Fully group graded algebras and a theorem of Fong

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Abstract

It is proved that any algebra fully graded by a finite group over a complete discrete valuation ring with an algebraically closed residue field of characteristic a prime $p$ is Morita equivalent to an embedded graded subalgebra which is a crossed product; and an explicit way to get a decomposition of unity with a bounded length is shown. When the finite group is $p$-solvable, a theorem of Fong’s type for fully graded algebras is obtained.

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1. Introduction

1.1. Let $\mathcal{O}$ be a complete discrete valuation ring with an algebraically closed residue field $k$ of characteristic $p$ where $p$ is a prime; it is allowed that $\mathcal{O} = k$. Any $\mathcal{O}$-algebra $A$ in this paper is unitary and $\mathcal{O}$-free of finite $\mathcal{O}$-rank; but note that a subalgebra $B$ of $A$ is not necessarily unitary and the unity $1_B$ of $B$ is an idempotent of $A$, and $B$ is called an embedded subalgebra of $A$ if $B = 1_B A 1_B$. Let $G$ be a finite group.

P. Fong in [4] showed that if $G$ is $p$-solvable, then every projective indecomposable $\mathcal{O}G$-module is isomorphic to an induced module from a module of a Hall $p'$-subgroup. Conversely, as illustrated in [2, Example 13], the induced module of an indecomposable module of a Hall $p'$-subgroup is not necessarily indecomposable; a natural question emerges: what is
about the induced modules? Reference [7] gave results in a character version of the Fong’s type for \( \pi \)-separable groups. And [5] exhibited a \( G \)-algebra approach of the questions. As shown in Ref. [2], it is more interesting to consider the questions of the Fong’s type for group graded algebras; and [2] gave a complete answer for crossed products, which covers the \( G \)-algebra approach of the questions. Later, [3] gave an answer for general graded algebras which are not necessarily fully graded.

In this paper we are concerned about the situation of fully graded algebras. The study of the question turns out a result of general interesting on group graded algebras: any fully \( G \)-graded algebra is Morita equivalent to an embedded graded subalgebra which is a crossed product of \( G \); and the decomposition of unity of the fully graded algebra can be obtained explicitly by means of such subalgebras. This fact allows to translate to fully graded algebras any result on crossed products in terms of Morita equivalence classes. Consequently, the result of [2] is translated to the fully graded case.

1.2. Recall that a point \( \alpha \) on an \( \Theta \)-algebra \( A \) means an \( A^* \)-conjugacy class of primitive idempotents on \( A \), where \( A^* \) denotes the multiplicative group consisting of the invertible elements of \( A \); and \( \mathcal{P}(A) \) denotes the set of the points on \( A \). For any non-zero idempotent \( e \) there is an orthogonal set \( I_e \subseteq \bigcup_{\alpha \in \mathcal{P}(A)} \alpha \) such that \( e = \sum_{i \in I_e} i \); and the number \( m^e_\alpha = |I_e \cap \alpha| \) is independent of the choice of the orthogonal set \( I_e \), and is called the multiplicity of the point \( \alpha \) at \( e \). If \( m^e_\alpha = m^e_\beta \) for any \( \alpha, \beta \in \mathcal{P}(A) \), we call \( e \) an isotypic idempotent of \( A \); at that case, the embedded subalgebra \( B = eAe \) is Morita equivalent to \( A \) in a natural way, see (2.2.2) below. In particular, if \( m^e_\alpha = 1 \) for any \( \alpha \in \mathcal{P}(A) \), we call \( e \) a basic idempotent of \( A \).

An \( \Theta \)-algebra \( A \) is said to be \( G \)-graded if \( A \) has \( \Theta \)-submodules \( A_x \) indexed by \( x \in G \) such that \( A = \bigoplus_{x \in G} A_x \) and \( A_x A_y \subseteq A_{xy} \) for all \( x, y \in G \). For \( S \subseteq G \) the submodule \( A_S = \bigoplus_{x \in S} A_x \) is called the \( S \)-component of \( A \); in particular, \( A_x \) for \( x \in G \) is named the \( x \)-component of \( A \). Observe that the 1-component \( A_1 \) must be an unitary subalgebra of \( A \), cf. [6]. A \( G \)-graded algebra \( A \) is said to be fully \( G \)-graded if \( A_x A_y = A_{xy} \) for all \( x, y \in G \); and \( A \) is said to be a crossed product of \( G \) over \( A_1 \) if for each \( x \in G \) there is an \( \hat{x} \in A_x \cap A^* \) such that \( A_x = A_1 \hat{x} \).

1.3. Theorem. Let \( A \) be a 1-graded \( \Theta \)-algebra and \( e_1 \) be an isotypic idempotent of the 1-component \( A_1 \). Then \( B = e_1 A e_1 \) is a graded subalgebra of \( A \) which is Morita equivalent to \( A \), and \( A \) is a fully \( G \)-graded if and only if \( B \) is a crossed product of \( G \) over \( B_1 \).

1.4. Let \( A \) be a fully \( G \)-graded algebra. For any \( x \in G \), inasmuch as \( A_x A_{x^{-1}} = A_1 \), there are \( a_x^{(1)}, \ldots, a_x^{(n)} \in A_x \) and \( b_{x^{-1}}^{(1)}, \ldots, b_{x^{-1}}^{(n)} \in A_{x^{-1}} \) such that \( 1 = \sum_{i = 1}^n a_x^{(i)} b_{x^{-1}}^{(i)} \), which is called a decomposition of unity at \( x \), and we denote \( n \) the length of the decomposition. In \( A_1 \) we consider the multiplicity \( m_\alpha^1 \) of any \( \alpha \in \mathcal{P}(A_1) \) at the unity element 1, and denote \( m_{\max} = \max\{m^1_\alpha \mid \alpha \in \mathcal{P}(A_1)\} \) and \( m_{\min} = \min\{m^1_\alpha \mid \alpha \in \mathcal{P}(A_1)\} \), and set

\[
d = \lceil m_{\max}/m_{\min} \rceil
\]

which denotes the least integer not less than \( m_{\max}/m_{\min} \). As a corollary, for any \( x \in G \) we show a way to find a decomposition of unity at \( x \) of length bounded by \( d \).

1.5. Theorem. With notation as above, at any \( x \in G \) there is a decomposition of unity of length at most \( d \).
1.6. Remark. The above boundary \( d \) for the length of decompositions of unity is strict in some sense; for example, let \( G = \{1, x\} \) be a group of order 2, and the matrix algebra \( A = \text{M}_4(k) = A_1 \oplus A_2 \) where

\[
A_1 = \left\{ \begin{pmatrix} a_{11} & a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{pmatrix} \bigg| a_{ij} \in k \right\}
\]

and

\[
A_x = \left\{ \begin{pmatrix} a_{12} & a_{13} & a_{14} \\ a_{21} & a_{31} \\ a_{41} \end{pmatrix} \bigg| a_{ij} \in k \right\};
\]

then it is easy to verify that \( A \) is a fully \( G \)-graded \( k \)-algebra, and the length of decomposition of unity at \( x \) is at least 3, since for any \( a, b \in A_x \) the bottom right \( 3 \times 3 \) block of \( ab \) has rank at most 1.

1.7. Let \( A = \bigoplus_{x \in G} A_x \) be a fully \( G \)-graded \( \Theta \)-algebra. Recall from [2] that the divisor monoid \( \mathcal{D}(A) \) of \( A \) is the free additive monoid generated by the set \( \mathcal{P}(A) \), and \( \mathcal{D}(A) \) is isomorphic to the positive part of the Grothendieck group of \( A \). Let \( K \) and \( H \) be any subgroups of \( G \) with \( K \subseteq H \). Then the inclusion map \( A_K \rightarrow A_H \) induces a monoid homomorphism

\[
\iota^A_{K,H} : \mathcal{D}(A_K) \rightarrow \mathcal{D}(A_H)
\]

which sends any point \( \beta \in \mathcal{P}(A_K) \) to \( \sum_{\alpha \in \mathcal{P}(A_H)} m^j_\alpha \cdot \alpha \in \mathcal{D}(A_H) \), where \( j \in \beta \); see 1.2 above or cf. [2, Section 4]. In particular, setting \( H_x = H \cap H^x \) for \( x \in G \) where \( H^x = x^{-1}Hx \), we have a monoid homomorphism

\[
\iota^A_{H} = \bigoplus_{x \in G} \iota^A_{H_x,H} : \bigoplus_{x \in G} \mathcal{D}(A_{H_x}) \rightarrow \mathcal{D}(A_{H}).
\]

Further, any \( \alpha \in \mathcal{P}(A_{K^x}) \) determines a minimal idempotent ideal \( m_\alpha = A_{K^x} \alpha A_{K^x} \) of \( A_{K^x} \), see (2.1.3) below. And \( A_x m_\alpha A_x^{-1} \) is obviously an idempotent ideal of \( A_K \); conversely, it is clear that \( A_{x^{-1}}(A_x m_\alpha A_{x^{-1}})A_x = m_\alpha \); that is, \( A_x m_\alpha A_{x^{-1}} \) is a minimal idempotent ideal of \( A_K \), and determines a point of \( A_K \), denoted by \( \alpha^{x^{-1}} \). In this way we have a bijection \( \mathcal{P}(A_{K^x}) \rightarrow \mathcal{P}(A_K) \), hence we get a monoid isomorphism

\[
\kappa_{K,x}^A : \mathcal{D}(A_{K^x}) \rightarrow \mathcal{D}(A_K).
\]

Then we have a monoid homomorphism

\[
\kappa_{H}^A = \bigoplus_{x \in G} \kappa_{H_x^{-1,H}}^A \circ \iota_{H_x^{-1,x}}^A : \bigoplus_{x \in G} \mathcal{D}(A_{H_x}) \rightarrow \mathcal{D}(A_H).
\]
1.8. Theorem. Let $G$ be a finite $p$-solvable group and $A = \bigoplus_{x \in G} A_x$ be a fully $G$-graded $\mathcal{O}$-algebra, and let $H$ be a Hall $p'$-subgroup of $G$. Then the following is a coequalizer diagram:

$$\bigoplus_{x \in G} \mathcal{D}(A_{H^x}) \xrightarrow{\iota^A_{H^x}} \mathcal{D}(A_H) \xrightarrow{\iota^A_{H^G}} \mathcal{D}(A_G). \quad (1.8.1)$$

1.9. In Subsections 2.1–2.3 we show necessary notation and preliminaries as preparations. After some lemmas, Theorem 1.3 follows in 2.7; and Theorem 1.5 will be proved in Subsection 2.8. At last, some explanations are made for 1.7.

2. Proofs

2.1. Let $p$, $\mathcal{O}$, $k$ and $G$ always be the same as in 1.1. Let $A$ be an $\mathcal{O}$-algebra, and $a \in A$. Denote $\bar{A} = A/J(A)$ where $J(A)$ denotes the Jacobson radical of $A$. Noting that $1 + r \in A^\ast$ (i.e. $1 + r$ is invertible) for any $r \in J(A)$, we have the following:

$$(2.1.1) \ a \in A^\ast \text{ if and only if its residual image } \bar{a} \in \bar{A}^\ast.$$ 

Further, the residual image $\bar{\alpha}$ of a point $\alpha \in \mathcal{P}(A)$ is a point of $\bar{A}$, and $\mathcal{P}(A) \to \mathcal{P}(\bar{A}), \alpha \mapsto \bar{\alpha}$, is a bijection. The residue algebra $A/J(A)$ is semisimple, i.e. a direct product of simple $k$-algebras; each of the simple factors corresponds exactly one point $\alpha \in \mathcal{P}(A)$, and the multiplicity $m_{\alpha} = m_{\alpha}^1$ of $\alpha$ at the unity 1, called the multiplicity of $\alpha$ in $A$, is preserved in the residue algebra $A/J(A)$; i.e.

$$A/J(A) \cong \prod_{\alpha \in \mathcal{P}(A)} M_{m_{\alpha}}(k), \quad (2.1.2)$$

where $M_{m_{\alpha}}(k)$ denotes the matrix algebra over $k$ of order $m_{\alpha}$; cf. [8, Section 2]. Namely, each point $\alpha$ on $A$ corresponds exactly to one irreducible $A$-module and $m_{\alpha}$ is the multiplicity of the irreducible module appearing in the regular $A$-module. For $\alpha \in \mathcal{P}(A)$, from (2.1.2) it is easy to see that $m_{\alpha} = A\alpha A = \{ \sum aia' \mid i \in \alpha, \ a, a' \in A \}$ is a minimal idempotent ideal of $A$ which corresponds to $M_{m_{\alpha}}(k)$ in (2.1.2), and in this way we have

$$(2.1.3) \text{ the } m_{\alpha} \text{'s for } \alpha \in \mathcal{P}(A) \text{ are exactly all minimal idempotent ideals of } A.$$ 

2.2. Let $B$ be a subalgebra of $A$. For $\beta \in \mathcal{P}(B)$ take $j \in \beta$, then in $A$ the multiplicity $m_{\alpha}^j$ for $\alpha \in \mathcal{P}(A)$ is independent of the choice of $j \in \beta$, hence we can denote it by $m_{\alpha}^\beta$; please consult [8, Section 2] for detail. In this way, the inclusion map $B \to A$ always induces a monoid homomorphism

$$\mathcal{D}(B) \to \mathcal{D}(A), \quad \beta \mapsto \sum_{\alpha \in \mathcal{P}(A)} m_{\alpha}^\beta \cdot \alpha. \quad (2.2.1)$$

We remark that the $\iota^A_{K^G}$ in (1.7.1) is just such a homomorphism.

Next, let $e$ be an idempotent of the algebra $A$ and assume $B = eAe$. For any $\alpha \in \mathcal{P}(A)$, if $m_{\alpha}^e \neq 0$ then $eAe \cap \alpha$ is a point on $eAe$ and $m_{\alpha}^e$ is just the multiplicity of the point $eAe \cap \alpha$
in $eAe$; otherwise, $eAe \cap \alpha = \emptyset$. In other words, in that case the homomorphism (2.2.1) maps a point $\beta \in \mathcal{P}(B)$ to a point $\alpha \in \mathcal{P}(A)$ fulfilling that $\beta \subseteq \alpha$. And then, the homomorphism in (2.2.1) is injective in this case. For the above, please refer to [8]; and the following is known in [8, Proposition 2.7]:

(2.2.2) The following three statements are equivalent:

(i) $eAe \cap \alpha \neq \emptyset$ for all $\alpha \in \mathcal{P}(A)$;

(ii) $A$ is Morita equivalent to $eAe$ by sending an $A$-module $M$ to the $eAe$-module $eM$;

(iii) $AeA = A$.

If $m_\alpha^e = m_\beta^e = m \neq 0$ for all $\alpha, \beta \in \mathcal{P}(A)$, then we call $e$ an isotypic idempotent of $A$ of multiplicity $m$. Further, if the unity 1 of $A$ is an isotypic idempotent of multiplicity $m$, then $A$ is named to be isotypic of multiplicity $m$.

2.3. Let $A = \bigoplus_{x \in G} A_x$ be a $G$-graded $\Theta$-algebra. It is easy to see that:

(2.3.1) $A$ is fully graded if and only if $A_xA_{x^{-1}} = A_1$ for all $x \in G$.

If $c \in A^*$, then multiplication by $c$ produces a bijection $A \to A$, $a \mapsto ac$. From this fact it is easy to descry that:

(2.3.2) If $a_x \in A_x \cap A^*$, then $a_x^{-1} \in A_{x^{-1}}$ and $A_x = A_1a_x$.

An ideal $I$ of $A$ is said to be graded if $I = \bigoplus_{x \in G} I_x$ with $I_x = A_x \cap I$; at that case, $A/I$ is a graded algebra with $x$-component isomorphic to $A_x/I_x$. It is well known that $AJ(A_1)A$ is a graded ideal of $A$ contained in $J(A)$; cf. [1, Lemma 7]. On the other hand, if $e$ is an idempotent of $A_1$, then $eAe = \bigoplus_{x \in G} eA_xe$ is an embedded and graded subalgebra of $A$; moreover, we have:

2.4. Lemma. If $A$ is a $G$-graded algebra and $e$ is an idempotent of $A_1$ such that $eA_1e$ is Morita equivalent to $A_1$, then

(i) $eAe$ is Morita equivalent to $A$;

(ii) $eAe$ is fully graded if and only if $A$ is fully graded.

Proof. By assumption we have $A_1eA_1 = A_1$, see (2.2.2); so $ AeA = A \cdot (A_1eA_1) \cdot A = AA_1A = A$, and by (2.2.2) again, $eAe$ is Morita equivalent to $A$.

If $A$ is fully graded, for any $x \in G$ we have $eA_xe \cdot eA_{x^{-1}}e = eA_x \cdot (A_1eA_1) \cdot A_{x^{-1}}e = eA_xA_1A_{x^{-1}}e = eA_1e$, therefore $eAe$ is fully graded with $x$-component $eA_xe$.

Conversely assume that $eAe = \bigoplus_{x \in G} eA_xe$ is fully graded. By the assumption of the lemma, for any $x \in G$ we have

$$A_x = A_1A_xA_1 = A_1eA_1A_xeA_1 = A_1e(A_1A_xeA_1)eA_1 = A_1eA_xeA_1;$$

hence for any $x, y \in G$ we have

$$A_xA_y = A_1eA_xeA_1A_yeA_1eA_1eA_1 = A_1eA_xeA_yeA_1 = A_1eA_xeA_1 = A_{xy};$$

that is, $A$ is fully graded. $\square$
2.5. Lemma. If $A$ is a fully $G$-graded $\Theta$-algebra and its 1-component $A_1$ is isotypic of multiplicity 1, then $A$ is a crossed product of $G$ over $A_1$.

Proof. First we assume that $A_1$ is semisimple. Then $\Theta = k$ and $A_1$ is commutative and every point of $A_1$ contains exactly one primitive idempotent $i$; cf. (2.1.2). Instead of $\{i\} \in \mathcal{P}(A_1)$, we write $i \in \mathcal{P}(A_1)$ for convenience. So we have that $A_1 = \bigoplus_{i \in \mathcal{P}(A_1)} ki$, where $ki$ is the 1-dimensional ideal of $A_1$ generated by $i$; and any ideal of $A_1$ is idempotent. So, for any $x \in G$, we gain that

$$A_x A_{x^{-1}} = A_1 = \bigoplus_{i \in \mathcal{P}(A_1)} ki = A_x \left( \bigoplus_{i \in \mathcal{P}(A_1)} ki \right) A_{x^{-1}}.$$

On one hand, let $i \in \mathcal{P}(A_1)$, if $A_x i A_{x^{-1}} = 0$, then $ki = A_1 i A_1 = A_{x^{-1}} A_x i A_{x^{-1}} A_x = 0$, which is a contradiction. Therefore, $A_x i A_{x^{-1}} \neq 0$ for all $i \in \mathcal{P}(A_1)$. On the other hand, if $i \neq i' \in \mathcal{P}(A_1)$ then $A_x i A_{x^{-1}} \cdot A_x i' A_{x^{-1}} = 0$. In virtue of (2.1.2) we get that

$$\bigoplus_{i \in \mathcal{P}(A_1)} A_x i A_{x^{-1}} = \bigoplus_{i \in \mathcal{P}(A_1)} ki$$

and $A_x i A_{x^{-1}} \neq 0$ for all $i \in \mathcal{P}(A_1)$. Namely, there exists a permutation $\rho$ of $\mathcal{P}(A_1)$ such that

$$A_x i \cdot i A_{x^{-1}} = A_x i A_{x^{-1}} = k \cdot \rho(i), \quad \forall i \in \mathcal{P}(A_1).$$

Hence there are $a_{x,i} \in A_x i$ and $b_{i,x^{-1}} \in i A_{x^{-1}}$ such that $a_{x,i} b_{i,x^{-1}} = \lambda \cdot \rho(i)$ with $0 \neq \lambda \in k$; replacing $a_{x,i}$ by $\lambda^{-1} a_{x,i}$, we acquire $a_{x,i} \in A_x i$ and $b_{i,x^{-1}} \in i A_{x^{-1}}$ such that

$$a_{x,i} b_{i,x^{-1}} = \rho(i).$$

Let

$$a_x = \sum_{i \in \mathcal{P}(A_1)} a_{x,i} \quad \text{and} \quad b_{x^{-1}} = \sum_{i \in \mathcal{P}(A_1)} b_{i,x^{-1}}.$$

Noting that the idempotents of $\mathcal{P}(A_1)$ are orthogonal to each other, one can check that

$$a_x b_{x^{-1}} = \sum_{i \in \mathcal{P}(A_1)} \rho(i) = \sum_{i \in \mathcal{P}(A_1)} i = 1.$$

Since $A$ is a finite-dimensional $k$-algebra, such $a_x$ is an invertible element of $A$. Therefore, $A_x = A_1 a_x$, and $A = \bigoplus_{x \in G} A_1 a_x$ is a crossed product of $G$.

For the general case, taking $I = A J(A_1) A$ which is a graded ideal contained in $J(A)$, we have the residue algebra $\bar{A} = A/I = \bigoplus_{x \in G} \bar{A}_x$ which is fully $G$-graded with semisimple 1-component $\bar{A}_1 = A_1 / J(A_1)$ which is isotypic of multiplicity 1. Thus, for any $x \in G$, there exists $\tilde{a}_x \in \bar{A}_x$, by the surjection $A_x \rightarrow \bar{A}_x$ we can choose an inverse image $a_x$ of $\tilde{a}_x$ in $A_x$. By using of (2.1.1) we see that $a_x \in A^* \cap A_x$. Then $A_x = A_1 \cdot a_x$ for any $x \in G$ and $A$ is a crossed product of $G$ over $A_1$. \qed
2.6. Corollary. If $A$ is a fully $G$-graded $\Theta$-algebra with isotypic 1-component $A_1$, then $A$ is a crossed product of $G$ over $A_1$.

Proof. Let $\mathcal{P}(A_1) = \{\alpha_1, \ldots, \alpha_h\}$. By the assumption, in $A_1$ we obtain an orthogonal primitive decomposition of 1 as follows:

$$1 = e_{11} + \cdots + e_{1m} + \cdots + e_{h1} + \cdots + e_{hm}$$ (2.6.1)

such that $e_{st} \in \alpha_s$ for $s = 1, \ldots, h$ and $t = 1, \ldots, m$. Consequently,

$$e_t = e_{1t} + \cdots + e_{ht}, \quad t = 1, \ldots, m,$$

are basic idempotents of $A_1$; hence, for each $t = 1, \ldots, m$, the embedded subalgebra $e_tA_t$ is fully $G$-graded too, see 2.4, and its 1-component $e_tA_1$ is isotypic of multiplicity 1. By the above lemma, for any $x \in G$ we have $a_{t,x} \in e_tA_t e_t$ and $b_{t,x-1} \in e_tA_{x-1} e_t$ fulfilling that $a_{t,x} b_{t,x-1} = e_t = b_{t,x-1} a_{t,x}$. Let

$$a_x = a_{1,x} + \cdots + a_{m,x}, \quad b_{x-1} = b_{1,x-1} + \cdots + b_{m,x-1}.$$

Since $a_{t,x} = e_t a_{t,x} e_t$ and $b_{t,x-1} = e_t b_{t,x-1} e_t$ for $t = 1, \ldots, m$, and $e_1, \ldots, e_m$ are orthogonal to each other, it is evident that

$$a_x b_{x-1} = a_{1,x}b_{1,x-1} + \cdots + a_{m,x}b_{m,x-1} = e_1 + \cdots + e_m = 1.$$

We can get $b_{x-1} a_x = 1$ in the same way. Namely, $a_x \in A_x \cap A^*$. As a result, $A = \bigoplus_{x \in G} A_1 a_x$ is a crossed product of $G$ over $A_1$. \square

2.7.

Proof of Theorem 1.3. It follows from Lemma 2.4 and Corollary 2.6. \square

2.8.

Proof of Theorem 1.5. By the assumption, in $A_1$ we have a set $I$ of orthogonal primitive idempotents such that $1 = \sum_{e \in I} e$ and $|\alpha \cap I| \leq m_{\text{max}}$ for all $\alpha \in \mathcal{P}(A_1)$.

First, we can take $I_1 \subseteq I$ such that for any $\alpha \in \mathcal{P}(A_1)$ we have $|\alpha \cap I_1| = m_{\text{min}}$. Let $I^{(1)} = I - I_1$ and denote

$$m_{\text{max}}^{(1)} = \max_{\alpha \in \mathcal{P}(A_1)} |\alpha \cap I^{(1)}| = m_{\text{max}} - m_{\text{min}}.$$

If $m_{\text{max}}^{(1)} > 0$, take $I_2 \subseteq I^{(1)}$ such that

$$|\alpha \cap I_2| = \begin{cases} m_{\text{min}}, & \text{if } |\alpha \cap I^{(1)}| \geq m_{\text{min}}, \\ |\alpha \cap I^{(1)}|, & \text{if } |\alpha \cap I^{(1)}| < m_{\text{min}}, \end{cases} \quad \forall \alpha \in \mathcal{P}(A_1).$$

Let $I^{(2)} = I^{(1)} - I_2$; then either $I^{(2)} = \emptyset$ or

$$m_{\text{max}}^{(2)} = \max_{\alpha \in \mathcal{P}(A_1)} |\alpha \cap I^{(2)}| = m_{\text{max}} - 2m_{\text{min}} > 0.$$
And we can repeat the process, up to getting a partition \( I_1, \ldots, I_d \) of \( I \) such that for any \( I_t \) with \( 1 \leq t \leq d \) and for any \( \alpha \in \mathcal{P}(A_1) \) we have \( |\alpha \cap I_t| \leq m_{\text{min}} \). So we can find \( J_t \) with \( I_t \subseteq J_t \subseteq I \) fulfilling that

\[
|\alpha \cap J_t| = m_{\text{min}}, \quad \forall \alpha \in \mathcal{P}(A_1).
\]

Let \( f_t = \sum_{e \in I_t} e \) and \( f'_t = \sum_{e \in J_t} e \); then \( f_t f'_t = f_t = f'_t f_t \), and \( f'_t \) is an isotypic idempotent on \( A_1 \) of multiplicity \( m_{\text{min}} \); by Corollary 2.6, for any \( x \in G \), we attain an \( a'_{t,x} \in A_x \) and a \( b'_{t,x}^{-1} \in A_{x^{-1}} \) such that \( f'_t = a'_{t,x} b'_{t,x}^{-1} \). Putting \( a_{t,x} = f_t a'_{t,x} \in A_x \) and \( b_{t,x}^{-1} = b'_{t,x}^{-1} f_t \in A_{x^{-1}} \), we have

\[
f_t = a_{t,x} b_{t,x}^{-1}.
\]

Thus we have

\[
1 = \sum_{e \in I} e = \sum_{t=1}^{d} \sum_{e \in I_t} e = \sum_{t=1}^{d} f_t = \sum_{t=1}^{d} a_{t,x} b_{t,x}^{-1},
\]

which is a decomposition at \( x \in G \) of the unity with length at most \( d \). And the proof of Theorem 1.5 is complete. \( \square \)

2.9. Let \( A = \bigoplus_{x \in G} A_x \) be a fully \( G \)-graded \( \mathcal{O} \)-algebra and \( e_1 \) be a basic idempotent on its 1-component \( A_1 \); set \( B = e_1 A e_1 \). Then, by Theorem 1.3, \( B \) is a crossed product of \( G \) over \( B_1 \); so for any \( x \in G \) we take \( \hat{x} \in B_x \cap B^* \) such that \( B = \bigoplus_{x \in G} B_1 \hat{x} \). Further, for any subgroup \( H \) of \( G \), by Theorem 1.3 again, the \( H \)-component \( B_H = e_1 A_H e_1 \) is Morita equivalent to \( A_H \). By (2.2.1) and (2.2.2) we have a natural isomorphism

\[
\delta_H : \mathcal{D}(B_H) \xrightarrow{\cong} \mathcal{D}(A_H).
\]

Let \( K \) be a subgroup of \( G \). In \( B = \bigoplus_{x \in G} B_1 \hat{x} \), each \( \hat{x} \) induces by conjugation an automorphism of \( B \), it maps \( B_K \), onto \( B_K \), and sends a point \( \beta \in \mathcal{P}(B_K) \) to a point \( \hat{x} \beta \hat{x}^{-1} = \{ \hat{x} j \hat{x}^{-1} \mid j \in \beta \} \) on \( B_K \), hence, it induces a monoid isomorphism

\[
\mathcal{D}(B_K) \to \mathcal{D}(B_K), \quad \beta \mapsto \hat{x} \beta \hat{x}^{-1},
\]

which is the conjugation map defined in [2]. In notation of 1.7, it is easy to see that

\[
B_1 m_\beta B_{x^{-1}} = B_1 B_K \beta B_{K^{-1}} B_{x^{-1}} = B_K \hat{x} \beta \hat{x}^{-1} B_K;
\]

that is, the map (2.9.2) coincides with the isomorphism \( \kappa^B_{K,x} \) defined in (1.7.3) for \( B \). Thus every thing in [2] is translated into fully graded algebras; in particular, we have a commutative diagram of monoids:
where the top line is, by the main result of [2], a coequalizer diagram; hereby the bottom line is a coequalizer too. That is the statement of Theorem 1.8.

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References