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Statistical summability and approximation by de la Vallée-Poussin mean

M. Mursaleen [a,*], A. Alotaibi [b]

[a] Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India

[b] Department of Mathematics, King Abdul Aziz University, Jeddah, Saudi Arabia

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ABSTRACT

In this paper we define a new type of summability method via statistical convergence by using the density and (V, λ)-summability. We further apply our new summability method to prove a Korovkin type approximation theorem.

1. Introduction and preliminaries

The concept of statistical convergence for sequences of real numbers was introduced by Fast [1] and Steinhaus [2] independently in the same year 1951 and since then several generalizations and applications of this notion have been investigated by various authors.

Let $K \subseteq \mathbb{N}$ and $K_n := \{k \le n : k \in K\}$. Then the *natural density* of K is defined by $\delta(K) = \lim_n n^{-1}|K_n|$ if the limit exists, where $|K_n|$ denotes the cardinality of K_n.

A sequence $x = (x_k)$ of real numbers is said to be *statistically convergent* to ℓ provided that for every $\epsilon > 0$ the set $K_\epsilon := \{k \in \mathbb{N} : |x_k - \ell| \ge \epsilon\}$ has natural density zero, i.e. for each $\epsilon > 0$,

$$\lim_n \frac{1}{n}|\{j \le n : |x_j - \ell| \ge \epsilon\}| = 0.$$

The idea of λ-statistical convergence was introduced in [3] as follows:

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{n+1} \le \lambda_n + 1, \lambda_1 = 0.$$

The generalized de la Vallée-Poussin mean is defined by

$$t_n(x) =: \frac{1}{\lambda_n} \sum_{j \in I_n} x_j$$

where $I_n = [n - \lambda_n + 1, n]$.

A sequence $x = (x_j)$ is said to be (V, λ)-*summable* to a number ℓ (see [4]) if

$$t_n(x) \to \ell \quad \text{as } n \to \infty.$$

In this case ℓ is called the λ-limit of x.

* Corresponding author. Tel.: +91 571 2720241.
E-mail addresses: mursaleenm@gmail.com (M. Mursaleen), aalotaibi@kau.edu.sa (A. Alotaibi).

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Let $K \subseteq \mathbb{N}$. Then

$$\delta_\lambda(K) = \lim_n \frac{1}{\lambda_n} |\{n - \lambda_n + 1 \le j \le n : j \in K\}|$$

is said to be λ-*density* of K.
In case $\lambda_n = n$, λ-density reduces to the natural density. Also, since $(\lambda_n/n) \le 1$, $\delta(K) \le \delta_\lambda(K)$ for every $K \subseteq \mathbb{N}$.

A sequence $x = (x_k)$ is said to be λ-*statistically convergent* (see [3]) to L if for every $\epsilon > 0$ the set $K_\epsilon := \{k \in \mathbb{N} : |x_k - L| \ge \epsilon\}$ has λ-density zero, i.e. $\delta_\lambda(K_\epsilon) = 0$. In this case we write $\lambda(\delta) - \lim x = L$. That is,

$$\lim_n \frac{1}{\lambda_n} | \{n - \lambda_n + 1 \le j \le n : |x_k - L| \ge \epsilon\} | = 0.$$

In this case we write $st_\lambda - \lim x = \ell$ and we denote the set of all λ-statistically convergent sequences by S_λ.

In this paper, we introduce *statistical* λ-*convergence* and *strongly* λ_q-*convergence* $(0 < q < \infty)$ and establish some relations between λ-statistical convergence, statistical λ-convergence, and strongly λ_q-convergence. We further apply our new type of convergence to prove a Korovkin type approximation theorem.

Now, we introduce some new concepts by using the notions of density and generalized de la Vallée-Poussin mean.

Definition 1.1. A sequence $x = (x_k)$ is said to be *statistically* λ-*convergent* to L if for every $\epsilon > 0$ the set $K_\epsilon(\lambda) := \{k \in \mathbb{N} : |t_k(x) - L| \ge \epsilon\}$ has natural density zero, i.e. $\delta(K_\epsilon(\lambda)) = 0$. In this case we write $\delta(\lambda) - \lim x = L$. That is,

$$\lim_n \frac{1}{n} | \{k \le n : |t_k(x) - L| \ge \epsilon\} | = 0.$$

Definition 1.2. A sequence $x = (x_k)$ is said to be *strongly* λ_q-*convergent* $(0 < q < \infty)$ to the limit L if $\lim_n \frac{1}{\lambda_n} \sum_{j \in I_n} |x_j - L|^q = 0$, and we write it as $x_k \longrightarrow L[V_\lambda]_q$. In this case L is called the $[V_\lambda]_q$-limit of x.

2. Statistical summability results

In our first theorem we establish the relation between our two newly defined concepts of λ-statistical convergence and statistical λ-convergence.

Theorem 2.1. *If a sequence $x = (x_k)$ is bounded and λ-statistically convergent to L then it is statistically λ-convergent to L but not conversely.*

Proof. Let $x = (x_k)$ be bounded and λ-statistically convergent to L. Write $K_\lambda(\epsilon) := \{n - \lambda_n + 1 \le j \le n : |x_k - L| \ge \epsilon\}$. Then

$$\begin{aligned}|t_k(x) - L| &= \left| \frac{1}{\lambda_k} \sum_{j \in I_k} x_j - L \right| = \left| \frac{1}{\lambda_k} \sum_{j=k-\lambda_k+1}^{k} x_j - L \right| = \left| \frac{1}{\lambda_k} \sum_{j=k-\lambda_k+1}^{k} (x_j - L) \right| \\ &\le \left| \frac{1}{\lambda_k} \sum_{j \in K_\lambda(\epsilon)} (x_j - L) \right| \le \frac{1}{\lambda_k} (\sup_j |x_j - L|) K_\epsilon \to 0,\end{aligned}$$

as $k \to \infty$, which implies that $t_k(x) \to L$ as $p \to \infty$. That is, x is (V, λ)-summable to L and hence statistically λ-convergent to L.

For converse, consider the case $\lambda_n = n$ and the sequence $x = (x_k)$ defined as

$$x_k = \begin{cases} 1; & \text{if } k \text{ is odd,} \\ -1; & \text{if } k \text{ is even.} \end{cases}$$

Of course this sequence is not λ-statistically convergent. On the other hand, x is (V, λ)-summable to 0 and hence statistically λ-convergent to 0.

This completes the proof of the theorem. □

Next theorem gives the relation between λ-statistical convergence and strong λ_q-convergence.

Theorem 2.2. (a) *If $0 < q < \infty$ and a sequence $x = (x_k)$ is strongly λ_q-convergent to the limit L, then it is λ-statistically convergent to L.*

(b) *If $x = (x_k)$ is bounded and λ-statistically convergent to L then $x_k \longrightarrow L[V_\lambda]_q$.*

Proof. (a) If $0 < q < \infty$ and $x_k \longrightarrow L[V_\lambda]_q$, then

$$0 \leftarrow \frac{1}{\lambda_n}\sum_{j\in I_n}|x_j - L|^q \geq \frac{1}{\lambda_n}\sum_{\substack{j\in I_n\\ |x_j-L|\geq\epsilon}}|x_j - L|^q$$
$$\geq \frac{\epsilon^q}{\lambda_n}|K_\epsilon|,$$

as $n \to \infty$. That is, $\lim_{n\to\infty}\frac{1}{\lambda_n}|K_\epsilon| = 0$ and so $\delta_\lambda(K_\epsilon) = 0$, where $K_\epsilon := \{k \leq n : |x_k - L| \geq \epsilon\}$. Hence $x = (x_k)$ is λ-statistically convergent to L.

(b) Suppose that $x = (x_k)$ is bounded and λ-statistically convergent to L. Then for $\epsilon > 0$, we have $\delta_\lambda(K_\epsilon) = 0$. Since $x \in l_\infty$, there exists $M > 0$ such that $|x_k - L| \leq M$ $(k = 1, 2, \ldots)$. We have

$$\frac{1}{\lambda_n}\sum_{j\in I_n}|x_j - L|^q = \frac{1}{\lambda_n}\sum_{\substack{k=n-\lambda_n+1\\ k\notin K_\epsilon}}^{n}|x_k - L|^q + \frac{1}{\lambda_n}\sum_{\substack{k=n-\lambda_n+1\\ k\in K_\epsilon}}^{n}|x_k - L|^q = S_1(n) + S_2(n),$$

where

$$S_1(n) = \frac{1}{\lambda_n}\sum_{\substack{k=n-\lambda_n+1\\ k\notin K_\epsilon}}^{n}|x_k - L|^q \quad\text{and}\quad S_2(n) = \frac{1}{\lambda_n}\sum_{\substack{k=n-\lambda_n+1\\ k\in K_\epsilon}}^{n}|x_k - L|^q.$$

Now if $k \notin K_\epsilon$ then $S_1(n) < \epsilon^q$. For $k \in K_\epsilon$, we have

$$S_2(n) \leq (\sup|x_k - L|)(|K_\epsilon|/\lambda_n) \leq M|K_\epsilon|/\lambda_n \to 0,$$

as $n \to \infty$, since $\delta_\lambda(K_\epsilon) = 0$. Hence $x_k \longrightarrow L[V_\lambda]_q$.

This completes the proof of the theorem. □

In the next result we characterize statistically λ-convergent sequences through the (V, λ)-summable subsequences.

Theorem 2.3. *A sequence $x = (x_k)$ is statistically λ-convergent to L if and only if there exists a set $K = \{k_1 < k_2 < \cdots < k_n < \cdots\} \subseteq \mathbb{N}$ such that $\delta(K) = 1$ and $\lambda - \lim x_{k_n} = L$.*

Proof. Suppose that there exists a set $K := \{k_1 < k_2 < \cdots < k_n < \cdots\} \subseteq \mathbb{N}$ such that $\delta(K) := 1$ and $\lambda - \lim x_{k_n} = L$. Then there is a positive integer N such that for $n > N$,

$$|t_n(x) - L| < \epsilon. \tag{2.3.1}$$

Put $K_\epsilon(\lambda) := \{n \in \mathbb{N} : |t_{k_n}(x) - L| \geq \epsilon\}$ and $K' := \{k_{N+1}, k_{N+2}, \ldots\}$. Then $\delta(K') = 1$ and $K_\epsilon(\lambda) \subseteq \mathbb{N} - K'$ which implies that $\delta(K_\epsilon(\lambda)) = 0$. Hence $x = (x_k)$ is statistically λ-convergent to L.

Conversely, let $x = (x_k)$ be statistically λ-convergent to L. For $r = 1, 2, 3, \ldots$, put $K_r(\lambda) := \{j \in \mathbb{N} : |t_{k_j}(x) - L| \geq 1/r\}$ and $M_r(\lambda) := \{j \in \mathbb{N} : |t_{k_j}(x) - L| < 1/r\}$. Then $\delta(K_r(\lambda)) = 0$ and

$$M_1(\lambda) \supset M_2(\lambda) \supset \cdots M_i(\lambda) \supset M_{i+1}(\lambda) \supset \cdots \tag{2.3.2}$$

and

$$\delta(M_r(\lambda)) = 1, \quad r = 1, 2, 3, \ldots. \tag{2.3.3}$$

Now we have to show that for $j \in M_r(\lambda)$, (x_{k_j}) is λ-convergent to L. Suppose that (x_{k_j}) is not λ-convergent to L. Therefore, there is $\epsilon > 0$ such that $|t_{k_j}(x) - L| \geq \epsilon$ for infinitely many terms. Let $M_\epsilon(\lambda) := \{j \in \mathbb{N} : |t_{k_j}(x) - L| < \epsilon\}$ and $\epsilon > 1/r$ $(r = 1, 2, 3, \ldots)$. Then

$$\delta(M_\epsilon(\lambda)) = 0, \tag{2.3.4}$$

and by (2.3.2), $M_r(\lambda) \subset M_\epsilon(\lambda)$. Hence $\delta(M_r(\lambda)) = 0$, which contradicts (2.3.3) and therefore (x_{k_j}) is λ-convergent to L.

This completes the proof of the theorem. □

Similarly we can prove the following dual statement:

Theorem 2.4. *A sequence $x = (x_k)$ is λ-statistically convergent to L if and only if there exists a set $K := \{k_1 < k_2 < \cdots < k_n < \cdots\} \subseteq \mathbb{N}$ such that $\delta_\lambda(K) = 1$ and $\lim x_{k_n} = L$.*

3. Statistical Korovkin type approximation theorem

In this section, we prove an analogue of the classical Korovkin theorem by using the concept of statistical λ-convergence. Recently, such types of approximation theorems are proved in [5–7] by using the notion of statistical convergence. The classical Korovkin approximation theorem states as follows (see [8,9]):

Let $C[a,b]$ be the space of all functions f continuous on $[a,b]$. Suppose that (T_n) be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$. Then

(i) $\lim_n \|T_n(f,x)-f(x)\|_\infty = 0$, for all $f \in C[a,b]$, if and only if
(ii) $\lim_n \|T_n(f_i,x)-f_i(x)\|_\infty = 0$, for $i = 0, 1, 2$,

where $f_0(x) = 1, f_1(x) = x$ and $f_2(x) = x^2$.

We know that $C[a,b]$ is a Banach space with norm $\|f\|_\infty := \sup_{a\le x\le b} |f(x)|, f \in C[a,b]$. We write $T_n(f,x)$ for $T_n(f(t),x)$ and we say that T is a positive operator if $T(f,x) \ge 0$ for all $f(x) \ge 0$.

Theorem 3.1. *Suppose that $T_n : C[a,b] \to C[a,b]$ is a sequence of positive linear operators. Then for any function $f \in C[a,b]$,*

$$\delta(\lambda) - \lim_n \|T_n(f,x)-f(x)\|_\infty = 0, \tag{3.0}$$

if and only if

$$\delta(\lambda) - \lim_n \|T_n(1,x)-1\|_\infty = 0, \tag{3.1}$$

$$\delta(\lambda) - \lim_n \|T_n(t,x)-x\|_\infty = 0, \tag{3.2}$$

$$\delta(\lambda) - \lim_n \|T_n(t^2,x)-x^2\|_\infty = 0. \tag{3.3}$$

Proof. Conditions (3.1)–(3.3) follow immediately from condition (3.0), since each of the functions $1, x, x^2$ belongs to $C[a,b]$. We prove the converse part. By the continuity of f on $[a,b]$, we can write

$$|f(x)| \le M, \quad -\infty < x < \infty.$$

Therefore,

$$|f(t)-f(x)| \le 2M, \quad -\infty < t, x < \infty. \tag{3.4}$$

Also, since $f \in C[a,b]$, for every $\epsilon > 0$, there is $\delta > 0$ such that

$$|f(t)-f(x)| < \epsilon, \quad \forall |t-x| < \delta. \tag{3.5}$$

Using (3.4), (3.5) and putting $\psi(t) = (t-x)^2$, we get

$$|f(t)-f(x)| < \epsilon + \frac{2M}{\delta^2}\psi, \quad \forall |t-x| < \delta.$$

This means

$$-\epsilon - \frac{2M}{\delta^2}\psi < f(t)-f(x) < \epsilon + \frac{2M}{\delta^2}\psi.$$

Now we could apply $T_n(1,x)$ to this inequality since $T_n(f,x)$ is monotone and linear. Hence,

$$T_n(1,x)\left(-\epsilon - \frac{2M}{\delta^2}\psi\right) < T_n(1,x)\,(f(t)-f(x)) < T_n(1,x)\left(\epsilon + \frac{2M}{\delta^2}\psi\right).$$

Note that x is fixed and so $f(x)$ is a constant number. Therefore,

$$-\epsilon T_n(1,x) - \frac{2M}{\delta^2}T_n(\psi,x) < T_n(f,x) - f(x)T_n(1,x) < \epsilon T_n(1,x) + \frac{2M}{\delta^2}T_n(\psi,x). \tag{3.6}$$

But

$$\begin{aligned} T_n(f,x) - f(x) &= T_n(f,x) - f(x)T_n(1,x) + f(x)T_n(1,x) - f(x) \\ &= [T_n(f,x) - f(x)T_n(1,x)] + f(x)[T_n(1,x) - 1]. \end{aligned} \tag{3.7}$$

Using (3.6) and (3.7), we have

$$T_n(f,x) - f(x) < \epsilon T_n(1,x) + \frac{2M}{\delta^2}T_n(\psi,x) + f(x)(T_n(1,x) - 1). \tag{3.8}$$

Now, let us estimate $T_n(\psi, x)$

$$\begin{aligned} T_n(\psi, x) &= T_n((t-x)^2, x) = T_n(t^2 - 2tx + x^2, x) \\ &= T_n(t^2, x) - 2xT_n(t, x) + x^2 T_n(1, x) \\ &= [T_n(t^2, x) - x^2] - 2x[T_n(t, x) - x] + x^2[T_n(1, x) - 1]. \end{aligned}$$

Using (3.8), we get

$$\begin{aligned} T_n(f, x) - f(x) &< \epsilon T_n(1, x) + \frac{2M}{\delta^2}\{[T_n(t^2, x) - x^2] - 2x[T_n(t, x) - x] + x^2[T_n(1, x) - 1]\} + f(x)(T_n(1, x) - 1) \\ &= \epsilon[T_n(1, x) - 1] + \epsilon + \frac{2M}{\delta^2}\{[T_n(t^2, x) - x^2] - 2x[T_n(t, x) - x] \\ &\quad + x^2[T_n(1, x) - 1]\} + f(x)(T_n(1, x) - 1). \end{aligned}$$

Since ϵ is arbitrary we can write

$$\begin{aligned} \|T_n(f, x) - f(x)\|_\infty &\le \left(\epsilon + \frac{2Mb^2}{\delta^2} + M\right)\|T_n(1, x) - 1\|_\infty + \frac{4Mb}{\delta^2}\|T_n(t, x) - x\|_\infty + \frac{2M}{\delta^2}\|T_n(t^2, x) - x^2\|_\infty \\ &\le K\left(\|T_n(1, x) - 1\|_\infty + \|T_n(t, x) - x\|_\infty + \|T_n(t^2, x) - x^2\|_\infty\right), \end{aligned} \tag{3.9}$$

where $K = \max\left(\epsilon + \frac{2Mb^2}{\delta^2} + M, \frac{4Mb}{\delta^2}\right)$.

Now replacing $T_n(t, x)$ by $B_k(t, x) = \frac{1}{\lambda_k}\sum_{n\in I_k} T_n(t, x)$, and for $\epsilon' > 0$, write

$$D := \left\{k \in \mathbb{N} : \|B_k(1, x) - 1\|_\infty + \|B_k(t, x) - x\|_\infty + \|B_k(t^2, x) - x^2\|_\infty \ge \frac{\epsilon'}{K}\right\},$$

$$D_1 := \left\{k \in \mathbb{N} : \|B_k(1, x) - 1\|_\infty \ge \frac{\epsilon'}{3K}\right\},$$

$$D_2 := \left\{k \in \mathbb{N} : \|B_k(t, x) - x\|_\infty \ge \frac{\epsilon'}{3K}\right\},$$

$$D_3 := \left\{k \in \mathbb{N} : \|B_k(t^2, x) - x^2\|_\infty \ge \frac{\epsilon'}{3K}\right\}.$$

Then $D \subset D_1 \cup D_2 \cup D_3$, and so $\delta(D) \le \delta(D_1) + \delta(D_2) + \delta(D_3)$. Therefore, using conditions (3.1)–(3.3), we get

$$\delta(\lambda) - \lim_n \|T_n(f, x) - f(x)\|_\infty = 0.$$

This completes the proof of the theorem. □

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