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An Inequality for the Chromatic Number of a Graph*

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Let G be a finite, connected, undirected graph, without loops or multiple edges. The degree $\rho(v)$ of a vertex v is the number of edges incident with v. R. L. Brooks [1] has shown that

$$k \leqslant 1 + \max_{v \in G} \rho(v), \tag{1}$$

where $k = k(G)$ is the chromatic number of G, with equality if and only if G is a complete graph or an odd circuit.

More recently Wilf [2] sharpened Brooks' inequality to

$$k(G) \leqslant 1 + \lambda, \tag{2}$$

where $\lambda = \lambda(G)$ is the largest eigenvalue of the vertex-adjacency matrix of G, i.e., the matrix whose i, j entry is 1 is vertices i and j are connected and 0 otherwise. In the case of a star graph with n vertices ($k = 2$), (1) gives only $k \leqslant n$ whereas (2) gives $0(\sqrt{n})$.

In examination it appears that the only properties of the function $\lambda(G)$ needed in the proof of (2) are

P_1. $\quad G' \subset G \Rightarrow \lambda(G') \leqslant \lambda(G)$.

P_2. $\quad \lambda(G) \geqslant \min_{v \in G} \rho(v)$.

In fact the following is true:

THEOREM. *Let $\lambda(G)$ be any real valued function on G with the properties P_1 and P_2 above. Then*

$$k(G) \leqslant \lambda(G) + 1. \tag{3}$$

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582/4/1-1

PROOF: Let the chromatic number of G be k. By removing a finite number of vertices, if necessary, we obtain a critical subgraph $G_c \subset G$ with the property that $k(G_c) = k$ and the removal of any vertex from G_c lowers the chromatic number. By property P_1, $\lambda(G_c) \leqslant \lambda(G)$. But clearly G_c cannot contain a vertex with $\rho(v) < k - 1$, therefore by P_2,

$$k - 1 \leqslant \min_{v \in G_c} \rho(v) \leqslant \lambda(G_c) \leqslant \lambda(G)$$

and (3) is proved.

Write $\mu(G) = \min_{v \in G} \rho(v)$. The function

$$\Lambda(G) = \max_{G' \subset G} \mu(G') \tag{4}$$

evidently satisfies conditions P_1, P_2, and is in fact the smallest such function. Thus the most favorable inequality obtainable from the theorem is

$$k(G) \leqslant \Lambda(G) + 1 = 1 + \max_{G' \in G} \min_{v \subset G'} \rho(v), \tag{5}$$

valid for all finite connected graphs (with or without loops or multiple edges). Note that for the star graph of any order, (5) supplies the correct value. For planar graphs it gives $k(G) \leqslant 6$ in general.

For given G the value of $\Lambda(G)$ can be determined by the following algorithm: Take an integer $\nu \geqslant \mu(G)$ and remove from $G_0 = G$ all stars of vertices with $\rho(v) \leqslant \nu$. From the subgraph G_1 so obtained again remove all vertices with $\rho(v) \leqslant \nu$ and repeat the process with the subgraph G_2 so obtained, etc. After a finite number of steps (say $p = p(\nu)$ steps) we either exhaust all vertices (i.e., G_p is empty), or we obtain a non-empty $G_p \subset G$ with $\mu(G_p) > \nu$.

In the second case we clearly have $\Lambda(G) > \nu$; in the first case we show that $\Lambda(G) \leqslant \nu$.

For let $S_1, \ldots, S_p$ denote the successive sets of vertices removed, and suppose that we had a subgraph $G' \subset G$ with $\mu(G') > \nu$. Clearly G' does not contain any vertices from S_1 since their orders are $\leqslant \nu$, hence $G' \subset G_1$. But S_2 is the set of those vertices of G_1 with $\rho(v) \leqslant \nu$, hence G' does not contain any vertices from S_2 either, $G' \subset G_2$ etc., $G' \subset G_p$ and so G' is empty. Thus $\mu(G') \leqslant \nu$ for every non-empty $G' \subset G$ and so $\Lambda(G) \leqslant \nu$. We conclude that $\Lambda(G)$ is equal to the smallest value of $\nu (\geqslant \mu(G))$ for which $G_{p(\nu)}$ is empty.

The algorithm supplies a second (constructive) proof of the theorem; by starting from the empty $G_{p(\nu)}$ and by adjoining successively (and in any order) the stars of the vertices of S_q to G_q, at no stage will more than $\nu + 1 = \Lambda(G) + 1$ colors be needed. Hence $k(G) \leqslant \Lambda(G) + 1$.

References

1. R. L. Brooks, On Colouring the Nodes of a Network, *Proc. Cambridge Philos. Soc.* **37** (1941), 194–197.
2. H. S. Wilf, The Eigenvalues of a Graph and Its Chromatic Number, *J. London Math. Soc.*, to appear.