ELSEVIER
Available online at www.sciencedirect.com
SCIENCE DIRECT
J. Differential Equations 209 (2005) 101–139
Journal of Differential Equations
www.elsevier.com/locate/jde

Global attractors for singular perturbations of the Cahn–Hilliard equations

Songmu Zheng[a], Albert Milani[b,*]

[a]*Institute of Mathematics, Fudan University, Shanghai 200433, PR China*
[b]*Department of Mathematics, University of Wisconsin, Milwaukee, WI 53211-3029, USA*

Received 18 September 2003; revised 27 August 2004

Available online 11 November 2004

Abstract

We consider the singular perturbations of two boundary value problems, concerning respectively the viscous and the nonviscous Cahn–Hilliard equations in one dimension of space. We show that the dynamical systems generated by these two problems admit global attractors in the phase space $H_0^1(0,\pi) \times H^{-1}(0,\pi)$, and that these global attractors are at least upper-semicontinuous with respect to the vanishing of the perturbation parameter.

MSC: 35L70; 37L30; 35Q53

Keywords: Cahn–Hilliard equations; Singular perturbation; Semiflow; Global attractor; Upper semicontinuity

1. Introduction

In this paper, we consider the singular perturbations of two boundary value problems, concerning respectively the viscous and the nonviscous Cahn–Hilliard equations in one dimension of space. Our goal is to show that, at least when the perturbation parameter is sufficiently small, the dynamical systems generated by these two problems admit

* Corresponding author. Fax: +1 414 2294907.
E-mail addresses: szheng@fudan.ac.cn (S. Zheng), ajmilani@csd.uwm.edu, ajmilani@uwm.edu (A. Milani).

0022-0396/$ - see front matter
doi:10.1016/j.jde.2004.08.026

global attractors in a suitable phase space X, and that these global attractors are at least upper-semicontinuous with respect to the vanishing of the perturbation parameter. In another paper [25], we have shown that these semiflows also admit an exponential attractor and an inertial manifold in X.

1.1. The differential equations

1. The equations we consider have the unified form

$$\varepsilon u_{tt} + u_t + \Delta\left(\Delta u - u^3 + u - \delta u_t\right) = 0, \tag{1.1}$$

where $\varepsilon \geqslant 0$ and $\delta \geqslant 0$. The unknown u is a function of the space and time variables (x, t), with $x \in]0, \pi[$ and $t > 0$, and $\Delta := \partial^2/\partial x^2$.

More specifically, we distinguish the following four cases, according to whether ε or δ vanish or not:

(1) The nonviscous Cahn–Hilliard equation (see [4]), corresponding to $\varepsilon = \delta = 0$, i.e.

$$u_t + \Delta\left(\Delta u - u^3 + u\right) = 0. \tag{1.2}$$

(2) The viscous Cahn–Hilliard equation (see [18]), corresponding to $\varepsilon = 0$, $\delta > 0$, i.e.

$$u_t + \Delta\left(\Delta u - u^3 + u - \delta u_t\right) = 0. \tag{1.3}$$

(3) The perturbed nonviscous Cahn–Hilliard equation, corresponding to $\varepsilon > 0$, $\delta = 0$, i.e.

$$\varepsilon u_{tt} + u_t + \Delta\left(\Delta u - u^3 + u\right) = 0. \tag{1.4}$$

(4) The perturbed viscous Cahn–Hilliard equation, corresponding to $\varepsilon > 0$, $\delta > 0$, i.e.

$$\varepsilon u_{tt} + u_t + \Delta\left(\Delta u - u^3 + u - \delta u_t\right) = 0. \tag{1.5}$$

2. The long-time behavior of the semiflows generated by the first two equations is relatively well understood; in the next section, we recall some of the main results that are of interest for the sequel. Here, we consider the semiflows generated by the other two equations, that is, (1.4) and (1.5), with the goal of establishing analogous results for these equations. Our motivations for this study reside in part in the fact that (1.4) and (1.5) are examples of nonlinear beam equations with viscous dissipation, which are hyperbolic. The qualitative properties of their solutions are quite different than those of the reduced equations (1.2) and (1.3), which are parabolic; for example, there is

no smoothing property for $t > 0$, and their orbits are not compact. However, in many situations it is found that the asymptotic properties of the solutions of the parabolic equations and those of their hyperbolic perturbations are similar; a typical case is given by models which exhibit the so-called diffusion phenomenon of hyperbolic waves. For example, this is the case for some initial-boundary value problems associated to the quantum mechanics equations (1.10) below, studied in [12]. Hence, it is of importance to be able to describe the long-time behavior of the solutions to nonlinear dissipative hyperbolic equations such as (1.4) and (1.5), specifically in terms of attracting sets such as the global attractor, the exponential attractor and the inertial manifolds. One of the goals of this investigation is that of comparing the asymptotic behavior of the "hyperbolic" solutions with that of the corresponding "parabolic" ones; for example, one of the results of this paper is the upper semicontinuity of the global attractors of the semiflows associated to these equations.

3. In all four equations written above, we subject u to homogeneous boundary conditions of Dirichlet type, i.e.

$$u(0,t) = u(\pi,t) = 0 \qquad \Delta u(0,t) = \Delta u(\pi,t) = 0, \qquad \text{for } t \geqslant 0. \tag{1.6}$$

We could also consider boundary conditions of Neumann or mixed type; the qualitative results would be the same, but their formulation, and their proof, is more involved.

As we have mentioned, Eqs. (1.2) and (1.3) are parabolic, and we impose the initial condition

$$u(x,0) = u_0(x), \qquad x \in]0,\pi[\,; \tag{1.7}$$

conversely, Eqs. (1.4) and (1.5) are nonlinear beam equations with viscous damping, i.e., they are "hyperbolic", and we impose the initial conditions

$$u(x,0) = u_0(x) \qquad u_t(x,0) = u_1(x) \qquad x \in]0,\pi[. \tag{1.8}$$

4. In conclusion, we shall refer to the following initial-boundary value problems (IBVP in short):

1. Problem $\mathcal{CH}_{00}$: The IBVP for the nonviscous Cahn–Hilliard equation, i.e. problem (1.2)+(1.7)+(1.6);
2. Problem $\mathcal{CH}_{0\delta}$: The IBVP for the viscous Cahn–Hilliard equation, i.e. problem (1.3)+(1.7)+(1.6);
3. Problem $\mathcal{CH}_{\varepsilon 0}$: The IBVP for the nonviscous, perturbed Cahn–Hilliard equation, i.e. problem (1.4)+(1.8)+(1.6);
4. Problem $\mathcal{CH}_{\varepsilon\delta}$: The IBVP for the viscous, perturbed Cahn–Hilliard equation, i.e. problem (1.3)+(1.8)+(1.6).

1.2. Statement of results

The parabolic problems $\mathcal{CH}_{00}$ and $\mathcal{CH}_{0\delta}$ have been extensively studied; in particular, the global existence and uniqueness of solutions to each problem is known, and the asymptotic behavior of these solutions as $t \to +\infty$ is also well understood. For the Cahn–Hilliard equation, we refer e.g. to [10,19,20,23]; see also [17,22, Chapter 3.4.2 or 21, Chapter 5.5.5] and the references cited in these books. These authors consider the case of Neumann boundary conditions, but the same type of results hold for Dirichlet boundary conditions, with proofs obtained along similar lines. For the viscous Cahn–Hilliard equation, we refer to [1,3,6,8,9]. In summary, we know that problems $\mathcal{CH}_{00}$ and $\mathcal{CH}_{0\delta}$ generate a semiflow in the space $H := L^2(0, \pi)$, and that these semiflows admit a compact global attractor in H. Moreover, the global attractors are lower- and upper-semicontinuous as $\delta \to 0$.

In this paper, we show, first, that analogous results hold for the perturbed equations; that is, that problems $\mathcal{CH}_{\varepsilon 0}$ and $\mathcal{CH}_{\varepsilon\delta}$ each generate a semiflow in the phase space $X := H_0^1(0, \pi) \times H^{-1}(0, \pi)$. This result is summarized in Theorem 2.1. We proceed then to show that these semiflows admit, for fixed ε and δ, a global attractor (Theorem 3.4) in X. We also give a regularity result for the attractors when $\delta > 0$ (Theorem 3.6). If $\delta > 0$, all the above-mentioned results hold without limitations on ε (i.e. for all $\varepsilon \in]0, 1]$), while if $\delta = 0$ they are guaranteed to hold at least if ε is sufficiently small. This difference is not surprising, since the term $-\delta \Delta u_t$ has a regularizing effect on the solution.

Our third goal is to compare, in a suitable sense, the global attractors of problems $\mathcal{CH}_{\varepsilon 0}$ and $\mathcal{CH}_{\varepsilon\delta}$ with the global attractors of the corresponding limit problems $\mathcal{CH}_{00}$ and $\mathcal{CH}_{0\delta}$. More specifically, we show that the global attractors of the "hyperbolic" semiflows, generated by problems $\mathcal{CH}_{\varepsilon 0}$ and $\mathcal{CH}_{\varepsilon\delta}$ are upper semicontinuous as $\varepsilon \to 0$; that is, they converge, in a sense to be specified, to corresponding limit sets $\mathcal{A}_{0\delta}$ and $\mathcal{A}_{00}$ in X, naturally constructed from the global attractors $A_{0\delta}$ and A_{00} of the semiflows generated respectively by the "parabolic" problems $\mathcal{CH}_{0\delta}$ and $\mathcal{CH}_{00}$. These results are presented in Sections 3.4.1 and 3.4.2, where we consider the upper semi-continuity of $\mathcal{A}_{\varepsilon\delta}$ as $\varepsilon \to 0$, respectively for fixed $\delta > 0$ and for $\delta = 0$. An earlier result on the existence and upper semicontinuity of the attractors for the semiflow $S_{\varepsilon 0}$ (i.e., in the nonviscous case) was given, under somewhat more restrictive conditions, in [5]. In addition, in Section 3.4.3 we also prove the upper semicontinuity of the global attractors $\mathcal{A}_{\varepsilon\delta}$ as $\delta \to 0$. Since the upper semicontinuity of $A_{0\delta}$ (the attractors of the semiflows generated by the viscous Cahn–Hilliard equation) to A_{00} (the attractor of the semiflow generated by the Cahn–Hilliard equation) has been proved in [6], combining this result with the ones we describe in Sections 3.4, we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathcal{A}_{\varepsilon\delta} & \longrightarrow & \mathcal{A}_{\varepsilon 0} \\ \downarrow & & \downarrow \\ \mathcal{A}_{0\delta} & \longrightarrow & \mathcal{A}_{00} \end{array}, \tag{1.9}$$

where the vertical arrows mean convergence as $\varepsilon \to 0$, and the horizontal arrows mean convergence as $\delta \to 0$.

Investigations on nonlinear parabolic equations of second order as limits of singularly perturbed nonlinear hyperbolic equations of second order, i.e., nonlinear damped wave equations (for example, the so-called "quantum mechanics" equation

$$\varepsilon u_{tt} + u_t - \Delta u + u^3 - u = 0, \tag{1.10}$$

often mentioned in the literature), have been extensively studied; see, e.g., [12–14], and the references cited therein. However, in contrast to the case of nonlinear damped wave equations, where the natural phase space for (u, u_t) is $H_0^1 \times L^2$, for Eq. (1.1), the natural phase space is $H_0^1 \times H^{-1}$. Dealing with distributions in H^{-1} introduces a higher degree of difficulty in our problem; this becomes apparent, when we try to establish estimates on the nonlinear terms of the equations, which depend only on bounds of u in H_0^1. In particular, the restriction to one space dimension seems to be essential.

Finally, we would like to mention that other types of perturbations, different than $\mathcal{CH}_{\varepsilon 0}$ and $\mathcal{CH}_{\varepsilon\delta}$, can of course be considered. For example, in [6,7] various phase field models are studied, in which the viscous and nonviscous parabolic Cahn–Hilliard equations are obtained as the limit of a perturbed system (which, in contrast to our perturbed system, is also a parabolic system), with coupled equations containing the temperature as an additional unknown.

1.3. Notations

In the sequel, j, k, m, n, will denote positive integers, unless otherwise specified. If X is a Banach space, we denote by X' its topological dual, and by $\langle \cdot, \cdot \rangle_{X' \times X}$ the duality pairing between X' and X. We set $H^0 := L^2(0,\pi) =: L^2$ and, for integer $m \geqslant 1$, $H^m := H^m(0,\pi) \cap H_0^1(0,\pi)$, $H_0^m := H_0^m(0,\pi)$, and $H^{-m} := \left(H_0^m\right)'$. We denote by $\|\cdot\|_m$ the norm in H^m, by $|\cdot|_p$ the norm in $L^p(0,\pi)$, $1 \leqslant p \leqslant +\infty$, and by $\langle \cdot, \cdot \rangle$ the scalar product in $L^2(0,\pi)$. We abbreviate $\|\cdot\|_0 = |\cdot|_2 = \|\cdot\|$. Because of Poincaré's inequality, we can, and in fact will, choose in H^1 the norm

$$\|u\|_1 = \|\nabla u\| . \tag{1.11}$$

We consider $-\Delta$ as an unbounded operator in $L^2(0,\pi)$, with domain H^2; since $-\Delta$ is a positive operator, for each $\alpha \in \mathrm{R}$ we can define the fractional powers $(-\Delta)^\alpha$ (see e.g. [22, Chapter 2.2]). We set then

$$H^\alpha := \mathcal{D}\left((-\Delta)^{\alpha/2}\right), \tag{1.12}$$

which is a Banach space with respect to the norm

$$\|u\|_\alpha := \|(-\Delta)^{\alpha/2} u\|, \qquad u \in H^\alpha \tag{1.13}$$

and, for u and $v \in H^{-1}$,

$$[u, v] := \left\langle v, (-\Delta)^{-1} u \right\rangle_{H^{-1} \times H^1}; \tag{1.14}$$

we easily check that

$$[u, v] \leqslant \|u\|_{-1} \|v\|_{-1}, \qquad [u, u] = \|u\|_{-1}^2. \tag{1.15}$$

In the sequel, we shall often consider the equation formally obtained from (1.1) by taking $(-\Delta)^{-1}$, that is, the equation

$$\varepsilon(-\Delta)^{-1} u_{tt} + (-\Delta)^{-1} u_t - \Delta u + u^3 - u + \delta u_t = 0. \tag{1.16}$$

Indeed, by establishing suitable energy estimates on u, seen as solution of (1.16), we shall see that Eq. (1.1) defines a semiflow in the space $X := H^1 \times H^{-1}$, which arises as the natural "energy'' space for Eq. (1.1). Correspondingly, (1.1) can be considered in the associated chain of spaces $H^{m+1} \times H^{m-1}$, $m \in N$; in particular, we shall establish regularity results in the spaces

$$X_1 := H^2 \times L^2 \quad \text{and} \qquad X_2 := Y \times H^1, \tag{1.17}$$

where

$$Y := \{u \in H^1 \mid -\Delta u \in H^1\}. \tag{1.18}$$

We shall also refer to the space

$$X_{-1} := H^{-1} \times Y'; \tag{1.19}$$

note that $H_0^3 \hookrightarrow Y \hookrightarrow H^3$ (the second inclusion being a consequence of standard elliptic theory), so that $Y' \hookrightarrow H^{-3}$. Finally, we recall that, since $n = 1$, the continuous imbedding $H^1 \hookrightarrow L^\infty$ holds; we reserve the letter K to denote a constant such that the inequalities

$$\|u\|_{-1} \leqslant K \ \|u\| \leqslant K^2 \ \|\nabla u\|, \qquad |u|_p \leqslant K \ \|u\|_1 = K \ \|\nabla u\|, \tag{1.20}$$

hold for all $u \in H^1$, and $1 \leqslant p \leqslant +\infty$. Without loss of generality, we can assume that $K \geqslant 1$.

2. Semiflows and a priori estimates

In this section, we show that problems $\mathcal{CH}_{\varepsilon\delta}$ and $\mathcal{CH}_{\varepsilon 0}$ define semiflows on the space $X := H^1 \times H^{-1}$, and establish various a priori estimates on weak solutions of these problems; that is, a bound on the norm of $\left(u(\cdot,t), \sqrt{\varepsilon}u_t(\cdot,t)\right)$ in X, independent of t when $t \geqslant 0$, and of ε and δ. There are several methods to establish existence and uniqueness results for these problems; for example, the semigroup approach, in which Eq. (1.16) is reduced into a first-order evolution equation in the unknown vector function $(u, u_t)^\top$, or the Faedo–Galerkin method, as described in [15]. Since these approaches are quite standard, we can omit the details here. In either case, the global existence of a weak solution to these problems can be deduced from the estimates we establish in the sequel. Since, in the light of our goals, we are eventually interested in small values of the parameters ε and δ, in the sequel we shall assume, without loss of generality, that $0 < \varepsilon \leqslant 1$ and $0 \leqslant \delta \leqslant 1$; however, we will indicate the necessary modifications for the case $\varepsilon > 1$.

Weak solutions to problems $\mathcal{CH}_{\varepsilon\delta}$ and $\mathcal{CH}_{\varepsilon 0}$ are defined as follows.

Definition 2.1. Let $u_0 \in H^1$, $u_1 \in H^{-1}$. A function $u : [0,\pi] \times [0,+\infty[\to \mathrm{R}$ is a weak solution of problem $\mathcal{CH}_{\varepsilon\delta}$, $0 \leqslant \delta \leqslant 1$, if $u \in C_{\mathrm{b}}([0,+\infty[; H^1) \cap C^1_{\mathrm{b}}([0,+\infty[; H^{-1})$, if it satisfies the initial conditions (1.8), and if for all test functions $\varphi \in L^2(0,+\infty; Y)$, with $\varphi_t \in L^2(0,+\infty; H^1)$ and compact support in $[0,+\infty[$,

$$\begin{aligned} &\int_0^{+\infty} \Big(-\varepsilon \langle u_t, \varphi_t \rangle_{H^{-1}\times H^1} + \langle u_t, \varphi \rangle_{H^{-1}\times H^1} \\ &\quad + \left\langle \Delta u - u^3 + u - \delta u_t, \Delta\varphi \right\rangle_{H^{-1}\times H^1} \Big)\, dt \\ &\quad = \varepsilon \langle u_1, \varphi(0) \rangle_{H^{-1}\times H^1} . \end{aligned} \tag{2.1}$$

2.1. A priori estimates

In this section, we establish a priori estimates on weak solutions to problems $\mathcal{CH}_{\varepsilon\delta}$ and $\mathcal{CH}_{\varepsilon 0}$. In these estimates, we call a constant "universal" if this constant is positive, and independent of ε, δ, t, and any solution u. We consider in X an equivalent norm, whose square is defined by

$$E_0(u,v) := \varepsilon \|v\|_{-1}^2 + \varepsilon[u,v] + \tfrac{1}{2}\|u\|_{-1}^2 + \|\nabla u\|^2, \quad (u,v) \in X. \tag{2.2}$$

If $\varepsilon \leqslant 1$, the square root of E_0 does define a norm: indeed, by (1.20) we immediately derive that for all $(u,v) \in X$,

$$\tfrac{1}{2}\left(\varepsilon \|v\|_{-1}^2 + \|\nabla u\|^2\right) \leqslant E_0(u,v) \leqslant \alpha \left(\varepsilon \|v\|_{-1}^2 + \|\nabla u\|^2\right) \tag{2.3}$$

with

$$\alpha := \max\left\{\frac{3}{2}, K^4+1\right\}. \tag{2.4}$$

We also introduce the function $\Phi_0 : X \to \mathrm{R}$ defined by

$$\Phi_0(u,v) := E_0(u,v) + \tfrac{1}{2}\,|u|_4^4 - \|u\|^2 + \tfrac{1}{2}\delta\,\|u\|^2\,; \tag{2.5}$$

it is easy to verify that Φ_0 is bounded from below; in fact, there exists $M_0 > 0$ such that for all $(u, v) \in X$,

$$\Phi_0(u,v) \geqslant E_0(u,v) - M_0 \geqslant -M_0. \tag{2.6}$$

Proposition 2.1. *Let u be a weak solution of problem $\mathcal{CH}_{\varepsilon\delta}$ or $\mathcal{CH}_{\varepsilon 0}$. There exists a universal constant M_1, such that for all $t \geqslant 0$,*

$$\Phi_0(u(t), u_t(t)) \leqslant (\Phi_0(u_0, u_1) - \alpha M_1)\,\mathrm{e}^{-t/\alpha} + \alpha M_1, \tag{2.7}$$

where α is as in (2.4).

Proof. We begin by multiplying Eq. (1.16) in H first by $2u_t$ and then by u. Recalling (1.15), this yields (omitting the variable t for convenience)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon\,\|u_t\|_{-1}^2 + \|\nabla u\|^2 + \frac{1}{2}|u|_4^4 - \|u\|^2\right) + 2\,\|u_t\|_{-1}^2 + 2\delta\,\|u_t\|^2 = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon[u,u_t] + \frac{1}{2}\,\|u\|_{-1}^2 + \frac{1}{2}\delta\,\|u\|^2\right) - \varepsilon\,\|u_t\|_{-1}^2 + \|\nabla u\|^2 + |u|_4^4 - \|u\|^2 = 0.$$

Adding these identities, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_0(u,u_t) + (2-\varepsilon)\,\|u_t\|_{-1}^2 + \|\nabla u\|^2 + |u|_4^4 - \|u\|^2 + 2\delta\,\|u_t\|^2 = 0, \tag{2.8}$$

from which, recalling that $\varepsilon \leqslant 1$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_0(u,u_t) + \varepsilon\,\|u_t\|_{-1}^2 + \|\nabla u\|^2 + |u|_4^4 \leqslant \|u\|^2\,. \tag{2.9}$$

From (2.3) and (2.5), since also $\delta \leqslant 1$,

$$\Phi_0(u,u_t) \leqslant \alpha\left(\varepsilon\,\|u_t\|_{-1}^2 + \|\nabla u\|^2\right) + \tfrac{1}{2}\,|u|_4^4 - \tfrac{1}{2}\,\|u\|^2\ ; \tag{2.10}$$

thus, from (2.8) and (2.10), and noting that $\alpha \geqslant \frac{3}{2}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_0(u,u_t)+\frac{1}{\alpha}\Phi_0(u,u_t)+\frac{1}{2\alpha}\|u\|^2+\frac{2}{3}|u|_4^4 \leqslant \|u\|^2 \leqslant C+\frac{2}{3}|u|_4^4. \tag{2.11}$$

From this, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_0(u,u_t)+\frac{1}{\alpha}\Phi_0(u,u_t)\leqslant M_1, \tag{2.12}$$

and (2.7) follows by integration. □

Recalling (2.6), Proposition 2.1 yields, as a first consequence, the desired time-independent estimates on weak solutions of problems $\mathcal{CH}_{\varepsilon\delta}$ and $\mathcal{CH}_{\varepsilon 0}$.

Corollary 2.1. *In the same conditions of Proposition* 2.1, *there exists a constant* $M_2 > 0$, *depending on the norm of the initial values* (u_0, u_1) *in* X, *such that for all* $t \geqslant 0$,

$$E_0(u(t),u_t(t))\leqslant M_2. \tag{2.13}$$

As we have stated, the corresponding estimates carried out on suitable Faedo–Galerkin approximations allow us to establish the global existence of a weak solution to these problems. In Section 2.3 below, we show that weak solutions to these problems are unique, and depend continuously on the data $\{u_0, u_1\}$; therefore, problems $\mathcal{CH}_{\varepsilon\delta}$ and $\mathcal{CH}_{\varepsilon 0}$ generate continuous semiflows $S_{\varepsilon\delta}=(S_{\varepsilon\delta}(t))_{t\geqslant 0}$ and $S_{\varepsilon 0}=(S_{\varepsilon 0}(t))_{t\geqslant 0}$ on X. We also remark that if $(u,u_t)\in C_{\mathrm{b}}([0,+\infty[;X)$, then

$$\varepsilon u_{tt}\in C_{\mathrm{b}}([0,+\infty[;Y'). \tag{2.14}$$

Indeed, from Eq. (1.1) we have that, at least as a distribution,

$$\varepsilon u_{tt} := \Delta(\delta u_t - u + u^3 - \Delta u) - u_t; \tag{2.15}$$

we now see that the right side of (2.15) is in fact in Y' (pointwise in t). In fact, we first note that, since $u \in H^1$, and H^1 is an algebra under pointwise multiplication because $n=1$, $u^3-u\in H^1$, so that $\Delta(u^3-u)\in H^{-1}$. Next, we note that, on one hand, $\Delta(\Delta u-\delta u_t)\in H^{-3}$, since Δu and $u_t\in H^{-1}$; on the other, for $\phi\in H_0^3$ we have

$$\left\langle \Delta(\Delta u-\delta u_t),\phi\right\rangle_{H^{-3}\times H_0^3}=\left\langle \Delta u-\delta u_t,\Delta\phi\right\rangle_{H^{-1}\times H_0^1}\leqslant \left\|\Delta u-\delta u_t\right\|_{-1}\left\|\phi\right\|_Y.$$

This shows that the right side of (2.15) is in Y' as claimed; therefore, (2.14) holds.

Finally, we would like to mention that we can obtain analogous estimates also if $\varepsilon > 1$. The only modification is that in this case we would consider, instead of E_0, the equivalent norm whose square is given, for $(u, v) \in X$, by

$$E_{0r}(u, v) := \varepsilon \|v\|_{-1}^2 + r\varepsilon[u, v] + \tfrac{1}{2} r \ \|u\|_{-1}^2 + \|\nabla u\|^2 , \tag{2.16}$$

where $r \in \left]0, \frac{1}{\varepsilon}\right]$ is suitably chosen. To estimate E_{0r} we would then multiply Eq. (1.16) by $2u_t$ and ru instead of $2u_t$ and u. We can proceed in the same way also for the estimates we establish in the next sections.

2.2. Absorbing sets

A second consequence of Proposition 2.1 is the existence of bounded, positively invariant absorbing sets for the semiflows $S_{\varepsilon\delta}$ and $S_{\varepsilon 0}$.

Proposition 2.2. *Assume that the same conditions of Proposition* 2.1 *hold, and let* M_0 *be as in* (2.6). *Then, for any* $R_0 > \alpha M_1 + M_0$, *the ball*

$$B_0 := \{(u, v) \in X \mid E_0(u, v) \leqslant R_0\}$$

is absorbing for $S_{\varepsilon\delta}$ *and* $S_{\varepsilon 0}$. *Moreover, for any* $R > \alpha M_1$, *the set*

$$B := \{(u, v) \in X \mid \Phi_0(u, v) \leqslant R\} \tag{2.17}$$

is bounded, positively invariant and absorbing for $S_{\varepsilon\delta}$ *and* $S_{\varepsilon 0}$.

Proof. The first claim follows from (2.6) and (2.7). In particular, for all $t \geqslant 0$,

$$E_0(u(t), u_t(t)) \leqslant (\Phi_0(u_0, u_1) - \alpha M_1)\mathrm{e}^{-t/\alpha} + \alpha M_1 + M_0. \tag{2.18}$$

Assume now that (u_0, u_1) is in a bounded set $G \subseteq X$. There exists then $\Gamma \geqslant 1$ such that $E_0(u_0, u_1) \leqslant \Gamma$. Now, from (1.20) and (2.5), recalling also (2.4),

$$\Phi_0(u_0, u_1) \leqslant E_0(u_0, u_1) + \tfrac{1}{4} |u_0|_4^4 \leqslant \Gamma + \tfrac{1}{4} K^4 \Gamma^2 \leqslant \alpha\Gamma^2; \tag{2.19}$$

thus, from (2.18) we deduce that for all $t \geqslant 0$,

$$E_0(u(t), u_t(t)) \leqslant \alpha(\Gamma^2 - M_1)\mathrm{e}^{-t/\alpha} + \alpha M_1 + M_0. \tag{2.20}$$

From this it follows that if $\alpha(\Gamma^2 - M_1) \leqslant R_0 - (\alpha M_1 + M_0)$, then $E_0(u(t), u_t(t)) \leqslant R_0$ for all $t \geqslant 0$, while if $\alpha(\Gamma^2 - M_1) > R_0 - (\alpha M_1 + M_0)$, then $E_0(u(t), u_t(t)) \leqslant R_0$ for all $t \geqslant T_G$, with

$$T_G := \alpha \ln \frac{\alpha(\Gamma^2 - M_1)}{R_0 - (\alpha M_1 + M_0)}.$$

This proves that the ball B_0 is absorbing. The boundedness of the set B follows from (2.6) and (2.7), and its positive invariance is a direct consequence of (2.7). In fact, if $\Phi_0(u_0, u_1) \leqslant R$, then for all $t \geqslant 0$

$$\Phi_0(u(t), u_t(t)) \leqslant (R - \alpha M_1)\mathrm{e}^{-t/\alpha} + \alpha M_1 \leqslant (R - \alpha M_1) + \alpha M_1 \leqslant R.$$

Finally, we prove that B is absorbing exactly in the same way as we did for B_0; we find that $\Phi_0(u(t), u_t(t)) \leqslant R$ for all $t \geqslant \tilde{T}_G$, where now

$$\tilde{T}_G := \begin{cases} 0 & \text{if } \alpha(\Gamma^2 - M_1) \leqslant R - \alpha M_1; \\ \dfrac{\alpha(\Gamma^2 - M_1)}{R - \alpha M_1} & \text{if } \alpha(\Gamma^2 - M_1) > R - \alpha M_1. \end{cases}$$

This concludes the proof of Proposition 2.2; we remark that the set B is not a ball of X. $\square$

2.3. Well-posedness and contractive estimates

In this section, we establish suitable estimates on the difference of two solutions of problems $\mathcal{CH}_{\varepsilon\delta}$ and $\mathcal{CH}_{\varepsilon 0}$. The first consequence of these estimates is that these problems are well-posed in X, and therefore they generate corresponding semiflows $S_{\varepsilon\delta}$ and $S_{\varepsilon 0}$ in X. In Section 3, we shall use these estimates to show that these semiflows admit a global attractor in X.

1. Let $z := u - \tilde{u}$ be the difference of two solutions of (1.1). Then z solves the equations

$$\varepsilon z_{tt} + z_t + \Delta\left(\Delta z - (u^3 - \tilde{u}^3) + z - \delta z_t\right) = 0, \tag{2.21}$$

$$\varepsilon(-\Delta)^{-1} z_{tt} + (-\Delta)^{-1} z_t - \Delta z + (u^3 - \tilde{u}^3) - z + \delta z_t = 0. \tag{2.22}$$

As in Section 2.1, we multiply (2.22) in H by $2z_t$ and z, and add the resulting identities. Setting $h := u^2 + u\tilde{u} + \tilde{u}^2$, we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon \|z_t\|_{-1}^2 + \varepsilon[z_t, z] + \frac{1}{2}\|z\|_{-1}^2 + \|\nabla z\|^2 + \left\langle u^3 - \tilde{u}^3, z\right\rangle + \frac{1}{2}\delta\|z\|^2\right) \\ &\quad + (2-\varepsilon)\|z_t\|_{-1}^2 + \|\nabla z\|^2 + \left\langle u^3 - \tilde{u}^3, z\right\rangle + 2\delta\|z_t\|^2 \qquad (2.23) \\ &= \langle h_t z, z\rangle + \langle z, 2z_t + z\rangle . \end{aligned}$$

Because of Corollary 2.1, we can assume that both u and $\tilde{u}$ are bounded in X; thus, by (2.13), we can estimate the first term of $\langle h_t z, z\rangle$ at the right side of (2.23) by

$$\langle uu_t z, z\rangle \leqslant \|u_t\|_{-1} \left\|uz^2\right\|_1 \leqslant C \left(\|\nabla u\| \, |z|_\infty^2 + 2\,|u|_\infty \, |z|_\infty \, \|\nabla z\|\right), \tag{2.24}$$

where C depends only on M_2 of (2.13). Resorting then to Agmon's inequality

$$|z|_\infty \leqslant C \; \|\nabla z\|^{1/2} \, \|z\|^{1/2} + C \; \|z\| \, , \tag{2.25}$$

we obtain from (2.24)

$$\begin{aligned} \langle uu_t z, z\rangle \; &\leqslant \; C \; \left(\|\nabla z\| \;\; \|z\| + \|\nabla z\|^{3/2} \, \|z\|^{1/2} + \|z\|^2\right) \\ &\leqslant \; \tfrac{1}{18} \, \|\nabla z\|^2 + C_2 \;\, \|z\|^2 \, . \end{aligned} \tag{2.26}$$

The other terms of $\langle h_t z, z\rangle$ are estimated in the same way.

We now proceed to establish further estimates, which differ for $\delta > 0$ and for $\delta = 0$.

In the case $\delta > 0$, we estimate

$$\langle z, 2z_t + z\rangle \leqslant \delta \, \|z_t\|^2 + \left(\frac{1}{\delta} + 1\right) \|z\|^2 \, . \tag{2.27}$$

Calling $\Psi(z, z_t)$ the differentiated term of the left side of (2.23), i.e.

$$\begin{aligned} \Psi(z, z_t) \; := \; & \varepsilon \, \|z_t\|_{-1}^2 + \varepsilon[z_t, z] + \tfrac{1}{2} \, \|z\|_{-1}^2 + \|\nabla z\|^2 \\ & + \left\langle u^3 - \tilde{u}^3, z\right\rangle + \tfrac{1}{2}\delta \, \|z\|^2 \, , \end{aligned} \tag{2.28}$$

by (2.26) and (2.27) we obtain from (2.23) that, for $\varepsilon \leqslant 1$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \Psi(z, z_t) + \varepsilon \, \|z_t\|_{-1}^2 + \frac{5}{6} \, \|\nabla z\|^2 + \left\langle u^3 - \tilde{u}^3, z\right\rangle + \delta \, \|z_t\|^2 \leqslant C_\delta \, \|z\|^2 \, , \tag{2.29}$$

where C_δ is a positive constant depending only on δ. Since the function $u \mapsto u^3$ is monotone, and $\varepsilon, \; \delta \leqslant 1$,

$$\Psi(z, z_t) \leqslant \tfrac{3}{2} \left(\varepsilon \, \|z_t\|_{-1}^2 + \|\nabla z\|^2 + \left\langle u^3 - \tilde{u}^3, z\right\rangle\right) + \left(\tfrac{1}{2} + K^2\right) \|z\|^2 \, . \tag{2.30}$$

From (2.29) and (2.30), it follows that, with a different C_δ,

$$\frac{\mathrm{d}}{\mathrm{d}t} \Psi(z, z_t) + \frac{1}{2} \Psi(z, z_t) \leqslant C_\delta \;\, \|z\|^2 \, ; \tag{2.31}$$

integrating (2.31), we obtain that for all $t \geqslant 0$

$$\Psi(z(t), z_t(t)) \leqslant \Psi(z(0), z_t(0))\mathrm{e}^{-t/2} + C_\delta \int_0^t \|z\|^2 \, ds. \tag{2.32}$$

From (2.28) and (2.3), we deduce that

$$\Psi(z, z_t) \geqslant \frac{1}{2}\left(\varepsilon \|z_t\|_{-1}^2 + \|\nabla z\|^2\right) \geqslant \frac{1}{2\alpha} E_0(z, z_t), \tag{2.33}$$

and we also have that

$$0 \leqslant \left\langle u^3 - \tilde{u}^3, z\right\rangle = \langle h\, z, z\rangle \leqslant \left(|u|_\infty^2 + |u|_\infty |\tilde{u}|_\infty + |\tilde{u}|_\infty^2\right) \|z\|^2 \leqslant C_3 \|\nabla z\|^2 ,$$

where C_3 depends only on M_2 and K. Hence, recalling (2.28), and that $\delta \leqslant 1$,

$$\begin{aligned} \Psi(z, z_t) &\leqslant \left(\varepsilon \|z_t\|_{-1}^2 + \varepsilon[z_t, z] + \tfrac{1}{2} \|z\|_{-1}^2 + (1 + C_3 + K) \|\nabla z\|^2\right) \\ &\leqslant (1 + C_3 + K) E_0(z, z_t) =: C_4 E_0(z, z_t). \end{aligned} \tag{2.34}$$

Finally, we recall that, by Schwartz' inequality, for all $(u, v) \in X$

$$E_0(u, v) \geqslant \tfrac{1}{4} \|u\|^2 . \tag{2.35}$$

Consequently, from (2.32), (2.33), (2.34) and (2.35),

$$E_0(z(t), z_t(t)) \leqslant 2\alpha C_4 E_0(z(0), z_t(0))\mathrm{e}^{-t/2} + 8\alpha C_3 \int_0^t E_0(z, z_t)\, ds. \tag{2.36}$$

In the case $\delta = 0$, we replace estimate (2.27) by

$$\langle z, 2z_t + z\rangle \leqslant \tfrac{8}{5} \|z_t\|_{-1}^2 + \tfrac{5}{8} \|\nabla z\|^2 + \|z\|^2 \tag{2.37}$$

and (2.29) becomes instead

$$\frac{\mathrm{d}}{\mathrm{d}t} \Psi(z, z_t) + \left(\frac{2}{5} - \varepsilon\right) \|z_t\|_{-1}^2 + \frac{5}{24} \|\nabla z\|^2 + \left\langle u^3 - \tilde{u}^3, z\right\rangle \leqslant C \, \|z\|^2 . \tag{2.38}$$

Assume now e.g. that $\varepsilon \leqslant \frac{1}{3}$. Then, $\frac{2}{5} - \varepsilon > \frac{1}{6}\varepsilon$, so that from (2.38) we deduce, instead of (2.31),

$$\frac{\mathrm{d}}{\mathrm{d}t} \Psi(z, z_t) + \tfrac{1}{6} \Psi(z, z_t) \leqslant C \, \|z\|^2 . \tag{2.39}$$

The rest of the proof, leading up to (2.36), proceeds then in the same way.

In conclusion, we obtain that, in either case, z satisfies the estimate

$$E_0(z(t), z_t(t)) \leqslant M\,\alpha E_0(z(0), z_t(0))\mathrm{e}^{-t/6} + M\,\alpha \int_0^t E_0(z, z_t)\,ds, \tag{2.40}$$

for a suitable constant M; we recall that M depends on the initial values of either solution u or $\bar{u}$.

2. We can now conclude with

Theorem 2.1. *For all* $\varepsilon \in]0, 1]$ *and* $\delta \in]0, 1]$, *problem* $\mathcal{CH}_{\varepsilon\delta}$ *is well-posed in* X, *and defines a corresponding continuous semiflow* $S_{\varepsilon\delta}$ *in* X. *If* $\delta = 0$, *the same is true if* ε *is sufficiently small*; *that is, there is* $\varepsilon_0 \in]0, 1]$, *such that for all* $\varepsilon \in]0, \varepsilon_0]$, *problem* $\mathcal{CH}_{\varepsilon 0}$ *is well-posed in* X, *and defines a corresponding continuous semiflow* $S_{\varepsilon 0}$ in X.

Proof. The existence of a weak solution to both problems can be obtained by a straightforward Faedo–Galerkin approximation method. The uniqueness of these solutions, as well as their continuous dependence on their initial data on compact time intervals, is a consequence of estimate (2.40). Indeed, by Gronwall's inequality, from (2.40) we deduce that for all $t \geqslant 0$,

$$E_0(z(t), z_t(t)) \leqslant M\,\alpha E_0(z(0), z_t(0))\mathrm{e}^{M\alpha t}. \tag{2.41}$$

In particular, (2.41) shows that, for each $t \geqslant 0$, the operators $S_{\varepsilon\delta}(t)$ and $S_{\varepsilon 0}(t)$ are locally Lipschitz continuous in X; consequently, the solution operators $S_{\varepsilon\delta} = (S_{\varepsilon\delta}(t))_{t\geqslant 0}$ and $S_{\varepsilon 0} = (S_{\varepsilon 0}(t))_{t \geqslant 0}$, defined in X by

$$S_{\varepsilon\delta}(t)(u_0, u_1) := (u(t), u_t(t)), \quad (u_0, u_1) \in X, \quad (0 \leqslant \delta \leqslant 1), \tag{2.42}$$

are semiflows, with the limitation $\varepsilon \leqslant \frac{1}{3} =: \varepsilon_0$ if $\delta = 0$. $\square$

3. Global attractors

In this section, we show that the semiflows $S_{\varepsilon\delta}$ and $S_{\varepsilon 0}$ admit global attractors in the space X, given by the ω-limit sets

$$\mathcal{A}_{\varepsilon\delta} := \omega_{\varepsilon\delta}(B) := \bigcap_{s \geqslant 0} \overline{\bigcup_{t \geqslant s} S_{\varepsilon\delta}(t) B}, \tag{3.1}$$

$$\mathcal{A}_{\varepsilon 0} := \omega_{\varepsilon 0}(B) := \bigcap_{s \geqslant 0} \overline{\bigcup_{t \geqslant s} S_{\varepsilon 0}(t) B}, \tag{3.2}$$

where B is the bounded, positively invariant absorbing set defined in (2.17).

In the sequel, we shall often refer to the following characterization of global attractors:

Lemma 3.1. *Let S be a semiflow on a Banach space X, and assume that S admits a global attractor A. Let $x \in X$. Then $x \in A$ if and only if there exists a complete orbit through x, contained in A.*

Proof. The "if" part is obvious, while the "only if" part is proven in Proposition 1.3 of [2, Chapter 3.1]. $\square$

3.1. Global attractors via α-contractions

Since problems $\mathcal{CH}_{\varepsilon\delta}$ and $\mathcal{CH}_{\varepsilon 0}$ are hyperbolic, following [12], we can establish the existence of a global attractor for the corresponding semiflows in one of two ways. The first is to show that the semiflows $S_{\varepsilon\delta}$ or $S_{\varepsilon 0}$ can be decomposed into the sum of two families of operators, one of which is uniformly compact, and the other, (which needs not be a semiflow) is uniformly decaying to 0 (see also [22, Chapter 4]). The second method consists in showing that the semiflows are α-contractions on X (see below). In our case, the situation is somewhat different, according to whether $\delta > 0$ or $\delta = 0$. If $\delta > 0$, we can prove that $S_{\varepsilon\delta}$ both admits the stated decomposition and is an α-contraction, and that the global attractor exists for all $\varepsilon \in]0, 1]$; in contrast, when $\delta = 0$ we can only prove that $S_{\varepsilon 0}$ is an α-contraction, and that the global attractor exists for all ε sufficiently small. In the sequel, we show that the semiflows $S_{\varepsilon\delta}$ and $S_{\varepsilon 0}$ are α-contractions in X, the latter at least if ε is small.

1. To this end, we first recall the notion of α-contraction, and state the main results, which guarantee that the ω-limit sets defined in (3.1) and (3.2) are indeed the global attractors for the semiflows $S_{\varepsilon\delta}$ and $S_{\varepsilon 0}$.

Definition 3.1. Let X be a Banach space, and α be a measure of compactness in X (see Definition A.1 of the Appendix A). Let $B \subseteq X$. A continuous map $T : B \to B$ is an α-contraction on B, if there exists a number $q \in]0, 1[$ such that for every subset $A \subseteq B$,

$$\alpha(T(A)) \leqslant q\, \alpha(A). \tag{3.3}$$

The following results describe the existence of an attractor, first for discrete semiflows generated by α-contractions, and then for continuous semiflows.

Theorem 3.1. *Assume that $B \subseteq X$ is closed and bounded, and that $T : B \to B$ is an α-contraction on B. Consider the semiflow generated by the iterations of T, i.e. $S := (T^n)_{n\in N}$. Then the set*

$$\omega(B) := \bigcap_{n\geqslant 0} \overline{\bigcup_{m\geqslant n} T^m(B)} \tag{3.4}$$

is compact, invariant, and attracts B.

Theorem 3.2. *Assume that S is a continuous semiflow on X, admitting a bounded, positively invariant absorbing set B, and that there exists $t_* > 0$ such that the operator $S_* := S(t_*)$ is an α-contraction on B. Let*

$$A_* := \bigcap_{n \geqslant 0} \overline{\bigcup_{m \geqslant n} S_*^m(B)} = \omega_*(B) \tag{3.5}$$

be the ω-limit set of B under the map S_*, and set

$$A := \bigcup_{0 \leqslant t \leqslant t_*} S(t)A_*. \tag{3.6}$$

Assume further that for all $t \in [0, t_*]$, the map $x \mapsto S(t)x$ is Lipschitz continuous from B to B, with Lipschitz constant $L(t)$, $L : [0, t_*] \to]0, +\infty[$ being a bounded function. Then $A = \omega(B)$, and this set is the global attractor of S in B.

Theorems 3.1 and 3.2 can be proven along the same lines of the results proven in [12, Chapters 2 and 3]. However, since these do not apply exactly to our situation, for the readers' convenience we include a self-contained proof in the Appendix A.

2. To apply Theorem 3.2 to problems $\mathcal{CH}_{\varepsilon\delta}$ and $\mathcal{CH}_{\varepsilon 0}$, we need an intermediary step, which assures that if an operator T fails to be contractive only because of a precompact pseudometric, it is still an α-contraction.

Definition 3.2. A pseudometric d in X is precompact in X if every bounded sequence has a subsequence which is a Cauchy sequence relative to d.

Theorem 3.3. *Let $B \subset X$ be bounded, let d be a precompact pseudometric in X, and let $T : B \to B$ be a continuous map. Suppose T satisfies the estimate*

$$\|Tx - Ty\|_X \leqslant q \; \|x - y\|_X + d(x, y) \tag{3.7}$$

for all $x, \; y \in B$ and some $q \in]0, 1[$ independent of x and y. Then T is an α-contraction.

3. We can now show that Theorems 3.2 and 3.3 can be applied to the semiflows $S_{\varepsilon\delta}$ and $S_{\varepsilon 0}$. That is, we show that there is $t_* > 0$ such that the operators $S_{\varepsilon\delta}(t_*)$, $S_{\varepsilon 0}(t_*)$, are α-contractions in X, up to a precompact pseudometric.

Theorem 3.4. *For all $\varepsilon, \delta \in]0, 1]$, the semiflow $S_{\varepsilon\delta}$ generated by problem $\mathcal{CH}_{\varepsilon\delta}$ admits a global attractor $\mathcal{A}_{\varepsilon\delta}$* in X, *given by* (3.1). *If $\varepsilon \leqslant \frac{1}{3}$, the semiflow $S_{\varepsilon 0}$ generated by problem $\mathcal{CH}_{\varepsilon 0}$ admits a global attractor $\mathcal{A}_{\varepsilon 0}$* in X, *given by* (3.2). *Moreover, these global attractors are compact and connected.*

Proof. It is sufficient to note that, as a second consequence of estimate (2.40), we can apply Theorem 3.2 to the semiflows $S_{\varepsilon\delta}$ and $S_{\varepsilon 0}$. Indeed, if e.g. we choose $t_* > 0$ such that

$$q_* := M\,\alpha e^{-t_*/6} < 1,$$

the operators $S_{\varepsilon\delta}(t_*)$ and $S_{\varepsilon 0}(t_*)$ are strict contractions in X, up to the pseudometric ψ_* defined by

$$\psi_*\left((u,v),(\tilde{u},\tilde{v})\right) := \left(M\,\alpha\int_0^{t_*}\|z(s)\|^2\,ds\right)^{1/2}, \tag{3.8}$$

where, for (u,v), $(\tilde{u},\tilde{v}) \in X$, $z := u - \tilde{u}$ is the difference of the solutions to problems $\mathcal{CH}_{\varepsilon\delta}$ and $\mathcal{CH}_{\varepsilon 0}$, with initial values (u,v) and $(\tilde{u},\tilde{v})$. This pseudometric is clearly precompact, because of the compactness of the injection

$$\left\{u \in L^2(0,t_*;H^1) \mid u_t \in L^2(0,t_*;H^{-1})\right\} \hookrightarrow L^2(0,t_*;L^2). \tag{3.9}$$

Thus, by Theorem 3.3, the maps $S_{\varepsilon\delta}(t_*)$ and $S_{\varepsilon 0}(t_*)$ are α-contractions. In turn, Theorem 3.2 implies that the ω-limit sets of the set B defined in (2.17) are the global attractors for the semiflows $S_{\varepsilon\delta}$ and $S_{\varepsilon 0}$. Finally, for the connectedness of the global attractors, we (refer to [22, Chapter 1, Lemma 1.3]). $\square$

3.2. *Uniform boundedness of the global attractors*

Since the global attractors $\mathcal{A}_{\varepsilon\delta}$ are compact in X, they are bounded sets. In this section, we show that, in fact, they are uniformly bounded with respect to ε and δ. In this section and the next, universal constants are also understood to be independent of any choice of initial values (u_0, u_1).

Theorem 3.5. *There exists a bounded set $G \subseteq X$ such that for all $\varepsilon \in]0,1]$ if $0 < \delta \leqslant 1$, or for all $\varepsilon \in \left]0, \frac{1}{3}\right]$ if $\delta = 0$, $\mathcal{A}_{\varepsilon\delta} \subseteq G$. More precisely, there exists a universal constant $M_3 > 0$, such that for all $(u_0, u_1) \in \mathcal{A}_{\varepsilon\delta}$,*

$$\|u_1\|_{-1}^2 + \|\nabla u_0\|^2 \leqslant M_3. \tag{3.10}$$

Proof. Let $(u_0, u_1) \in \mathcal{A}_{\varepsilon\delta}$. By Lemma 3.1, (u_0, u_1) belongs to a complete orbit $(u(t), u_t(t))_{t\in\mathbb{R}}$, contained in $\mathcal{A}_{\varepsilon\delta}$. Arguing as in (2.14), we note that $\varepsilon u_{tt}(t) \in Y' \hookrightarrow H^{-3}$ for all $t \in \mathbb{R}$. We have then the following preliminary result:

Proposition 3.1. *There are* $M_4 > 0$ *and* $\varepsilon_1 \in]0, \varepsilon_0]$ *such that for all* $\varepsilon \leqslant \varepsilon_1$, all $\delta \in [0, 1]$, *if* $(u(t), u_t(t))_{t\in\mathrm{R}}$ *is a complete orbit contained in* $\mathcal{A}_{\varepsilon\delta}$, *then for all* $t \in \mathrm{R}$,

$$\varepsilon \, \|u_{tt}(t)\|^2_{-3} + \|u_t(t)\|^2_{-1} \leqslant M_4. \tag{3.11}$$

Proof. By (3.1), $\mathcal{A}_{\varepsilon\delta} \subseteq B$, where B is the bounded absorbing set defined in (2.17); since the constants R, α and M_1 appearing in the definition of B are independent of ε, δ and (u_0, u_1), we deduce that there is a universal constant $C_1 > 0$, such that for all $t \in \mathrm{R}$,

$$\varepsilon \, \|u_t(t)\|^2_{-1} + \|\nabla u(t)\|^2 \leqslant C_1. \tag{3.12}$$

1. As a preliminary step, we show that there exists a universal constant $C > 0$, such that for all $t \in \mathrm{R}$,

$$\|u_t(t)\|_{-3} \leqslant C. \tag{3.13}$$

Multiplying Eq. (1.16) by $2(-\Delta)^{-2}u_t$ we obtain, recalling (1.15),

$$\begin{aligned}
&\varepsilon \, \frac{\mathrm{d}}{\mathrm{d}t} \, \|u_t\|^2_{-3} + 2\, \|u_t\|^2_{-3} + 2\delta\, \|u_t\|^2 = 2\left\langle \Delta u - u^3 + u, (-\Delta)^{-2}u_t \right\rangle \\
&\quad = 2\left\langle (-\Delta)^{-1/2}(\Delta u - u^3 + u), (-\Delta)^{-3/2}u_t \right\rangle \\
&\quad \leqslant 2\|\Delta u - u^3 + u\|_{-1} \, \|u_t\|_{-3} \, .
\end{aligned} \tag{3.14}$$

Because of (3.12),

$$\|\Delta u - u^3 + u\|_{-1} \leqslant C, \tag{3.15}$$

uniformly in $t \in \mathrm{R}$; hence, we obtain from (3.14)

$$\varepsilon \, \frac{\mathrm{d}}{\mathrm{d}t} \, \|u_t\|^2_{-3} + \|u_t\|^2_{-3} \leqslant C_2. \tag{3.16}$$

Integrating this inequality on an arbitrary interval $[t_0, t]$, and recalling (3.12) again, we deduce the estimate

$$\|u_t(t)\|^2_{-3} \leqslant \mathrm{e}^{-(t-t_0)/\varepsilon} \, \|u_t(t_0)\|^2_{-3} + C_2 \leqslant C_3 \left(\frac{1}{\varepsilon} \mathrm{e}^{-(t-t_0)/\varepsilon} + 1 \right). \tag{3.17}$$

We can then deduce (3.13) by letting $t_0 \to -\infty$ in (3.17).

2. We now differentiate Eq. (1.16) with respect to t, and multiply the resulting equation by $2(-\Delta)^{-2}u_{tt}$ and $\rho(-\Delta)^{-2}u_t$, with $\rho > 0$ to be determined (sufficiently

large). Adding the resulting identities, as well as a common term to both sides, we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\varepsilon\,\|u_{tt}\|_{-3}^2+\rho\varepsilon\Big\langle(-\Delta)^{-1}u_{tt},(-\Delta)^{-2}u_t\Big\rangle \\
&\qquad +\frac{1}{2}\,\rho\,\|u_t\|_{-3}^2+\|u_t\|_{-1}^2+\frac{1}{2}\delta\rho\,\|u_t\|_{-2}^2\Big) \\
&\qquad +(2-\rho\varepsilon)\,\|u_{tt}\|_{-3}^2+\rho\,\|u_t\|_{-1}^2+2\delta\,\|u_{tt}\|_{-2}^2+\frac{1}{8}\delta\rho\,\|u_t\|_{-2}^2 \\
&\quad =-\Big\langle(3u^2-1)u_t,2(-\Delta)^{-2}u_{tt}+\rho(-\Delta)^{-2}u_t\Big\rangle \\
&\qquad +\frac{1}{8}\delta\rho\,\|u_t\|_{-2}^2=:R_\rho.
\end{aligned}
\tag{3.18}
$$

By the interpolation inequality

$$
\|u_t\|_{-2}^2\leqslant C\ \|u_t\|_{-1}\ \|u_t\|_{-3}
$$

and recalling (3.12), (3.13), we can estimate the right side of (3.18) by

$$
\begin{aligned}
R_\rho\ &\leqslant\ 2\ \left\|(3u^2-1)u_t\right\|_{-1}\big(\|u_{tt}\|_{-3}+\rho\ \|u_t\|_{-3}\big)+C\rho\,\|u_t\|_{-1}\,\|u_t\|_{-3} \\
&\leqslant\ \big(C_4+\tfrac{1}{4}\,\rho\big)\,\|u_t\|_{-1}^2+\tfrac{1}{2}\ \|u_{tt}\|_{-3}^2+C\rho.
\end{aligned}
\tag{3.19}
$$

3. We assume now that $\rho\geqslant 4C_4$ and, correspondingly, ε is so small that $\rho\varepsilon\leqslant 1$. Then, replacing (3.19) into (3.18), and denoting $E_\rho(u_t,u_{tt})$ the quantity under differentiation at the left side of (3.18), we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}E_\rho(u_t,u_{tt})+\tfrac{1}{2}\,\rho\left(\varepsilon\,\|u_{tt}\|_{-3}^2+\|u_t\|_{-1}^2\right)+\frac{1}{8}\delta\rho\,\|u_t\|_{-2}^2\leqslant C\rho. \tag{3.20}
$$

We can easily verify that, if in addition $\rho\geqslant\max\{2,K^4\}$ and $\rho\varepsilon\leqslant\frac{1}{2}$,

$$
\begin{aligned}
\tfrac{1}{2}\left(\varepsilon\,\|u_{tt}\|_{-3}^2+\|u_t\|_{-1}^2\right)\ &\leqslant\ E_\rho(u_t,u_{tt}) \\
&\leqslant\ 2\rho\left(\varepsilon\,\|u_{tt}\|_{-3}^2+\|u_t\|_{-1}^2\right)+\tfrac{1}{2}\,\delta\rho\,\|u_t\|_{-2}^2\,.
\end{aligned}
\tag{3.21}
$$

Consequently, we obtain from (3.19) and (3.20) that

$$
\frac{\mathrm{d}}{\mathrm{d}t}E_\rho(u_t,u_{tt})+\frac{1}{4}E_\rho(u_t,u_{tt})\leqslant\rho\,C_5. \tag{3.22}
$$

Integrating (3.22) on an arbitrary interval $[t_0, t] \subset \mathrm{R}$, we obtain

$$E_\rho(u_t(t), u_{tt}(t)) \leqslant \mathrm{e}^{-(t-t_0)/4} E_\rho(u_t(t_0), u_{tt}(t_0)) + 4\rho C_5. \tag{3.23}$$

4. From (1.1) for $t = t_0$ we have

$$\varepsilon u_{tt}(t_0) = -\Delta\left(u(t_0) - (u(t_0))^3 + \Delta u(t_0) - \delta u_t(t_0)\right) - u_t(t_0); \tag{3.24}$$

hence, because of (3.12),

$$E_\rho(u_t(t_0), u_{tt}(t_0)) \leqslant C_6 \frac{1}{\varepsilon}. \tag{3.25}$$

Consequently, we obtain from (3.23)

$$E_\rho(u_t(t), u_{tt}(t)) \leqslant C_7 \left(\frac{1}{\varepsilon}\mathrm{e}^{-(t-t_0)/4} + 1\right), \tag{3.26}$$

from which, letting $t_0 \to -\infty$ and recalling (3.21), we deduce (3.11), with $M_4 := C_7$. This completes the proof of Proposition 3.1, with $\varepsilon_1 := \frac{1}{2\rho}$ and $\rho := \max\{4C_4, 2, K^4\}$; note that both ε_1 and ρ are universal constants. □

We can then conclude the proof of Theorem 3.5: Indeed, estimate (3.10) follows from (3.11) and (3.12) for $t = 0$. □

3.3. *Regularity of the attractors*

In this section, we prove a regularity result for the global attractors $\mathcal{A}_{\varepsilon\delta}$ when $\delta > 0$. More precisely, we show that $\mathcal{A}_{\varepsilon\delta}$ is contained, and actually bounded, in the "more regular" space $X_2 = Y \times H^1$ defined in (1.17). This result is hardly unexpected, given the presence of the damping term $-\delta\,\Delta u_t$, which has a regularizing effect on the solution. In contrast, we have not been able to prove an analogous result for $\mathcal{A}_{\varepsilon 0}$; in fact, we are not even able to show an inclusion of the type $\mathcal{A}_{\varepsilon 0} \subset X_\gamma$, with $X_2 \hookrightarrow X_\gamma \hookrightarrow X$, $X_\gamma := H^{\gamma+1} \times H^{\gamma-1}$, $0 < \gamma < 2$ (the factor spaces being defined as in (1.12)).

Theorem 3.6. *For all* $\varepsilon, \delta \in]0, 1]$, *the global attractor* $\mathcal{A}_{\varepsilon\delta}$ *is contained in a bounded set of* X_2. *For each fixed* $\delta \in]0, 1]$, *this set is independent of* ε.

Proof. We follow a method presented by Grasselli and Pata [11] for a class of damped semilinear wave equations. We proceed in three steps: we show at first that $\mathcal{A}_{\varepsilon\delta}$ is bounded in $X_1 = H^2 \times L^2$, and then, bootstrapping the argument, we show that, in fact, $\mathcal{A}_{\varepsilon\delta}$ is bounded in X_2. Finally, we establish the bound

$$\|\nabla u_1\|^2 + \|\nabla \Delta u_0\|^2 \leqslant M, \tag{3.27}$$

where $M > 0$ is independent of ε and $(u_0, u_1) \in \mathcal{A}_{\varepsilon\delta}$, but depends on δ.

1. Let $(u_0, u_1) \in \mathcal{A}_{\varepsilon\delta}$, and consider the corresponding solution u of (1.1). We decompose $u = v + w$, where v and $w \in C_{\mathrm{b}}([0, +\infty[; X)$ are the solutions of the initial boundary value problems

$$\begin{cases} \varepsilon v_{tt} + v_t + \Delta\left(\Delta v - \delta v_t\right) = 0, \\ v(\cdot, 0) = u_0, \quad v_t(\cdot, 0) = u_1, \\ v(0, t) = v(\pi, t) = 0, \end{cases} \tag{3.28}$$

$$\begin{cases} \varepsilon w_{tt} + w_t + \Delta\left(\Delta w - w^3 - \delta w_t\right) = \Delta h, \\ w(\cdot, 0) = 0, \quad w_t(\cdot, 0) = 0, \\ w(0, t) = w(\pi, t) = 0 \end{cases} \tag{3.29}$$

with $h := v^3 + (v+w)(3vw-1)$. We next show that the function v decays exponentially, while w is more regular than u and v. More precisely:

Proposition 3.2. *Let $\delta \in [0, 1]$, and v, w be the solutions of* (3.28) *and* (3.29).

(1) *There exist universal positive constants R_1 and β, such that for all $\varepsilon \in]0, 1]$ if $0 < \delta \leqslant 1$, or for all $\varepsilon \in \left]0, \frac{1}{3}\right]$ if $\delta = 0$, and all $t \geqslant 0$,*

$$E_0(v(t), v_t(t)) \leqslant R_1\, \mathrm{e}^{-t/\beta}, \tag{3.30}$$

where E_0 is the square of the norm in X defined in (2.2).

(2) *If $\delta > 0$, the function $t \mapsto (w(t), \sqrt{\varepsilon} w_t(t))$ is bounded from* $[0, +\infty[$ into X_1; *more precisely, $(w(t), \sqrt{\varepsilon} w_t(t)) \in X_1$ for all $t \geqslant 0$, and there is a universal constant $R_2 > 0$ such that for all $t \geqslant 0$,*

$$\varepsilon\, \|w_t(t)\|^2 + \|\Delta w(t)\|^2 \leqslant R_2^2. \tag{3.31}$$

Proof. The first claim is proven with estimates analogous to those of Section 2.1. Indeed, in analogy to (2.8) we first obtain the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E_0(v, v_t) + \frac{\delta}{2}\|v\|^2\right) + (2 - \varepsilon)\|v_t\|_{-1}^2 + 2\delta\|v_t\|^2 + \|\nabla v\|^2 = 0. \tag{3.32}$$

Setting

$$\beta := \max\left\{\frac{3}{2}, K^4 + K^2 + 1\right\}, \tag{3.33}$$

we easily see that

$$E_0(v, v_t) + \frac{\delta}{2} \|v\|^2 \leqslant \beta \left(\varepsilon \|v_t\|_{-1}^2 + \|\nabla v\|^2 \right); \tag{3.34}$$

consequently, from (3.32) we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(E_0(v, v_t) + \frac{\delta}{2} \|v\|^2 \right) + \frac{1}{\beta} \left(E_0(v, v_t) + \frac{\delta}{2} \|v\|^2 \right) \leqslant 0. \tag{3.35}$$

The decay estimate (3.30) follows then by integration of (3.35); note that the corresponding constant R_1 is indeed universal, because, by Theorem 3.5, $\mathcal{A}_{\varepsilon\delta}$ is contained in a bounded set of X, independently of ε and δ.

To show the additional regularity of w, we multiply the equation of (3.29) in L^2 by $2w_t$ and w. Adding the resulting identities, and setting

$$\Phi_1(w, w_t) := \varepsilon \|w_t\|^2 + \varepsilon \langle w, w_t \rangle + \tfrac{1}{2} \|w\|^2 + \left\| \Delta w \right\|^2 + Q(w) + \frac{\delta}{2} \|\nabla w\|^2 ,$$

where $Q(w) := 3 \left\langle w^2 \nabla w, \nabla w \right\rangle$, we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \Phi_1(w, w_t) + (2 - \varepsilon) \|w_t\|^2 + \left\| \Delta w \right\|^2 + 3 \left\langle w^2 \nabla w, \nabla w \right\rangle + 2\delta \|\nabla w_t\|^2 \\ &\quad = \left\langle \Delta h, 2w_t + w \right\rangle_{H^{-1} \times H^1} + 3 \langle w\, w_t \nabla w, \nabla w \rangle =: \rho_1. \end{aligned} \tag{3.36}$$

We start the estimate of ρ_1 as follows:

$$\begin{aligned} \rho_1 \ &\leqslant\ 2 \left\| \Delta h \right\|_{-1} (\|w_t\|_1 + \|w\|_1) + 3 \, |w|_\infty \, |w_t|_\infty \ \|\nabla w\|^2 \\ &\leqslant\ C \ \|\nabla h\| \, (\|\nabla w_t\| + \|\nabla w\|) + C \ \|\nabla w\|^3 \, \|\nabla w_t\| . \end{aligned} \tag{3.37}$$

Since

$$\begin{aligned} \|\nabla h\| \ \leqslant\ & 3 \, |v|_\infty \, \|\nabla v\| + 6 \, |v|_\infty \, \|\nabla v\| \ |w|_\infty \\ & + 3 \, |v|_\infty^2 \, \|\nabla w\| + 6 \, |v|_\infty \, |w|_\infty \, \|\nabla w\| \\ & + 3 \, \|\nabla v\| \ |w|_\infty^2 + \|\nabla v\| + \|\nabla w\| \end{aligned} \tag{3.38}$$

and both $v(\cdot, t)$ and $w(\cdot, t)$ are uniformly bounded in $H^1 \hookrightarrow L^\infty$, the function $t \mapsto \|\nabla h(\cdot, t)\|$ is bounded. Hence, we deduce from (3.37) and (3.38) that

$$\rho_1 \leqslant C \, (1 + \|\nabla w_t\|) \leqslant C_\delta + \delta \ \|\nabla w_t\|^2 , \tag{3.39}$$

where C is universal and C_δ is a positive constant depending on δ, but not on ε nor on (u_0, u_1). Replacing (3.39) into (3.36), and adding $\|\nabla w\|^2$ to both sides, we obtain (for a different C_δ)

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_1(w, w_t) + \varepsilon \|w_t\|^2 + \left\|\Delta w\right\|^2 + 3\left\langle w^2\nabla w, \nabla w\right\rangle + \|\nabla w\|^2 \leqslant C_\delta. \tag{3.40}$$

From this we deduce, as usual, that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_1(w, w_t) + \frac{2}{3}\,\Phi_1(w, w_t) \leqslant C_\delta \tag{3.41}$$

and since $\Phi_1(w(0), w_t(0)) = 0$, we conclude that, for all $t \geqslant 0$,

$$\Phi_1(w(t), w_t(t)) \leqslant \frac{3}{2}\,C_\delta. \tag{3.42}$$

The conclusion of Proposition 3.2 then follows. $\square$

2. We now show that $\mathcal{A}_{\varepsilon\delta}$ is bounded in X_1. Let $(u_0, u_1) \in \mathcal{A}_{\varepsilon\delta}$. Because of (3.1), there are sequences $(t_n)_{n\geqslant 1} \subset [0, +\infty[$ and $\big((\phi_n, \psi_n)\big)_{n\geqslant 1} \subseteq B$, such that $t_n \to +\infty$ and $S_{\varepsilon\delta}(t_n)(\phi_n, \psi_n) \to (u_0, u_1)$ in X. Let v_n and w_n be the solutions of (3.28) and (3.29), corresponding to the initial values (ϕ_n, ψ_n): then, Proposition 3.2 implies that, as $n \to \infty$,

$$(v_n(t_n), (v_n)_t(t_n)) \to 0 \tag{3.43}$$

in X, while $(w_n(t_n), (w_n)_t(t_n))$ is in a bounded set of X_1. Thus, there is a subsequence, still denoted by $((w_n(t_n), (w_n)_t(t_n)))_n$, converging to a limit $(\bar{u}_0, \bar{u}_1)$ weakly in X_1 and strongly in X. Since (3.43) implies that $S_{\varepsilon\delta}(t_n)(\phi_n, \psi_n) \to (\bar{u}_0, \bar{u}_1)$, it follows that $(u_0, u_1) = (\bar{u}_0, \bar{u}_1)$ is in a bounded set of X_1. Thus, $\mathcal{A}_{\varepsilon\delta}$ is bounded in X_1, as claimed.

3. We now bootstrap this argument, and show that, in fact, $\mathcal{A}_{\varepsilon\delta}$ is bounded in X_2. For $(u, v) \in X_1$ we set

$$E_1(u, v) := \varepsilon \|v\|^2 + \varepsilon \langle u, v\rangle + \tfrac{1}{2}\|u\|^2 + \left\|\Delta u\right\|^2 \tag{3.44}$$

and claim:

Proposition 3.3. *Let ε, $\delta \in]0, 1]$, $(u_0, u_1) \in \mathcal{A}_{\varepsilon\delta}$, and v, w be the solutions of* (3.28) *and* (3.29) *corresponding to the initial values (u_0, u_1).*

(1) *There is a universal constant $R_3 > 0$, such that for all $\varepsilon \in]0, 1]$ and all $t \geqslant 0$,*

$$E_1(v(t), v_t(t)) \leqslant R_3\,\mathrm{e}^{-t/\beta}, \tag{3.45}$$

where β is as in (3.33).

(2) *The function* $t \mapsto (w(t), \sqrt{\varepsilon} w_t(t))$ *is bounded from* $[0, +\infty[$ *into* X_2; *more precisely,* $(w(t), \sqrt{\varepsilon} w_t(t)) \in X_2$ *for all* $t \geqslant 0$, *and there is a universal constant* $R_4 > 0$ *such that for all* $t \geqslant 0$,

$$\varepsilon \|\nabla w_t(t)\|^2 + \|\nabla \Delta w(t)\|^2 \leqslant R_4^2. \tag{3.46}$$

Proof. The proof is similar to that of Proposition 3.2. Multiplying the equation of (3.28) in L^2 by $2v_t$ and v, and adding the resulting identities, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E_1(v, v_t) + \frac{\delta}{2}\|\nabla v\|^2\right) + (2-\varepsilon)\|v_t\|^2 + \left\|\Delta v\right\|^2 + 2\delta\|\nabla v_t\|^2 \leqslant 0. \tag{3.47}$$

Since

$$\|\nabla v\|^2 = \left\langle -\Delta v, v\right\rangle \leqslant \left\|\Delta v\right\| \, \|v\| \leqslant K \, \left\|\Delta v\right\| \, \|\nabla v\|,$$

we have that $\|\nabla v\| \leqslant K \left\|\Delta v\right\|$; consequently,

$$\begin{aligned} E_1(v, v_t) + \frac{\delta}{2} \|\nabla v\|^2 &\leqslant \frac{3}{2}\varepsilon\|v_t\|^2 + \|v\|^2 + \left\|\Delta v\right\|^2 + \|\nabla v\|^2 \\ &\leqslant \frac{3}{2}\varepsilon\|v_t\|^2 + (K^2+1)\|\nabla v\|^2 + \left\|\Delta v\right\|^2 \\ &\leqslant \frac{3}{2}\varepsilon\|v_t\|^2 + (K^4+K^2+1)\left\|\Delta v\right\|^2. \end{aligned} \tag{3.48}$$

Inserting (3.48) into (3.47) we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E_1(v, v_t) + \frac{\delta}{2}\|\nabla v\|^2\right) + \frac{1}{\beta}\left(E_1(v, v_t) + \frac{\delta}{2}\|\nabla v\|^2\right) \leqslant 0, \tag{3.49}$$

from which (3.45) follows.

To show the additional regularity of *w*, we multiply the equation of (3.29) in L^2 by $-2\Delta w_t$ and $-\Delta w$. Adding the resulting identities, and setting, for $(u, v) \in X_2$,

$$\Phi_2(u, v) := \varepsilon\|\nabla v\|^2 + \varepsilon\langle \nabla u, \nabla v\rangle + \tfrac{1}{2}\|\nabla u\|^2 + \left\|\nabla \Delta u\right\|^2 + \frac{\delta}{2}\left\|\Delta u\right\|^2, \tag{3.50}$$

we obtain, as usual,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Phi_2(w, w_t) + (2-\varepsilon)\|\nabla w_t\|^2 + \left\|\nabla\Delta w\right\|^2 + 2\delta\left\|\Delta w_t\right\|^2 \\ &\quad = -\left\langle \Delta h + \Delta(w^3), 2\Delta w_t + \Delta w\right\rangle. \end{aligned} \tag{3.51}$$

By Proposition 3.2, we know that both $v(\cdot,t)$ and $w(\cdot,t)$ are uniformly bounded in H^2; therefore, as we can easily verify, $\Delta h + \Delta(w^3)$ is uniformly bounded in L^2. Consequently, we obtain from (3.51)

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_2(w,w_t) + (2-\varepsilon)\,\|\nabla w_t\|^2 + \left\|\nabla\Delta w\right\|^2 + 2\delta\,\left\|\Delta w_t\right\|^2 \leqslant C_\delta + \delta\,\left\|\Delta w_t\right\|^2.$$

From this, recalling that, in the usual way, for all $(u,v)\in X_2$,

$$\tfrac{1}{2}\left(\varepsilon\,\|\nabla v\|^2 + \left\|\nabla\Delta u\right\|^2\right) \leqslant \Phi_2(u,v) \leqslant \beta\left(\varepsilon\,\|\nabla v\|^2 + \left\|\nabla\Delta u\right\|^2\right), \tag{3.52}$$

we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_2(w,w_t) + \frac{1}{\beta}\,\Phi_2(w,w_t) \leqslant C_\delta. \tag{3.53}$$

Since $\Phi_2(w(0),w_t(0)) = 0$, integration of (3.53) allows us to conclude the proof of Proposition 3.3. □

4. We can now show that $\mathcal{A}_{\varepsilon\delta}$ is bounded in X_2. With the same notations of part (2) of the present proof we now have that (3.43) holds also in X_1, while $(w_n(t_n),(w_n)_t(t_n))$ is in a bounded set of X_2. Thus, there is a second subsequence, still denoted by $((w_n(t_n),(w_n)_t(t_n)))_n$, converging to a limit $(\bar u_0,\bar u_1)$ weakly in X_2 and strongly in X_1. Since (3.43) implies that $S_{\varepsilon\delta}(t_n)(\phi_n,\psi_n)\to(\bar u_0,\bar u_1)$, it follows that $(u_0,u_1)=(\bar u_0,\bar u_1)$ is in a bounded set of X_2. Thus, $\mathcal{A}_{\varepsilon\delta}$ is bounded in X_2, as claimed.

5. We now proceed to show that, in fact, $\mathcal{A}_{\varepsilon\delta}$ can be bounded in X_2 independently of ε. Let $(u_0,u_1)\in\mathcal{A}_{\varepsilon\delta}$. By Lemma 3.1 (u_0,u_1) lies on a complete orbit $(u(t),u_t(t))_{t\in\mathrm{R}}$, contained in $\mathcal{A}_{\varepsilon\delta}$; without loss of generality, we can assume that $(u_0,u_1)=(u(0),u_t(0))$. Since all the constants C_δ appearing in the proof of the boundedness of $\mathcal{A}_{\varepsilon\delta}$ in X_1 and X_2 depend only on δ (i.e., they are otherwise universal), we deduce from the uniform estimate (3.12) (which also holds for $\delta>0$), that the estimate

$$\varepsilon\,\|\nabla u_1\|^2 + \left\|\nabla\Delta u_0\right\|^2 \leqslant C_2, \tag{3.54}$$

holds, uniformly with respect to ε and $(u_0,u_1)\in\mathcal{A}_{\varepsilon\delta}$ (however, C_2 depends on δ). This provides part of (3.27); to remove the dependence of the term with u_1 on ε, we prove

Proposition 3.4. *Let $(u(t),u_t(t))_{t\in\mathrm{R}}$ be a complete orbit contained in $\mathcal{A}_{\varepsilon\delta}$. There exists a positive constant C_3, dependent on δ but not on ε, such that for all $t\in\mathrm{R}$ and $\varepsilon\in$* $]0,1]$,

$$\varepsilon\,\|u_{tt}(t)\|^2 + \|\nabla u_t(t)\|^2 \leqslant C_3. \tag{3.55}$$

Proof. The proof is similar to that of Proposition 3.1; note that (3.55) involves a higher regularity of the orbit than (3.11). In the sequel, we denote by C_δ various different positive constants, depending on δ but not on ε, nor on t. Note that, since $\mathcal{A}_{\varepsilon\delta}$ is invariant, and (3.54) holds uniformly with respect to $(u_0, u_1) \in \mathcal{A}_{\varepsilon\delta}$, we have that for all $t \in \mathrm{R}$,

$$\varepsilon \|\nabla u_t(t)\|^2 + \left\|\nabla \Delta u(t)\right\|^2 \leqslant C_\delta. \tag{3.56}$$

As a preliminary step, we show that there is $\bar{C}_\delta > 0$, independent of t and ε, such that for all $t \in \mathrm{R}$ and $\varepsilon \in]0, 1]$,

$$\|u_t(t)\|_{-1} \leqslant \bar{C}_\delta. \tag{3.57}$$

Multiplying Eq. (1.16) by $2u_t$ we obtain, by (3.56),

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}t} \|u_t\|_{-1}^2 + \|u_t\|_{-1}^2 \leqslant \left\|\Delta u - u^3 + u\right\|_1^2 \leqslant C_\delta. \tag{3.58}$$

Integrating (3.58) on an arbitrary interval $[t_0, t] \subset \mathrm{R}$, and recalling (3.56) again, we obtain

$$\|u_t(t)\|_{-1}^2 \leqslant C_\delta \left(\frac{1}{\varepsilon}\, \mathrm{e}^{-(t-t_0)/\varepsilon} + 1 \right), \tag{3.59}$$

from which we deduce (3.57) by letting $t_0 \to -\infty$.

We now differentiate Eq. (1.16) with respect to t, and multiply the resulting equation by $2u_{tt} + u_t$, to obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Big(\varepsilon \|u_{tt}\|_{-1}^2 + \varepsilon[u_{tt}, u_t] + \frac{1}{2}\, \|u_t\|_{-1}^2 + \|\nabla u_t\|^2 + \frac{1}{2}\delta\, \|u_t\|^2\Big) \\ &\quad + (2-\varepsilon)\, \|u_{tt}\|_{-1}^2 + \|\nabla u_t\|_{-1}^2 + 2\delta\, \|u_{tt}\|^2 \\ &= -\Big\langle (3u^2 - 1)u_t, 2u_{tt} + u_t \Big\rangle =: R_1. \end{aligned} \tag{3.60}$$

By the interpolation inequality

$$\|u_t\|^2 \leqslant C\ \|u_t\|_1\ \|u_t\|_{-1}$$

and recalling (3.12) and (3.57), we can estimate the right side of (3.60) by

$$R_1 \leqslant \delta\ \|u_{tt}\|^2 + C_\delta\ \|u_t\|^2 \leqslant \delta\ \|u_{tt}\|^2 + \tfrac{1}{2}\, \|\nabla u_t\|^2 + C_\delta. \tag{3.61}$$

We now denote by $\Phi_3(u_t, u_{tt})$ the term under differentiation in (3.60). Replacing (3.61) into (3.60), and recalling (3.57), we obtain, as usual, the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_3(u_t, u_{tt}) + \frac{1}{2\beta}\Phi_3(u_t, u_{tt}) \leqslant C_\delta. \tag{3.62}$$

Integrating this inequality on an arbitrary interval $[t_0, t] \subset \mathrm{R}$, we obtain

$$\Phi_3(u_t(t), u_{tt}(t)) \leqslant \mathrm{e}^{-(t-t_0)/2\beta}\Phi_3(u_t(t_0), u_{tt}(t_0)) + 2\beta C_\delta. \tag{3.63}$$

From (3.24) and (3.56) we have

$$\Phi_3(u_t(t_0), u_{tt}(t_0)) \leqslant C_\delta \frac{1}{\varepsilon}; \tag{3.64}$$

consequently, we obtain from (3.63)

$$\Phi_3(u_t(t), u_{tt}(t)) \leqslant C_\delta \left(\frac{1}{\varepsilon}\mathrm{e}^{-(t-t_0)/2\beta} + 1\right). \tag{3.65}$$

Letting $t_0 \to -\infty$ in (3.65) we can finally deduce (3.55). This completes the proof of Proposition 3.4. □

We can now conclude the proof of Theorem 3.6: Indeed, (3.27) follows from (3.55), taking $t = 0$. □

3.4. *Upper semicontinuity of the global attractors*

In this section, we present some results on the upper semicontinuity of the global attractors $\mathcal{A}_{\varepsilon\delta}$, either as $\varepsilon \to 0$ for fixed δ, or as $\delta \to 0$ for fixed ε. As a byproduct, we also deduce some results on the convergence of solutions of problems $\mathcal{CH}_{\varepsilon\delta}$ to those of problem $\mathcal{CH}_{0\delta}$ when $\varepsilon \to 0$, or of problem $\mathcal{CH}_{\varepsilon 0}$, when $\delta \to 0$. We shall loosely follow the arguments developed by Hale [12, Chapter 4.10], from which we recall the following definition of upper semicontinuity of a family of sets.

Definition 3.3. Let X be a complete metric space, $\Lambda \subseteq \mathrm{R}$, and $(C_\lambda)_{\lambda\in\Lambda}$ a family of subsets of X. Let $\lambda_0 \in \Lambda$. Then, $(C_\lambda)_{\lambda\in\Lambda}$ is upper semicontinuous at λ_0 if

$$\lim_{\lambda\to\lambda_0} \operatorname{dist}\left(C_\lambda, C_{\lambda_0}\right) = 0, \tag{3.66}$$

where dist is the semidistance in X defined on subsets A, $B \subseteq X$, by

$$\operatorname{dist}(A, B) := \sup_{a\in A}\, \inf_{b\in B}\, \|a - b\|_X\,. \tag{3.67}$$

In the sequel, for $\eta > 0$ and integer m we set

$$X_{m-\eta} := H^{m+1-\eta} \times H^{m-1-\eta}, \tag{3.68}$$

with the factor spaces defined in accord to (1.12).

3.4.1. The case $\delta > 0$, $\varepsilon \to 0$

In this section, we consider the upper semicontinuity of the attractors $\mathcal{A}_{\varepsilon\delta}$ as $\varepsilon \to 0$, for fixed $\delta \in]0, 1]$. By Theorem 3.6, we know that these attractors are bounded in X_2, uniformly with respect to ε. For $\delta \in]0, 1]$, let $A_{0\delta}$ be the global attractors of the semiflows $S_{0\delta}$ generated by the parabolic problems $\mathcal{CH}_{0\delta}$. As stated in the introduction, these attractors are known to exist and, by well known parabolic regularity results, to be bounded in H^3. Hence, we can introduce the sets

$$\mathcal{A}_{0\delta} := \left\{(u, v) \in X : u \in A_{0\delta}, \quad v = -\Delta(I - \delta\Delta)^{-1} \left(u - u^3 + \Delta u\right)\right\}, \tag{3.69}$$

which we consider as "natural'' imbeddings of $A_{0\delta}$ in X. We have then the following result:

Theorem 3.7. *Let ε_1 be as in Proposition* 3.1. *For $0 < \varepsilon \leqslant \varepsilon_1$ and $0 < \delta \leqslant 1$, let $\mathcal{A}_{\varepsilon\delta}$ be the global attractor of the semiflow $S_{\varepsilon\delta}$ generated by the hyperbolic problem $\mathcal{CH}_{\varepsilon\delta}$. Let $\mathcal{A}_{0\delta}$ be as in* (3.69). *Then for any $\delta \in]0, 1]$ and $\eta > 0$, the family $(\mathcal{A}_{\varepsilon\delta})_{0 \leqslant \varepsilon \leqslant \varepsilon_1}$ is upper-semicontinuous at $\varepsilon = 0$, with respect to the topology of $X_{2-\eta}$.*

Proof. Recalling (3.66), we must show that

$$\sup_{(u,v)\in\mathcal{A}_{\varepsilon\delta}} \inf_{(\bar{u},\bar{v})\in\mathcal{A}_{0\delta}} \left(\|u - \bar{u}\|^2_{3-\eta} + \|v - \bar{v}\|^2_{1-\eta}\right)^{1/2} \to 0 \tag{3.70}$$

as $\varepsilon \to 0$. We reason by contradiction. Assuming (3.70) did not hold, we could find $r_0 > 0$, and sequences $(\varepsilon_n)_{n\in N}$, $\big((\varphi_n, \psi_n)\big)_{n\in N} \subseteq \mathcal{A}_{\varepsilon_n\delta}$, such that $\varepsilon_n \to 0$, and for all $n \in N$,

$$\inf_{(\bar{u},\bar{v})\in\mathcal{A}_{0\delta}} \left(\|\varphi_n - \bar{u}\|^2_{3-\eta} + \|\psi_n - \bar{v}\|^2_{1-\eta}\right) \geqslant r_0^2. \tag{3.71}$$

By (3.27), we have the uniform estimate

$$\|\psi_n\|^2_1 + \|\varphi_n\|^2_3 \leqslant M \tag{3.72}$$

with M independent of n; thus, there is a subsequence, still denoted by $\big((\varphi_n, \psi_n)\big)_{n\in N}$, converging to a limit (φ_*, ψ_*) weakly in X_2 and, by compactness, strongly in $X_{2-\eta}$. We now claim that $(\varphi_*, \psi_*) \in \mathcal{A}_{0\delta}$: if true, this would contradict (3.71).

By Lemma 3.1, for each $n \in N$ there is a complete orbit

$$\big(u^n(t), u_t^n(t)\big)_{t\in \mathrm{R}} := \big(S_{\varepsilon_n\delta}(t)(\varphi_n, \psi_n)\big)_{t\in \mathrm{R}} \tag{3.73}$$

contained in $\mathcal{A}_{\varepsilon_n\delta}$ and passing through (φ_n, ψ_n). In particular, we can assume that

$$(\varphi_n, \psi_n) = (u^n(0), u_t^n(0)). \tag{3.74}$$

From (3.55) and (3.56) we have the uniform estimates

$$\varepsilon_n \left\|u_{tt}^n(t)\right\|_{-1}^2 + \left\|u_t^n(t)\right\|_1^2 + \left\|u^n(t)\right\|_3^2 \leqslant M_5^2, \tag{3.75}$$

with M_5 independent of t and ε_n. From this it follows that for all $T > 0$, the functions $u^{\varepsilon_n\delta}$, $u_t^{\varepsilon_n\delta}$ and $\sqrt{\varepsilon_n}u_{tt}^{\varepsilon_n\delta}$ are, respectively, in a bounded set of $L^\infty(-T, T; H^3)$, $L^\infty(-T, T; H^1)$ and $L^\infty(-T, T; H^{-1})$. Consequently, for each $\delta \in]0, 1]$ there are a function u^δ, and a subsequence, still denoted $(\varepsilon_n)_{n\in N}$, such that

$$u^{\varepsilon_n\delta} \to u^\delta \qquad \text{in} \quad L^\infty(-T, T; H^3) \quad \text{weakly}^*, \tag{3.76}$$

$$u_t^{\varepsilon_n\delta} \to u_t^\delta \qquad \text{in} \quad L^\infty(-T, T; H^1) \quad \text{weakly}^*, \tag{3.77}$$

$$\varepsilon_n u_{tt}^{\varepsilon_n\delta} \to 0 \qquad \text{in} \quad L^\infty(-T, T; H^{-1}) \quad \text{weakly}^*. \tag{3.78}$$

We now show that, for each $\delta \in]0, 1]$, u^δ is a weak solution of the parabolic problem $\mathcal{CH}_{0\delta}$ on R (which are defined similarly to Definition 2.1, further replacing the interval $[0, +\infty[$ with all R).

Proposition 3.5. *Let u^δ be defined as the limit in (3.76). Then u^δ is a weak solution of problem $\mathcal{CH}_{0\delta}$ in* R, *with initial value $u^\delta(0) = u_0$. In fact, u^δ is a complete orbit for problem $\mathcal{CH}_{0\delta}$.*

Proof. Let φ be a test function, as per Definition 2.1, and fix T so that $\operatorname{supp}(\varphi) \subseteq [-T, T]$. As in (3.9), the injection

$$\{u \in L^2(-T, T; H^3) \ : \ u_t \in L^2(-T, T; H^1)\} \hookrightarrow \ L^2(-T, T; H^2) \tag{3.79}$$

is compact. Since the restriction operator $u \mapsto u_{|[-T,T]}$ is continuous, denoting restrictions $u_{|[-T,T]}$ still by u we deduce from (3.76) and (3.77) that, taking if necessary a further subsequence $(\varepsilon_n)_{n\in N} \to 0$,

$$u^{\varepsilon_n\delta} \to u^\delta \qquad \text{in} \quad L^2(-T, T; H^2) \quad \text{strongly}. \tag{3.80}$$

From (3.80), recalling (3.12), it is easy deduce that also

$$(u^{\varepsilon_n\delta})^3 \to (u^\delta)^3 \qquad \text{in} \quad L^2(-T,T;L^2) \quad \text{strongly.} \tag{3.81}$$

Thus, we can let $\varepsilon_n \to 0$ in Eq. (2.1), and deduce that u^δ is a weak solution of $\mathcal{CH}_{0\delta}$, provided that we can show that $u^\delta(0) = u_0$. To this end, we recall from Lions–Magenes [16, Chapter 1, Theorem 3.1], that the space

$$\{u \in L^2(-T,T;H^2) \ : \ u_t \in L^2(-T,T;H^1)\}$$

is continuously injected in $C([-T,T];H^1)$. Hence, (3.77) and (3.80) imply that

$$\max_{-T\leqslant t\leqslant T} \|u^{\varepsilon_n\delta}(t) - u^\delta(t)\|_1 \ \to \ 0; \tag{3.82}$$

that is,

$$u^{\varepsilon_n\delta} \to u^\delta \qquad \text{in} \quad C([-T,T];H^1) \quad \text{strongly,} \tag{3.83}$$

as $\varepsilon_n \to 0$. In particular, (3.82) implies that $u^\delta(0) = u_0$, as claimed. With this, the proof of Proposition 3.5 is complete. □

We can now conclude the proof of Theorem 3.7. By (3.82), $\varphi_n = u^n(0) \to u^\delta(0)$ in H^1; hence, $u^\delta(0) = \varphi_*$, and, therefore, $u^\delta(0) \in H^3$. Moreover, since u^δ is a complete orbit of $S_{0\delta}$ passing through φ_*, Lemma 3.1 yields that $\varphi_* \in A_{0\delta}$. By (3.75),

$$\left\|\varepsilon_n u^n_{tt}(0)\right\|_{-1} = \sqrt{\varepsilon_n}\ \left\|\sqrt{\varepsilon_n}u^n_{tt}(0)\right\|_{-1} \leqslant \sqrt{\varepsilon_n}\, M_5; \tag{3.84}$$

hence, $\varepsilon_n u^n_{tt}(0) \to 0$ in H^{-1}. Consequently,

$$\begin{aligned} u^n_t(0) - \delta\Delta u^n_t(0) &= -\Delta\left(u^n(0) - (u^n(0))^3 + \Delta u^n(0)\right) - \varepsilon_n u^n_{tt}(0) \\ &= -\Delta\left(\varphi_n - \varphi_n^3 + \Delta\varphi_n\right) - \varepsilon_n u^n_{tt}(0) \\ &\to -\Delta\left(\varphi_* - \varphi_*^3 + \Delta\varphi_*\right) \end{aligned} \tag{3.85}$$

in H^{-1} weakly. Since $u^n_t(0) = \psi_n$, from (3.85) we deduce that

$$\psi_* = -\Delta(I - \delta\Delta)^{-1}\left(\varphi_* - \varphi_*^3 + \Delta\varphi_*\right). \tag{3.86}$$

Since $\varphi_* \in A_{0\delta}$, (3.86) implies that $(\varphi_*, \psi_*) \in \mathcal{A}_{0\delta}$, as claimed. Having thus reached the desired contradiction with (3.71), the proof of Theorem 3.7 is complete. □

3.4.2. The case $\delta = 0$, $\varepsilon \to 0$

In this section, we consider the upper semicontinuity of the attractors $\mathcal{A}_{\varepsilon 0}$ as $\varepsilon \to 0$. By Theorem 3.5, these attractors are bounded in X, uniformly with respect to ε. Let A_{00} be the global attractor of the semiflow S_{00} generated by the parabolic problem $\mathcal{CH}_{00}$, which, as we have recalled above, is a bounded set of H^3. We introduce the set

$$\mathcal{A}_{00} := \left\{(u, v) \in X \ : \ u \in A_{00}, \quad v = -\Delta\left(u - u^3 + \Delta u\right)\right\}, \tag{3.87}$$

as a “natural” imbedding of A_{00} in X. We have then the following result:

Theorem 3.8. *Let ε_1 be as in Proposition* 3.1. *For* $0 < \varepsilon \leqslant \varepsilon_1$, *let $\mathcal{A}_{\varepsilon 0}$ be the global attractor of the semiflow $S_{\varepsilon 0}$ generated by the hyperbolic problem $\mathcal{CH}_{\varepsilon 0}$. Let $\mathcal{A}_{00}$ be as in* (3.87). *Then for any $\eta > 0$, the family $(\mathcal{A}_{\varepsilon 0})_{0 \leqslant \varepsilon \leqslant \varepsilon_1}$ is upper-semicontinuous at $\varepsilon = 0$, with respect to the topology of $X_{-\eta}$.*

Proof. The proof is identical to that of Theorem 3.7, except that the spaces H^3, H^1 and H^{-1} are replaced, respectively, by H^1, H^{-1} and H^{-3}. Note that the analogous of the uniform estimate (3.75), i.e. the estimate

$$\varepsilon_n \left\|u^n_{tt}(t)\right\|^2_{-3} + \left\|u^n_t(t)\right\|^2_{-1} + \left\|u^n(t)\right\|^2_1 \leqslant M_6^2 \tag{3.88}$$

with M_6 independent of t and ε_n, is a consequence of (3.11) and (3.12). Moreover, in the proof of (3.81), we only need the strong convergence $u^n \to u^\delta$ in $L^2(-T, T; L^2)$, which holds also when $\delta = 0$. We can therefore omit the details of the proof. □

We remark that the weak solution u^δ found in Theorems 3.7 and 3.8 is actually in C^∞ for $t > 0$, as can be easily shown by standard parabolic regularity techniques (see e.g. [24]).

3.4.3. The case $\delta \to 0$, $\varepsilon > 0$

Our last goal is to prove the upper semicontinuity of the attractors $\mathcal{A}_{\varepsilon\delta}$ as $\delta \to 0$, for fixed $\varepsilon \in]0, \varepsilon_1]$. By Theorem 3.5, these attractors are bounded in X, uniformly also with respect to δ. We have then the following result:

Theorem 3.9. *Let ε_1 be as in Proposition* 3.1. *For* $0 < \varepsilon \leqslant \varepsilon_1$ *and* $0 \leqslant \delta \leqslant 1$, *let $\mathcal{A}_{\varepsilon\delta}$ be the attractors of the semiflows $S_{\varepsilon\delta}$ generated by the hyperbolic problems $\mathcal{CH}_{\varepsilon\delta}$. For any $\eta > 0$ and $\varepsilon \in]0, \varepsilon_1]$, the family $(\mathcal{A}_{\varepsilon\delta})_{0 \leqslant \delta \leqslant 1}$ is upper-semicontinuous at $\delta = 0$, with respect to the topology of $X_{-\eta}$.*

Proof. The proof proceeds in the same spirit of that of Theorem 3.7. We must show that

$$\sup_{(u,v)\in\mathcal{A}_{\varepsilon\delta}} \inf_{(\bar u,\bar v)\in\mathcal{A}_{\varepsilon 0}} \left(\|u-\bar u\|^2_{1-\eta}+\|v-\bar v\|^2_{-1-\eta}\right)^{1/2} \to 0 \tag{3.89}$$

as $\delta\to 0$. Assuming otherwise, we could find $r_0>0$, and sequences $\big(\delta_n\big)_{n\in N}$, $\big((\varphi_n,\psi_n)\big)_{n\in N}\subseteq\mathcal{A}_{\varepsilon\delta}$, such that $\delta_n\to 0$, and for all $n\in N$,

$$\inf_{(\bar u,\bar v)\in\mathcal{A}_{\varepsilon 0}} \left(\left\|\varphi_n-\bar u\right\|^2_{1-\eta}+\left\|\psi_n-\bar v\right\|^2_{-1-\eta}\right) \geqslant r_0^2. \tag{3.90}$$

As in Theorem 3.7, we see that, since the constant M_3 of (3.10) is also independent of δ, the sequence $\big((\varphi_n,\psi_n)\big)_{n\in N}$ admits a subsequence, still denoted by $\big((\varphi_n,\psi_n)\big)_{n\in N}$, converging weakly to a limit (φ_*,ψ_*) in X. We now claim that $(\varphi_*,\psi_*)\in\mathcal{A}_{\varepsilon 0}$: if true, this would contradict (3.89).

By Lemma 3.1, for each $n\in N$ there is a complete orbit

$$\big(u^n(t),u^n_t(t)\big)_{t\in\mathrm{R}} := \big(S_{\varepsilon\delta_n}(t)(\varphi_n,\psi_n)\big)_{t\in\mathrm{R}} \tag{3.91}$$

contained in $\mathcal{A}_{\varepsilon\delta_n}$ and passing through (φ_n,ψ_n); again, we can assume that (3.74) holds. We can establish an estimate analogous to (3.75); from this estimate, recalling that ε is now fixed, we deduce that for any $T>0$, the functions $u^{\varepsilon\delta_n}_{tt}$ are in a bounded set of $L^2(-T,T;H^{-3})$. Consequently, for each $\varepsilon\in]0,\varepsilon_1]$ there are a function u^ε, and a sequence $\big(\delta_n\big)_{n\in N}$, such that $\delta_n\to 0$ and

$$u^{\varepsilon\delta_n}\to u^\varepsilon \quad \text{in} \quad L^\infty(-T,T;H^1) \quad \text{weakly}^*, \tag{3.92}$$

$$u^{\varepsilon\delta_n}_t\to u^\varepsilon_t \quad \text{in} \quad L^\infty(-T,T;H^{-1}) \quad \text{weakly}^*, \tag{3.93}$$

$$u^{\varepsilon\delta_n}_{tt}\to u^\varepsilon_{tt} \quad \text{in} \quad L^\infty(-T,T;H^{-3}) \quad \text{weakly}^*. \tag{3.94}$$

Proceeding as in Section 3.4.1, it is easy to see that u^ε is a weak solution of the hyperbolic problem $\mathcal{CH}_{\varepsilon 0}$ on R, with the same initial values u_0 and u_1. As in (3.80), by passing if necessary to a further subsequence $(\delta_n)_n\to 0$,

$$\begin{aligned} u^{\varepsilon\delta_n}&\to u^\varepsilon \quad \text{in} \quad L^2(-T,T;L^2) \quad \text{strongly},\\ u^{\varepsilon\delta_n}_t&\to u^\varepsilon_t \quad \text{in} \quad L^2(-T,T;H^{-2}) \quad \text{strongly}; \end{aligned} \tag{3.95}$$

hence, we deduce, as in (3.82), that

$$\max_{-T\leqslant t\leqslant T}\left\|u^n(t)-u^\varepsilon(t)\right\|_{-1}+\max_{-T\leqslant t\leqslant T}\left\|u^n_t(t)-u^\varepsilon_t(t)\right\|_{-3} \to 0. \tag{3.96}$$

Thus, $\varphi_n = u^n(0) \to u^\varepsilon(0)$ in H^{-1} and $\psi_n = u_t^n(0) \to u_t^\varepsilon(0)$ in H^{-3}. But, since $\varphi_n \to \varphi_*$ in H^1 weakly, and $\psi_n \to \psi_*$ in H^{-1} weakly, we deduce that $u^\varepsilon(0) = \varphi_* \in H^1$, and $u_t^\varepsilon(0) = \psi_* \in H^{-1}$. Since also, obviously, $-\delta_n \Delta u_t^n(0) \to 0$ in H^{-1}, we conclude that $(u^\varepsilon(0), u_t^\varepsilon(0)) = (\varphi_*, \psi_*)$. Thus, $(u^\varepsilon(0), u_t^\varepsilon(0)) \in X$; moreover, since $\left(u^\varepsilon(t), u_t^\varepsilon(t)\right)_{t\in\mathrm{R}}$ is a bounded complete orbit through (φ_*, ψ_*), we conclude that $(\varphi_*, \psi_*) \in \mathcal{A}_{\varepsilon 0}$, as claimed. Thus, we reach a contradiction with (3.90), and the proof of Theorem 3.9 is complete. $\square$

As a final remark, we mention that, when $\varepsilon = 0$, the upper-semicontinuity of the family $(\mathcal{A}_{0\delta})_{0\leqslant\delta\leqslant 1}$ as $\delta \to 0$ is a consequence of that of the attractors $A_{0\delta}$, which has been proven in [6]. Hence, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{A}_{\varepsilon\delta} & \longrightarrow & \mathcal{A}_{\varepsilon 0} \\ \downarrow & & \downarrow \\ \mathcal{A}_{0\delta} & \longrightarrow & \mathcal{A}_{00} \end{array}, \tag{3.97}$$

where the vertical arrows mean convergence, in the sense of (3.67), as $\varepsilon \to 0$, and the horizontal arrows mean convergence as $\delta \to 0$.

Appendix A

In this section, we give a self-contained proof of Theorems 3.1 and 3.2, loosely following [12, Chapters 2.2 and 3.2], with some important modifications.

1. We first recall some preliminary definitions, notations, and results. In the sequel we denote by E a complete metric space with distance d.

Given $M \subset E$, we denote by $I(M)$ the set consisting of all those numbers $\beta > 0$ such that M has a finite covering of sets, each having diameter not exceeding β.

Definition A.1. Let $\mathcal{P}(E)$ denote power set of E (that is, the set of its subsets). A measure of compactness on E is the map $\alpha\colon \mathcal{P}(E) \to [0, +\infty]$ defined by

$$E \supseteq A \;\mapsto \alpha(A) := \begin{cases} +\infty & \text{if } \; A \; \text{ has no finite covering,} \\ \inf I(A) & \text{otherwise.} \end{cases} \tag{A.1}$$

The following Proposition, of immediate proof, lists the main properties of measures of compactness.

Proposition A.1. *Let α be a measure of compactness on E. Then*:

1. *If $A \subset E$ is bounded, $\alpha(A) < +\infty$*;
2. *If $A \subseteq B$, $\alpha(A) \leqslant \alpha(B)$* (monotonicity);
3. *If $\alpha(A) = 0$, then A is totally bounded*;

4. *If* $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq \ldots$ *is a decreasing sequence of nonempty closed sets such that* $\alpha(A_n) \to 0$ *as* $n \to +\infty$, *then the set*

$$A := \bigcap_{n \geqslant 1} A_n$$

is compact.

In the sequel, we shall need the following technical result, which is a consequence of Lemma 2.1.1 of [12, Chapter 2].

Proposition A.2. *Let S be a semiflow on X, and* $B \subseteq X$ *be such that* $\omega(B)$ *is compact, and attracts B. Then* $\omega(B)$ *is invariant under S.*

2. We now proceed to sketch the proof of Theorem 3.1 of Section 3.
   2.1. For $n \in N$, set $A_n := \overline{T^n(B)}$. Clearly, $A_n \supseteq A_{n+1}$ for each n. We show that, as a consequence,

$$\omega(B) = \bigcap_{n \geqslant 0} A_n =: A.$$

To see this, note first that, since obviously

$$T^n(B) \subseteq \bigcup_{m \geqslant n} T^m(B),$$

we immediately deduce that

$$A \subseteq \omega(B) = \bigcap_{n \geqslant 0} \overline{\bigcup_{m \geqslant n} T^m(B)}.$$

Conversely, let $z \in \omega(B)$. Then, there are sequences $(n_j)_{j \in N}$ and $(z_j)_{j \in N} \subseteq B$, such that $n_j \to \infty$ and $T^{n_j} z_j \to z$ as $j \to \infty$. Now, for each $n \in N$ there is $j_n \in N$ such that $n_j \geqslant n$ for all $j \geqslant j_n$. Hence, for $j \geqslant j_n$,

$$T^{n_j} z_j \in T^{n_j}(B) \subseteq A_{n_j} \subseteq A_n.$$

Letting $j \to \infty$, it follows that $z \in A_n$ for all $n \in N$. Consequently, $z \in A$, and $\omega(B) = A$.
   2.2. Since B is bounded, there is $M > 0$ such that $\alpha(B) \leqslant M$. A repeated application of (3.3) yields then

$$\alpha(A_n) = \alpha(T^n(B)) \leqslant q^n \alpha(B) \leqslant q^n M;$$

thus, $\alpha(A_n) \to 0$. Since each A_n is closed, part (4) of Proposition A.1 implies that $\omega(B) = A$ is compact.

2.3. To see that $\omega(B)$ attracts B, we show that

$$\lim_{n\to\infty} d(T^n x, \omega(B)) = 0, \tag{A.2}$$

uniformly in $x \in B$; that is, for any $\varepsilon > 0$ there exists N such that for all integer $n \geqslant N$ and all $x \in B$,

$$d(T^n x, \omega(B)) < \varepsilon.$$

Proceeding by contradiction, assume there is $\varepsilon_0 > 0$ such that for all integers j it is possible to find another integer $n_j \geqslant j$, and a point $x_j \in B$, such that

$$d(T^{n_j}(x_j), \omega(B)) \geqslant \varepsilon_0. \tag{A.3}$$

This process defines a bounded sequence $\zeta_* := (T^{n_j} x_j)_{j\in N} \subset B$. If we can show that ζ_* contains a convergent subsequence, we reach the desired contradiction, because by (A.3) the limit z of this subsequence would on the one hand be in $\omega(B)$, and on the other hand z would satisfy $d(z, \omega(B)) \geqslant \varepsilon_0$.
To show that ζ_* does contain a convergent subsequence, let Σ be the subset of B consisting of all the sequences of the form $\zeta = (T^{m_j} x_j)_{j\in N}$, with $x_j \in B$, $m_j \in N$ and $m_j \to \infty$ as $j \to \infty$. Since $\alpha(\zeta) \leqslant \alpha(B)$ for all $\zeta \in \Sigma$,

$$0 \leqslant \alpha_0 := \sup_{\zeta\in\Sigma} \alpha(\zeta) < +\infty.$$

We claim that $\alpha_0 = 0$. Otherwise, we could first choose $\theta > 0$ such that $\theta < (1-q)\alpha_0$, and then a sequence $\zeta_0 \in \Sigma$ such that $\alpha_0 - \theta < \alpha(\zeta_0)$. Let $\zeta_0 = (T^{m_j} x_j)_{j\in N}$. Since $m_j \to \infty$, there is $j_0 \in N$ such that $m_j \geqslant 1$ for all $j \geqslant j_0$. Consider then the sequence $\zeta_1 := (T^{m_j - 1} x_j)_{j \geqslant j_0}$. Since ζ_1 can be written as $\zeta_1 = (T^{n_k} y_k)_{k\in N}$, with $n_k = m_{j_0+k} - 1 \to \infty$ as $k \to \infty$, and $y_k = x_{j_0+k} \in B$, it follows that $\zeta_1 \in \Sigma$; therefore, $\alpha(\zeta_1) \leqslant \alpha_0$. Next, setting

$$\tilde{\zeta}_0 := T\zeta_1 = (T^{m_j} x_j)_{j \geqslant j_0},$$

we see that the sequence $T\zeta_1$ coincides with the sequence ζ_0, deprived of its first j_0 terms. We now check that dropping this finite number of terms does not affect the measure of α-compactness of ζ_0. Indeed, from part (2) of Proposition A.1 we first have that $\alpha(\tilde{\zeta}_0) \leqslant \alpha(\zeta_0)$. To show the opposite inequality, it is sufficient to show that $I(\tilde{\zeta}_0) \subseteq I(\zeta_0)$. Now, if $\beta \in I(\tilde{\zeta}_0)$ and $C_1, \ldots, C_r$ is a finite covering of $\tilde{\zeta}_0$, such that $\operatorname{diam}(C_i) \leqslant \beta$, the addition to this covering of

the j_0 balls $B(Tx_i, \frac{1}{2}\beta)$, $0 \leqslant i \leqslant j_0$, produces a finite covering of ζ_0 with sets whose diameter does not exceed β. Thus, $\beta \in I(\zeta_0)$, as claimed.
In conclusion, we have the chain of inequalities

$$\alpha_0 - \theta < \alpha(\zeta_0) = \alpha(\tilde{\zeta}_0) = \alpha(T\zeta_1) \leqslant q\alpha(\zeta_1) \leqslant q\alpha_0 < \alpha_0 - \theta,$$

which yields a contradiction. This means that $\alpha_0 = 0$ and, therefore, $\alpha(\zeta) = 0$ for all $\zeta \in \Sigma$. In particular, $\alpha(\zeta_*) = 0$, which implies, by part (3) of Proposition A.1, that ζ_* is totally bounded. Hence, ζ_* is compact, and contains a convergent subsequence, as claimed. Finally, the invariance of $\omega(B)$ follows Proposition A.2. This concludes the proof of Theorem 3.1. $\square$

3. We now prove Theorem 3.2 of Section 3.
   3.1. To show that A is compact, note that the function $F: [0, +\infty[\times B \to B$ defined by $F(t, x) := S(t)x$ is continuous on $[0, t_*] \times A_*$. To see this, set

$$R := \sup_{0 \leqslant t \leqslant t_*} L(t) \tag{A.4}$$

and fix $(t_0, x_0) \in [0, t_*] \times A_*$. Since the map $t \mapsto S(t)x$ is continuous for each $x \in X$, for any given $\eta > 0$, there is $\delta_1 > 0$ such that if $|t - t_0| \leqslant \delta_1$,

$$d(S(t)x_0, S(t_0)x_0) \leqslant \frac{1}{2}\eta; \tag{A.5}$$

note that δ_1 depends on η and (x_0, t_0). Let then $\delta_2 := \frac{\eta}{2R}$, and $\delta := \min\{\delta_1, \delta_2\}$. Then if (t, x) is such that

$$(d(x, x_0))^2 + |t - t_0|^2 \leqslant \delta^2,$$

by (A.4) and (A.5) we have that

$$\begin{aligned} d(S(t)x, S(t_0)x_0) &\leqslant d(S(t)x, S(t)x_0) + d(S(t)x_0, S(t_0)x_0) \\ &\leqslant R\, d(x, x_0) + d(S(t)x_0, S(t_0)x_0) \\ &\leqslant R\delta_2 + \frac{1}{2}\eta \leqslant \eta. \end{aligned} \tag{A.6}$$

This shows the continuity of F. It is then immediate to verify that

$$A = A_1 := F([0, t_*] \times A_*);$$

thus, A is compact, because F is continuous and $[0, t_*] \times A_*$ is compact.

3.2. We now show that A attracts all bounded subsets of B. Let $G \subseteq B$ be bounded, and fix $t \geqslant t_*$. Given any $x \in S(t)G$ and $a_* \in A_*$, let $g \in G$ be such that $x = S(t)g$, and decompose $t = nt_* + \theta_t$, for suitable $n \in N$ and $\theta_t \in [0, t_*]$. Let $\bar{a} := S(\theta_t)a_*$. Then, $\bar{a} \in A$, and recalling (A.4), we can estimate

$$\begin{aligned} d(x, \bar{a}) &= d(S(\theta_t)S(t-\theta_t)g, S(\theta_t)a_*) \leqslant R\, d(S(t-\theta_t)g, a_*) \\ &\leqslant R\, d(S(nt_*)g, a_*) = R\, d(S_*^n g, a_*). \end{aligned}$$

From this, it follows that

$$\inf_{a \in A} d(x, a) \leqslant d(x, \bar{a}) \leqslant R\, d(S_*^n g, a_*)$$

and, since a_* is arbitrary in A_*,

$$\inf_{a \in A} d(x, a) \leqslant R \inf_{a_* \in A_*} d(S_*^n g, a_*). \tag{A.7}$$

Since $g \in G \subseteq B$, and B is positively invariant, $S_*^n g \in B$. Thus, recalling the definition of semidistance, we can proceed from (A.7) with

$$\inf_{a \in A} d(x, a) \leqslant R \sup_{b \in S_*^n B} \inf_{a_* \in A_*} d(b, a_*) = R\, \mathrm{dist}\, (S_*^n B, A_*). \tag{A.8}$$

Since (A.8) is true for arbitrary $x \in S(t)G$, it follows that

$$\sup_{x \in S(t)G} \inf_{a \in A} d(x, a) = \mathrm{dist}(S(t)G, A) \leqslant R\, \mathrm{dist}\, (S_*^n B, A_*). \tag{A.9}$$

Since A_* attracts B under S_*, (A.9) implies that A attracts G under S, as claimed.

3.3. We next show that $A = \omega(B)$. Let $a \in A$. There are then $\theta \in [0, t_*]$ and $a_* \in A_*$, such that $a = S(\theta)a_*$. Since $A_* = \omega_*(B)$, there are sequences $(m_j)_{j \in N} \subseteq N$ and $(z_j)_{j \in N} \subseteq B$, such that $m_j \to \infty$ and $S_*^{m_j} z_j \to a_*$ as $j \to \infty$. Let $t_j := \theta + m_j t_*$. Then, $t_j \to \infty$, and

$$a = S(\theta)a_* = \lim_{j \to \infty} S(\theta + m_j t_*) z_j = \lim_{j \to \infty} S(t_j) z_j.$$

Thus, $a \in \omega(B)$. This proves that $A \subseteq \omega(B)$. Conversely, let $z \in \omega(B)$. Then, there are sequences $(t_j)_{j \in N} \subseteq [0, +\infty[$ and $(z_j)_{j \in N} \subseteq B$, such that $t_j \to \infty$ and $S(t_j)z_j \to z$ as $j \to \infty$. For each $j \in N$, we can write $t_j = m_j t_* + \theta_j$, with $m_j \in N$, $\theta_j \in [0, t_*]$, and $m_j \to \infty$ as $j \to \infty$. Since B is positively invariant, $S(\theta_j)z_j =: \tilde{z_j} \in B$ for all j. Hence,

$$z = \lim_{j\to\infty} S(t_j)z_j = \lim_{j\to\infty} S(m_j t_*)S(\theta_j)z_j = \lim_{j\to\infty} S_*^{m_j} \tilde{z_j}.$$

This means that $z \in \omega_*(B) = A_*$. Since $A_* \subseteq A$, it follows that $\omega(B) \subseteq A$. Thus, $A = \omega(B)$.

3.4. Since A is compact and attracts B, and $\omega(B) = A$, Proposition A.2 implies that A is invariant. This concludes the proof of Theorem 3.2. $\square$

Acknowledgments

This paper was written while the second author was visiting the Institute of Mathematics of Fudan University. The financial support provided by the Institute of Mathematics is gratefully acknowledged. The first author is supported by the NSF of China, grants No. 19831060 and No. 10371022. Finally, we are grateful to the anonymous referee of this paper, who provided very useful and detailed suggestions.

References

[1] L. An, Global dynamics of the viscous Cahn–Hilliard equations, J. Lanzhou Univ. 36/5 (2000) 17–23.
[2] A.V. Babin, M.I. Vishik, Attractors of Evolution Equations, North Holland, Amsterdam, 1992.
[3] E. Bonetti, W. Dreyer, G. Schimperna, Global solutions to a generalized Cahn–Hilliard equation with viscosity, Adv. Differential Equations 8/2 (2003) 231–256.
[4] J.W. Cahn, E. Hilliard, Free energy of a nonuniform system, I, Interfacial free energy, J. Chem. Phys. 28 (1958) 258–367.
[5] A. Debussche, A singular perturbation of the Cahn–Hilliard equations, Asymptotic Anal. 4/2 (1991) 161–185.
[6] C. Dupaix, D. Hilhorst, A singularly perturbed phase field problem, Upper semicontinuity of the attractor, Gakuto Internat. Ser. Math. Sci. Appl. 7 (1995) 101–112.
[7] C. Dupaix, D. Hilhorst, I.N. Kostin, The viscous Cahn–Hilliard equations as a limit of the phase field model: the lower semicontinuity of the attractors, J. Dyn. Differential Equations 11/2 (1999) 333–353.
[8] C.M. Elliott, I.K. Kostin, Lower semicontinuity of a non-hyperbolic attractor for the viscous Cahn–Hilliard equation, Nonlinearity 9/3 (1996) 687–702.
[9] C.M. Elliott, A.M. Stuart, The viscous Cahn–Hilliard equation, II, J. Differential Equations 128/2 (1996) 387–414.
[10] C.M. Elliott, S. Zheng, On the Cahn–Hilliard equations, Arch. Rational Mech. Anal. 96 (1986) 339–357.
[11] M. Grasselli, V. Pata, On the damped semilinear wave equation with critical exponent, Proceedings of the IV International Conference on Dynamic Systems and Differential Equations, Wilmington, NC, May 24–27, 2002, pp. 1–9.
[12] J. Hale, Asymptotic Behavior of Dissipative Systems, AMS Mathematics, Surveys and Monographs, #25, Providence, Rhode Island, 1988.
[13] J.K. Hale, G. Raugel, Upper semicontinuity of the attractor for a singularly perturbed hyperbolic equation, J. Differential Equations 73/2 (1988) 197–214.

[14] J.K. Hale, G. Raugel, Lower semicontinuity of the attractor for a singularly perturbed hyperbolic equation, J. Dyn. Differential Equations 2/1 (1990) 19–67.
[15] J.L. Lions, Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires, Dunod, Gauthier-Villars, Paris, 1969.
[16] J.L. Lions, E. Magenes, Non-Homogeneous Boundary value Problems, vol. I, Springer, New York, 1972.
[17] B. Nicolaenko, B. Scheurer, R. Temam, Some global dynamical properties of a class of pattern formation equations, Comm. Partial Differential Equations 14 (1989) 245–297.
[18] A. Novick-Cohen, On the viscous Cahn–Hilliard equation, in: J. Ball (Ed.), Material Instabilities in Continuum Mechanics and Related Mathematical Problems, Clarendon Press, Oxford, 1988.
[19] A. Novick-Cohen, L. Peletier, Steady states of the one-dimensional Cahn–Hilliard equation, Proc. Roy. Soc. Edinburgh Sect. A 123/6 (1993) 1071–1098.
[20] P. Rybka, L.K.-H. Hoffmann, Convergence of Solutions to Cahn–Hilliard equation, Comm. Partial Differential Equations 24/5&6 (1999) 1055–1077.
[21] G. Sell, Y. You, Dynamics of Evolutionary Equations, Springer, New York, 2001.
[22] R. Temam, Infinite Dimensional Dynamical Systems in Mechanics and Physics, Springer, New York, 1988.
[23] S. Zheng, Asymptotic behavior of solutions to the Cahn–Hilliard equation, Appl. Anal. 3 (1986) 165–184.
[24] S. Zheng, Nonlinear parabolic equations and hyperbolic-parabolic coupled systems, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 76, Longman, New York, 1995.
[25] S. Zheng, A. Milani, Exponential attractors and inertial manifolds for singular perturbations of the Cahn–Hilliard equations, NonLinear Anal. TMA 57/5&6 (2004) 843–877.