector $\mu = (\mu_1, \mu_2, \ldots, \mu_m)' \in \mathbb{R}^m$ with

$$\mu_i = \Lambda_{b(i)} + \frac{1}{2}\sum_{j=1}^{d} \Lambda_{a_j(i)}.$$

Also let

$$\beta := -\pi\mu = -\sum_{i=1}^{m} \pi_i \left(\Lambda_{b(i)} + \frac{1}{2}\sum_{j=1}^{d} \Lambda_{a_j(i)}\right).$$

Note that $\beta > 0$ by (4.4). It follows from assumption (A4) and Lemma 2.3 that the equation

$$\widehat{Q}c = \mu + \beta\mathbb{1}$$

has a solution $c = (c_1, c_2, \ldots, c_m)' \in \mathbb{R}^m$. Thus we have

$$\mu_i - \sum_{j=1}^{m} \widehat{q}_{ij} c_j = -\beta, \quad i \in \mathcal{M}. \tag{4.6}$$

For each $i \in \mathcal{M}$, consider the Lyapunov function

$$V(x, i) = (1 - \alpha c_i)|x|^\alpha,$$

where $0 < \alpha < 1$ is sufficiently small so that $1 - \alpha c_i > 0$ for each $i \in \mathcal{M}$. It is readily seen that for each $i \in \mathcal{M}$, $V(\cdot, i)$ is continuous, nonnegative, and vanishes only at $x = 0$. Detailed calculations reveal that for $x \neq 0$, we have

$$\nabla V(x, i) = (1 - \alpha c_i)\alpha|x|^{\alpha-2}x,$$
$$\nabla^2 V(x, i) = (1 - \alpha c_i)\alpha\left[|x|^{\alpha-2}I + (\alpha - 2)|x|^{\alpha-4}xx'\right].$$

Meanwhile, it follows from (4.1) that

$$a(x, i) = \sigma(x, i)\sigma'(x, i) = \sum_{j=1}^{d} \sigma_j(i)xx'\sigma_j'(i) + o(|x|^2).$$

Note that for any matrix $A \in \mathbb{R}^{n\times n}$, we have

$$\mathrm{tr}(\sigma_j(i)xx'\sigma_j'(i)A) = x'\sigma_j'(i)A\sigma_j(i)x.$$

Therefore, we have that

$$\begin{aligned}
\mathcal{L}V(x, i) &= \frac{1}{2}\mathrm{tr}\left[\left(\sum_{j=1}^{d} \sigma_j(i)xx'\sigma_j'(i) + o(|x|^2)\right)\nabla^2 V(x, i)\right] \\
&\quad + (\nabla V(x, i))'(b(i)x + o(|x|)) - \sum_{j\neq i} q_{ij}(x)|x|^\alpha \alpha(c_j - c_i) \\
&= \frac{\alpha}{2}(1 - \alpha c_i)\sum_{j=1}^{d}\left(x'\sigma_j'(i)|x|^{\alpha-2}I\sigma_j(i)x + x'\sigma_j'(i)(\alpha - 2)|x|^{\alpha-4}xx'\sigma_j(i)x\right) \\
&\quad + (1 - \alpha c_i)\alpha|x|^{\alpha-2}x'b(i)x + o(|x|^\alpha) - \sum_{j\neq i} q_{ij}(x)|x|^\alpha \alpha(c_j - c_i)
\end{aligned}$$

$$= \alpha(1-\alpha c_i)|x|^\alpha \left\{ \frac{1}{2}\sum_{j=1}^{d} \left(\frac{x'\sigma_j'(i)\sigma_j(i)x}{|x|^2} + (\alpha-2)\frac{(x'\sigma_j'(i)x)^2}{|x|^4} \right) + \frac{x'b(i)x}{|x|^2} - \sum_{j\neq i} q_{ij}(x)\frac{c_j - c_i}{1-\alpha c_i} + o(1) \right\}. \tag{4.7}$$

By virtue of Remark 4.2 (in particular (4.2)), it follows that

$$\frac{x'b(i)x}{|x|^2} = \frac{x'(b'(i)+b(i))x}{2|x|^2} \leq \Lambda_{b(i)}. \tag{4.8}$$

Similarly we have

$$\frac{x'\sigma_j'(i)\sigma_j(i)x}{|x|^2} \leq \Lambda_{a_j(i)}. \tag{4.9}$$

Next, it follows from condition (A4) that when $|x|$ is sufficiently small,

$$\begin{aligned} \sum_{j\neq i} q_{ij}(x)\frac{c_j - c_i}{1-\alpha c_i} &= \sum_{j=1}^{m} q_{ij}(x)c_j + \sum_{j\neq i} q_{ij}(x)\frac{c_i(c_j-c_i)}{1-\alpha c_i}\alpha \\ &= \sum_{j=1}^{m} \widehat{q}_{ij}c_j + O(\alpha) + o(1), \end{aligned} \tag{4.10}$$

when $x \to 0$ and $\alpha \to 0$. Hence it follows from (4.7)–(4.10) that when $|x| < r$ with r and $0 < \alpha < 1$ sufficiently small, we have

$$\mathcal{L}V(x,i) \leq \alpha(1-\alpha c_i)|x|^\alpha \left\{ \Lambda_{b(i)} + \frac{1}{2}\sum_{j=1}^{d}\Lambda_{a_j(i)} - \sum_{j=1}^{m}\widehat{q}_{ij}c_j + o(1) + O(\alpha) \right\}.$$

Furthermore, by virtue of (4.6), we have

$$\begin{aligned} \mathcal{L}V(x,i) &\leq \alpha(1-\alpha c_i)|x|^\alpha \left\{ \mu_i - \sum_{j=1}^{m}\widehat{q}_{ij}c_j + o(1) + O(\alpha) \right\} \\ &= \alpha(1-\alpha c_i)|x|^\alpha(-\beta + o(1) + O(\alpha)) \leq -\kappa(\varepsilon) < 0, \end{aligned}$$

for any $(x,i) \in N \times \mathcal{M}$ with $\varepsilon < |x| < r$, where $N \subset \mathbb{R}^n$ is a small neighborhood of 0 and $\kappa(\varepsilon)$ is a positive constant. Therefore we conclude from Lemma 3.3 and Remark 3.5 that the equilibrium point $x = 0$ is asymptotically stable in probability.

(b) Now we prove that the equilibrium point $x = 0$ is unstable in probability if (4.5) holds. Define the column vector $\theta = (\theta, \theta_2, \ldots, \theta_m)' \in \mathbb{R}^m$ by

$$\theta_i := \lambda_{b(i)} + \frac{1}{2}\sum_{j=1}^{d}[\lambda_{a_j(i)} - 2(\rho_{\sigma_j(i)})^2],$$

and set

$$\delta := -\pi\theta = -\sum_{i=1}^{m}\pi_i\left(\lambda_{b(i)} + \frac{1}{2}\sum_{j=1}^{d}[\lambda_{a_j(i)} - 2(\rho_{\sigma_j(i)})^2]\right) < 0.$$

As in part (a), assumption (A4), the definition of δ, and Lemma 2.3 imply that the equation $\widehat{Q}c = \theta + \delta\mathbb{1}$ has a solution $c = (c_1, c_2, \dots, c_m)' \in \mathbb{R}^m$ and

$$\theta_i - \sum_{j=1}^{d} \widehat{q}_{ij} c_j = -\delta > 0, \quad i \in \mathcal{M}. \tag{4.11}$$

For $i \in \mathcal{M}$, consider the Lyapunov function

$$V(x, i) = (1 - \alpha c_i)|x|^\alpha,$$

where $-1 < \alpha < 0$ is sufficiently small so that $1 - \alpha c_i > 0$ for each $i \in \mathcal{M}$. Obviously the nonnegative function $V(\cdot, i), i \in \mathcal{M}$ satisfies (3.4). Like the arguments in part (a), Remark 4.2 implies that

$$\frac{x'b(i)x}{|x|^2} \ge \lambda_{b(i)}, \qquad \frac{x'\sigma_j'(i)\sigma_j(i)x}{|x|^2} \ge \lambda_{a_j(i)}.$$

Also note that for any symmetric matrix A with real eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$, using the transformation $x = Uy$, where U is a real orthogonal matrix such that $U'AU = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ ([5, Theorem 8.1.1]), we have

$$|x'Ax| = |\lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2| \le \rho_A |y|^2 = \rho_A |x|^2. \tag{4.12}$$

Thus by applying (4.12) to the matrix $A = \frac{\sigma_j'(i)+\sigma_j(i)}{2}$, we obtain that

$$\frac{(x'\sigma_j'(i)x)^2}{|x|^4} = \frac{(x'(\sigma_j'(i) + \sigma_j(i))x)^2}{4|x|^4} \le (\rho_{\sigma_j(i)})^2.$$

Therefore, detailed computations, as in part (a) (taking into account the extra term involving $(\rho_{\sigma_j(i)})^2$), show that for any sufficiently small $0 < \varepsilon < r$, we have

$$\mathcal{L}V(x, i) \le -\kappa(\varepsilon) < 0, \quad \text{for any } (x, i) \in N \times \mathcal{M} \text{ with } \varepsilon < |x| < r,$$

where $N \subset \mathbb{R}^n$ is some small neighborhood of 0 and $\kappa(\varepsilon)$ is some positive constant. Therefore Lemma 3.4 and Remark 3.5 imply that the equilibrium point $x = 0$ is unstable in probability. This completes the proof of the theorem. □

Remark 4.4. Suppose that for all $i \in \mathcal{M}$ and $j = 1, 2, \dots, d$, the matrices $\frac{\sigma_j'(i)+\sigma_j(i)}{2}$ are nonnegative definite; then we have $\rho_{\sigma_j(i)} = \Lambda_{\sigma_j(i)} \ge \lambda_{\sigma_j(i)} \ge 0$. Consequently a close examination of the proof of Theorem 4.3 reveals that the conditions (4.4) and (4.5) can be replaced by

$$\sum_{i=1}^{m} \pi_i \left(\Lambda_{b(i)} + \frac{1}{2} \sum_{j=1}^{d} [\Lambda_{a_j(i)} - 2(\lambda_{\sigma_j(i)})^2] \right) < 0 \tag{4.13}$$

and

$$\sum_{i=1}^{m} \pi_i \left(\lambda_{b(i)} + \frac{1}{2} \sum_{j=1}^{d} [\lambda_{a_j(i)} - 2(\Lambda_{\sigma_j(i)})^2] \right) > 0, \tag{4.14}$$

respectively. In a sense, the above two equations, in particular (4.13), strengthen the corresponding results in Theorem 4.3.

Theorem 4.3 gives sufficient conditions in terms of the maximum and minimum eigenvalues of the matrices for stability and instability of the equilibrium point $x = 0$. Since there is a "gap" between the maximum and minimum eigenvalues, a natural question arises: Can we obtain necessary and sufficient condition for stability? If the component $X(t)$ is one dimensional, we have the following result.

To proceed, we replace the first and second equations of (4.1) in assumption (A4) by

$$\begin{aligned} b(x,i) &= b_i x + o(x), \\ \sigma(x,i) &= \sigma_i x + o(|x|). \end{aligned} \tag{4.15}$$

where $x \in \mathbb{R}$, and b_i, σ_i^2 are real constants with $\sigma_i^2 \geq 0$, $i \in \mathcal{M}$. Then we immediately have the following corollary from Theorem 4.3 and Remark 4.4.

Corollary 4.5. *Let assumption* (A4) *and* (4.15) *be valid. Then the equilibrium point* $x = 0$ *is asymptotically stable in probability if*

$$\sum_{i=1}^{m} \pi_i \left(b_i - \frac{\sigma_i^2}{2} \right) < 0,$$

and is unstable in probability if

$$\sum_{i=1}^{m} \pi_i \left(b_i - \frac{\sigma_i^2}{2} \right) > 0.$$

4.2. An example

Example 4.6. To illustrate Theorem 4.3 and Corollary 4.5, we consider a real-valued process given by

$$\begin{cases} \mathrm{d}X(t) = b(X(t),\gamma(t))\mathrm{d}t + \sigma(X(t),\gamma(t))\mathrm{d}w(t) \\ \mathbf{P}\{\gamma(t+\Delta t) = j \mid \gamma(t) = i, X(s), \gamma(s), s \leq t\} = q_{ij}(X(t))\Delta t + o(\Delta t), \quad j \neq i, \end{cases} \tag{4.16}$$

where the jump process $\gamma(t)$ has three states and is generated by

$$Q(x) = \begin{pmatrix} -3 - \sin x \cos x + \sin x^2 & 1 + \sin x \cos x & 2 - \sin x^2 \\ 2 & -2 - \dfrac{x^2}{2+x^2} & \dfrac{x^2}{2+x^2} \\ 4 - \sin x & \sin^2 x & -4 + \sin x - \sin^2 x \end{pmatrix},$$

and the drift and diffusion coefficients are given by

$$b(x,1) = x - x\sin x, \qquad b(x,2) = x - x^2 \cos x, \qquad b(x,3) = 4x + x \sin x,$$
$$\sigma(x,1) = -\frac{3x}{1+x^2}, \qquad \sigma(x,2) = x + \frac{1}{3}x \sin x, \qquad \sigma(x,3) = x - \frac{1}{2}x\sin^2 x.$$

Associated with (4.16), there are three diffusions

$$\mathrm{d}X(t) = (X(t) - X(t)\sin X(t))\mathrm{d}t - \frac{3X(t)}{1+X^2(t)}\mathrm{d}w(t), \tag{4.17}$$

$$\mathrm{d}X(t) = \left(X(t) - X^2(t)\cos X(t)\right)\mathrm{d}t + \left(X(t) + \frac{1}{3}X(t)\sin X(t)\right)\mathrm{d}w(t), \tag{4.18}$$

$$\mathrm{d}X(t) = (4X(t) + X(t)\sin X(t))\mathrm{d}t + \left(X(t) - \frac{1}{2}X(t)\sin^2 X(t)\right)\mathrm{d}w(t), \tag{4.19}$$

switching back and forth from one to another according to the movement of the jump process $\gamma(t)$. It is readily seen that as $x \to 0$, the constants $b_i, \sigma_i^2, i = 1, 2, 3$, as in (4.15) are given by

$$\begin{array}{lll} b_1 = 1, & b_2 = 1, & b_3 = 4 \\ \sigma_1^2 = 9, & \sigma_2^2 = 1, & \sigma_3^2 = 1. \end{array} \tag{4.20}$$

Also as $x \to 0$, $Q(x)$ tends to

$$\widehat{Q} = \begin{pmatrix} -3 & 1 & 2 \\ 2 & -2 & 0 \\ 4 & 0 & -4 \end{pmatrix}.$$

By solving the equations $\pi\widehat{Q} = 0$ and $\mathbb{1}\pi = 1$, we obtain the stationary distribution π associated with $\widehat{Q}$:

$$\pi = (0.5, 0.25, 0.25). \tag{4.21}$$

Finally, by virtue of (4.20) and (4.21) we examine

$$\sum_{i=1}^{3} \pi_i \left(b_i - \frac{\sigma_i^2}{2}\right) = -0.75 < 0.$$

Therefore, we conclude from Corollary 4.5 that the equilibrium point $x = 0$ of (4.16) is asymptotically stable in probability.

This example is interesting and provides insight. It was proven in [6, pp. 171–172] that a one-dimensional nonlinear diffusion is stable if and only if its linear approximation is stable. Hence we can readily check that the equilibrium point $x = 0$ of (4.17) is stable in probability while (4.18) and (4.19) are unstable in probability. Therefore the jump process $\gamma(t)$ is a stabilizing factor. Note that similar examples were demonstrated in [9] under the assumptions that the jump component $\gamma(\cdot)$ is generated by constant Q and that the Markov chain $\gamma(\cdot)$ is independent of the Brownian motion $w(\cdot)$. Note also that Examples 4.1 and 4.2 in [4] are also concerned with stability of switching systems. Their result indicates that if the switching takes place sufficiently fast, the system will be stable even if the individual mode may be unstable. Essentially, it is related to singularly perturbed systems. Due to the fast variation, there is a limit system that is an average with respect to the stationary distribution of the Markov chain and that is stable. Then if the rate of switching is fast enough the original system will also be stable. Such an idea was also used in an earlier paper [2].

Remark 4.7. As can be seen in Section 4.1, if the continuous component of the system is of one dimensional, we obtain a nearly necessary and sufficient condition for stability. One question of particular interest is: Will we be able to obtain a similar condition for a multi-dimensional counterpart. In [6, pp. 220–224] for linear systems of stochastic differential equations with constant coefficients such a condition was obtained. The main ingredient is the use of the transformations $y = x/|x|$ and $\ln|x|$. The result is sharp and nearly necessary and sufficient. In [10], inspired by the approach of [6], regime-switching diffusion is considered, and a nearly necessary and sufficient condition is obtained for exponential stability. Such an approach can be adopted to the current problem with x-dependent $Q(x)$ used.

5. Further remarks

This work has been devoted to stability of regime-switching diffusions. As alluded to in the introduction, such models are becoming increasingly important owing to the needs of modeling and analysis of complex systems for random environments.

In this paper, we obtained sufficient conditions for stability and instability in probability for regime-switching diffusions. We also obtained necessary and sufficient conditions for exponential p-stability of linear systems. For a special class of linear systems, we obtained that stability implies p-stability for sufficiently small p. Concerning stability and instability of nonlinear systems, easily verifiable conditions were provided.

A number of questions deserve further consideration. It will be useful to extend our results to more general systems such as non-autonomous systems. We will work on these issues in the near future.

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