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Existence and a priori bounds for steady stagnation flow toward a stretching cylinder

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ABSTRACT

We investigate the nonlinear boundary value problem (BVP) that is derived from a similarity transformation of the Navier–Stokes equations governing fluid flow toward a stretching permeable cylinder. Existence of a solution is proven for all values of the Reynolds number and for both suction and injection, and uniqueness results are obtained in the case of a monotonic solution. A priori bounds on the skin friction coefficient are also obtained. These bounds achieve any desired order of accuracy as the injection parameter tends to negative infinity.

1. Introduction

The fluid dynamics of radial stagnation flow on a cylinder were first reviewed by Wang [1]. Since then there have been a number of extensions to the problem that incorporate various physically relevant phenomena. For example, Cunning et al. [2] considered rotation of the cylinder in addition to the suction or injection of a fluid through the bounding surface. Studies of radial stagnation flow toward a stretching cylinder were first performed by Crane [3] and Wang [4]. For a general review of the similarity solutions to the various forms of stagnation flow that have been studied, see [5]. Recent work by Ishak et al. [6] has included the effects of suction or injection of fluid for flow toward a stretching cylinder. In this paper, we follow the formulation of [6] and obtain existence and uniqueness results for their model. We also obtain a priori bounds on the skin friction coefficient.

The results on existence of a solution to this problem are similar to those for the case of an unstretched cylinder investigated by Paullet and Weidman [7]. However, some of the fundamental details of the argument differ and are presented here for completeness. Also our results show that there cannot be more than one *monotonic* solution for *all* values of the parameters. The results of [7] did so for only a limited range of parameter values. We note that the existence or nonexistence of nonmonotonic solutions remains an open question. Finally, the current work obtains much stricter bounds on the skin friction coefficient.

The similarity transformation of the Navier–Stokes equation for this problem leads to the boundary value problem

$$\eta f''' + f'' + R\left[ff'' - (f')^2\right] = 0, \quad 1 < \eta < \infty, \tag{1.1}$$

subject to

$$f(1) = \gamma, \tag{1.2}$$

$$f'(1) = 1, \tag{1.3}$$

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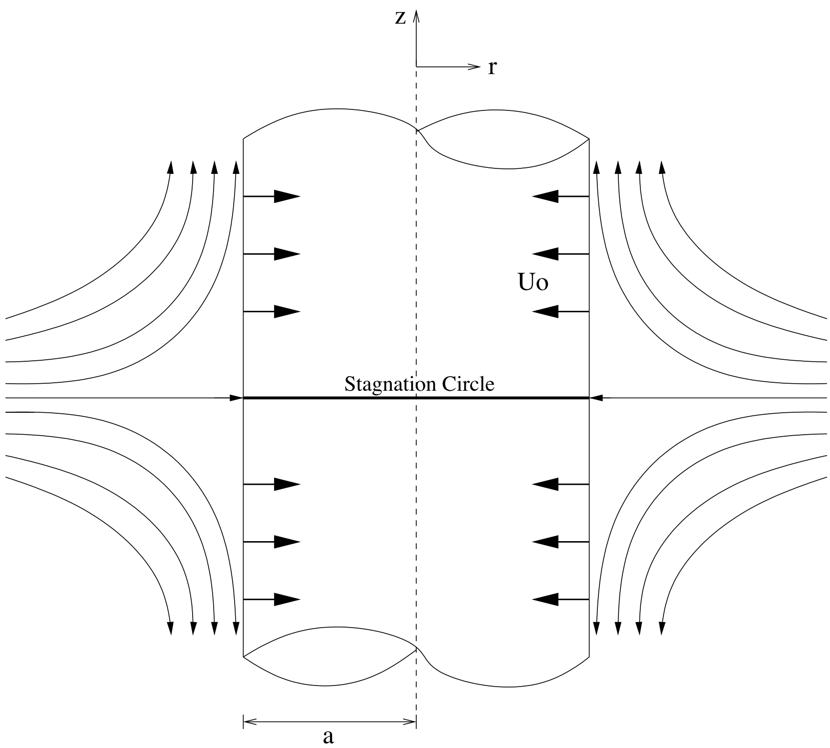

Fig. 1. Schematic diagram of radial stagnation flow toward a circular cylinder of radius a with uniform transpiration U_0 through the cylinder wall.

$$f'(\infty) = 0 \tag{1.4}$$

where $R > 0$ and $-\infty < \gamma < \infty$. Here the Reynolds number R and the dimensionless wall transpiration rate γ are defined by

$$R = \frac{ca^2}{2\nu}, \qquad \gamma = \frac{U_0}{ca}$$

in which c is the strain rate of the oncoming radial flow, a is the cylinder radius, ν is the kinematic viscosity of the fluid, and U_0 is the radially inward wall transpiration velocity as depicted in Fig. 1. With this notation, $\gamma > 0$ corresponds to wall suction and $\gamma < 0$ corresponds to wall injection.

2. Existence

To investigate the existence of a solution to this BVP we will study a related initial value problem (IVP), namely Eq. (1.1) subject to boundary conditions (1.2)–(1.3) along with

$$f''(1) = \alpha, \tag{2.5}$$

where α is a free parameter corresponding to the skin friction coefficient. We will denote the solution of the IVP by $f(\eta; \alpha)$. Occasionally the dependence on α will be dropped for notational convenience. By standard theory for initial value problems, this IVP will have a unique solution, at least locally. Using a topological shooting argument, we will show that α can be chosen so that $f'(\eta; \alpha)$ exists for all $\eta > 1$ and also satisfies Eq. (1.4), giving a solution to the BVP. Toward this end, consider the two sets,

$$\mathcal{A} = \left\{ \alpha < 0 \;\middle|\; \text{there is a first point } \eta_{\mathcal{A}} > 1 \text{ such that } f''(\eta_{\mathcal{A}}) = 0 \text{ and } f'(\eta) > 0 \text{ on } [1, \eta_{\mathcal{A}}] \right\}$$

and

$$\mathcal{B} = \left\{ \alpha < 0 \;\middle|\; \text{there is a first point } \eta_{\mathcal{B}} > 1 \text{ such that } f'(\eta_{\mathcal{B}}) = 0 \text{ and } f''(\eta) < 0 \text{ on } [1, \eta_{\mathcal{B}}] \right\}.$$

The next two lemmas will show that these sets are non-empty and open.

Lemma 1. *The set $\mathcal{A}$ is non-empty and open.*

Proof. First note that

$$f'''(1) = R - \alpha(1 + R\gamma). \tag{2.6}$$

If $\alpha = 0$ then $f'''(1) = R > 0$ and therefore we initially have $f' > 1$ and $f'' > 0$ on $(1, 1+\epsilon]$ for some $\epsilon > 0$. By continuity of the solutions of the initial value problem in its initial conditions, for $\alpha < 0$ sufficiently close to 0, $f'(\eta;\alpha)$ will stay close to $f'(\eta;0)$, i.e., will satisfy $f'(\eta;\alpha) > 0$ on $[1, 1+\epsilon]$ with $f'(1+\epsilon;\alpha) > 1$. But $f'(\eta;\alpha)$ is less than 1 and *decreasing* for $\eta \in (1, 1+\delta)$ for some $0 < \delta < \epsilon$. So for f' to rise above 1, it must have a minimum. Thus there exists a first point $\eta_{\mathcal{A}}$ such that $f''(\eta_{\mathcal{A}};\alpha) = 0$ with $f'(\eta;\alpha) > 0$ on $[1, \eta_{\mathcal{A}}]$. Thus for $\alpha < 0$ sufficiently close to 0, we have $\alpha \in \mathcal{A}$.

To show that $\mathcal{A}$ is open, let $\bar{\alpha} \in \mathcal{A}$. We will show that all α sufficiently close to $\bar{\alpha}$ are in $\mathcal{A}$. At $\eta_{\mathcal{A}}(\bar{\alpha})$ we have $0 < f' < 1$ and $f'' = 0$. Evaluating Eq. (1.1) at $\eta_{\mathcal{A}}(\bar{\alpha})$ implies that

$$f'''(\eta_{\mathcal{A}}) = R\big(f'(\eta_{\mathcal{A}})\big)^2/\eta_{\mathcal{A}} > 0.$$

Thus, by continuity of the solution of the IVP in its initial conditions, for α sufficiently close to $\bar{\alpha}$, $f''(\eta;\alpha)$ will also have a root, $\eta_{\mathcal{A}}(\alpha)$, near $\eta_{\mathcal{A}}(\bar{\alpha})$ with $f'(\eta;\alpha) > 0$. Thus $\alpha \in A$. This leaves only the possibility that $f' = 0$ and $f'' = 0$ simultaneously; however, substituting this information into Eq. (1.1) gives $f''' = 0$ implying that $f'(\eta) \equiv 0$ for all η, contradicting Eq. (1.3). $\square$

Lemma 2. *The set $\mathcal{B}$ is non-empty and open.*

Proof. An integration of Eq. (1.1) results in

$$\eta f''(\eta) = \alpha + R\int_1^\eta \big(f'(t)\big)^2 - f(t)f''(t)\,dt, \tag{2.7}$$

and a subsequent integration by parts yields

$$\eta f''(\eta) = \alpha + 2R\int_1^\eta \big(f'(t)\big)^2\,dt + R\big[\gamma - f(\eta)f'(\eta)\big]. \tag{2.8}$$

We will show that there are values of $\alpha < 0$, namely α large in magnitude, such that f' will equal 0 in the interval $(1, 2]$, say, strictly before $f'' = 0$. Suppose that this assertion is false, and consider the following cases.

Case (1A). $0 < f' < 1$, $f'' < 0$ for $\eta \in (1, 2]$, $\gamma \geqslant 0$: Integrating $0 < f' < 1$ yields $\gamma < f < \gamma + \eta - 1$ on $(1, 2]$. Using these bounds in Eq. (2.8) results in

$$f'' < \frac{\alpha}{2} + 2R + R\gamma, \quad \eta \in (1, 2].$$

Choosing $\alpha < -2 - 4R - 2R\gamma$ yields $f'' < -1$ on $(1, 2]$ and therefore $f'(2) < 0$ contradicting the assumption that $f' > 0$ on $(1, 2]$.

Case (1B). $0 < f' < 1$, $f'' < 0$ for $\eta \in (1, 2]$, $\gamma < 0$: Again, since $0 < f' < 1$ on $(1, 2]$ we have $\gamma < f < \gamma + \eta - 1$ on $(1, 2]$. Using these bounds in Eq. (2.8) results in

$$f'' < \frac{\alpha}{2} + 2R, \quad \eta \in (1, 2].$$

Choosing $\alpha < -2 - 4R$ yields $f'' < -1$ on $(1, 2]$ and therefore $f'(2) < 0$ contradicting the assumption that $f' > 0$ on $(1, 2]$.

Case (2). Next suppose there exists a first point $\eta_1 \in (1, 2]$ such that $f''(\eta_1) = 0$ with $f'' < 0$ on $(1, \eta_1)$. Using the bounds on f'' from case (1), we have

$$f'' < \begin{cases} \frac{\alpha}{2} + 2R + R\gamma, & \text{if } \gamma \geqslant 0, \\ \frac{\alpha}{2} + 2R, & \text{if } \gamma < 0, \end{cases}$$

for $\eta \in (1, \eta_1]$. Choosing

$$\alpha < \begin{cases} -(4R + 2R\gamma), & \text{if } \gamma \geqslant 0, \\ -4R, & \text{if } \gamma < 0, \end{cases}$$

results in $f''(\eta_1) < 0$ contradicting $f''(\eta_1) = 0$.

Case (3). This leaves the possibility that $f'' = 0$ and $f' = 0$ simultaneously, but as in Lemma 1, this implies that $f' \equiv 0$ contradicting Eq. (1.3).

Thus $\mathcal{B}$ is non-empty. To show that $\mathcal{B}$ is open, let $\bar{\alpha} \in \mathcal{B}$. Then there exists a first point $\eta_{\mathcal{B}}(\bar{\alpha})$ such that $f'(\eta_{\mathcal{B}}(\bar{\alpha})) = 0$ and $f''(\eta_{\mathcal{B}}(\bar{\alpha})) < 0$. By continuity of the solution of the IVP in its initial conditions, for α sufficiently close to $\bar{\alpha}$, there exists $\eta_{\mathcal{B}}(\alpha)$ with $f'(\eta_{\mathcal{B}}(\alpha)) = 0$ and $f''(\eta_{\mathcal{B}}(\alpha)) < 0$, and so, $\mathcal{B}$ is open.

Thus the sets $\mathcal{A}$ and $\mathcal{B}$ are non-empty, disjoint, and open, but the interval $(-\infty, 0)$ is connected and thus $\mathcal{A} \cup \mathcal{B} \neq (-\infty, 0)$. Therefore, there exists some α^* such that $\alpha^* \notin \mathcal{A}$ and $\alpha^* \notin \mathcal{B}$. As previously observed, we cannot have $f' = 0$ and $f'' = 0$ simultaneously; thus, the only other possibility is $f'(\eta; \alpha^*) > 0$ and $f''(\eta; \alpha^*) < 0$ for all $\eta > 1$.

Since f' is bounded below and decreasing, $f'(\infty; \alpha^*) = C$ exists where $0 \leqslant C < 1$. We now show that we must have $C = 0$. We begin with the supposition $0 < C < 1$. Since $f'' < 0$ for $\eta > 1$, f' is bounded below by $C > 0$, and so, f tends to positive infinity. Thus the term ff'' is eventually negative. From Eq. (1.1), we have

$$\eta f'''(\eta) = -f''(\eta) + R\big[\big(f'(\eta)\big)^2 - f(\eta)f''(\eta)\big] \geqslant RC^2 = K > 0$$

for η large enough. Thus there exists a point $\eta_2 > 1$ such that for $\eta > \eta_2$ we have

$$\eta f'''(\eta) > \frac{K}{2}.$$

Dividing by η and integrating results in

$$f''(\eta) > f''(\eta_2) + \frac{K}{2}[\ln \eta - \ln \eta_2] \quad \text{for } \eta > \eta_2,$$

and letting $\eta \to \infty$ implies that $f'' \to \infty$ contradicting the fact that $f'' < 0$. Thus we must have $f'(\infty; \alpha^*) = 0$ establishing the following theorem. □

Theorem 1. *For any $R > 0$ and $-\infty < \gamma < \infty$, there exists a solution to the boundary value problem. This solution satisfies $f'(\eta) > 0$ and $f''(\eta) < 0$ for all $\eta > 1$.*

3. Uniqueness results

In this section we prove the following:

Theorem 2. *If $R > 0$ and $-\infty < \gamma < \infty$, then there cannot be two solutions to the BVP with the property $f'(\eta) > 0$.*

Proof. Note that from Eq. (1.1), $f'(\eta; \alpha^*)$ cannot have a maximum. Thus for any solution satisfying $f'(\eta; \alpha^*) > 0$, we must also have $f''(\eta; \alpha^*) < 0$. Thus $0 < f'(\eta; \alpha^*) < 1$ for any positive solution.

Next consider the function $v = \partial f/\partial\alpha$. Differentiating Eq. (1.1) with respect to α gives

$$\eta v''' + v'' + R\big[vf'' + fv'' - 2f'v'\big] = 0 \tag{3.9}$$

subject to

$$v(1) = v'(1) = 0, \qquad v''(1) = 1. \tag{3.10}$$

Thus for $\eta > 1$ we have v' positive and increasing.

We now show that $v'(\eta; \alpha^*)$ cannot have a positive maximum. Suppose a first such maximum exists; then at such a point we have $v > 0$, $v' > 0$, $v'' = 0$, and $v''' \leqslant 0$. Substituting $v'' = 0$ into Eq. (3.9) yields

$$\eta v''' = R\big[2f'v' - vf''\big] > 0 \tag{3.11}$$

which is a contradiction. Thus v' cannot have a maximum and therefore $v' = \partial f'/\partial\alpha > 0$.

Now suppose that there are two solutions $f'(\eta; \alpha^*)$ and $f'(\eta; \alpha^{**})$ with $\alpha^{**} > \alpha^*$. By the Mean Value Theorem we have

$$f'\big(\eta; \alpha^{**}\big) - f'\big(\eta; \alpha^*\big) = \left(\frac{\partial f'}{\partial\alpha}\right)_{\alpha=\hat{\alpha}}\big(\alpha^{**} - \alpha^*\big) = v'(\eta; \hat{\alpha})\big(\alpha^{**} - \alpha^*\big) \tag{3.12}$$

where $\alpha^* < \hat{\alpha} < \alpha^{**}$. Now v' cannot have a maximum and so is bounded below by $L > 0$ for η large. Thus letting $M = L(\alpha^{**} - \alpha^*)$ and $\eta \to \infty$ in Eq. (3.12) we obtain

$$0 = 1 - 1 = f'\big(\eta; \alpha^{**}\big) - f'\big(\eta; \alpha^*\big) = v'(\eta; \hat{\alpha})\big(\alpha^{**} - \alpha^*\big) > M > 0$$

which is a contradiction. □

4. Bounds on skin friction coefficient

As a quantity of physical interest, we now derive bounds on the skin friction coefficient $f''(1) = \alpha^*$. Since $f'(\eta;\alpha^*)$ cannot have a maximum, any solution of the BVP must satisfy $f''(1;\alpha^*) = \alpha^* < 0$. Next we claim that in order for a solution to satisfy the boundary condition (1.4), we must have

$$f'''(1) = R - \alpha(1 + R\gamma) > 0. \tag{4.13}$$

Consider the following cases.

Case (1). Solutions with $f'(\eta;\alpha^*) > 0$ for $\eta > 1$: Suppose $f'''(1) < 0$ so that f' is initially concave down. In order to satisfy Eq. (1.4), f' must at some point change concavity. Thus there exists a point η_3 such that $f'(\eta_3) > 0$, $f''(\eta_3) < 0$, and $f'''(\eta_3) = 0$ with $f^{iv}(\eta_3) \geqslant 0$. Differentiating Eq. (1.1) results in

$$\eta f^{(iv)} + (2 + Rf) f''' - Rf' f'' = 0, \quad 1 < \eta < \infty, \tag{4.14}$$

and evaluating Eq. (4.14) at η_3 yields

$$\eta_3 f^{(iv)}(\eta_3) = Rf'(\eta_3) f''(\eta_3) < 0$$

since we cannot have $f'''(\eta_3) = f''(\eta_3) = 0$ as observed in the proof of Lemma 1, and thus, a contradiction has been established. Next suppose $f'''(1) = 0$. Substituting in Eq. (4.14) results in

$$f^{(iv)}(1) = R\alpha < 0 \tag{4.15}$$

and so we initially have $f''' < 0$ for $\eta > 1$. From the analysis given above, f''' cannot change sign as would be necessary to satisfy Eq. (1.4).

Case (2). Solutions for which $f'(\eta;\alpha^*)$ becomes negative: Suppose $f'''(1) < 0$ so that f' is initially concave down. Since f' is not positive for all η, there exists a first point η_4 such that $f'(\eta_4) = 0$ and $f''(\eta_4) < 0$. To satisfy Eq. (1.4), f' must become concave up for some $\eta > \eta_4$ and attain a minimum. Since f' cannot have a maximum, f' must monotonically increase from its minimum, approach 0 from below, and thus, become eventually concave down.

From this analysis, f''' must change sign twice, that is, from negative to positive and back to negative. Thus there exists a point η_5 such that f''' has a positive max, i.e., $f'''(\eta_5) > 0$, $f^{(iv)}(\eta_5) = 0$, and $f^{(v)}(\eta_5) \leqslant 0$. Differentiating Eq. (4.14) and evaluating at η_5 results in

$$\eta_5 f^{(v)}(\eta_5) = R\big(f''(\eta_5)\big)^2 \geqslant 0.$$

If $f''(\eta_5) \neq 0$, we get an immediate contradiction; so, consider the case $f''(\eta_5) = f^{(v)}(\eta_5) = 0$. Differentiating Eq. (4.14) twice and evaluating at η_5 implies that $f^{(vi)}(\eta_5) = 0$. Finally, differentiating Eq. (4.14) three times and evaluating at η_5 results in

$$\eta_5 f^{(vii)}(\eta_5) = 2R\big(f'''(\eta_5)\big)^2 > 0.$$

In summary we have $f^{(iv)}(\eta_5) = f^{(v)}(\eta_5) = f^{(vi)}(\eta_5) = 0$ with $f^{(vii)}(\eta_5) > 0$. Thus for η just greater than η_5, $f^{(iv)}$ is positive and f''' is increasing which is a contradiction if f''' is to have a maximum at η_5. Thus for any solution we must have

$$f'''(1) = R - \alpha^*(1 + R\gamma) > 0. \tag{4.16}$$

If $\gamma \geqslant -\frac{1}{R}$, this bound gives us nothing useful. However, we have

$$\frac{R}{1 + R\gamma} < \alpha^*, \quad \text{if } \gamma < -\frac{1}{R}. \tag{4.17}$$

Note also that an upper bound on α^* can be obtained if one assumes $\gamma \leqslant -\frac{2}{R}$. To this end, we claim that

$$f^{(iv)}(1) = R\alpha - (2 + R\gamma)\big[R - (1 + R\gamma)\alpha\big] < 0. \tag{4.18}$$

First if $f^{(iv)}(1) > 0$, then exists a first point η_6 such that $f^{(iv)}(\eta_6) = 0$ with $f^{(v)}(\eta_6) \leqslant 0$; otherwise,

$$f^{(iv)}(\eta) > 0 \quad \text{for } \eta > 1 \tag{4.19}$$

which we show leads to a contradiction. Integrating (4.19) results in

$$f'''(\eta) > K \quad \text{for } \eta > 1 \tag{4.20}$$

where $K = R - (1 + R\gamma)\alpha > 0$. Another integration yields

$$f''(\eta) > \alpha + K(\eta - 1) \quad \text{for } \eta > 1. \tag{4.21}$$

Letting $\eta \to \infty$ implies that $f'' \to \infty$ and we cannot have $f' \to 0$ as required by Eq. (1.4).

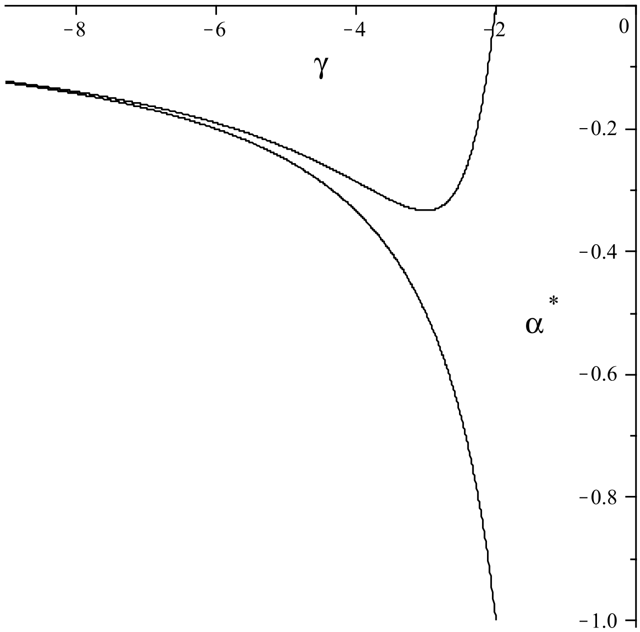

Fig. 2. Upper and lower bounds on the skin friction coefficient $f''(1)$ as a function of γ for $R = 1$.

Table 1
The values of the skin friction coefficient $f''(1)$ for $R = 1$.

γ	Lower bound on $f''(1)$	$f''(1)$ num. approx.	Upper bound on $f''(1)$
−0.5	NA	−0.96221	NA
−1	NA	−0.78818	NA
−2	−1	−0.54109	0
−4	−0.33333	−0.29388	−0.28571
−6	−0.2	−0.19101	−0.19048
−8	−0.14286	−0.13962	−0.13953
−10	−0.11111	−0.10961	−0.10959

Thus $f^{(iv)}$ must decrease through 0 at some point η_6. Differentiating Eq. (4.14) and evaluating at η_6 results in

$$\eta_6 f^{(v)}(\eta_6) = R\big(f''(\eta_6)\big)^2 \geqslant 0.$$

If $f''(\eta_6) \neq 0$, we get an immediate contradiction. If $f''(\eta_6) = 0$, then analysis similar to that given above yields $f^{(vi)}(\eta_6) = 0$ and $f^{(vii)}(\eta_6) > 0$. Thus $f^{(iv)}$ will be positive on a right interval of η_6, not negative as required, and we conclude that $f^{(iv)}(1) \not> 0$.

If $f^{(iv)}(1) = 0$ then Eq. (4.16) becomes $f^{(v)}(1) = R\alpha^2 > 0$ and the argument given above yields a contradiction. Solving for α in Eq. (4.18) and using Eq. (4.17) results in

$$\frac{R}{1 + R\gamma} < \alpha^* < \frac{R(2 + R\gamma)}{R + (1 + R\gamma)(2 + R\gamma)}, \quad \text{if } \gamma \leqslant -\frac{2}{R}. \tag{4.22}$$

We have provided a graphical representation of these bounds in Fig. 2 for $R = 1$. Note that both bounds converge to zero, and thus, the skin friction coefficient converges to zero as the injection parameter approaches negative infinity. Our results corroborate the remark in [6] that an increase in injection reduces the skin friction coefficient. In Table 1 we provide a numerical computation of the skin friction coefficient $f''(1) = \alpha^*$ [8]. This table elucidates the sharpening of the bounds on $f''(1)$ as the injection parameter γ increases in magnitude for a fixed Reynolds number.

The bounds given above are valid for all solutions to the BVP if $\gamma \leqslant -\frac{2}{R}$. In the analysis that follows, we determine bounds on the skin friction coefficient for $\gamma > -\frac{1}{2R}$ (unfortunately, not $\gamma > -\frac{2}{R}$ as would be desirable), first for solutions with $f'(\eta; \alpha^*) > 0$ for $\eta > 1$, and then for solutions where $f'(\eta; \alpha^*)$ becomes negative, in case such solutions exist. We begin with a lemma that will be needed in our proof of Theorem 3.

Lemma 3. *Let $f'(\eta; \alpha^*) > 0$ be a solution to Eq.* (1.1) *subject to boundary conditions* (1.2)–(1.4)*. If $\gamma > -\frac{1}{2R}$, then*

$$\lim_{\eta \to \infty} \left[-\eta\big(f''(\eta)\big)^2 + \frac{2R}{3}\big(f'(\eta)\big)^3 \right] = 0.$$

Proof. To prove the existence of the limit, recall that from Theorem 1 a solution with $f'(\eta;\alpha^*)>0$ for $\eta>1$ also satisfies $f''(\eta;\alpha^*)<0$ for $\eta>1$. Thus f is an increasing function and f' is a decreasing function. Since $\gamma>-\frac{1}{2R}$, we have $1+2Rf>0$ for $\eta>1$. Multiplying Eq. (1.1) by f'' and integrating results in

$$\int_1^\eta \big(1+2Rf(t)\big)\big(f''(t)\big)^2\,dt-\alpha^2+\frac{2R}{3}=-\eta\big(f''(\eta)\big)^2+\frac{2R}{3}\big(f'(\eta)\big)^3. \tag{4.23}$$

Note that the left-hand side of Eq. (4.23) is an increasing function. Thus the right-hand side is increasing, and in particular, the term $-\eta(f'')^2$ is increasing since the term $\frac{2R}{3}(f')^3$ is decreasing. Since $f'(\eta;\alpha^*)$ is a solution to the BVP, we have $f'\to 0$ as $\eta\to\infty$. Also, since $-\eta(f''(\eta))^2$ is increasing and bounded above by 0, its limit as $\eta\to\infty$ exists.

Next suppose that the limit is $L\neq 0$. Since $\lim_{\eta\to\infty}f'(\eta)=0$ and $-\eta(f''(\eta))^2<0$ for $\eta>1$, we must have $L<0$. Let $L=-m$. Since the right-hand side of Eq. (4.23) is increasing, we have

$$-\eta\big(f''(\eta)\big)^2+\frac{2R}{3}\big(f'(\eta)\big)^3<-m\quad\text{for }\eta\geqslant 1$$

or, ignoring the positive term on the left and multiplying by -1,

$$\eta\big(f''(\eta)\big)^2>m\quad\text{for }\eta\geqslant 1.$$

Further algebra results in

$$\left(f''(\eta)-\sqrt{\frac{m}{\eta}}\right)\left(f''(\eta)+\sqrt{\frac{m}{\eta}}\right)>0\quad\text{for }\eta\geqslant 1,$$

and since the first factor on the left is negative, we must have

$$f''(\eta)<-\sqrt{\frac{m}{\eta}}\quad\text{for }\eta\geqslant 1.$$

Integrating this inequality results in

$$f'(\eta)<1-2\sqrt{m}(\sqrt{\eta}-1)\quad\text{for }\eta\geqslant 1,$$

and letting $\eta\to\infty$ implies that $f'\to-\infty$ contradicting Eq. (1.4). □

Theorem 3. *Let $f'(\eta;\alpha^*)>0$ be a solution to Eq.* (1.1) *subject to boundary conditions* (1.2)–(1.4)*. If $\gamma>-\frac{1}{2R}$, then $\alpha^*<-\sqrt{\frac{2R}{3}}$.*

Proof. Letting $\eta\to\infty$ in Eq. (4.23) and using Lemma 3 results in

$$\int_1^\infty \big(1+2Rf(t)\big)\big(f''(t)\big)^2\,dt=\alpha^2-\frac{2R}{3}>0,$$

since $1+2Rf>0$ for $\eta>1$. Thus

$$\alpha^*<-\sqrt{\frac{2R}{3}}.\qquad □$$

Recall that the existence of solutions where $f'(\eta;\alpha^*)$ becomes negative remains an open problem. However, if such solutions exist, a useful bound on the skin friction coefficient is obtained in Theorem 4 below. The proof of this bound will require two lemmas.

Lemma 4. *Suppose there exists a solution to Eq.* (1.1) *subject to boundary conditions* (1.2)–(1.4) *where $f'(\eta;\alpha^*)$ becomes negative. Then* $\lim_{\eta\to\infty}\eta f''(\eta)=0$.

Proof. Recall from the discussion of the case $\gamma\leqslant-\frac{2}{R}$ that if a solution to the BVP exists where $f'(\eta;\alpha^*)$ becomes negative, then the solution must have very specific behavior. The function f' must have one negative minimum and then turn concave down as f' approaches 0 from below. Thus there exists a point η_7 such that $f'<0$, $f''>0$, and $f'''<0$ for $\eta>\eta_7$. By using these inequalities and rearranging Eq. (1.1) into the form

$$\eta f'''+(1+Rf)f''-R\big(f'\big)^2=0,\quad 1<\eta<\infty, \tag{4.24}$$

we conclude that

$$f(\eta) > -\frac{1}{R} \quad \text{for } \eta > \eta_7.$$

Thus f is bounded below and decreasing for $\eta > \eta_7$, and so, $f(\infty) = L \geqslant -\frac{1}{R}$ where L is finite. This results in

$$\lim_{\eta \to \infty} Rf(\eta)f'(\eta) = 0. \tag{4.25}$$

Thus for all $\epsilon > 0$, there exists $\bar{\eta} > \eta_7$ such that

$$-\frac{\epsilon}{4} < Rf(\eta)f'(\eta) < \frac{\epsilon}{4} \quad \text{for } \eta > \bar{\eta}. \tag{4.26}$$

Now for the sake of contradiction, suppose that $\lim_{\eta \to \infty} \eta f''(\eta) \neq 0$. Then there exists an $\epsilon > 0$ and a sequence $\eta_i \to \infty$ such that

$$\left|\eta_i f''(\eta_i)\right| \geqslant \epsilon \quad \text{for } i = 1, 2, \ldots$$

and since $f'' > 0$ for $\eta > \eta_7$, we have

$$\eta_i f''(\eta_i) \geqslant \epsilon \quad \text{for } \eta_i > \eta_7. \tag{4.27}$$

Now choose a positive integer N so that the inequalities (4.26)–(4.27) hold where $\eta_N > \bar{\eta} > \eta_7$. It follows that

$$\eta_i f''(\eta_i) + Rf(\eta_i)f'(\eta_i) > \epsilon - \frac{\epsilon}{4} = \frac{3\epsilon}{4} \quad \text{for } \eta_i \geqslant \eta_N. \tag{4.28}$$

By rearranging Eq. (2.8) into the form

$$2R\int_1^\eta \left(f'(t)\right)^2 dt + R\gamma + \alpha = \eta f''(\eta) + Rf(\eta)f'(\eta), \tag{4.29}$$

we note that the left-hand side of Eq. (4.29) is an increasing function. We thus conclude that the inequality (4.28) is valid for all $\eta \geqslant \eta_N$. Thus (4.28) becomes

$$\eta f''(\eta) \geqslant \frac{3\epsilon}{4} - Rf(\eta)f'(\eta) \quad \text{for } \eta \geqslant \eta_N, \tag{4.30}$$

and using (4.26) in (4.30) yields

$$\eta f''(\eta) \geqslant \frac{\epsilon}{2} \quad \text{for } \eta \geqslant \eta_N.$$

Dividing by η and integrating results in

$$f'(\eta) \geqslant f'(\eta_N) + \frac{\epsilon}{2}[\ln \eta - \ln \eta_N] \quad \text{for } \eta \geqslant \eta_N.$$

Finally, letting $\eta \to \infty$ implies that $f' \to \infty$ contradicting Eq. (1.4) and the lemma is proved. □

Lemma 5. *Suppose there exists a solution to Eq.* (1.1) *subject to boundary conditions* (1.2)–(1.4) *where* $f'(\eta; \alpha^*)$ *becomes negative. If* $\gamma > -\frac{1}{2R}$, *then* $\int_1^\infty (1 + 2Rf(t))(f''(t))^2\,dt > 0$.

Proof. It suffices to show that $1 + 2Rf > 0$ for $\eta \geqslant 1$. From the proof of Lemma 4, we have $f' < 0$, $f'' > 0$, and $f''' < 0$ for $\eta > \eta_7$. Thus f'' is positive and decreasing, and since $f'(\infty)$ exists, we have $f''(\infty) = 0$. Now letting $\eta \to \infty$ and using Lemma 4 in Eq. (4.23) results in

$$\lim_{\eta \to \infty}\left[-\eta\left(f''(\eta)\right)^2 + \frac{2R}{3}\left(f'(\eta)\right)^3\right] = 0,$$

and thus

$$\int_1^\infty \left(1 + 2Rf(t)\right)\left(f''(t)\right)^2 dt = \alpha^2 - \frac{2R}{3}. \tag{4.31}$$

Furthermore, we have

$$\int_1^{\eta} \big(1+2Rf(t)\big)\big(f''(t)\big)^2\,dt - \alpha^2 + \frac{2R}{3} = -\eta\big(f''(\eta)\big)^2 + \frac{2R}{3}\big(f'(\eta)\big)^3 < 0 \quad \text{for } \eta > \eta_7 \tag{4.32}$$

since both terms on the right are negative, and so,

$$\int_1^{\eta} \big(1+2Rf(t)\big)\big(f''(t)\big)^2\,dt < \alpha^2 - \frac{2R}{3} \quad \text{for } \eta > \eta_7. \tag{4.33}$$

Thus $\int_1^{\eta}(1+2Rf(t))(f''(t))^2\,dt$ approaches its limit at infinity from below, and $1+2Rf$ must be positive for arbitrarily large values of η. Since $\gamma > -\frac{1}{2R}$, the function $1+2Rf$ starts out positive and since f' has only one sign change – from positive to negative – f has one maximum and so does $1+2Rf$. Thus we must have $1+2Rf > 0$ for $\eta \geqslant 1$ and the lemma is proved. $\square$

Theorem 4. *Suppose there exists a solution to Eq.* (1.1) *subject to boundary conditions* (1.2)–(1.4) *where* $f'(\eta;\alpha^*)$ *becomes negative. If* $\gamma > -\frac{1}{2R}$, *then* $\alpha^* < \min\{-\sqrt{\frac{2R}{3}}, -R\gamma\}$.

Proof. Letting $\eta \to \infty$, and using Lemma 4 and (4.25) in Eq. (4.29) results in

$$\int_1^{\infty} \big(f'(t)\big)^2\,dt = -\left(\frac{\alpha + R\gamma}{2R}\right) > 0,$$

and we have

$$\alpha^* < -R\gamma. \tag{4.34}$$

Using Lemma 5 in Eq. (4.31) results in

$$\alpha^* < -\sqrt{\frac{2R}{3}}, \quad \text{if } \gamma > -\frac{1}{2R}, \tag{4.35}$$

and combining the inequalities (4.34)–(4.35) we have

$$\alpha^* < \min\left\{-\sqrt{\frac{2R}{3}}, -R\gamma\right\}, \quad \text{if } \gamma > -\frac{1}{2R}. \qquad \square$$

5. Conclusions

In this paper, we have studied radial stagnation flow toward a stretching cylinder with wall transpiration. We have shown that there exists at most one *monotonic* solution for all values of the parameters by using a topological shooting argument where the shooting parameter, $f''(1) = \alpha$, corresponds to the skin friction coefficient. However, we have not been able to eliminate the possibility of nonmonotonic solutions. Numerical integration of the differential equation suggests that no such solutions exist, but analytically, this remains an open question.

From numerical calculations, we present the following conjecture in regard to the characterization of solutions of the IVP involving Eqs. (1.1)–(1.3) and Eq. (2.5). For $\alpha \geqslant 0$, $f'(\eta;\alpha)$ increases to positive infinity. For $\alpha^* < \alpha < 0$, $f'(\eta;\alpha)$ attains a positive minimum and then increases to positive infinity. For $\alpha = \alpha^*$, $f'(\eta;\alpha^*)$ gives a solution to the BVP. For $\alpha < \alpha^*$, $f'(\eta;\alpha)$ attains a negative minimum and then increases to positive infinity.

From our analysis, we have been able to determine useful bounds on the skin friction coefficient. In particular we have determined that these bounds become arbitrarily narrow and that the value of the skin friction coefficient approaches zero as the injection parameter ($\gamma < 0$) approaches negative infinity. Also, in the case of suction ($\gamma > 0$), we have shown that the skin friction coefficient becomes arbitrarily large in magnitude as the Reynolds number R goes to infinity.

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