Asymptotic values of polynomial mappings of the real plane✩

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Abstract

Using algebraically constructible functions we give a generically effective method to detect asymptotic values of polynomial mappings with finite fibers defined on the real plane. By asymptotic variety we mean the set of points at which the polynomial mapping fails to be proper.

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1. Introduction

In the study of geometrical or topological properties of polynomial mappings, the asymptotic variety plays a very important role. For example, if we can prove that the asymptotic variety is empty for mappings with non-vanishing Jacobian then such a mapping will necessarily be a polynomial isomorphism. Recently, Jelonek and Kurdyka [6] using geometrical properties of asymptotic varieties gave a beautiful estimation of the number of bifurcation points of a complex polynomial. Other interesting examples can be found in [5]. The asymptotic variety of a map \( F : \mathbb{R}^n \to \mathbb{R}^m \) can be defined as follows:

\[
J_F := \left\{ y \in \mathbb{R}^m \mid \exists \{x_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^n \text{ such that } \lim_{k \to \infty} |x_k| = \infty \text{ and } \lim_{k \to \infty} F(x_k) = y \right\}
\]

We also call it the set of asymptotic values of \( F \) or the Jelonek set. We remark that the asymptotic set is just the set of points at which \( F \) fails to be proper. The geometry of this set has been investigated by Jelonek in the nineties (see for example [4,5]).

For many applications it would be helpful to have an algorithm to obtain an explicit description of the asymptotic set. In the case of algebraically closed fields such a method, involving Gröbner bases, exists (e.g. [10]). In the real case the situation is much more complicated, and until now no satisfactory algorithm to detect the set of asymptotic

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values exists, even in the simplest case, for a polynomial mapping defined on the real plane. The difficulties arise from the fact that it is not known how to distinguish the real roots among the complex ones.

The aim of this paper is to give a generically effective method to detect the asymptotic variety of a polynomial map \( F: \mathbb{R}^2 \to \mathbb{R}^2 \) with finite fibers. We show that in this case this set is equal as a set to the support of an algebraically constructible function, which can be generically obtained effectively. Algebraically constructible functions were introduced by McCrory and Parusiński [7] to express some topological invariants of real algebraic sets.

Our proof is purely topological; we use results due to Coste and Kurdyka [2], and Parusiński and Szafraniec [9]. To detect the points at which the map under consideration is not proper, the main idea is to examine how the number of points of the fibers varies.

In the sequel all sets considered are real algebraic.

2. Algebraically constructible functions

Due to the following result, the Euler characteristic (with respect to the Borel–Moore homology) is a very good tool to examine how the fibers change.

**Theorem 2.1.** (Coste–Kurdyka [2]; Parusiński–Szafraniec [9]) Let \( F: X \to Y \) be a polynomial function between algebraic sets. Then there exists a finite family of polynomials \( g_i \in \mathbb{R}[Y] \) such that for any \( y \) we have

\[
\chi(F^{-1}(y)) = \sum_{i=1}^{k} \text{sgn} \ g_i(y). \tag{2.1}
\]

The functions which are finite sums of signs of some polynomials are called *algebraically constructible*. Let us recall that these functions actually were introduced in another form by McCrory and Parusiński in [7]. Later it turned out [9] that both definitions were equivalent. Let us quickly recall some basic facts, for details see [7] or [8,9]. In particular, each algebraically constructible function is constructible in the sense that there exist a finite family of semi-algebraic subsets \( X_i \subset X \) and integers \( m_i \) such that the considered function is of the form \( \sum m_i 1_{X_i} \), where \( 1_X \) denotes the characteristic function of the set \( X \). Hence, according to Viro’s definition [11], these functions can be integrated with respect to the Euler characteristic. Let us recall that if \( \varphi = \sum m_i 1_{X_i} \) then the Euler integral of \( \varphi \) is defined as:

\[
\int \varphi = \sum m_i \chi(X_i),
\]

where \( \chi \) denotes Euler characteristic.

Let \( S(x, \epsilon) \) denotes a sphere at \( x \) of radius \( \epsilon \). To any algebraically constructible function there is an associated function:

\[
\Lambda \varphi(x) = \int_{S(x, \epsilon)} \varphi
\]

called a *link* of \( \varphi \) at \( x \), where \( \epsilon \) is small enough. It is again an algebraically constructible function [7].

Now we will introduce a function which controls the behavior of the germs of polynomial functions under small deformations. For this denote by

\[
\varphi_F: Y \ni y \to \chi(F^{-1}(y)) \in \mathbb{Z}
\]

a function, which due to Theorem 2.1 is algebraically constructible.

**Definition 2.2.** Let \( F: X \to Y \) be a polynomial map between algebraic sets of the same dimension. The local link at \( a \) of the map \( F \) is the following number

\[
\lambda(a) := \Lambda(\varphi_{F|_{U_a}})(F(a)),
\]

where \( U_a \) is a sufficiently small neighborhood of \( a \).
Remark 2.3. If all the fibres of $F$ are finite, then the fibre $(F|_{U_a})^{-1}(F(a))$ of the restriction of $F$ to $U_a$ consists only of the point $a$.

Observe that in this situation at any regular point $a$ of the map $F$, we have $\lambda(a) = \chi(S^{n-1})$, where by a regular point we mean a point at which the differential of $F$ is of maximal rank and $n$ is the dimension of $X$. In fact, in this case $F$ is a local diffeomorphism at $a$.

From now on we take $X = Y = \mathbb{R}^2$. At the beginning let us note that the local link vanishes outside some finite set. To see this, let $\Sigma_F$ be the set of critical points of the map $F$ (i.e. the set of points $a$ such that $\text{Jac}(F)(a) = 0$). Denote by $\text{Reg} \Sigma_F$ the set of regular points of $\Sigma_F$. Suppose now, that the set of critical points is not finite; then take $g := F|_{\text{Reg} \Sigma_F}$ the restriction of $F$ to $\text{Reg} \Sigma_F$. Let

$$A := \{x \in \text{Reg} \Sigma_F \text{ such that rank } d_xg = 0\}.$$  

The set $A$ is finite, since all the fibres of $F$ are finite. Put

$$\Gamma_F := \text{Reg} \Sigma_F \setminus A,$$

which, by definition, is a one-dimensional smooth variety. Hence, the set

$$S(F) := \Sigma_F \setminus \Gamma_F$$

is finite.

If the set of critical points is finite we set $S(F) = \Sigma_F$.

**Lemma 2.4.** Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map with finite fibres. Then, there is a discrete set $\tilde{S}(F)$ such that at any point $a \in \mathbb{R}^2 \setminus \tilde{S}(F)$ the local link $\lambda(a)$ vanishes.

The set $\tilde{S}(F)$ contains the set $S(F)$ and perhaps some discrete subset $D$ of $\Gamma_F$ which comes from the following auxiliary lemma.

**Lemma 2.5.** There is a discrete subset $D \subset \Gamma_F$ such that if $a \in \Gamma_F \setminus D$ then, after suitable analytic change of variables in the neighborhoods of points $a$ and $F(a)$, the map $F$ is of the following form:

$$F(v, w) = (v, w^kH(v, w)),$$  \hspace{1cm} (2.3)

where $H$ is an analytic map such that $H(0, 0) > 0$, and $k \in \mathbb{N}$.

**Proof of Lemma 2.4.** As we have seen the local link vanishes at regular points. Suppose then that the point $a$ belongs to $\Gamma_F$. Due to the auxiliary lemma $F$ is of the form (2.3) in the neighborhood of $0$, hence for a sufficiently small $\varepsilon$ the map $\varphi_F$ is given by

$$\varphi_F(x, y) = \begin{cases} 
1 & \text{if } k \text{ odd or } y = 0, \\
2 & \text{if } k \text{ even and } y > 0, \\
0 & \text{if } k \text{ even and } y < 0.
\end{cases} \hspace{1cm} (2.4)$$

where the points $(x, y)$ satisfy $\|(x, y)\| \leq \varepsilon$. So, $\lambda(a) = \lambda(\varphi_F)(0, 0) = 0$. \hfill \Box

**Proof of Lemma 2.5.** Without loss of generality we can assume that $a = (0, 0)$ and $F(a) = (0, 0)$. Let $(x, y)$ be a local linear system of coordinate at $0$. In our situation the set of critical points in neighborhood of $a$ is a smooth one-dimensional variety we can assume that in fact it is locally a line $y = 0$. We can also assume that the multiplicity of $F$ is constant on the critical locus. Observe as well that we can assume that the partial derivative $\frac{\partial f}{\partial x}$ does not vanish at $0$, since the differential is of rank 1. Hence, the map

$$z: W \ni (x, y) \to (f(x, y), y) \in \mathbb{R}^2$$

is a diffeomorphism in the neighborhood $W$ of $0$. Consequently,

$$F(z^{-1}(v, w)) = (v, h(v, w)),$$
where $h$ is an analytic function. In what follows we identify $F$ with the composition $F \circ z^{-1}$. We get then
\[
\text{Jac}(F) = \frac{\partial h}{\partial w},
\] (2.5)
but according to our assumptions $\text{Jac}(F) = 0$ iff $w = 0$. We claim that $\text{Jac}(F) = w^n A(v, w)$, where $A$ is an analytic function not vanishing at the origin. Indeed
\[
\text{Jac}(F) = w^n (\varphi(v) + w^j B(v, w)),
\]
with $j > 0$, $\varphi$ and $B$ analytic. If now $\varphi(0) = 0$ then the multiplicity is equal to $n$ if $v \neq 0$ and strictly greater than $n$ if $v = 0$, in contradiction with the fact that the multiplicity is locally constant at the origin. Hence $\varphi(0) \neq 0$ as required.

Finally, by (2.5) we obtain that
\[
F(v, w) = (v, \theta(v) + w^k H(v, w))
\]
for some non-negative integer $k$ and some function $\theta$ analytic at 0, so up to the change of coordinates at the target we have the desired form of the map $F$. \qed

3. Description of asymptotic values

Let us start by recalling a property of the asymptotic set, which we will use in the proof of our result. At the beginning we recall the definition of the Jelonek set of the map $F$:

**Definition 3.1.**
\[
J_F := \left\{ y \in \mathbb{R}^m \mid \exists \{x_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^n \text{ such that } \lim_{k \to \infty} |x_k| = \infty \text{ and } \lim_{k \to \infty} F(x_k) = y \right\}.
\]

**Theorem 3.2.** (Jelonek [5]) Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a non-constant polynomial mapping. Then all the connected components of the set $J_F$ are at most $(n-1)$-dimensional. Moreover for any $x \in J_F$ there exists a non-constant polynomial map $\phi : \mathbb{R} \to J_F$ such that $\phi(0) = x$.

In particular the asymptotic set does not admit isolated points. This property does not hold in the general case of affine varieties. For instance, Jelonek gave the following example. Let $X = \mathbb{R}^2 \setminus \{(0, 0)\}$ and $f : X \to \mathbb{R}^2$ an embedding, then $J_f = \{(0, 0)\}$. However this is true for the varieties which are connected at infinity.

In what follows we will denote the link of the function $\varphi_F$ (see (2.2)) by $A_F$.

**Theorem 3.3.** Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial function with finite fibers, then:
\[
J_F = \text{supp } A_F = \left\{ y \in \mathbb{R}^2 \text{ such that } A_F(y) \neq 0 \right\}.
\] (3.1)

**Proof.** The proof consists of two steps. In the first we show that the sets $J_F$ and $\text{supp } A_F$ coincide except for a discrete set of points. In the second, we prove that $\text{supp } A_F$ is closed. Then, because both sets, as we will see, do not have isolated points we can conclude they are equal.

**First step.** In our situation $\chi(F^{-1}(y)) = \#F^{-1}(y)$. Denote by $C_0$ the closure of the set of critical values of the map $F$. Let $g_1, \ldots, g_m$ be the polynomials satisfying the equality (2.1) of Theorem 2.1 for the function $\varphi_F$. Put
\[
C_i := \left\{ y \in \mathbb{R}^2 \mid g_i(y) = 0 \right\} \quad \text{for } i \in \{1, \ldots, m\}, \quad Z := \bigcup_{i=0}^m C_i \quad \text{and} \quad \bar{C}_i := C_i \setminus \bigcup_{j \neq i} C_j.
\]
The complement $\mathbb{R}^2 \setminus Z$ is a disjoint union of open sets $U_j$ on which the function $\varphi_F$ is constant, say equal to $k_j$. Since $\chi(S(y, \varepsilon)) = 0$, directly by definition we get $A_F|_{U_j} \equiv 0$ for each $j$. It means that $\text{supp } A_F \subset Z$. The restriction of $F$ to $F^{-1}(U_j)$ is a local homeomorphism and the number of points in the fibers is constant, hence it is proper. It means that $J_F \subset Z$. It suffices now to examine the points of $Z$.

Let $y \in \bar{C}_i$ and put $\{t_1, \ldots, t_k\} := F^{-1}(y)$. Take sufficiently small open neighborhoods $V_i$ of points $t_i$ such that $V_i \cap V_j = \emptyset$ for $i \neq j$. Then put $V := \bigcup_{i=1}^k V_i$. 

Now, assume that \( y \) is a regular point of the variety \( \widetilde{C}_i \) and does not belong to the image of the set \( \widetilde{S}(F) \) by the map \( F \). Due to Lemma 2.4:

\[
0 = \sum_{i=1}^{k} \lambda(t_i) = \Lambda_{F|V}(y). \tag{3.2}
\]

Let \( U_x \) and \( U_t \) be such that \( y \in \overline{U}_x \cap \overline{U}_t \). As the set \( \widetilde{S}(F) \) is discrete, the equality (3.2) shows that

\[
0 \geq 2k - (k_x + k_t) = \Lambda_F(y). \tag{3.3}
\]

(1) If \( \Lambda_F(y) = 0 \) then \( k_x + k_t = 2k \). In this case equality (3.2) shows that all sufficiently closed fibers to \( F^{-1}(y) \) are included in \( V \). Consequently \( F \) is proper at \( y \).

(2) If \( y \in \text{supp} \Lambda_F \) then \( k_x + k_t > 2k \). Equality (3.2) shows that there exists a sequence \( \{x_n\} \subset \mathbb{R}^2 \), such that

1. \( x_n \notin V \) for \( n \in \mathbb{N} \),
2. \( F(x_n) \in S(y, \frac{1}{n}) \).

Suppose now that the sequence \( \{x_n\} \) is bounded. We can find a subsequence \( \{x_{n_k}\} \) converging to some \( x \). Since \( F \) is continuous then \( F(x_{n_k}) \to F(x) = y \), but by the assumption on \( V_i \), \( F^{-1}(y) \subset V \) which contradicts \( x_n \notin V \). So we conclude that \( \lim_{n \to \infty} |x_n| = \infty \); thus, \( y \) is a point in the asymptotic set. Since

\[
\bigcup_{i=0}^{m} \text{Sing} \widetilde{C}_i \cup \left( \mathbb{Z} \setminus \bigcup_{i=0}^{m} \widetilde{C}_i \right)
\]

is finite, the sets \( J_F \) and \( \text{supp} \Lambda_F \) coincide outside a finite set.

**Second step.** We check now that \( \text{supp} \Lambda_F \) is closed. Indeed, let \( y \in \overline{\text{supp}} \Lambda_F \). Changing the order, we can suppose that the sets \( U_1, \ldots, U_r \) restricted to a sufficiently small ball \( B(y, \varepsilon) \) with center at \( y \) are all connected components of the set \((\mathbb{R}^2 \setminus \mathbb{Z}) \cap B(y, \varepsilon)\) such that \( y \in \overline{U}_i \cap B(y, \varepsilon), i \in \{1, \ldots, r\} \). Order these sets in such a way that \( L_i := \overline{U}_i \cap \overline{U}_{i+1} \setminus \{y\} \) is not empty for all \( i \) (under the convention that \( U_{r+1} = U_1 \)).

The number of points in the fiber of \( F \) over \( L_i \) is constant, say equal to \( l_i \). Hence, by (3.3) we get that

\[
k_i + k_{i+1} \geq 2l_i \quad \text{for } 1 \leq i \leq r, \tag{3.4}
\]

where \( k_i \) denotes the number of points in the fiber over a point of \( U_i \).

On the other hand, the link of \( \varphi_F \) at \( y \) satisfies

\[
\Lambda_F(y) = l_1 + \cdots + l_r - (k_1 + \cdots + k_r).
\]

If \( y \in \text{supp} \Lambda_F \) then there exists \( i \in \{1, \ldots, r\} \) such that \( k_i + k_{i+1} > 2l_i \). Hence, we get

\[
k_1 + \cdots + k_r > l_1 + \cdots + l_r,
\]

which shows that \( \Lambda_F(y) < 0 \). Consequently \( y \) belongs to \( \text{supp} \Lambda_F \) which means that this set is closed.

To finish our proof, remark that the same arguments show that the support of the set \( \Lambda_F \) does not admit isolated points. Theorem 3.2 implies that the asymptotic variety does not admit any isolated point either.

Knowing that the closed sets \( J_F \) and \( \text{supp} \Lambda_F \) coincide except for a discrete set and that they have no isolated points we get that they are equal. \( \square \)

Let us remark that this theorem gives us an important corollary.

**Corollary 3.4.** Let \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) be a polynomial map as in the assumptions of the previous theorem. If the number of points in each fiber over some open set \( U \subset \mathbb{R}^2 \) is constant then \( F \) is proper at \( U \).

**Remark 3.5.** Let us point out that due to the remark just after Theorem 3.2 the equality (3.1) holds in a more general setting of polynomials maps defined on algebraic surfaces with isolated singularities and such that these are connected at infinity.
Example 3.6. Let \( F(x, y) = (y^2(xy - 1), x) \). On Fig. 1 we show the values of \( \varphi_F \) and the values of its link: hence the asymptotic set is exactly the line \( \{ y = 0 \} \).

Now, to finish, we observe that to get an effective algorithm to determine the set of asymptotic values it suffices to determine the polynomials \( g_i \) of Theorem 2.1, generically one can do it using Hermite’s method [3]. We give here a sketch of the method (for details see [1]).

Let \( F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2 \) be as in Theorem 3.3. By assumption \( \chi(F^{-1}(u, v)) = \# V_{\mathbb{R}}(f - u, g - v) \), where \( V_{\mathbb{R}}(f - u, g - v) \) is the set of real points \((x, y)\) satisfying equations \( f(x, y) = u \) and \( g(x, y) = v \). Denote by \( \mathcal{A} \) an \( \mathbb{R} \)-algebra \( \mathbb{R}[x, y]/(f - u, g - v) \). Generically its complexification \( \overline{\mathcal{A}} := \mathbb{C}[x, y]/(f - u, g - v) \) is a finite-dimensional vector space, as the complex set \( V(f - u, g - v) \) is finite generically. Fix an element \( h \) in \( \mathcal{A} \) then \( A_h : \mathcal{A} \ni t \to ht \in \mathcal{A} \) is \( \mathbb{R} \)-linear endomorphism. Denote by \( \text{tr}(A_h) \) its trace. Hermite’s method says that the number of real points in the set \( V(f - u, g - v) \) is equal to the signature of the following quadratic form:

\[
H : \mathcal{A} \ni h \to \text{tr}(A_h^2) \in \mathbb{R}.
\] (3.5)

For our purpose, we need to consider a family of quadratic forms parameterized by \((u, v)\) so the entries of the matrix of \( H \) are two-variable rational functions. After diagonalization we multiply the coefficients by their common denominator. Then the signature of our form is the sum of the signs of the polynomials which appear on the diagonal of the matrix under consideration.

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References