Some existence results for a class of degenerate semilinear elliptic systems✩

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Abstract
Using a variational approach, we investigate a class of degenerate semilinear elliptic systems with measurable, unbounded nonnegative weights, where the domain is bounded or unbounded. Some existence results are obtained.
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1. Introduction
In this paper, we deal with a class of degenerate semilinear elliptic systems of the form

\[
\begin{align*}
-\text{div}(a(x)\nabla u) &= F_u(x, u, v), & x \in \Omega, \\
-\text{div}(b(x)\nabla v) &= F_v(x, u, v), & x \in \Omega, \\
u &= 0, & v = 0, & x \in \partial \Omega,
\end{align*}
\]

(1.1)

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where the domain $\Omega$ is a bounded or an unbounded domain in $\mathbb{R}^N$, $N \geq 2$, the weights $a(x)$, $b(x)$ are measurable nonnegative weights on $\Omega$ and $F_u(x,u,v)$, $F_v(x,u,v)$ are the gradients of a $C^1$-functional $F(x,u,v)$.

Recently, many authors have studied the existence of nontrivial solutions for such problems (equations or systems) because several physical phenomena related to equilibrium of continuous media are modeled by these elliptic problems, see [5, p. 79] by Dautray and Lions. Caldiroli and Musina [3] investigated a variational degenerate elliptic problem of the form

$$
-\text{div}(a(x)\nabla u) = f(x,u), \quad x \in \Omega,
$$

$$
u = 0, \quad x \in \partial \Omega.
$$

They allowed the weight $a(x)$ to vanish somewhere or to be unbounded and proved some existence results by using a variational approach based on the Mountain Pass Lemma. Yu [12] obtained sufficient conditions on the nonlinearity for the existence of positive solutions for some nonlinear equations of the form

$$
-\text{div}(a(x)|\nabla u|^{p-2}\nabla u) = b(x)|u|^{p-2}u + f(x,u),
$$

defined on a smooth exterior domain, $a(x)$ and $b(x)$ are smooth and bounded functions. Zoographopoulos [13] studied a class of degenerate potential semilinear elliptic systems of the form

$$
\begin{aligned}
-\text{div}(a(x)\nabla u) &= \lambda \mu(x)|u|^{\gamma-1}|v|^\delta u, \quad x \in \Omega, \\
-\text{div}(b(x)\nabla v) &= \lambda \mu(x)|u|^{\gamma+1}|v|^{-\delta-1}v, \quad x \in \Omega, \\
u &= 0, \quad v = 0, \quad x \in \partial \Omega,
\end{aligned}
$$

where $\lambda > 0$, $\gamma \geq 0$, $\delta \geq 0$ and $\mu(x)$ may change sign. He proved the existence of at least one solution for the system (1.4) under suitable assumptions on the data.

In this work, we obtain some existence results for Problem (P) under subcritical growth conditions. In particular, the weights $a(x)$, $b(x)$ are allowed to vanish somewhere or to be unbounded, the primitive $F(x,u,v)$ being intimately related to with the first eigenvalue of a corresponding linear system. Our main goal is to illustrate how the idea introduced in [2,6,10,13] can be applied to obtain the existence of a nontrivial solution for Problem (P).

The present paper is organized as follows. In Section 2, we define the function space and operator settings, state our basic assumptions and collect some Sobolev and Rellich embedding theorems (even for unbounded domain). In Section 3, we obtain some existence results on bounded domain. In Section 4, an existence result on unbounded domain is also proved. In Section 5, we give three examples to illustrate our main theorems.

2. Preliminaries and functional setting

In this section, we will state some Sobolev and Rellich embedding theorems, which are the key results for the treatment of Problem (P) via variational methods.

Throughout this work, $c$ denotes a generic positive constant. Let the weights $a(x)$, $b(x) \in L^1_{\text{loc}}(\Omega)$ be given functions defined on a domain $\Omega$, and let $\alpha \in [0, +\infty)$. We introduce the following assumptions:

(H) $\liminf_{x \to z} |x - z|^{-\alpha}a(x) > 0$ and $\liminf_{x \to z} |x - z|^{-\beta}b(x) > 0$, for every $z \in \overline{\Omega}$;

(H$^\infty$) $\liminf_{|x| \to \infty} |x|^{-\alpha}a(x) > 0$ and $\liminf_{|x| \to \infty} |x|^{-\beta}b(x) > 0$. 
Remark.

(1) The assumption (H\(\infty\)) is meaningful only in the case that \(\Omega\) is unbounded;
(2) the assumptions (H) and (H\(\infty\)) imply that (see [3, Lemma 2.2])
(i) the sets of zeros \(Z_a = \{z \in \Omega : a(z) = 0\}\) and \(Z_b = \{z \in \Omega : b(z) = 0\}\) are finite;
(ii) the weights \(a(x), b(x)\) could be nonsmooth.

Let \(a(x), b(x)\) be nonnegative weights in \(L^1_{\text{loc}}(\Omega)\). For \(\varphi, \psi \in C_0^\infty(\Omega)\) let us define
\[
\|\varphi\|_a^2 = \int_\Omega a(x)|\nabla \varphi|^2\,dx, \quad \|\psi\|_b^2 = \int_\Omega b(x)|\nabla \psi|^2\,dx,
\]
and the spaces
\[
D^1_0(\Omega; a) = \text{closure of } C_0^\infty(\Omega) \text{ with respect to the } \|\|_a \text{ norm},
\]
\[
D^1_0(\Omega; b) = \text{closure of } C_0^\infty(\Omega) \text{ with respect to the } \|\|_b \text{ norm}.
\]

By Lemma 3.1 in [3], we obtain that:

If the weights \(a(x), b(x)\) satisfy (H) and (H\(\infty\)) for some \(\alpha \in (0, 2), \beta \in (0, 2)\), then we have
\[
\left(\int_\Omega |u|^{2_\alpha}a\,dx\right)^{\frac{2}{2_\alpha}} \leq c \int_\Omega a(x)|\nabla u|^2\,dx \quad \text{for every } u \in D^1_0(\Omega; a), \tag{2.1}
\]
\[
\left(\int_\Omega |v|^{2_\beta}b\,dx\right)^{\frac{2}{2_\beta}} \leq c \int_\Omega b(x)|\nabla v|^2\,dx \quad \text{for every } v \in D^1_0(\Omega; b), \tag{2.2}
\]
where
\[
2_\alpha = \frac{2N}{N - 2 + \alpha}, \quad 2_\beta = \frac{2N}{N - 2 + \beta},
\]
and the spaces \(D^1_0(\Omega; a)\) and \(D^1_0(\Omega; b)\) are Hilbert spaces with respect to the following scalar products (respectively):
\[
\langle u, \varphi \rangle_a = \int_\Omega a(x)\nabla u \cdot \nabla \varphi\,dx, \quad \langle v, \psi \rangle_b = \int_\Omega b(x)\nabla v \cdot \nabla \psi\,dx.
\]

Now, we state some embedding theorems for the spaces \(D^1_0(\Omega; a)\) and \(D^1_0(\Omega; b)\).

**Lemma 2.1.** [3] Assume that \(\Omega\) is a bounded domain, \(a(x), b(x) \in L^1_{\text{loc}}(\Omega)\) satisfy (H) for some \(\alpha \in (0, 2), \beta \in (0, 2)\). Then the following embeddings hold:

(i) \(D^1_0(\Omega; a) \subset L^{2_\alpha}(\Omega), D^1_0(\Omega; b) \subset L^{2_\beta}(\Omega)\) continuously;
(ii) \(D^1_0(\Omega; a) \subset L^p(\Omega), D^1_0(\Omega; b) \subset L^q(\Omega)\) with compact inclusion if \(p \in [1, 2_\alpha), q \in [1, 2_\beta)\).

**Lemma 2.2.** [3] Assume that \(\Omega\) is an unbounded domain, \(a(x), b(x) \in L^1_{\text{loc}}(\Omega)\) satisfy (H) for some \(\alpha \in (0, 2), \beta \in (0, 2)\) and
\[ (H^\infty) \liminf_{|x| \to \infty} |x|^{-\alpha} a(x) > 0 \text{ for some } \alpha' > \alpha, \\
\liminf_{|x| \to \infty} |x|^{-\beta} b(x) > 0 \text{ for some } \beta' > \beta. \]

Then the following embeddings hold:

(i) \( D_0^1(\Omega; a) \subset L^p(\Omega), D_0^1(\Omega; b) \subset L^q(\Omega) \) continuously for every \( p \in [2_{\alpha'}^*, 2_\alpha^*], q \in [2_{\beta'}^*, 2_\beta^*]; \)

(ii) \( D_0^1(\Omega; a) \subset L^p(\Omega), D_0^1(\Omega; b) \subset L^q(\Omega) \) with compact inclusion if \( p \in (2_\alpha^*, 2_{\alpha'}^*), q \in (2_{\beta'}^*, 2_\beta^*). \)

Now, we define the Cartesian product of Hilbert space \( D_0^1(\Omega; a) \times D_0^1(\Omega; b). \ Let W = D_0^1(\Omega; a) \times D_0^1(\Omega; b). \) Consider the linear eigenvalue problem

\[
(LP) \begin{cases}
- \text{div}(a(x)\nabla u) = \lambda \mu(x) v, & x \in \Omega, \\
- \text{div}(b(x)\nabla v) = \lambda \mu(x) u, & x \in \Omega, \\
u = 0, & x \in \partial \Omega.
\end{cases}
\]

\[ (2.3) \]

Lemma 2.3. [3] Assume that \( \Omega \) is a bounded domain, \( a(x), b(x) \in L^1_{\text{loc}}(\Omega) \) satisfy \((H)\) for some \( \alpha \in (0, 2), \beta \in (0, 2), \mu(x) \geq 0 \) and \( \mu(x) \in L^\infty(\Omega), \) or assume that \( \Omega \) is an unbounded domain, \( a(x), b(x) \in L^1_{\text{loc}}(\Omega) \) satisfy \((H)\) and \((H^\infty)\) for some \( \alpha \in (0, 2), \beta \in (0, 2), \mu(x) \geq 0 \) and \( \mu(x) \in L^\infty(\Omega) \cap L^\omega(\Omega), \) where

\[ \omega = \frac{2_{\alpha'}^* 2_{\beta'}^*}{2_\alpha^* 2_\beta^* - (\gamma + 1) 2_{\beta'}^* - (\delta + 1) 2_{\alpha'}^*}. \]

Then Problem \((LP)\) admits a positive principal eigenvalue \( \lambda_1 \) given by

\[ \lambda_1 = \inf_{(u, v) \in W \setminus \{(0,0)\}} \frac{\int_\Omega (a(x)|\nabla u|^2 + b(x)|\nabla v|^2) \, dx}{\int_\Omega \mu(x) |u||v| \, dx}. \]

The associated eigenfunction \((u_0, v_0)\) is componentwise nonnegative and is unique (up to multiplication by a nonzero scalar).

Definition 2.4. Let \( V \) be a real Banach space with the norm \( \|\| \) and \( E : V \to \mathbb{R} \) be a \( C^1 \) functional. We say the functional \( E \) satisfies condition \((PS)\) if every sequence \((u_n)\) in \( V \) satisfying

\[ |E(u_n)| \leq c \quad \text{for some constant } c, \quad \|E'(u_n)\| \to 0 \quad \text{as } n \to \infty, \]

possesses a convergent subsequence. We say the functional \( E \) satisfies condition \((C)\) if every sequence \((u_n)\) in \( V \) such that

\[ |E(u_n)| \to c, \quad (1 + \|u_n\|) \|E'(u_n)\| \to 0 \quad \text{as } n \to \infty, \]

has a convergent subsequence.

3. Bounded domain

In this section, we prove two existence results for Problem \((P)\) defined on a bounded domain. Throughout this section, the domain \( \Omega \) is bounded. Now, we state our main theorems in this section.

Assume
(F1) $F$ is a function in $C^1$ and
\[
|F_u(x, u, v)| \leq c \left(1 + |u|^{2^*_a-1} + |v|^{\frac{2^*_a}{2^*_a-1}(2^*_a-1)}\right),
\]
\[
|F_v(x, u, v)| \leq c \left(1 + |v|^{2^*_b-1} + |u|^{\frac{2^*_b}{2^*_b-1}(2^*_b-1)}\right);
\]
(F2) $F(x, 0, 0) = F_u(x, 0, 0) = F_v(x, 0, 0)$ for all $x \in \Omega$.

**Theorem 3.1.** Assume that $a(x), b(x) \in L^1_{\text{loc}}(\Omega)$ satisfy (H) for some $\alpha \in (0, 2), \beta \in (0, 2)$, $F$ satisfies (F1), (F2) and
\[
|F(x, u, v)| \leq c \left(1 + |u|^r + |v|^s\right),
\]
where $2 < r < 2^*_a, 2 < s < 2^*_b$ (“superlinear-like” in the terminology of [2]);
(F4) there exist $R > 0, \theta_a$ and $\theta_b$ with $\frac{1}{2^*_a} < \theta_a < \frac{1}{2}, \frac{1}{2^*_b} < \theta_b < \frac{1}{2}$ such that
\[
0 < F(x, u, v) \leq \theta_a u F_u(x, u, v) + \theta_b v F_v(x, u, v),
\]
for all $x \in \Omega$ and $|u| \geq R, |v| \geq R$;
(F5) there exist $\tilde{r} > 2, \tilde{s} > 2$ and $\varepsilon > 0$ such that
\[
|F(x, u, v)| \leq c \left(|u|^\tilde{r} + |v|^\tilde{s}\right),
\]
for all $x \in \Omega$ and $|u| \leq \varepsilon, |v| \leq \varepsilon$.

Then Problem $(P)$ has a nontrivial solution.

**Theorem 3.2.** Assume that $a(x), b(x) \in L^1_{\text{loc}}(\Omega)$ satisfy (H) for some $\alpha \in (0, 2), \beta \in (0, 2)$, $F$ satisfies (F1), (F2) and
\[
|F(x, u, v)| \leq c \left(1 + |u|^2 + |v|^2\right)
\]
(“of resonant-type” in the terminology of [2]);
(F7) there exist $R > 0, 0 < \mu, \nu < 2$ such that
\[
\frac{1}{2} \left(u F_u(x, u, v) + v F_v(x, u, v)\right) - F(x, u, v) \geq c \left(|u|^\mu + |v|^\nu\right),
\]
for all $x \in \Omega$ and $|u| \geq R, |v| \geq R$;
(F8) $\limsup_{|U| \to 0} \frac{2F(x, u, v)}{\mu(x)|u||v|} < \lambda_1 < \liminf_{|U| \to \infty} \frac{2F(x, u, v)}{\mu(x)|u||v|}$,
where $U = (u, v)$ and $\lambda_1$ is defined in Lemma 2.3.

Then Problem $(P)$ has a nontrivial solution.

**Remark.** The hypothesis (F8) is related to the interaction of the potential $F$ and $\lambda_1$. Costa [4] was the first to introduce such an assumption. A variant of this condition appeared in Do Ó’s article [7].
Now, we define the functional
\[ I(u,v) = \frac{1}{2} \int_{\Omega} (a(x)|\nabla u|^2 + b(x)|\nabla v|^2) \, dx - \int_{\Omega} F(x,u,v) \, dx, \quad \forall (u,v) \in W. \]

From Lemma 2.3 in [13], we see that the functional \( I(u,v) \) is well defined and is of class \( C^1 \) in \( W \) under the hypothesis (F1), and that critical points of the functional \( I(u,v) \) are precisely the weak solutions of Problem (P). Furthermore, if \( F \) satisfies (F2), it is obvious that \((u,v) \equiv (0,0)\) is a trivial solution of Problem (P).

**Lemma 3.3.** Assume that \( a(x), b(x) \in L^1_{\text{loc}}(\Omega) \) satisfy (H) for some \( \alpha \in (0,2), \beta \in (0,2) \), and \( F \) satisfies (F1), (F3), (F4). Then the functional \( I(u,v) \) satisfies condition (PS).

**Proof.** Let \((u_n, v_n) \subset W\) be a sequence such that
\[ |I(u_n, v_n)| \leq c, \quad I'(u_n, v_n) \to 0 \quad \text{as} \ n \to +\infty. \] (3.1)

From (3.1), we have
\[ \left| \frac{1}{2} \int_{\Omega} (a(x)|\nabla u_n|^2 + b(x)|\nabla v_n|^2) \, dx - \int_{\Omega} F(x,u_n, v_n) \, dx \right| \leq c, \] (3.2)
\[ \left| \int_{\Omega} a(x)|\nabla u_n|^2 \, dx - \int_{\Omega} F_u(x,u_n, v_n) u_n \, dx \right| \leq \varepsilon_n \|u_n\|_a, \] (3.3)
\[ \left| \int_{\Omega} b(x)|\nabla v_n|^2 \, dx - \int_{\Omega} F_v(x,u_n, v_n) v_n \, dx \right| \leq \varepsilon_n \|v_n\|_b. \] (3.4)

By (3.2)–(3.4), we get
\[ \left( \frac{1}{2} - \theta_\alpha \right) \int_{\Omega} a(x)|\nabla u_n|^2 \, dx + \left( \frac{1}{2} - \theta_\beta \right) \int_{\Omega} b(x)|\nabla v|^2 \, dx \]
\[ - \int_{\Omega} (F(x,u_n, v_n) - \theta_\alpha F(x,u_n, v_n) - \theta_\beta F(x,u_n, v_n)) \, dx \]
\[ \leq c + c \left( \|u_n\|^2_a + \|v_n\|^2_b \right). \]

Then, taking (F4) into account, we obtain that \((u_n, v_n)\) is bounded in \( W \). The existence of convergent subsequences follows in a standard way, since the growth of \( F \) is below the critical exponents \( 2_\alpha^* \) and \( 2_\beta^* \) by the hypothesis (F3). \( \square \)

**Lemma 3.4.** Assume that \( a(x), b(x) \in L^1_{\text{loc}}(\Omega) \) satisfy (H) for some \( \alpha \in (0,2), \beta \in (0,2) \), and \( F \) satisfies (F1), (F6), (F7). Then the functional \( I(u,v) \) satisfies condition (C).

**Proof.** Let \((u_n, v_n) \subset W\) be a sequence such that
\[ |I(u_n, v_n)| \leq c, \quad (1 + \|u_n\|_a + \|v_n\|_b) I'(u_n, v_n) \to 0 \quad \text{as} \ n \to +\infty. \] (3.5)

From the hypothesis (F6), we obtain that the growth of \( F \) is below the critical exponents \( 2_\alpha^* \) and \( 2_\beta^* \). Hence, we only need to prove that \( \|u_n\|_a \) and \( \|v_n\|_b \) are bounded, as remarked in the proof of Lemma 3.3. By (3.5), we obtain
\[ \varepsilon_n + c \geq I'(u_n, v_n) \left( \frac{u_n}{2}, \frac{v_n}{2} \right) - I(u_n, v_n) = \int_{\Omega} \left( \frac{1}{2} \left( F_u(x, u_n, v_n)u_n - F_v(x, u_n, v_n)v_n \right) - F(x, u_n, v_n) \right) \ dx. \]

Then using (F7), we have
\[ \int_{\Omega} (|u_n|^\mu + |v_n|^\nu) \ dx \leq c. \] (3.6)

Next, by using the interpolation inequality in [9], we obtain
\[ \int_{\Omega} |u_n|^2 \ dx \leq c \left( \int_{\Omega} |u_n|^{2^*_\alpha} \ dx \right)^{\frac{2-\mu}{2^*_\alpha-\mu}}, \]
\[ \int_{\Omega} |v_n|^2 \ dx \leq c \left( \int_{\Omega} |v_n|^{2^*_\beta} \ dx \right)^{\frac{2-\nu}{2^*_\beta-\nu}}. \]

So using (3.6), we get
\[ \int_{\Omega} |u_n|^2 \ dx \leq c \left( \int_{\Omega} |u_n|^{2^*_\alpha} \ dx \right)^{\frac{2-\mu}{2^*_\alpha-\mu}}, \quad \int_{\Omega} |v_n|^2 \ dx \leq c \left( \int_{\Omega} |v_n|^{2^*_\beta} \ dx \right)^{\frac{2-\nu}{2^*_\beta-\nu}}, \]
which implies by Lemma 2.1 that
\[ \left( \int_{\Omega} |u_n|^{2^*_\alpha} \ dx \right)^{\frac{2-\mu}{2^*_\alpha-\mu}} \leq c \| u_n \|^\tilde{a}_a, \quad \left( \int_{\Omega} |v_n|^{2^*_\beta} \ dx \right)^{\frac{2-\nu}{2^*_\beta-\nu}} \leq c \| v_n \|^\tilde{b}_b, \] (3.7)
where
\[ \tilde{a} = \frac{2 - \mu}{2^*_\alpha - \mu} 2^*_\alpha, \quad \tilde{b} = \frac{2 - \nu}{2^*_\beta - \nu} 2^*_\beta. \]

On the other hand, by (F6), we obtain
\[ I(u_n, v_n) \geq \frac{1}{2} \left( \| u_n \|^\tilde{a}_a + \| v_n \|^\tilde{b}_b \right) - c \int_{\Omega} (|u_n|^2 + |v_n|^2) \ dx \] (3.8)
which leads to (estimated by using (3.7))
\[ I(u_n, v_n) \geq \frac{1}{2} \left( \| u_n \|^\tilde{a}_a + \| v_n \|^\tilde{b}_b \right) - c \left( \| u_n \|^\tilde{a}_a + \| v_n \|^\tilde{b}_b \right). \]

Since \( I(u_n, v_n) \) is bounded and \( \tilde{a} < 2, \tilde{b} < 2 \), it follows that \( \| u_n \|_a \) and \( \| v_n \|_b \) are bounded. \( \square \)

**Proof of Theorem 3.1.** We will apply the Mountain Pass Lemma [11] to obtain a nontrivial critical point of the functional \( I(u, v) \). By Lemma 3.3, the functional \( I(u, v) \) satisfies condition (PS) (compactness condition). So we only need to check that the functional \( I(u, v) \) has the geometry of the Mountain Pass Lemma.
(i) From (F3) and (F5), one obtains
\[ |F(x, u, v)| \leq c \left( |u|^r + |v|^s + |u|^\bar{r} + |v|^\bar{s} \right), \]
for all \( x \in \Omega \) and \((u, v) \in \mathbb{R}^2\), where \( 2 < r, \bar{r} < 2^*_a, 2 < s, \bar{s} < 2^*_b \). From Lemma 2.1, one obtains the Sobolev embedding inequality
\[ \int_{\Omega} |u|^r dx \leq c \|u\|_a^r, \quad \int_{\Omega} |u|^\bar{r} dx \leq c \|u\|_a^{\bar{r}} \]
and
\[ \int_{\Omega} |v|^s dx \leq c \|v\|_b^s, \quad \int_{\Omega} |v|^\bar{s} dx \leq c \|v\|_b^{\bar{s}}. \]
Hence, we obtain
\[ \int_{\Omega} F(x, u, v) dx \leq c \left( \|u\|_a^r + \|u\|_a^{\bar{r}} + \|v\|_b^s + \|v\|_b^{\bar{s}} \right). \]

Now, we can estimate the functional \( I(u, v) \) by
\[ I(u, v) \geq \frac{1}{2} \left( \|u\|_a^2 + \|v\|_b^2 \right) - c \left( \|u\|_a^r + \|u\|_a^{\bar{r}} + \|v\|_b^s + \|v\|_b^{\bar{s}} \right). \]
Since \( r > 2, \bar{r} > 2, s > 2, \bar{s} > 2 \), we can fix positive constants \( \sigma, \rho > 0 \) such that \( \|u\|_a + \|v\|_b = \rho \) implies \( I(u, v) \geq \sigma > 0 \).

(ii) By Theorem 4.1 in [3], we have
\[ \begin{cases} -\text{div}(a(x) \nabla \varphi) = \lambda_1(a) \varphi, & x \in \Omega, \\ \varphi = 0, & x \in \partial \Omega, \end{cases} \]
and
\[ \begin{cases} -\text{div}(b(x) \nabla \psi) = \lambda_1(b) \psi, & x \in \Omega, \\ \psi = 0, & x \in \partial \Omega. \end{cases} \]
Using (F4), we have
\[ \frac{d}{dt} F(x, t^{\theta_a} u, t^{\theta_b} v) = \theta_a u F_u(x, t^{\theta_a} u, t^{\theta_b} v) t^{\theta_a - 1} + \theta_b v F_v(x, t^{\theta_a} u, t^{\theta_b} v) t^{\theta_b - 1} \]
\[ \geq \frac{1}{t} F(x, t^{\theta_a} u, t^{\theta_b} v) \]
which implies that there exists some function \( m(x, u, v) \) such that
\[ F(x, t^{\theta_a} u, t^{\theta_b} v) \geq t m(x, u, v). \quad (3.9) \]
From (3.9), we obtain
\[ I(t^{\theta_a} \varphi, t^{\theta_b} \psi) = \frac{1}{2} \left( t^{2\theta_a} \|\varphi\|_a^2 + t^{2\theta_b} \|\psi\|_b^2 \right) - \int \Omega F(x, t^{\theta_a} \varphi, t^{\theta_b} \psi) dx \]
\[ \leq \frac{1}{2} \left( t^{2\theta_a} \|\varphi\|_a^2 + t^{2\theta_b} \|\psi\|_b^2 \right) - t \int \Omega m(x, \varphi, \psi) dx. \]
Since $2\theta_\alpha < 1$, $2\theta_\beta < 1$, we conclude that
\[ I(t^{\theta_\alpha} \varphi, t^{\theta_\beta} \psi) \to -\infty \text{ as } t \to +\infty, \]
and thus there exists a constant $t_0$ such that $I(t_0^{\theta_\alpha} \varphi, t_0^{\theta_\beta} \psi) < 0$. □

**Proof of Theorem 3.2.** By Lemma 3.4, we obtain the functional $I(u, v)$ satisfies condition (C) (compactness condition). Now we verify that the functional $I(u, v)$ satisfies the geometry of the Mountain Pass Lemma.

(i) From the left-hand side of (F8), there exists $\rho > 0$ such that
\[ \|u\|_a + \|v\|_b = \rho \implies F(x, u, v) \leq \frac{1}{2}\lambda_1 \mu(x)|u||v|. \]
By Lemma 2.3 and the variational characterization of the principle eigenvalue $\lambda_1$, we have
\[ \int_\Omega F(x, u, v) < \frac{1}{2}(\|u\|_a^2 + \|v\|_b^2). \]
Then there exist $\sigma, \rho > 0$ such that $I(u, v) \geq \sigma > 0$ if $\|u\|_a + \|v\|_b = \rho$.

(ii) From the right-hand side of (F8), we get for $\varepsilon > 0$ and $t$ sufficiently large that
\[ F(x, tu_0, tv_0) \geq (\lambda_1 + \varepsilon)t^2 \mu(x)|u_0||v_0|, \]
where $(u_0, v_0)$ is the eigenfunction pair corresponding to the principle eigenvalue $\lambda_1$ of Problem (LP). Hence
\[ I(tu_0, tv_0) = \int_\Omega F(x, tu_0, tv_0) \, dx \]
\[ \leq \frac{1}{2}(\|u_0\|_a^2 + \|v_0\|_b^2) - (\lambda_1 + \varepsilon)t^2 \int_\Omega \mu(x)|u_0||v_0| \, dx \]
\[ \leq -t^2\varepsilon \int_\Omega \mu(x)|u_0||v_0| \, dx \]
which goes to $-\infty$ as $t \to +\infty$. So we obtain $I(tu_0, tv_0) < 0$, for $t$ large enough.

Consequently, the functional $I(u, v)$ has a nonzero critical point, and the nonzero critical point of $I(u, v)$ is precisely the nontrivial solution of Problem (P). □

**4. Unbounded domain**

In this section, an existence result for Problem (P) in unbounded domain is obtained. Throughout this section, the domain $\Omega$ is unbounded. Assume

(H1) $F(x, u, v) \in C^1(\Omega \times \mathbb{R}^2, \mathbb{R})$ and $F(x, 0, 0) = 0$;
(H2) for all $x \in \partial \Omega$ and $U = (u, v) \in \mathbb{R}^2$
\[ |F_u(x, U)| \leq a_1(x)|U|^{p_1-1} + a_2(x), \]
\[ |F_v(x, U)| \leq b_1(x)|U|^{q_1-1} + b_2(x), \]
where \(2 < p_1, q_1 < \min(2^*_\alpha, 2^*_\beta)\), \(a_1(x) \in L^{\delta_1}(\Omega) \cap L^{\delta_2}(\Omega)\), \(a_2(x) \in L^{\frac{2^*_\alpha}{p_1-1}}(\Omega) \cap L^{\infty}(\Omega)\), \(b_1(x) \in L^{\delta_3}(\Omega) \cap L^{\delta_4}(\Omega)\), \(b_2(x) \in L^{\frac{2^*_\beta}{q_1-1}}(\Omega) \cap L^{\infty}(\Omega)\),

\[
\delta_1 = \frac{2^*_\alpha}{2^*_\alpha - 1}, \quad \delta_3 = \frac{2^*_\beta}{2^*_\beta - 1}, \quad \delta_2 = \frac{2^*_\alpha 2^*_\beta}{2^*_\alpha 2^*_\beta - 2^*_\alpha (p_1 - 1) - 2^*_\beta}, \quad \delta_4 = \frac{2^*_\alpha 2^*_\beta}{2^*_\alpha 2^*_\beta - 2^*_\beta (q_1 - 1) - 2^*_\alpha};
\]

(H3) there exist a measurable function \(h(x) \in L^{\infty}(\Omega)\) and \(0 < \mu, \nu < 2\) such that

\[
\frac{1}{2} u F_u(x, u, v) + \frac{1}{2} v F_v(x, u, v) - F(x, u, v) \geq h(x)(|u|^\mu + |v|^\nu);
\]

(H4) there exists a measurable function \(h(x) \in L^{\infty}(\Omega)\), which may be different from the \(h(x)\) in (H3), such that

\[
F(x, u, v) \leq h(x)(|u|^2 + |v|^2).
\]

**Theorem 4.1.** Assume that \(a(x), b(x) \in L^1_{\text{loc}}(\Omega)\) satisfy (H) and \((H^\infty)\) for some \(\alpha \in (0, 2), \beta \in (0, 2)\), \(F\) satisfies (H1)–(H4) and

(H5) there exist \(\mu_1(x) \geq 0, \mu_1(x) \in L^{\infty}(\Omega) \cap L^{\omega}(\Omega)\) and \(\varepsilon > 0\) such that

\[
F(x, u, v) \leq \frac{1}{2} \mu_1(x)|u||v|, \quad \forall |u|, |v| \leq \varepsilon,
\]

where \(\lambda_1(\mu_1(x)) > 1\); (H6) there exist \(\mu_2(x) \geq 0, \mu_2(x) \in L^{\infty}(\Omega) \cap L^{\omega}(\Omega)\) and \(R > 0\) such that

\[
F(x, u, v) \geq \frac{1}{2} \mu_2(x)|u||v|, \quad \forall |u|, |v| \geq R,
\]

where \(\lambda_1(\mu_2(x)) < 1\).

Then Problem (P) has a nontrivial solution.

Now, we define the functional

\[
I(u, v) = \frac{1}{2} \int_{\Omega} \left( a(x)|\nabla u|^2 + b(x)|\nabla v|^2 \right) \, dx - \int_{\Omega} F(x, u, v) \, dx
\]

\[
= J(u, v) - N(u, v),
\]

where

\[
J(u, v) = \frac{1}{2} \int_{\Omega} \left( a(x)|\nabla u|^2 + b(x)|\nabla v|^2 \right) \, dx \quad \text{and} \quad N(u, v) = \int_{\Omega} F(x, u, v) \, dx.
\]

**Lemma 4.2.** Under hypotheses (H1) and (H2), the functional \(N(u, v)\) is well defined and is of class \(C^1\) in \(W\). Moreover, its derivative \(N'(u, v)\) is compact.
**Proof.** Using Lemma 2.2 and Hölder inequality, we obtain that the functional $N(u, v)$ is well defined and is of class $C^1$ in $W$, as remarked in the proof of Lemma 2.3 in [13]. By simple computations, its derivative is as follows:

$$N'(u, v)(\varphi, \psi) = \int_\Omega \left( F_u(x, u, v)\varphi + F_v(x, u, v)\psi \right) dx, \quad \forall (u, v), (\varphi, \psi) \in W.$$ 

Now, we prove the compactness of $N'(u, v)$. Let $(u_n, v_n)$ be a bounded sequence in $W$. Then there is a subsequence denoted again by $(u_n, v_n)$ weakly convergent to $(u, v)$ in $W$. So we have

$$|N'(u_n, v_n)(\varphi, \psi) - N'(u, v)(\varphi, \psi)| \leq \int_\Omega \left| F_u(x, u_n, v_n)\varphi - F_u(x, u, v)\varphi \right| dx + \int_\Omega \left| F_v(x, u_n, v_n)\psi - F_v(x, u, v)\psi \right| dx.$$

Let $B_R$ be a ball in $\Omega$ with radius $R > 0$, and we write

$$\int_\Omega \left( F_u(x, u_n, v_n)\varphi - F_u(x, u, v)\varphi \right) dx = \int_{B_R} \left( F_u(x, u_n, v_n)\varphi - F_u(x, u, v)\varphi \right) dx + \int_{\Omega \setminus B_R} \left( F_u(x, u_n, v_n)\varphi - F_u(x, u, v)\varphi \right) dx.$$

Taking (H1), (H2) into account, we can obtain that $\int_{B_R} (F_u(x, u_n, v_n)\varphi - F_u(x, u, v)\varphi) dx$ is compact (see [7]). Hence we have

$$\int_{B_R} \left( F_u(x, u_n, v_n)\varphi - F_u(x, u, v)\varphi \right) dx \to 0 \quad \text{as } n \to \infty. \quad (4.2)$$

On the other hand, by the growth hypothesis (H2) and the fact that

$$|a_1(x)|_{L^{\delta_1}(\Omega \setminus B_R)} + |a_1(x)|_{L^{\delta_2}(\Omega \setminus B_R)} \to 0,$$

$$|a_2(x)|_{L^{\frac{2\gamma}{\gamma - 1}}(\Omega \setminus B_R)} + |a_2(x)|_{L^\infty(\Omega \setminus B_R)} \to 0,$$

as $R \to \infty$, we obtain that for $R$ sufficiently large

$$\int_{\Omega \setminus B_R} \left( F_u(x, u_n, v_n)\varphi - F_u(x, u, v)\varphi \right) dx \to 0 \quad \text{as } n \to \infty. \quad (4.3)$$

Combining (4.2) and (4.3), we obtain

$$\int_\Omega \left( F_u(x, u_n, v_n)\varphi - F_u(x, u, v)\varphi \right) dx = \int_{B_R} \left( F_u(x, u_n, v_n)\varphi - F_u(x, u, v)\varphi \right) dx$$

$$+ \int_{\Omega \setminus B_R} \left( F_u(x, u_n, v_n)\varphi - F_u(x, u, v)\varphi \right) dx \to 0 \quad \text{as } n \to \infty. \quad (4.4)$$
Similarly, we can also obtain
\[
\int_{\Omega} \left( F_v(x, u_n, v_n) \psi - F_v(x, u, v) \psi \right) \, dx \to 0 \quad \text{as } n \to \infty. \tag{4.5}
\]
From (4.4) and (4.5), we have
\[
\left| N'(u_n, v_n)(\varphi, \psi) - N'(u, v)(\varphi, \psi) \right|
\leq \left| \int_{\Omega} \left( F_u(x, u_n, v_n) \varphi - F_u(x, u, v) \varphi \right) \, dx \right| + \left| \int_{\Omega} \left( F_v(x, u_n, v_n) \psi - F_v(x, u, v) \psi \right) \, dx \right| \to 0 \quad \text{as } n \to \infty.
\]
So we conclude that \( N'(u, v) \) is compact. \( \square \)

Assume that \( a(x), b(x) \) satisfy (H) and (H\(^\infty\)) for some \( \alpha \in (0, 2), \beta \in (0, 2) \). Then the functional \( J(u, v) \) is well defined and is of class \( C^1 \) in \( W \) by Lemma 2.2. Hence, by Lemma 4.2, we obtain that the critical points of the functional \( I(u, v) \) are weak solutions of Problem (P).

**Lemma 4.3.** Assume that \( a(x), b(x) \in L^1_{\text{loc}}(\Omega) \) satisfy (H) and (H\(^\infty\)) for some \( \alpha \in (0, 2) \) and \( \beta \in (0, 2) \), \( F \) satisfies (H1)–(H4). Then the functional \( I(u, v) \) satisfies condition (C).

**Proof.** Let \( (u_n, v_n) \subset W \) be a sequence such that
\[
|I(u_n, v_n)| \leq c, \quad \left( 1 + \|u_n\|_a + \|v_n\|_b \right)|I'(u_n, v_n)| \to 0 \quad \text{as } n \to +\infty. \tag{4.6}
\]
By Lemma 4.2, we get
\[
I'(u_n, v_n)(u_n, v_n) = \|u_n\|^2_a + \|v_n\|^2_b - \int_{\Omega} \left( F_u(x, u_n, v_n)u_n + F_v(x, u_n, v_n)v_n \right) \, dx.
\]
Using (4.6) and (H3), we obtain
\[
\varepsilon_n + c \geq I'(u_n, v_n) \left( \frac{u_n}{2}, \frac{v_n}{2} \right) - I(u_n, v_n)
\geq \int_{\Omega} \left( \frac{1}{2} \left( F_u(x, u_n, v_n)u_n - F_v(x, u_n, v_n)v_n \right) - F(x, u_n, v_n) \right) \, dx
\geq \int_{\Omega} h(x) \left( |u_n|^{\mu} + |v_n|^{\nu} \right) \, dx
\]
for sufficiently large \( n \). Thus we have
\[
\int_{\Omega} h(x) \left( |u_n|^{\mu} + |v_n|^{\nu} \right) \, dx \leq c.
\]
Now we define the Lebesgue spaces
\[
L^\mu_h(\Omega) = \left\{ u: \int_{\Omega} h(x)|u_n|^{\mu} \, dx < \infty \right\}, \quad L^\nu_h(\Omega) = \left\{ v: \int_{\Omega} h(x)|v_n|^{\nu} \, dx < \infty \right\}.
\]
From the interpolation inequality in [9], we have
\[ \left( \int_{\Omega} h(x)|u_n|^2 \, dx \right)^{\frac{1}{2}} \leq \left( \int_{\Omega} h(x)|u_n|^\mu \, dx \right)^{\frac{1}{\mu}} \left( \int_{\Omega} h(x)|u_n|^\nu \, dx \right)^{\frac{1}{\nu}} \left( \int_{\Omega} h(x)|u_n|^2 \, dx \right)^{\frac{1}{2}}, \]
\[ \left( \int_{\Omega} h(x)|v_n|^2 \, dx \right)^{\frac{1}{2}} \leq \left( \int_{\Omega} h(x)|v_n|^\nu \, dx \right)^{\frac{1}{\nu}} \left( \int_{\Omega} h(x)|v_n|^2 \, dx \right)^{\frac{1}{2}}, \]
where \( t_1, t_2 \in (0, 1), \) \( \frac{1}{2} = \frac{1-t_1}{\mu} + \frac{t_1}{2} \) and \( \frac{1}{2} = \frac{1-t_2}{\nu} + \frac{t_2}{2}. \)

On the other hand, by Proposition 3.4 in [3], we have
\[ \left( \int_{\Omega} h(x)|u_n|^\nu \, dx \right)^{\frac{1}{\nu}} \leq c \left( \int_{\Omega} |u_n|^\nu \, dx \right)^{\frac{1}{\nu}} \leq c \left( \int_{\Omega} |u_n|^\nu \, dx \right)^{\frac{1}{\nu}} \left( \int_{\Omega} |v_n|^\nu \, dx \right)^{\frac{1}{\nu}}, \]
\[ \left( \int_{\Omega} h(x)|v_n|^\nu \, dx \right)^{\frac{1}{\nu}} \leq c \left( \int_{\Omega} |v_n|^\nu \, dx \right)^{\frac{1}{\nu}} \leq c \left( \int_{\Omega} |v_n|^\nu \, dx \right)^{\frac{1}{\nu}}, \]
where \( c \) is a positive constant.

Hence the sequence \((u_n, v_n)\) is bounded in \( W \). Thus, there is a subsequence denoted again by \((u_n, v_n)\) weakly converging in \( W \). Since \( N'(u, v) \) is compact, \( N'(u_n, v_n) \) is a Cauchy’s sequence, we have \( J'(u, v) = I'(u, v) + N'(u, v), \forall (u, v) \in W \) and
\[ \|u_n - u_m\|^2_a = (J'(u_n, v_m) - J'(u_n, v_m))(u_n - u_m, 0). \]

Since \((u_n)\) is bounded in \( D^1_0(\Omega; a) \) and \((J'(u_n, v_m) - J'(u_n, v_m))(u_n - u_m, 0) \to 0 \), as \( n, m \to \infty \), \((u_n)\) is a Cauchy’s sequence in \( D^1_0(\Omega; a) \). Hence \((u_n)\) strongly converges in \( D^1_0(\Omega; a) \). Similarly we can prove that \((v_n)\) strongly converges in \( D^1_0(\Omega; b) \).

**Proof of Theorem 4.1.** By Lemma 4.3, the functional \( I(u, v) \) satisfies condition (C) (compactness condition). Now we verify that the functional \( I(u, v) \) satisfies the geometry of the Mountain Pass Lemma.

(i) Using the growth hypothesis (H2), (H5), the variational characterization of the principle eigenvalue \( \lambda_1 \) and Lemma 2.2, we obtain
\[ I(u, v) \geq \frac{1}{2} \left( \int_{\Omega} \alpha(x)|\nabla u|^2 + b(x)|\nabla v|^2 \, dx - \mu \frac{1}{2} \left( \int_{\Omega} \mu_1(x)uv \, dx \right) - C \int_{\Omega} \left( |u|^{2s} + |v|^{2\beta} \right) \, dx \right) \]
\[ \geq \frac{1}{2} \left( 1 - \frac{1}{\lambda_1}(\mu_1(x)) \right) \left( \|u\|^2_a + \|v\|^2_b \right) - \hat{C} \left( \|u\|^2_a + \|v\|^2_b \right), \]
for positive constants \( C_\delta, \hat{C}_\delta \). Since \( \lambda_1(\mu_1(x)) > 1 \), we can fix \( \sigma, \rho > 0 \) such that \( I(u, v) \geq \sigma > 0 \) if \( \|u\|_a + \|v\|_b = \rho \).
(ii) Consider \( \varepsilon > 0 \) such that \( \lambda_1(\mu_2(x)) + \varepsilon < 1 \), and choose \( (\varphi_0, \psi_0) \in W \) satisfying

\[
\int_{\Omega} \left( a(x) |\nabla \varphi_0|^2 + b(x) |\nabla \psi_0|^2 \right) \, dx \leq \left( \lambda_1(\mu_2(x)) + \varepsilon \right) \int_{\Omega} \mu_2(x) |\varphi_0| |\psi_0| \, dx.
\]

From (H6), we have

\[
\int_{\Omega} F(x, t\varphi_0, t\psi_0) \, dx \geq t^2 \left( \|\varphi_0\|_a^2 + \|\psi_0\|_b^2 \right) - \int_{\Omega} F(x, t\varphi_0, t\psi_0) \, dx
\]

for some positive constants \( c \) and \( t \) sufficiently large. Hence

\[
I(t\varphi_0, t\psi_0) = \frac{t^2}{2} (\|\varphi_0\|_a^2 + \|\psi_0\|_b^2) - \int_{\Omega} F(x, t\varphi_0, t\psi_0) \, dx
\]

\[
\leq \frac{t^2}{2} \left( \lambda_1(\mu_2(x)) + \varepsilon - 1 \right) \int_{\Omega} \mu_2(x) |\varphi_0| |\psi_0| \, dx - c,
\]

goes to \(-\infty\) as \( t \to +\infty \). So we obtain \( I(tu_0, tv_0) < 0 \), for \( t \) sufficiently large.

Consequently, we conclude that Problem \((P)\) has a nontrivial solution by the version of the Mountain Pass Lemma [1].

5. Examples

In this section, we present three examples of the function \( F \) satisfying hypotheses of Theorems 3.1, 3.2 and 4.1, respectively.

Example 5.1. Let \( F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be given by

\[
F(x, u, v) = n(x) |u|^{m_1} |v|^{m_2},
\]

where \( n(x) \in L^\infty(\Omega) \), \( m_1 \geq 1, m_2 \geq 1 \), \( \frac{m_1 + m_2}{2} > 1 \) and \( \frac{m_1}{m_2} + \frac{m_2}{m_1} < 1 \), with \( \theta_a m_1 + \theta \beta m_2 \geq 1 \). By a simple computation, we can obtain that \( F \) satisfies hypotheses (F1)–(F5) of Theorem 3.1.

Example 5.2. Let \( F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be given by

\[
F(x, u, v) = n_1(x) |u| + n_2(x) uv + n_3 |v|,
\]

where \( n_i(x) \in L^\infty(\Omega) \), \( i = 1, 2, 3 \). Similar to Section 2 in [2], we can obtain that \( F \) satisfies hypotheses (F1), (F2), (F6)–(F8) of Theorem 3.2.

Example 5.3. Let \( F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be given by

\[
F(x, u, v) = \begin{cases} \frac{1}{2} \mu_2(x)|u||v| \ln(|u||v|), & \text{if } (u, v) \neq (0, 0), \\ 0, & \text{if } (u, v) = (0, 0), \end{cases}
\]

where \( \mu_2(x) \geq 0, \lambda_1(\mu_2(x)) < 1 \). Since

\[
u F_u(x, u, v) + v F_v(x, u, v) - 2 F(x, u, v) \geq \frac{1}{2} \mu_2(x)|u||v|,
\]

we can obtain that the hypothesis (H3) is satisfied. It is not difficult to see that hypotheses (H1), (H2), (H4)–(H6) of Theorem 4.1 are satisfied.
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