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Measurable dynamics of maps on profinite groups

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ABSTRACT

We study the measurable dynamics of transformations on profinite groups, in particular of those which factor through sufficiently many of the projection maps; these maps generalize the 1-Lipschitz maps on $\mathbb{Z}_p$.

1. INTRODUCTION

Several authors have studied the measurable dynamics of polynomial maps that define Haar measure-preserving transformations on balls or spheres in the (locally compact) field of p-adic numbers, see for example [1,3,5,7]. Anashin [1] has studied a class of maps on $\mathbb{Z}_p^k$ that are 1-Lipschitz and that he calls *compatible*; Anashin stated that if a compatible (i.e., 1-Lipschitz) map is measure-preserving, then it is bijective, and moreover it is an isometry of $\mathbb{Z}_p^k$ (under the p-adic metric). It is also true that if it is bijective then it is measure-preserving, hence an isometry (see [3, Lemma 4.5]). It was also shown in [3] that an isometry on a compact-open subset of $\mathbb{Q}_p$ is never totally ergodic, in contrast to the real case where, for example, irrational rotations on the circle are totally ergodic. In this paper we introduce a class of maps called *quotient-preserving maps* that generalize the asymptotically compatible (and compatible) maps of Anashin and classify their

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measurable dynamics. However, rather then studying these maps on $\mathbb{Z}_p$ we find that their natural setting is in the context of profinite groups. We now outline the contents of the various sections.

Section 2 reviews inverse limits and states the basic properties of profinite groups that we will use. Section 3 is a review of applications of these notions, in particular of inverse limits, to the context of measurable dynamics. In Section 4 we introduce the notion of quotient-preserving maps and prove the following theorem on the dynamics of these maps.

Theorem 1.1. *Let G be a second-countable profinite group, μ normalized Haar measure on G, and $T : G \to G$ a quotient-preserving map. Define the* finite factor set *of T as*

$$\mathcal{F}(T) = \{N \lhd_O G\colon\ T \text{ factors through } \pi_N : G \to G/N\}.$$

Let $\mathcal{F} \subseteq \mathcal{F}(T)$ be a base for the neighborhoods of $\mathrm{e} \in G$. For each $N \in \mathcal{F}(T)$ let T_N denote the induced map $G/N \to G/N$. Then, the following are equivalent:

(i) *T is measure-preserving (equivalently nonsingular) with respect to μ;*
(ii) *T_N is bijective for each $N \in \mathcal{F}$;*
(iii) *T is surjective;*
(iv) *There exists a translation invariant metric d inducing the topology on G such T is an isometry with respect to d and the subset of $\mathcal{F}$ consisting of sets that are balls of some radius with respect to d is a base for the neighborhoods of $\mathrm{e} \in G$.*

Also, the following are equivalent:

(i) *T is measure-preserving and ergodic with respect to μ;*
(ii) *T_N is measure-preserving and ergodic with respect to $\mu_{G/N}$ for each $N \in \mathcal{F}$;*
(iii) *T_N is minimal with respect to $\mu_{G/N}$ for each $N \in \mathcal{F}$.*

Section 5 applies our methods to the case of continuous homomorphisms, where the additional structure allows us to give a simpler characterization of quotient-preserving maps. Finally, Section 6 applies our results to products of quotient-preserving maps. The prototypical examples of such products are given by products of 1-Lipschitz maps on $\mathbb{Z}_p$, for possibly different primes p. The main result of Section 6 is Theorem 6.3.

2. INVERSE LIMITS

For our purposes, we are primarily interested in inverse limits in two categories:

(i) The category **TopGp**: The objects of **TopGp** are topological groups, and the morphisms are continuous group homomorphisms.

(ii) The category **MD**: The objects are measurable dynamical systems, and the morphisms are measure-preserving maps commuting (almost everywhere) with the action of the dynamical systems (identifying two morphisms if they agree almost-everywhere).

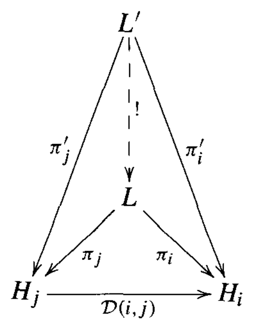
In the following, we let $\mathfrak{C}$ be an arbitrary category; in light of the above, the reader should feel free to replace it with either of the above.

An *inverse system in* $\mathfrak{C}$, denoted $\mathcal{D} : (I, \leqslant) \to \mathfrak{C}$ consists of the following data:

(i) A directed set $(I, \leqslant)$ (i.e. $(I, \leqslant)$ is a partially ordered set, such that each finite subset has an upper bound in I);
(ii) A collection $\{\mathcal{D}(i) \in \mathrm{Ob}_{\mathfrak{C}} \colon i \in I\}$ of objects of $\mathfrak{C}$;
(iii) A collection $\{\mathcal{D}(i, j) \in \mathrm{Hom}_{\mathfrak{C}}(\mathcal{D}(j), \mathcal{D}(i)) \colon i \leqslant j\}$ of morphisms such that for all $i \leqslant j \leqslant k \in I$ we have $\mathcal{D}(i, j) \circ \mathcal{D}(j, k) = \mathcal{D}(i, k)$ and such that $\mathcal{D}(i, i) = \mathrm{id}_i$ for all $i \in I$.

A pair $(L, \{\pi_i\})$ with $L \in \mathrm{Ob}_{\mathfrak{C}}$ and with $\{\pi_i \in \mathrm{Hom}_{\mathfrak{C}}(L, \mathcal{D}(i)) \colon i \in I\}$ a collection of morphisms such that $\mathcal{D}(i, j) \circ \pi_j = \pi_i$ for all $i \leqslant j \in I$ is said to satisfy the *defining property of an inverse limit* for the inverse system $\mathcal{D}$.

An *inverse limit* for the inverse system $\mathcal{D}$ is a pair $(L, \{\pi_i\})$ satisfying the defining property of an inverse limit and the following universal property: For any pair $(L', \{\pi_i'\})$ satisfying the defining property of an inverse limit there must exist a unique morphism $L' \to L$ making the following diagram commute for all $i \leqslant j \in I$:



Such an object, which is unique if it exists, is denoted by

$$\varprojlim_{i \in I} \mathcal{D}(i).$$

If $\mathfrak{C}$ is **TopGp** then each directed system in $\mathfrak{C}$ has an inverse limit, given by the following construction:

$$\varprojlim_{i \in I} \mathcal{D}(i) = \left\{ x \in \prod_{i \in I} \mathcal{D}(i) \colon \pi_i(x) = \mathcal{D}(i, j)(\pi_j(x)) \text{ for all } i \leqslant j \in I \right\},$$

with the subspace topology from the product topology and with projection maps given by the projection maps from the product.

We are now ready to define a *profinite group*. We say that a topological group G is *profinite* if it is isomorphic, as a topological group, to an inverse limit of finite groups. That is, if

$$G \cong \varprojlim_{i \in I} \mathcal{D}(i)$$

for $\mathcal{D} : (I, \leqslant) \to \mathbf{TopGp}$ an inverse system of *finite* (topological via the discrete topology) groups.

Let us sketch and cite some standard results on profinite groups:

Proposition 2.1. *Let G be a profinite group. Then:*

(i) *G is a compact Hausdorff totally-disconnected topological group. Moreover, these properties characterize profinite groups.*
(ii) *Every open subgroup $U \leq_O G$ is also closed (this in fact holds for all topological groups).*
(iii) *Every open subgroup $U \leq_O G$ has finite index.*
(iv) *The normal open subgroups form a base for the neighborhoods of $\mathrm{e} \in G$ (equivalently, their translates form a base for the topology on G).*
(v) *Let $\mathcal{F}$ be a collection of open normal subgroups of G such that $\mathcal{F}$ is a base for the neighborhoods of $\mathrm{e} \in G$. Then, we may order $\mathcal{F}$ by inclusion, and for $N \supseteq N'$ we have a projection $G/N' \to G/N$. This makes the system of quotients G/N into an inverse system, with*

$$G \cong \varprojlim_{N \in \mathcal{F}} G/N,$$

where the inverse limit and isomorphism are **TopGp**.
(vi) *Let $\mathcal{B}$ be smallest σ-algebra containing the compact subsets of G. Then, there is a unique measure μ on $\mathcal{B}$ such that $\mu(gS) = \mu(sG) = \mu(S)$ for $g \in G$ and $S \in \mathcal{B}$, μ is regular, and $\mu(G) = 1$. We call μ the (normalized) Haar measure on G.*

Proof. For (i), note that the product space in the construction given above is compact Hausdorff. Then, G corresponds to a closed subgroup of the product, and so is also a compact Hausdorff topological group. That G is totally disconnected then follows from (ii) and (iv). For the converse, it suffices to show that (iv) holds for such a space and then use the proof of (v); for this see the reference below.

Distinct cosets of U are disjoint; so the union of the cosets different from U is just $G \setminus U$, and this set must be open. This proves claim (ii). Claim (iii) follows by compactness.

Say $G \cong \varprojlim_{i \in I} \mathcal{D}(i)$, $\mathcal{D}(i)$ finite groups with the discrete topology, and let $\pi_i : G \to \mathcal{D}(i)$ be the projection map. Then, $\ker \pi_i$ is a normal open subgroup of G for each $i \in I$. We readily check that these form a base for the neighborhoods of $\mathrm{e} \in G$

(indeed, their cosets are just the restriction of the standard base for the product topology on the inverse limit). This proves (iv).

Now, say $\mathcal{F}$ forms a base for the neighborhoods of $e \in G$. Let $\pi_N : G \to G/N$ be the quotient maps. Then, $(G, \{\pi_N\})$ satisfies the defining property of the inverse limit, so by the universal property of the inverse limit we have a canonical map

$$\phi : G \longrightarrow \varprojlim_{N \in \mathcal{F}} G/N$$

such that the appropriate diagram must commute. Note that this map must be an injection, for

$$\bigcap_{N \in \mathcal{F}} \ker \pi_N = \bigcap_{N \in \mathcal{F}} N = \{1\}.$$

Furthermore, the image of ϕ must be dense, and must be compact as G is compact, ϕ continuous, and the inverse limit Hausdorff. So, ϕ is surjective. So, ϕ is a continuous bijection. But, ϕ must take closed, hence compact, sets to compact, hence closed, sets; so ϕ^{-1} is continuous. So, ϕ is an isomorphism of topological groups. This proves (v). For more on the general theory of topological groups see for instance [13]. For complete proofs of the above claims, see for instance [14, pp. 17–20].

Finally, G is a compact topological group, so it is unimodular and has a unique (left and right) Haar measure. This proves (vi). For more details on Haar measure on locally compact groups and the unimodularity of compact groups see for instance [6, pp. 36–47]. □

Example 2.2. Let $I = \mathbb{N}$, and for $k \in I$ let $\mathcal{D}(i) = \mathbb{Z}/p^i\mathbb{Z}$. For $i \leqslant j \in I$ let $\mathcal{D}(i, j) : \mathbb{Z}/p^j\mathbb{Z} \to \mathbb{Z}/p^i\mathbb{Z}$ be the reduction mod p^i map. Then, we have

$$\mathbb{Z}_p \cong \varprojlim_{i \in I} \mathcal{D}(i) = \varprojlim_{k \geqslant 1} \mathbb{Z}/p^k\mathbb{Z},$$

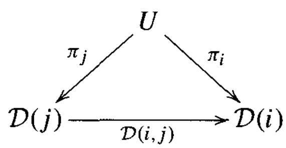
where $\mathbb{Z}_p$ refers to the additive group of the ring of p-adic integers.

3. MEASURABLE DYNAMICAL STRUCTURE

By a *measurable dynamical system* we mean a 4-tuple $(X, \mu, \mathcal{B}, T)$ where X is a set, $\mathcal{B}$ is a σ-algebra of subsets of X, μ is a probability measure on $\mathcal{B}$, and T is a $\mathcal{B}$-measurable function. We define a *morphism of measurable dynamical systems* $(X, \mu, \mathcal{B}, T) \to (X', \mu', \mathcal{B}', T')$ to be an equivalence class of maps $\phi : X \to X'$ such that ϕ is measurable and measure-preserving, and $\phi \circ T(x) = T' \circ \phi(x)$ holds outside a set of μ-measure 0; our equivalence relation is to identify $\phi : X \to X'$ and $\phi' : X \to X'$ when $\phi(x) = \phi'(x)$ holds outside a set of μ-measure 0. These definitions define a category, which we shall denote **MD**.

Inverse limits need not always exist in **MD**; indeed even when the inverse system consists just of finite direct products, there need not be a measure on the topological inverse limit [8, p. 214]. There are significant existence results, such as in the case of standard spaces [12] or of topological measures on compact spaces [4]. Even without these topological restrictions, we may sometimes be guaranteed that an inverse system has an inverse limit; furthermore when an inverse limit exists its dynamics are closely related to the dynamics of the systems in the inverse system:

Proposition 3.1. *Let $(I, \leqslant)$ be a directed set, and $\mathcal{D} : I \to$ **MD** an inverse system in* **MD**. *Moreover, assume there is an object $U = (X, \mu, \mathcal{B}, T)$ and morphisms $\{\pi_i \in \mathrm{Hom}_{\mathbf{MD}}(U \to \mathcal{D}(i))\colon\ i \in I\}$ such that the following diagram commutes for each $i \leqslant j \in I$*



For each $i \in I$, let $\mathcal{B}_i$ denote the σ-algebra of measurable sets of $\mathcal{D}(i)$, and let $\widetilde{\mathcal{B}}$ be the smallest σ-algebra containing

$$\bigcup_{i \in I} \pi_i^{-1}(\mathcal{B}_i).$$

Then, $(X, \mu, \widetilde{\mathcal{B}}, T)$ is an inverse limit for $\mathcal{D}$.

Moreover, if L is an inverse limit for $\mathcal{D}$ then L is measure-preserving if and only if $\mathcal{D}(i)$ is measure-preserving for each $i \in I$. The previous sentence still holds when one adds to "measure-preserving" any of the following additional conditions: ergodic, weakly mixing, mixing.

Proof. See [2]. □

Now, Proposition 2.1(vi) turns each profinite group, in a natural way, into a probability space. Say G is a profinite group, μ Haar measure on G, and $\mathcal{B}$ the σ-algebra of $\mathcal{B}$-measurable sets. Then, for any μ-measurable map $T : G \to G$ we have that the 4-tuple $\Sigma = (G, \mu, \mathcal{B}, T)$ is an object of **MD**. The final statement of Proposition 2.1 combined with Proposition 3.1 suggests that we may be able to study the dynamics of a system on G by looking at systems on some finite quotients of G. Unfortunately, for $N \lhd_O G$ an open normal subgroup, T need not induce a well-defined map $G/N \to G/N$. We may recover some such information through the following construction.

For $N \lhd_O G$ we define the following objects:

- Let

$$X_N = \prod_{k \geqslant 0} G/N,$$

let $\pi_N : G \to G/N$ be the quotient map, and let the map $\Phi_N : G \to X_N$ be given by

$$x \mapsto \left(\pi_N(x), \pi_N(Tx), \pi_N\left(T^2x\right), \pi\left(T^3x\right), \ldots\right) \quad \text{that is}$$
$$\varpi_k \circ \Phi = \pi_N \circ T^k,$$

where $\varpi_k : X_N \to G/N$ is projection to the kth slot.

- We may define a measure on X_N such that Φ_N is measure-preserving; specifically, let $\mu_N = \mu \circ \Phi_N^{-1}$, let $\mathcal{B}_N$ the σ-algebra of μ_N-measurable sets.
- Finally, let T_N be the left-shift map on X_N. Then, we may define the following measurable dynamical system:

$$\Sigma_N = (X_N, \mu_N, \mathcal{B}_N, T_N).$$

Lemma 3.2. *Let $\Sigma = (G, \mu, \mathcal{B}, T)$ be a measurable dynamical system with G a profinite group and μ Haar measure on G. Let Σ_N, Φ_N be as above.*

Say $I \subseteq \{N \lhd_O G\}$ is ordered by set-inclusion. For $N \supseteq N' \in I$, we have a natural projection $G/N' \to G/N$; this induces a morphism (of **MD***) $\Sigma_{N'} \to \Sigma_N$. Now, we may define $\mathcal{D} : (I, \supseteq) \to$* **MD** *by*

$$\mathcal{D}(N) = \Sigma_N, \qquad \mathcal{D}\left(N, N'\right) = \textit{the above morphism } \Sigma_{N'} \to \Sigma_N$$

for all $N, N' \in I$.

Then:

(i) *$\mathcal{D}$ is an inverse system in* **MD**;
(ii) *$(\Sigma, \{\Phi_N\})$ satisfies the defining property for the inverse limit of $\mathcal{D}$*;
(iii) *$\mathcal{D}$ has an inverse limit in* **MD**;
(iv) *If G is second-countable and I forms a base for the neighborhoods of $e \in G$, then $(\Sigma, \{\Phi_N\})$ is an inverse limit for $\mathcal{D}$.*

Proof. The commutativity of the appropriate diagrams for (i) and (ii) are routine verifications. We note that the maps π_N, as well as the maps $\mathcal{D}(N, N')$ are surjective continuous group homomorphisms. It is a standard result that surjective continuous group homomorphisms preserve Haar measure. Also, for each $N \in I$, the map Φ_N is continuous and is measure-preserving by construction of μ_N. So, all relevant maps are indeed morphisms in **MD** and claims (i) and (ii) are complete. Then, claim (iii) follows by Proposition 3.1.

Now, by Proposition 3.1, letting $\widetilde{\mathcal{B}}$ be the smallest σ-algebra containing

$$\bigcup_{N \in I} \Phi_N^{-1}(\mathcal{B}_N),$$

we have that $(X, \mu, \widetilde{\mathcal{B}}, T)$ is an inverse limit for $\mathcal{D}$. Noting that the maps Φ_N are measurable we have $\widetilde{\mathcal{B}} \subseteq \mathcal{B}$.

Say G is second-countable. Each element of I is a compact-open set, and is thus a finite union of elements of the countable base of G. As the collection of finite subsets of a countable set is itself countable, we have that I must be at most countable. Moreover, each $N \subseteq I$ has finitely many distinct translates. So, if I forms a base for the neighborhoods of $e \in G$, then the collection of translates of the elements of I form a countable base for the topology of G.

For $N \subseteq I$, the cosets of N are contained in $\Phi_N^{-1}(\mathcal{B}_N)$. So, $\widetilde{\mathcal{B}}$ contains all translates of I, and hence a countable base for the open sets of G. By countable unions, $\widetilde{\mathcal{B}}$ contains the open sets of G, and by taking complements it contains the closed sets of G and so the compact sets. Recalling that $\mathcal{B}$ was generated by the compact sets, we have $\mathcal{B} \subseteq \widetilde{\mathcal{B}}$. With the above, this implies that $\mathcal{B} = \widetilde{\mathcal{B}}$ and proves our claim. □

Example 3.3. Let $G = \mathbb{Z}_p$. Note that each element of $\mathbb{Z}_p$ has a unique expression of the form $c + pd$ with $c \in \{0, \dots, p-1\}$ and $d \in \mathbb{Z}_p$. Then, we may define $T : G \to G$ by

$$T(c + pd) = d \quad \text{for } c \in \{0, \dots, p-1\}, d \in \mathbb{Z}_p.$$

Then, T is a surjective, p-to-1, measure-preserving map. Take $N = p\mathbb{Z}_p$. Then, Σ_N is a Bernoulli shift on p symbols. Moreover, one can show that the map $\Phi_N : G \to X_N$ is a measurable (and topological) isomorphism.

Example 3.4. Let $G = \mathbb{Z}_p$, and define the transformation $f : G \to G$ by

$$f(x) = \binom{x}{p} = \frac{x(x-1)\cdots(x-p+1)}{p!}.$$

Take $N = p\mathbb{Z}_p$. It is possible to check that Σ_N is a Bernoulli shift on p symbols, and that $\Phi_N : G \to X_N$ is a measurable (and topological) isomorphism. Details of this construction are worked out in [10].

4. FACTORING THROUGH PROJECTIONS

Let G, H be compact topological groups. For a transformation $T : G \to G$ we say that T *factors through* a surjective continuous group homomorphism $\phi : G \to H$ if there exists a transformation $T' : H \to H$ such that the following diagram commutes

$$\begin{array}{ccc} G & \xrightarrow{T} & G \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ H & \xrightarrow{T'} & H \end{array}$$

Let us relate this to the situation of Lemma 3.2.

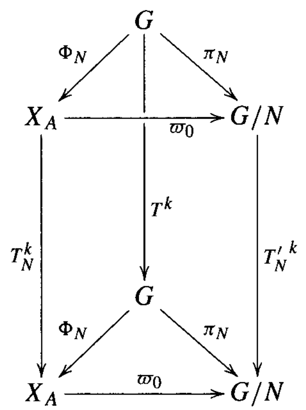
Lemma 4.1. *Let G be a profinite group, μ normalized Haar measure on G, and $T : G \to G$ a transformation on G. Let $N \triangleleft_O G$ be such that T factors* *through the quotient map* $\pi_N : G \to G/N$. *Let* $T'_N : G/N \to G/N$ *denote the factor transformation. Let* $\Sigma_N = (X_N, \mu_N, \mathcal{B}_N, T_N)$ *be as defined in Lemma* 3.2. *Define*

$$\Sigma'_N = \big(G/N, \mu_{G/N}, \mathcal{B}_{G/N}, T'_N\big)$$

where $\mu_{G/N}$ *is Haar measure on the finite group* G/N *(i.e. normalized counting measure), and* $\mathcal{B}_{G/N}$ *its* σ*-algebra (i.e. the power set of* G/N*). Then, projection to the first coordinate* $X_N \to G/N$ *gives an isomorphism*

$$\Sigma_N \cong \Sigma'_N.$$

Proof. For $k \geqslant 0$, let $\varpi_k : X_N \to G/N$ be the projection to the kth coordinate. Then, by the definition of T_N and T'_N we have the commutative diagram



for each $k \geqslant 0$, where T^k, T_N^k, and $T_N'^{\,k}$ denote the k-fold composites of T, T_N, T'_N respectively.

Note that for $x \in G/N$,

$$\mu_N\big(\varpi_0^{-1}(x)\big) = \mu\big(\Phi_N^{-1}\varpi_0^{-1}(x)\big) = \mu\big(\pi_N^{-1}(x)\big) = \mu_{G/N}(x).$$

So, ϖ_0 is measure-preserving and Σ'_N is a measurable factor of Σ_N. Moreover, note that $\varpi_k = \varpi_0 \circ T_N^k = \varpi_0 \circ T_N'^{\,k}\varpi_0$; so each element of X_N is uniquely determined by its first entry. It follows that $\varpi_0^{-1}(\mathcal{B}_{G/N}) = \mathcal{B}_N$. Then,

$$\Sigma'_N \cong \big(X_N, \mu_N, \varpi_0^{-1}(\mathcal{B}_{G/N}), T_N\big) = (X_N, \mu_N, \mathcal{B}_N, T_N) = \Sigma_N. \qquad \square$$

For $T : G \to G$, define the *finite factor set of* T as

$$\mathcal{F}(T) = \{N \lhd_O G\colon\ T \text{ factors through } \pi_N : G \to G/N\}.$$

Note that each π_N is a continuous surjective group homomorphism, thus measure-preserving with respect to Haar measure.

Remark 4.2. The notion of $\mathcal{F}(T)$ has another natural description. Denote

$$\mathcal{F}'(T) = \{\pi \in \mathrm{Hom}_{\mathbf{TopGp}}(G, H) \text{ surjective: } H \text{ is a finite group, } T \text{ factors through } \pi\}/\{\sim\},$$

where $\pi_1 \sim \pi_2$ if there exists an isomorphism $\operatorname{im}\pi_1 \cong \operatorname{im}\pi_2$ conjugating the two maps.

That is, $\mathcal{F}'(T)$ is the set of all finite group factors of $T : G \to G$. The relationship between $\mathcal{F}(T)$ and $\mathcal{F}'(T)$ is clear: for each $N \in \mathcal{F}(T)$ we have $G \to G/N \in \mathcal{F}'(T)$, and conversely for each $\pi \in \mathcal{F}'(T)$ we have $\ker\pi \in \mathcal{F}(T)$.

Definition 4.3. For a profinite group G, we say that $T : G \to G$ is a *quotient-preserving map* if the cosets of $\mathcal{F}(T)$ form a base for the topology of G.

If G is known to be second-countable, then Lemma 3.2 and Lemma 4.1 give us that

$$\Sigma \overset{\text{def}}{=} (G, \mu, \mathcal{B}, T) \cong \varprojlim_{N \in \mathcal{F}(T)} \Sigma'_N,$$

where Σ'_N is in the sense of Lemma 4.1, and Σ'_N is in particular a measurable dynamical system on a finite set.

We invite the reader to prove the following alternate characterization of the quotient-preserving maps:

Lemma 4.4. *Let G be a profinite group. Then a map $T : G \to G$ is a quotient-preserving map if and only if there exists a directed set $(I, \leqslant)$ and an inverse system $\mathcal{D} : I \to$ **TopGp** of finite groups such that*

$$G \cong \varprojlim_{i \in I} \mathcal{D}(i)$$

and T factors through the projection $G \to \mathcal{D}(i)$ for each $i \in I$. The inverse system may be assumed surjective. In addition, instead of T factoring through each projection, it suffices that for each $i \in I$ there exists a $j \in I$ with $i \leqslant j$ such that T factors through the projection $G \to \mathcal{D}_j$.

Example 4.5. Let $G = \mathbb{Z}_p$. We note that

$$\mathbb{Z}_p \cong \varprojlim_{k \in \mathbb{N}} \mathbb{Z}/p^k\mathbb{Z}.$$

The open normal subgroups of $\mathbb{Z}_p$ are all of the form $p^k\mathbb{Z}_p$. Then, we see that $T : \mathbb{Z}_p \to \mathbb{Z}_p$ is a quotient-preserving map if and only if there is an infinite subset I of $\mathbb{N}$ such that $k \in I$ and $|x - y|_p \leqslant p^{-k}$ implies that $|Tx - Ty|_p \leqslant p^{-k}$. In particular, this holds for all maps satisfying $|Tx - Ty|_p \leqslant |x - y|_p$ (i.e., the 1-Lipschitz maps). In this context, our notion of quotient-preserving maps may be viewed as generalizing the notions of (asymptotically) compatible maps found in [1], and our Proposition 4.9 generalizes Lemma 4.5 of [3].

Example 4.6. Let $G = \mathbb{Z}_p \times \mathbb{Z}_p$. We note that

$$G \cong \varprojlim_{k_1,k_2 \in \mathbb{N}} \mathbb{Z}/p^{k_1}\mathbb{Z} \times \mathbb{Z}/p^{k_2}\mathbb{Z}.$$

Let T be given by multiplication by an element of $\mathrm{GL}_2(\mathbb{Z}_p)$. Given $k_1, k_2 \in \mathbb{N}$ it need not be the case that T factors through the projection to $\mathbb{Z}/p^{k_1}\mathbb{Z} \times \mathbb{Z}/p^{k_2}\mathbb{Z}$. However, T does factor through the projection for $k_1 = k_2$. The kernels of these projections form a base for the neighborhoods of $e \in G$, so T is a quotient-preserving map.

Lemma 4.7. *Let G be a profinite group and $T : G \to G$ a quotient-preserving map. Then, T is continuous.*

Proof. Say T factors through $\pi_N : G \to G/N$ as T_N for each $N \in \mathcal{F}(T)$.

For $N \in \mathcal{F}(T)$ and $h \in G/N$, then

$$T^{-1}\big(\pi_N^{-1}(h)\big) = \pi_N^{-1}\big(T_N^{-1}(h)\big) = \bigcup_{h' \in T_N^{-1}(h)} \pi_N^{-1}(h').$$

As the sets

$$\big\{\pi_N^{-1}(h) \colon N \in \mathcal{F}(T), h \in G/N\big\}$$

are precisely the cosets of the elements of $\mathcal{F}(T)$ they form a base for the topology on G. As T^{-1} takes each set in this base to an open set, continuity of T follows. □

Lemma 4.8. *Let G be a compact Hausdorff topological group, μ normalized Haar measure on G, and $T : G \to G$ continuous. If T is nonsingular with respect to μ, then T is surjective.*

Proof. As T is continuous, $T(G)$ is the continuous image of a compact set, thus compact and so closed in the Hausdorff space G.

But,

$$\mu\big(T^{-1}(G \setminus T(G))\big) = \mu(\emptyset) = 0,$$

and by nonsingularity

$$\mu\big(G \setminus T(G)\big) = 0.$$

Note that μ is positive on non-empty open sets, so this implies that $G \setminus T(G)$ does not contain a non-empty open set and hence that $T(G)$ is dense in G. As $T(G)$ is closed in G, this implies $T(G) = G$. So, T is surjective. □

Proposition 4.9. *Let G be a second-countable profinite group, μ normalized Haar measure on G, and $T : G \to G$ a quotient-preserving map. Let $\mathcal{F} \subseteq \mathcal{F}(T)$ be a base* *for the neighborhoods of* $\mathrm{e} \in G$. *For each* $N \in \mathcal{F}(T)$ *let* T_N *denote the induced map* $G/N \to G/N$. *Then, the following are equivalent*:

(i) T_N *is bijective on* G/N *for all* $N \in \mathcal{F}$;
(ii) T_N *is nonsingular with respect to* $\mu_{G/N}$ *for all* $N \in \mathcal{F}$;
(iii) T_N *is measure-preserving with respect to* $\mu_{G/N}$ *for all* $N \in \mathcal{F}$;
(iv) T *is measure-preserving with respect to* μ;
(v) T *is nonsingular with respect to* μ;
(vi) T *is surjective.*

Proof. We prove the following implications:

$$\text{(i)} \Leftrightarrow \text{(ii)} \Leftrightarrow \text{(iii)} \Leftrightarrow \text{(iv)} \Leftrightarrow \text{(v)} \Leftrightarrow \text{(vi)} \Leftrightarrow \text{(i)}.$$

The implications (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) follow as G/N is finite and $\mu_{G/N}$ is counting measure. The implications (iii)$\Leftrightarrow$(iv) follow by Lemma 3.2 and Lemma 4.1. The implication (iv)$\Rightarrow$(v) is true by definition. The implication (v)$\Rightarrow$(vi) follows by Lemma 4.8. Finally, (vi) implies that each T_N is surjective; as a surjective map of a finite set to itself is bijective, we have (vi)$\Rightarrow$(i). □

Lemma 4.10. *Let* G *be a second-countable profinite group,* μ *normalized Haar measure on* G*, and* $T : G \to G$ *a transformation. Then,* T *is a measure-preserving quotient-preserving map if and only if there exists a metric* $d : G^2 \to \mathbb{R}_{\geqslant 0}$ *on* G *such that the following conditions hold*:

(i) d *induces the usual topology on* G;
(ii) d *is left translation invariant in the sense that* $d(gx, gy) = d(x, y)$ *for all* $x, y, g \in G$;
(iii) T *is an isometry with respect to* d;
(iv) *the set of open-subgroups of* G *which are (closed) balls with respect to* d*, i.e.*

$$\{N \lhd_O G\colon\ N = \{x \in G\colon\ d(\mathrm{e}, x) \leqslant r_N\} \text{ for some } r_N > 0\}$$

is a base for the neighborhoods of $\mathrm{e} \in G$.

Proof. $\Rightarrow$: By the proof of Lemma 3.2 we note that $\mathcal{F}(T)$ is countable and that the translates of the elements of $\mathcal{F}(T)$ give a countable base for the topology on G. Say $\mathcal{F}(T) = \{N_1', N_2', N_3', \ldots\}$. Set $N_1 = N_1'$, and for $k > 1$ let $N_k \in \mathcal{F}(T)$ be such that $N_k \subseteq N_{k-1} \cap N_k'$. Note that $N_{k-1} \cap N_k'$ is open and contains e for each $k > 1$, so such an N_k must exist. Then, set $\mathcal{F} = \{N_1, N_2, \ldots\}$. Note that $\mathcal{F}$ is countable, nested, and forms a base for the neighborhoods of $\mathrm{e} \in G$.

For $N \lhd_O G$, let $\pi_N : G \to G/N$ be the quotient map. Then, we may define $d : G^2 \to \mathbb{R}_{\geqslant 0}$ for $x, y \in G$ by

$$d(x, y) = 2^{-\ell} \quad \text{where } \ell = \min\{k\colon\ \pi_{N_k}(x) = \pi_{N_k}(y)\},$$

and $d(x, y) = 0$ if $\pi_{N_k}(x) = \pi_{N_k}(y)$ for all $k \geqslant 0$.

We claim that d is a metric, and that it moreover satisfies the conditions in the lemma:

- It is clear by construction that d is symmetric and non-negative. Note that

$$\bigcap_{k\geqslant 0} N_k = \{\mathrm{e}\},$$

so $d(x,y)=0 \Leftrightarrow x=y$. Moreover, $d(x,y)\leqslant 2^{-k}$ and $d(y,z)\leqslant 2^{-k}$ implies $d(x,z)\leqslant 2^{-k}$, so d satisfies the strong triangle inequality. So, we see that d is indeed a metric.
- The set of balls with respect to d is precisely $\mathcal{F}$ and the empty set. So, d satisfies condition (iv) of the Lemma, and moreover it induces the same topology as $\mathcal{F}$ and so satisfies (i).
- As π_{N_k} is a homomorphism for each $k\geqslant 0$, we see immediately that $\pi_{N_k}(x)=\pi_{N_k}(y) \Leftrightarrow \pi_{N_k}(gx)=\pi_{N_k}(gy) \Leftrightarrow \pi_{N_k}(xg)=\pi_{N_k}(yg)$ for all $x,y,g\in G$. So, d is (left and right) translation invariant, and satisfies (ii).
- For $N\in\mathcal{F}\subseteq\mathcal{F}(T)$ we have that $T_N : G/N\to G/N$ is a bijection by Proposition 4.9. So, $\pi_{N_k}(x)=\pi_{N_k}(y) \Leftrightarrow \pi_{N_k}(T(x))=\pi_{N_k}(T(y))$ for all $k\geqslant 0$. So, we see that T is an isometry with respect to d, hence condition (iii).

$\Leftarrow$: Let $\mathcal{F}$ be the collection in (iv). For each $N\in\mathcal{F}$, let $r_N>0$ be as in the definition of $\mathcal{F}$.

Using the fact that $N=\{x\in G : d(\mathrm{e},x)=d(x,\mathrm{e})\leqslant r_N\}$ and the fact that d is left translation invariant we confirm that

$$\pi_N(x)=\pi_N(y) \Leftrightarrow x^{-1}y\in N \Leftrightarrow d(x,y)=d\big(x^{-1}y,\mathrm{e}\big)\leqslant r_N.$$

Then, the fact that T is an isometry with respect to d implies that $\pi_N(x)=\pi_N(y) \Leftrightarrow \pi_N(T(x))=\pi_N(T(y))$. So, T induces a well-defined injective, hence bijective as G/N is finite, map $T_N : G/N\to G/N$.

As this holds for arbitrary $N\in\mathcal{F}$, we have that $\mathcal{F}\subseteq\mathcal{F}(T)$ is a base for the neighborhoods of $\mathrm{e}\in G$ with T_N bijective on G/N for all $N\in\mathcal{F}$. This implies immediately that T is a quotient-preserving map, and by Proposition 4.9 that T is measure-preserving. □

Proposition 4.11. *Let G be a second-countable profinite group, μ normalized Haar measure on G, and $T : G\to G$ a quotient-preserving map. Let $\mathcal{F}\subseteq\mathcal{F}(T)$ be a base for the neighborhoods of $\mathrm{e}\in G$. For each $N\in\mathcal{F}(T)$ let T_N denote the induced map $G/N\to G/N$. Then, the following are equivalent:*

(i) *T is measure-preserving and ergodic with respect to μ;*
(ii) *T_N is measure-preserving and ergodic with respect to $\mu_{G/N}$ for all $N\in\mathcal{F}$;*
(iii) *T_N is minimal for all $N\in\mathcal{F}$.*

By Proposition 4.9, *we may replace "measure-preserving" with "nonsingular" in one or both of the above occurrences.*

Proof. The equivalence (i)$\Leftrightarrow$(ii) follows by Lemma 3.2 and Lemma 4.1. The equivalence (ii)$\Leftrightarrow$(iii) holds as each G/N is finite with $\mu_{G/N}$ the normalized counting measure. $\square$

Proposition 4.12. *Let G be a compact Hausdorff topological group, μ normalized Haar measure on G, and $T : G \to G$ a transformation. Say $\mathcal{F}(T) \supsetneq \{G\}$. Then, T is not totally ergodic . In particular, if T is a quotient-preserving map then it is not totally ergodic unless $|G| = 1$.*

Proof. Let $N \in \mathcal{F}(T) \setminus \{G\}$. Then, $1 < |G/N| < \infty$, and $T : G \to G$ factors through G/N as

$$\begin{array}{ccc} G & \xrightarrow{T} & G \\ \downarrow{\scriptstyle \pi_N} & & \downarrow{\scriptstyle \pi_N} \\ G/N & \xrightarrow{T_N} & G/N \end{array}$$

If T is ergodic then T_N is ergodic, hence minimal. In particular for $h \in G/N$ we have that $T^{\ell}(h) = h$ if and only if $|G/N| \mid \ell$. Then, $T^{|G/N|}$ factors through the projection as $T_N^{|G/N|}$; but this is just the identity map on G/N. So, T is not totally ergodic.

If T is a quotient-preserving map, then

$$G \cong \varprojlim_{N \in \mathcal{F}(T)} G/N$$

by Proposition 2.1. In particular $\mathcal{F}(T) = \{G\}$ implies $|G| = 1$. $\square$

Remark 4.13. Recall also that weakly mixing implies totally ergodic. So, the above also gives negative results for weak mixing.

Now, the results of the propositions yield the proof of Theorem 1.1.

5. HOMOMORPHISMS

We begin by recalling a result on when a continuous group endomorphism is measure-preserving:

Lemma 5.1. *Let G be a compact Hausdorff topological group, μ normalized Haar measure on G, and $T : G \to G$ a homomorphism of topological groups. Then, the following are equivalent:*

(i) *T is nonsingular with respect to μ;*

(ii) *T is surjective*;
(iii) *T is measure-preserving with respect to* μ.

Proof. The assertion (i)⇒(ii) follows from Proposition 4.9. The assertion (ii)⇒(iii) is true as $\mu \circ T^{-1}$ can be shown to be regular, translation invariant, and normalized. The assertion (iii)⇒(i) is true by definition. □

Now, in the case of continuous group endomorphisms, we may give an alternate characterization of the collection $\mathcal{F}(T)$ in the definition of a quotient-preserving map:

Lemma 5.2. *Let G be a compact Hausdorff topological group and* $T : G \to G$ *a homomorphism of topological groups. Then*

$$\mathcal{F}(T) = \left\{N \lhd_O G \colon\ N \subseteq T^{-1}(N)\right\}.$$

If T is surjective then in fact

$$\mathcal{F}(T) = \left\{N \lhd_O G \colon\ N = T^{-1}(N)\right\}.$$

Proof. Note that for $N \lhd_O G$, any T_N making the following diagram commute must be a group homomorphism

$$\begin{array}{ccc} G & \xrightarrow{\ T\ } & G \\ \downarrow{\scriptstyle \pi_N} & & \downarrow{\scriptstyle \pi_N} \\ G/N & \xrightarrow{\ T_N\ } & G/N \end{array}$$

Furthermore, such a T' exists if and only if $N = \ker \pi_N \subseteq \ker \pi_N \circ T = T^{-1}(N)$. If T is in addition surjective, then by Lemma 5.1 it is measure-preserving. Then, $\mu(N) = \mu(T^{-1}(N))$ and so $\mu(T^{-1}(N) \setminus N) = 0$; as $T^{-1}(N) \setminus N$ is open, this implies that it is empty and so $T^{-1}(N) = N$.

So,

$$\mathcal{F}(T) = \{N \lhd_O G \colon\ T(N) \subseteq N\},$$

and if T is in addition surjective then we may replace the constraint by $T(N) = N$. □

Remark 5.3. Note that if $\mathcal{F}(T) \neq \{G\}$ then T is not ergodic. This follows because the factor transformation would be a group homomorphism on a finite group, which can not be ergodic (for it maps e to itself).

For many profinite groups, the following criterion suffices to show that all group endomorphisms are quotient-preserving maps:

Proposition 5.4. *Let G be a profinite group such that G has finitely many open normal subgroups of each finite index. If $T : G \to G$ is a (Haar) nonsingular homomorphism of topological groups (i.e. a surjective continuous group homomorphism), then T is a quotient-preserving map.*

In particular, if G has a finitely-generated dense subgroup then the any such T is a quotient-preserving map.

Proof. Say $N \lhd_O G$. Then, $T^{-1}(N) \lhd_O G$. Taking measures and noting that T is measure-preserving with respect to Haar measure by Lemma 5.1 we observe that

$$1/[G : N] = \mu(N) = \mu\big(T^{-1}(N)\big) = 1/\big[G : \mu\big(T^{-1}(N)\big)\big].$$

Now, for $N \lhd_O G$ consider the collection

$$\big\{T^{-k}(N)\colon k \geqslant 0\big\}.$$

Each element of the this collection must be an open normal subgroup of the same index in G, so the collection must be finite by hypothesis. Set

$$N' = \bigcap_{k \geqslant 0} T^{-k}(N),$$

where the intersection is over finitely many distinct sets; so $N' \lhd_O G$. Note that $N \cap T^{-1}(N') = N'$, so $N' \subseteq T^{-1}(N')$ and $N' \in \mathcal{F}(T)$ by Lemma 5.2. Moreover, $N' \subseteq N$ and N may be written as a union of cosets of N'. As this holds for arbitrary $N \lhd_O G$, we see that $\mathcal{F}(T)$ forms a base for the neighborhoods of $e \in G$, and T is a quotient-preserving map.

By [14, Lemma 4.1.2], if G has a finitely-generated dense subgroup then G has finitely many open normal subgroups of a given index, and the final assertion of the proposition follows. □

We may apply Proposition 5.4 to several groups of interest:

Corollary 5.5. *Let $G = \prod_{i=1}^{g} \mathbb{Z}_{p_i}^{e_i}$ with the p_i rational primes and $e_i \in \mathbb{N}$. Then, any continuous homomorphism $T : G \to G$ is a quotient-preserving map and is not ergodic.*

Proof. We note that G contains a dense finitely-generated subgroup

$$\prod_{i=1}^{g} \mathbb{Z}^{e_i}.$$

Then, G has finitely many open normal subgroups of a given index, and in particular for each open normal subgroup $N \lhd_O G$ we have that $\{T^{-k}(N)\}$ must be finite (for each element of this set has index equal to the index of N). Applying Proposition 5.4 proves that T is a quotient-preserving map, and applying Remark 5.3 yields that T is not ergodic. □

Corollary 5.6. *Let $G = \mathbb{Z}_p^k$. Then, the nonsingular continuous homomorphisms $T : G \to G$ are given by multiplication by elements of* $\mathrm{GL}_k(\mathbb{Z}_p)$. *Any such homomorphism is a quotient-preserving map and is not ergodic.*

Proof. We note that $\mathbb{Z}^k$ is dense in G, and so a continuous homomorphism is defined by its values on a basis for $\mathbb{Z}^k$. In particular, this implies that any continuous homomorphism must be given by multiplication by some $T \in \mathrm{Mat}_{k\times k}(\mathbb{Z}_p)$. By Proposition 4.9 we must have T surjective. In particular, the image of T must contain the generators for $\mathbb{Z}_p^k$, so there must exist a $S \in \mathrm{Mat}_{k\times k}(\mathbb{Z}_p)$ such that $TS = \mathrm{id}_{k\times k} \in \mathrm{Mat}_{k\times k}(\mathbb{Z}_p)$. Then, $T \in \mathrm{GL}_k(\mathbb{Z}_p)$ (and of course, the converse holds by reversing this logic). Now, the previous corollary gives that this map must be a quotient-preserving map and is not ergodic. □

Remark 5.7. In this context we mention that Juzvinskiĭ [9] showed that ergodic group endomorphisms have completely positive entropy and Lind proves in [11] that ergodic automorphisms of compact metrizable groups are measurably isomorphic to Bernoulli shifts.

6. PRODUCTS

Lemma 6.1. *Let A be an index set. For each $\alpha \in A$ let G_α be a profinite group and $T_\alpha : G_\alpha \to G_\alpha$ a quotient-preserving map. Then,*

$$G = \prod_{\alpha\in A} G_\alpha$$

is a profinite group, and

$$T = \prod_{\alpha\in A} T_\alpha$$

is a quotient-preserving map on G.

Proof. Note that for each $\alpha \in A$ we have that $\mathcal{F}(T_\alpha)$ is a base for the neighborhoods of $\mathrm{e} \in G_\alpha$. Then, the collection

$$\mathcal{F} = \left\{\prod_{\alpha\in A} N_\alpha\colon\ N_\alpha \in \mathcal{F}(T_\alpha),\ N_\alpha = G_\alpha \text{ for all but finitely many } \alpha \in A\right\}$$

forms a base for the neighborhoods of $\mathrm{e} \in G$. Moreover, observe that each element of $\mathcal{F}$ is a normal subgroup of G.

We claim that

$$G \cong \varprojlim_{N\in\mathcal{F}} G/N.$$

Indeed, the natural projections induce a homomorphism

$$\phi : G \xrightarrow{\phi} \varprojlim_{N \in F} G/N.$$

Observe that ϕ is injective as G is Hausdorff. Moreover, ϕ is continuous and G compact (by Tychonoff's Theorem), so the image of ϕ is closed; but the image of ϕ is also dense in the codomain. So, ϕ is surjective. Then, ϕ is a continuous bijection with compact domain, so a homeomorphism, and G is indeed profinite.

Now, note that for any $N \in \mathcal{F}$, T factors through the projection $G \to G/N$ as the product of the factor transformations in each coordinate. So, T is a quotient-preserving map. □

Lemma 6.2. *Let S_k be a finite non-empty set and $T_k : S_k \to S_k$ a transformation for $k = 1, \ldots, n$. Let*

$$S = \prod_{k=1}^{n} S_k, \qquad T = \prod_{k=1}^{n} T_k.$$

Then, T is minimal on S if and only if each T_k is minimal on S_k and the $|S_k|$ are pairwise coprime.

Proof. Note that the general case follows from $n = 2$ case by induction. So, we may assume $n = 2$.

We have that T minimal implies T_1, T_2 minimal. By the minimality of T_k, each point of S_k must have full orbit. So we have $T_k^\ell(x) = x$ if and only if $|S_k| \mid \ell$. Let $\ell = |S_1||S_2|/(|S_1|, |S_2|)$ be the least common multiple of $|S_1|, |S_2|$. Then,

$$T^\ell(s_1, s_2) = \left(T_1^\ell(s_1), T_2^\ell(s_2)\right) = (s_1, s_2).$$

So, T minimal requires $(|S_1|, |S_2|) = 1$, that is that the cardinalities be coprime.

Conversely, say $(|S_1|, |S_2|) = 1$. In particular, given $s_k \in T_k$, $\ell_k \in \mathbb{N}$ for $k = 1, 2$, the Chinese Remainder Theorem gives us a $\ell \in \mathbb{N}$ such that $\ell \equiv \ell_k \pmod{|S_k|}$ for $k = 1, 2$. Then,

$$T^\ell(s_1, s_2) = \left(T_1^\ell(s_1), T_2^\ell(s_2)\right) = \left(s_1^{\ell_1}, s_2^{\ell_2}\right).$$

Then, T_1, T_2 minimal implies T minimal. □

Then:

Theorem 6.3. *Let $A, G_\alpha, T_\alpha, G, T$ be as in Lemma* 6.1. *Moreover, assume each G_α is second-countable and A is countable. Then, G is second-countable and*

(i) *T is nonsingular if and only if T_α is nonsingular for each $\alpha \in A$.*

(ii) *Denote*

$$D_\alpha = \{|G_\alpha/N_\alpha| \colon N_\alpha \in \mathcal{F}(T_\alpha)\}.$$

Then, T is ergodic if and only if T_α is ergodic for each $\alpha \in A$ and for all $\alpha, \beta \in A$ distinct and all $n \in D_\alpha, m \in D_\beta$ we have $(n, m) = 1$.

Proof. For each $\alpha \in A$ let C_α be a countable base for G_α. We may assume without loss of generality that $G_\alpha \in C_\alpha$ for each $\alpha \in A$. Then the set

$$\left\{\prod_{\alpha \in A} S_\alpha \colon S_\alpha \in C_\alpha, S\alpha = G_\alpha \text{ for all but finitely many } \alpha \in A\right\}$$

is a countable base for G. So, G is second-countable.

Note that T and each T_α are quotient-preserving maps. So, by Proposition 4.9, they are nonsingular if and only if they are surjective. Now, the product of a set of maps is surjective if and only if each map is surjective. The first claim follows.

Applying Proposition 4.11 to each T_α we see that T_α is ergodic if and only if each of the factor transformations $\{T_\alpha^{N_\alpha} \colon N_\alpha \in \mathcal{F}(T_\alpha)\}$ is minimal.

For $N_\alpha \in \mathcal{F}(T_\alpha)$, let $T_\alpha^{N_\alpha}$ denote the map making the following diagram commute

$$\begin{array}{ccc} G_\alpha & \xrightarrow{T_\alpha} & G_\alpha \\ \downarrow & & \downarrow \\ G_\alpha/N_\alpha & \xrightarrow{T_\alpha^{N_\alpha}} & G_\alpha/N_\alpha \end{array}$$

We note that $\mathcal{F}(T_\alpha)$ is a base for the open sets containing e $\in G_\alpha$, and so,

$$\left\{\prod_{\alpha \in A} N_\alpha \colon N_\alpha \in \mathcal{F}(T_\alpha), N_\alpha = G_\alpha \text{ for all but finitely many } \alpha \in A\right\}$$

is a base for the open sets containing e $\in G$. Given

$$N = \prod_{\alpha \in A} N_\alpha$$

in this base, we have that T factors through the projection $G \to G/N$ as

$$T_N = \prod_{\alpha \in A} T_\alpha^{N_\alpha}.$$

Applying Proposition 4.11, we see that T is ergodic if and only if each of these factor transformations is minimal on the finite quotient

$$G/N = \prod_{\alpha \in A} G_\alpha/N_\alpha \cong \prod_{N_\alpha \neq G_\alpha} G_\alpha/N_\alpha.$$

Dropping trivial factors and applying Lemma 6.2, we have that T is ergodic if and only if each $T_\alpha^{N_\alpha}$ is ergodic for all $\alpha \in A$ and $N_\alpha \in \mathcal{F}(T_\alpha)$ and the elements of the

D_α are pairwise co-prime (for different subscripts). Applying Proposition 4.11 to the T_α, this yields our desired result. $\square$

Remark 6.4. As a consequence, we get an alternate proof that no quotient-preserving maps are weakly mixing.

Corollary 6.5. *Let $T_p : \mathbb{Z}_p \to \mathbb{Z}_p$ be an ergodic quotient-preserving map for each rational prime p. Then, the map $T = \prod_p T_p$ on $G = \prod_p \mathbb{Z}_p$ is an ergodic quotient-preserving map.*

Proof. Follows immediately by Theorem 6.3 after noting that $\mathbb{Z}_p$ has quotients of p-power orders. $\square$

Corollary 6.6. *The maps $x \mapsto x \pm 1$ on*

$$\widehat{\mathbb{Z}} = \varprojlim_{n,|} \mathbb{Z}/n\mathbb{Z} \cong \prod_p \mathbb{Z}_p$$

is ergodic.

Proof. Each maps factors through all the projections and so is a quotient-preserving map. Note that the maps $x \mapsto x \pm 1$ are certainly minimal on $\mathbb{Z}/n\mathbb{Z}$ for each $n > 0$. In light of Proposition 4.11 this gives a direct proof that the induced map on $\widehat{\mathbb{Z}}$ is ergodic. Alternatively, we may use Proposition 4.11 to show that the induced map on $\mathbb{Z}_p$ is ergodic for each p, and then use the previous corollary.

Also, observe that for $n > 2$, the maps $x \mapsto -x \pm 1$ are *not* minimal on $\mathbb{Z}/n\mathbb{Z}$ $[1-0=1, 1-1=0; -1-0=-1, -1-(-1)=0]$. $\square$

Remark 6.7. Let $K = \mathbb{F}_p$ be the finite field of p elements. Let L be an algebraic closure of K. Then,

$$G = \mathrm{Gal}(L/K) \cong \widehat{\mathbb{Z}}.$$

The pth power map (the "Frobenius automorphism"), denoted Frob $\in G$, generates a dense cyclic subgroup of G. Indeed, the map $\mathbb{Z} \to G$ given by $n \mapsto \mathrm{Frob}^n$ induces the above isomorphism. So, the map $x \mapsto x+1$ on $\widehat{\mathbb{Z}}$ may be reinterpreted as the map on G given by $\sigma \mapsto \sigma \circ \mathrm{Frob}$.

Alternatively, we could let $K = \mathbb{C}(t)$, and $L = \mathbb{C}(t, t^{1/2}, t^{1/3}, t^{1/4}, \ldots, t^{1/n}, \ldots)$. Then,

$$G = \mathrm{Gal}(L/K) \cong \widehat{\mathbb{Z}}.$$

Take $\tau \in G$ defined by

$$\tau t^{1/m} = e^{2\pi\sqrt{-1}/m} t^{1/m}.$$

Then, τ generates a dense cyclic subgroup of G, and the map $\sigma \mapsto \sigma \circ \tau$ is the equivalent of $x \mapsto x+1$.

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