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On Sabidussi–Fawcett subdirect representation

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Abstract

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Sabidussi's representation theorem for symmetric graphs is generalized to fairly general concrete categories. As applications, the lists of the irreducible objects in several cases (for instance, symmetric or directed graphs with or without loops, n-partite graphs, posets) are presented.

Introduction

The classical subdirect representation of an algebra A by means of algebras B_i as defined by Birkhoff [1] is a one–one homomorphism $\mu : A \to \prod B_i$ such that for all the projections we have $p_i(\mu(A)) = B_i$. Categories of algebras have the special feature that all one–one homomorphisms are embeddings of subobjects and all homomorphisms onto are quotient maps. This property is typically not shared by more general categories, notably not by categories of combinatorial objects (graphs, digraphs, special graphs, posets, etc.). Therefore, when extending the notion of subdirect representation, one has to make clear that, first, it is necessary to require something of the monomorphism μ. It has to be a subobject embedding or else the A would not be represented in the product in any reasonable sense (one could, of course, require more, as, e.g., in [2] where Hell studies the representation of bipartite graphs as isometric embeddings; here we restrict ourselves to the basic requirement). Secondly, one has to decide on the onto morphisms $p_i\mu$. Here, the decision as to whether one should really require anything more depends on what one wants to have: already with plain onto morphisms (provided it is an embedding) one has a reasonable representation;

one may, however, wish to have the representation particularly nice in some other sense.

In the pioneering works by Fawcett [3] and Sabidussi [10]–concerning subdirect representations of symmetric graphs without loops—the requirement of $p_i\mu$ being quotients was adopted. Later on [6–9], the subdirect representation was discussed in some generality with $p_i\mu$ just onto. Although this latter approach has some arguments in its favour (it concerns the representation in the most basic form, one has the nice fact that an object is subdirectly irreducible iff prohibiting it as a subobject yields a complete category), the importance of Fawcett's and Sabidussi's approach is not diminished. In fact, nowadays the interest in particularly nice representations (sometimes with even stronger types of epimorphisms, notably retractions) has increased.

In this paper we present a generalization of Sabidussi's representation theorem to fairly general categories. First (in Section 2) we restrict ourselves to a case where the analogy with Sabidussi's theorem is quite immediately apparent, namely to that where the structures of the given type on a set form a complete lattice. Even so, we obtain various concrete examples. Those of categories of G-coloured graphs, being perhaps of a special interest, are treated separately in Section 3. Then, in Section 4, the general theorem is presented, and a few more examples are added. The last section, Section 5, contains a few remarks and open problems.

1. Preliminaries

1.1. A *concrete category* $(\mathscr{C}, U)$ consists of a category $\mathscr{C}$ and a faithful functor $U:\mathscr{C}\to\text{Set}$ (the category of all sets and mappings). A monomorphism $\mu:A\to B$ is called *subobject morphism* if for each $f:U(C)\to U(A)$ such that there is a $\psi:C\to B$ with $U\mu\circ f=U\psi$, we have $f=U(\varphi)$ for a $\varphi:C\to A$. A *quotient* is a morphism $\varepsilon:A\to B$ such that $U\varepsilon$ is onto, and for each $f:U(B)\to U(C)$ such that there is a $\psi:A\to C$ with $f\circ U\varepsilon=U\psi$, we have $f=U(\varphi)$ for a $\varphi:B\to C$.

The cardinality of an object A is that of $U(A)$ and is denoted by $|A|$.

1.2. The class $\{A \mid U(A)=X\}$ preordered by the relation

$$A\leq B \text{ iff } \exists\iota:A\to B,\ U\iota=\text{id}_x$$

will be denoted by $\mathscr{C}(X)$ (more exactly, $\mathscr{C}_U(X)$), and we abbreviate $\mathscr{C}(U(A))$ to $\mathscr{C}(A)$.

1.3. A concrete category $(\mathscr{C}, U)$ is said to be *regular* (cf. [9]) if:

(i) U preserves limits,

(ii) each $\varphi:A\to B$ has a decomposition

$$A\xrightarrow{\varphi_2}C\xrightarrow{\varphi_1}B$$

with φ_1 a subobject and $U(\varphi_2)$ onto,

(iii) $\mathscr{C}(X)$ are sets, and are finite for finite X,

(iv) subcategories closed with respect to products and subobjects are reflective,

(v) if $U\varphi = \mathrm{id}$ for an isomorphism $\varphi: A \to A$ then $\varphi = 1_A$,

(vi) for each invertible $f: X \to U(A)$ there is an isomorphism $\varphi: B \to A$ such that $U(\varphi) = f$.

A *weakly regular* category is allowed to fail in (iv).

Remark. Everyday concrete categories (graphs or special graphs, posets, spaces, relational systems, algebras) are typically regular. There is a distinction in the conditions: while (i)–(iv) (resp. (i)–(iii)) are essential, (v) and (vi) merely serve technical convenience: (v) makes the preorders in $\mathscr{C}(X)$ orders, and (vi) allows to transfer a structure from an underlying set onto another.

1.4. A *congruence* (more exactly, $(\mathscr{C}, U)$-congruence, but the specification will always be obvious) on an object A is an equivalence E on $U(A)$ obtained from a fixed quotient $\varepsilon: A \to B$ by putting

$$x \mathrm{E} y \text{ iff } \varepsilon(x) = \varepsilon(y).$$

Recalling the definition of quotient we see that if E is obtained this way from $\varepsilon: A \to B$ and also from $\varepsilon': A \to B'$, then there is an isomorphism $\beta: B \to B'$ such that $\beta\varepsilon = \varepsilon'$. This justifies speaking of the 'congruence ε' where convenient, rather then E.

The following is a trivial observation:

The intersection $\bigcap_J E_i$ is trivial iff the respective system of quotients $(\varepsilon_i)_{i\in J}$ is collectionwise monomorphic (that is, if $(\forall i\ \varepsilon_i\alpha = \varepsilon_i\beta) \Rightarrow \alpha = \beta$).

An object A is said to be *congruence trivial* (CT) if whenever for a system of congruences $(E_i)_{i\in J}$ the intersection $\bigcap E_i$ is trivial then at least one of the E_i is trivial. (In other words, if a system $(\varepsilon_i: A \to B_i)_i$ of quotients is collectionwise monomorphic then some of the ε_i is monomorphic—and hence an isomorphism).

1.5. A congruence $\varepsilon: A \to B$ is *critical* if there is no $\varepsilon': A' \to B$ with $A < A'$ and $U\varepsilon = U\varepsilon'$. (In more intuitive words: whenever the respective equivalence is a congruence also with respect to a strictly stronger structure, the resulting quotient object is strictly stronger than in the original situation).

1.6. A *subdirect representation* of an object A is a subobject morphism

$$\mu: A \to \prod_{i\in j} B_i$$

such that all the $p_i\mu: A \to B_i$, where $p_i: \prod B_j \to B_i$ are the projections, are onto. Thus, in other words, a subdirect representation is given by a collectionwise monomorphic system $(\mu_i: A \to B_i)_i$ of onto morphisms, such that the $\mu: A \to \prod B_i$ given by $p_i\mu = \mu_i$ is a subobject morphism.

An object A is said to be *subdirectly irreducible* (abbreviated, SI) if in each subdirect representation $(\mu_i : A \to B_i)_i$ some of the μ_i is an isomorphism.

Remark. In general, one should distinguish between the subdirect irreducibility with respect to finite and general representations. Since we are mainly concerned with finite objects, and since for those (due to the condition (iii)) a nontrivial representation can be replaced by a nontrivial finite representation (see [9], cf. also Remark 2.1 below), we do not really need to make the distinction here.

1.7. It is an easy well-known observation (see [1]) that in varieties of algebras the subdirectly irreducible objects are exactly the congruence trivial ones. In our more general case the situation is not so simple. We have (see [8, 9]) the following.

Theorem. *Let $(\mathscr{C}, U)$ be a regular category with finite products. Then a finite object A is subdirectly irreducible iff either it is maximal (in $\mathscr{C}(A)$) and CT, or it is nonmaximal meet-irreducible in $\mathscr{C}(A)$ and admits no critical congruence.*

The relation of the conditions SI and CT will be discussed below in connection with a further irreducibility condition (which is, in fact, the main concern of this paper).

2. Sabidussi–Fawcett representation, and the respective irreducibles

2.1. A *Sabidussi–Fawcett representation* ([3, 10], briefly, SF-representation) is a subdirect representation (recall 1.6) in which all the $p_i\mu$ are quotients.

The objects with no nontrivial SF-representation will be called Sabidussi–Fawcett irreducibles (briefly, SFI).

Thus, each SI is an SFI. Of course one can expect that there are often more SFI's. This indeed is the case, as we will see below.

Remark. If A is finite, and $(\mu_i : A \to B_i)_{i \in J}$ is an SF-representation, we have, if (iii) holds, a finite $K \subseteq J$, a correspondence $(i \mapsto \bar{i}) : J \to K$ and isomorphisms $\beta_i : B_{\bar{i}} \to B_i$ such that $\beta_i \mu_{\bar{i}} = \mu_i$. It is easy to check that $(\mu_i : A \to B_i)_{i \in K}$ is then again an SF-representation. Thus, our concern being mostly with finite objects, we do not need to go into distinguishing the finite and general irreducibility.

2.2. A concrete category $(\mathscr{C}, U)$ is said to be *latticed* if each diagram $(\iota_i : A \leqslant A_i)_{i \in J}$ has a colimit $(\bar{\iota}_i : A_i \leqslant \bar{A})_{i \in J}$. Then of course, $\bar{A} = \bigvee_{i \in J} A_i$ in $\mathscr{C}(A)$.

Unlike the conditions in the definition of regular category the condition to be latticed is rather restrictive (e.g., the category of posets is not latticed). Although we will be able to be more general later, we will prove a characterization theorem for this special case first, mainly to enable the reader acquainted with the paper [10] to see how our proof follows the lines of Sabidussi's theorem on the special case of symmetric graphs without loops. Even so we will be able to discuss a variety of examples substantially differing from this case. In Section 4 we will get rid of the restriction.

2.3. Let $\varepsilon : A \to B$ be a congruence in a latticed category. Consider the system of all the $\varepsilon_i : A_i \to B$ with $A_i \geq A$ and $U\varepsilon_i = U\varepsilon$, and put

$$A\varepsilon = \bigvee A_i.$$

By definition of latticed category we have a morphism

$$\bar{\varepsilon} : A\varepsilon \to B$$

such that $U\bar{\varepsilon} = U\varepsilon$.

Remarks. (1) Thus, for each decomposition

$$\left(A \leq A' \xrightarrow{\varepsilon'} B\right) = \left(A \xrightarrow{\varepsilon} B\right)$$

we have $A' \leq A\varepsilon$ and $(A' \leq A\varepsilon \xrightarrow{\bar{\varepsilon}} B) = \varepsilon$.

(2) It is easy to check that $\bar{\varepsilon} : A\varepsilon \to B$ is a quotient.

(3) Note that in a latticed category, ε is a critical congruence iff $\varepsilon = \bar{\varepsilon}$.

The following is an immediate generalization of the above mentioned theorem by Sabidussi.

Theorem 2.4. *Let $(\mathscr{C}, U)$ be a latticed weakly regular category with products. A family $(\varepsilon_i : A \to B_i)_{i \in J}$ of quotients induces an SF-representation of A in $(\mathscr{C}, U)$ iff:*

(1) $(\varepsilon_i)_{i \in J}$ is collectionwise monomorphic, and

(2) $\bigwedge_{i \in J} A\varepsilon_i = A$.

Proof. Define $\varepsilon : A \to \prod B_i$ by $p_i \varepsilon = \varepsilon_i$.

(I) Let (1) and (2) hold. By (1), ε is a monomorphism. Decompose it as

$$A \xrightarrow[\iota]{\leq} A' \xrightarrow{\varepsilon'} B_i$$

with ε' a subobject. Considering $p_i \varepsilon' \iota$ we see that $A' \leq A\varepsilon_i$ and hence, by (2), $A' \leq \bigwedge A\varepsilon_i = A \leq A'$. Thus, ε is a subobject.

(II) Let $(\varepsilon_i)_i$ be an SF-representation. Thus, (1) holds true and ε is a subobject. Put $A' = \bigwedge A\varepsilon_i$, $\iota_i' : A' \leq A\varepsilon_i$ and define $\varepsilon' : A' \to \prod B_i$ by $p_i \varepsilon' = \bar{\varepsilon}_i \iota_i'$. Since $U(\varepsilon') = U(\varepsilon)$, we have $A' \leq A$. Trivially, $A \leq A'$. $\square$

Corollary 2.5. *Thus, A is SFI iff for each nontrivial system of congruences $(\varepsilon_i)_{i \in J}$ with trivial meet, $\bigwedge_{i \in J} A\varepsilon_i \neq A$. In other words, iff either there is no nontrivial system of congruences with trivial intersection, or*

$$\{A\varepsilon \mid \varepsilon \textit{ nontrivial}\} \neq A.$$

Observation 2.6. *All the A's with $|A| \leq 2$ are SFI.*

Notation 2.7. The congruence identifying a with b will be denoted by $[ab]$, that identifying a_i with b_i, $i = 1, 2$, by $[a_1b_1, a_2b_2]$ etc.

Example 2.8. We will use the symbol K_n for the complete graph without loops (in the directed case interpret it as having between a and b both the edges $\overrightarrow{ab}$ and $\overrightarrow{ba}$, in the symmetric case we have the unoriented edge ab).

Consider first the Sabidussi and Fawcett case of symmetric graphs without loops. By [9] we have here that

SI's are exactly the K_n, $n = 1, 2, \ldots$

Now let us use 2.5. The congruences are exactly those equivalences which do not identify vertices joined by edges. We easily see that in case of more than one edge missing in A, $A = \bigwedge \{A[ab] \mid a, b \text{ not joined}\}$, while in case of at most one missing edge there is no nontrivial system of congruences with trivial intersection. Thus, we have that

SFI's are exactly the K_n and $K_n \backslash \{ab\}$, $n = 1, 2, \ldots, a \neq b$.

In the case of directed graphs without loops we have, by [9],

SI's: K_n and $K_n \backslash \{\overrightarrow{ab}\}$,

while the SFI are very many, including, e.g., all tournaments, all $K_n \backslash \{\overrightarrow{ab}, \overrightarrow{ba}\}$, and many others. It is not a very lucid system. We shall do better in other examples.

2.9. *The category of all directed graphs* (loops allowed): In this and the next examples in this section we deal with systems where all equivalences are congruences. Observe, first, that in such a case, the SFI have at most three points. Indeed, $A[ab]$ can differ from A only in the edges meeting $\{a, b\}$ and hence, if a, b, c, d are distinct vertices in A, $A = A[ab] \wedge A[cd]$. Thus, according also to 2.6, it suffices to discuss the case of $|A| = 3$.

Thus, let a, b, c, be the vertices of A. Our task is to decide when $A' = A[ab] \wedge A[ac] \wedge A[bc]$ equals A and when not. First, we easily check that if there are arrows between at most two of the vertices, or exactly $\overrightarrow{ab}$ and $\overrightarrow{ac}$, or exactly $\overrightarrow{ba}$ and $\overrightarrow{ca}$, or exactly $\overrightarrow{ca}$, $\overrightarrow{ab}$, $\overrightarrow{cb}$ or finally if there are all of the arrows, we have $A = A'$ no matter what loops are present. On the other hand, if the system of arrows is not complete but contains a circuit, obviously $A \neq A'$. Thus the only case to discuss is that of arrows $\overrightarrow{ca}$, $\overrightarrow{ab}$ and perhaps more, but none between b and c. Then we see that $A \neq A'$ iff there is a loop on either b or c. Thus, the complete list of SFI is as indicated in Fig. 1 (the list of SI from [9]). Hence, we have here 54 SF-irreducibles, out of which 6 are subdirectly irreducible.

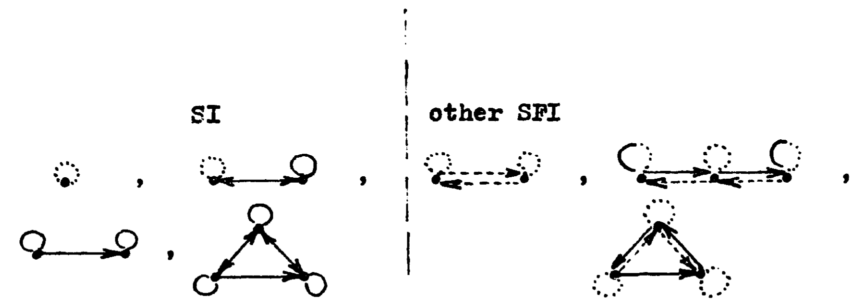

Fig. 1. The dotted loops and arrows indicate free choice, the half-dotted loop stands for one of the couple of loops out of which at least one has to be present.

2.10. *The category of all symmetric graphs*: Analogous to 2.9, only much simpler. We find that if there are 0, 1 or 3 edges, $A = A'$, and if there are just two edges, the answer depends on the presence of a loop at the free end. We end up with the list as indicated in Fig. 2 (SI, again, from [9]). Thus we have here 10 SF irreducibles, out of which 6 are subdirectly irreducible.

2.11. *Graphs with all loops*: The checking is now easy, after the practice from the two preceding cases. The list for the oriented case is indicated in Fig. 3(a), that for the symmetric case in Fig. 3(b).

3. An important case: G-coloured graphs

3.1. Let G be a graph. A graph A is said to be *G-coloured* if there is a morphism $A \to G$ (see [5]). The G-coloured graphs form a latticed complete regular category, which will be denoted by $\to G$ (cf. [6], to be more exact, to achieve the completeness one must add a formal singleton object, as in the case of graphs without loops). Thus, e.g., $\to K_2$ is the category of bipartite graphs, $\to K_n$ is that of n-partite ones.

We will be concerned with finite G only. Taking into account the fact that a one-one morphism $\varphi : G \to G$ is then an isomorphism, we easily see that

(i) if $\bar{G}$ designates the smallest retract of G, then $\to G = \to H$ iff $\bar{G}$ is isomorphic to $\bar{H}$, and

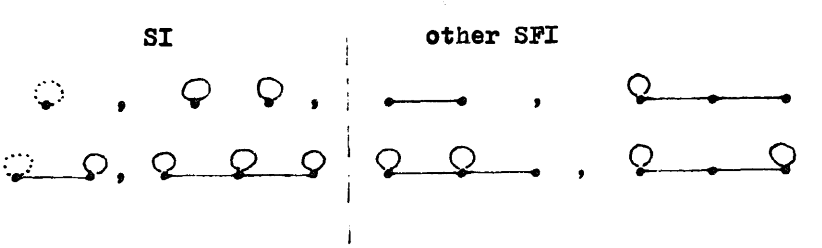

Fig. 2.

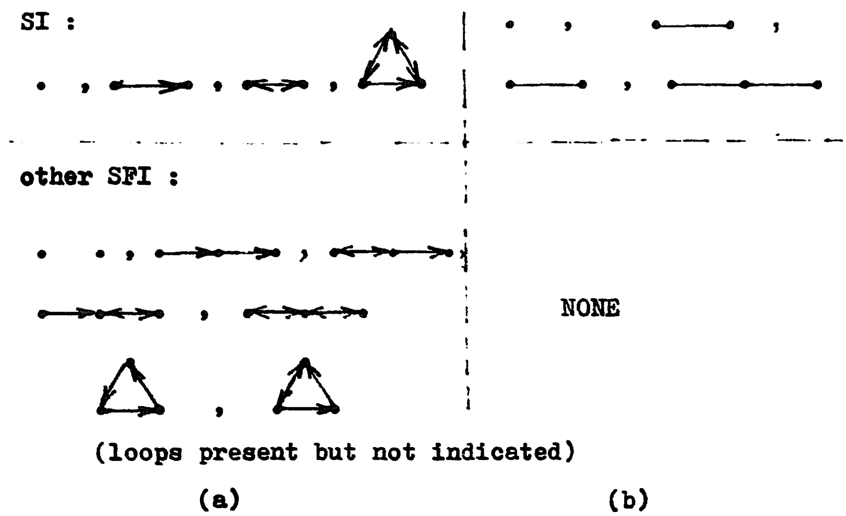

Fig. 3.

(ii) it suffices to consider the G such that each endomorphism is an automorphism (which we will, from now on, do).

Consider a graph A from $\to G$ and a fixed morphism $f:A\to G$.

Lemma 3.2. *In the following three cases, A is necessarily SF reducible*:

(a) $|f^{-1}(x)|\geqslant 3$ *for some* $x\in G$,

(b) $|f^{-1}(x_i)|=2$ *for three distinct* $x_1, x_2, x_3\in G$,

(c) $|f^{-1}(x_i)|=2$ *for two distinct* x_1, $x_2\in G$ *and no vertex from* $f^{-1}(x_1)$ *is joined with one of* $f^{-1}(x_2)$.

Proof. (a) Let a, b, c be distinct in $f^{-1}(x)$. Obviously

$$A=A[ab]\wedge A[bc]\wedge A[ac].$$

(b) Let a_i, b_i be distinct in $f^{-1}(x_i)$. Then

$$A=\bigwedge_{i=1}^{3} A[a_ib_i]$$

(c) Again, $A=A[a_1b_1]\wedge A[a_2b_2]$. □

Corollary 3.3. *If G is finite, $\to G$ has only finitely many SFI, and all of them are finite. Consequently, the same holds for the SI.*

Notation 3.4. P_n is the path with n edges, D is the discrete graph with two vertices, C_n is the cycle of length n. If A, B are graphs, $A*B$ is obtained from the disjoint union of A and B by adding all the edges between the vertices of A and the vertices of B.

Lemma 3.5. *Let $n \geq 2$. Then $K_{n-2} * P_2$ is SF representable in $K_n \times K_{n+1}$, and $K_{n-2} * P_3$ is SF representable in $K_n \times (K_{n-1} * P_2)$.*

Proof. Denote by $1, \ldots, n$ the vertices of K_n, similarly for K_{n-1} and K_{n-2}, and by $\bar{1}, \bar{2}, \bar{3}$ resp. $\bar{1}, \bar{2}, \bar{3}, \bar{4}$ the subsequent vertices of P_2 resp. P_3. In both cases send k to (k, k) for $k \leq n-2$. Then, in the former case send $\bar{1}$ to $(n-1, n-1)$, $\bar{2}$ to (n, n) and $\bar{3}$ to $(n+1, n-1)$. In the latter case send $\bar{1}$ to $(n-1, \bar{1})$, $\bar{2}$ to $(n, \bar{2})$, $\bar{3}$ to $(n-1, n-1)$ and $\bar{4}$ to $(n, \bar{3})$. □

Theorem 3.6. *Let $n \geq 2$. Then the SFI in the category of n-partite graphs are*

$$D, \quad K_m \text{ with } m \leq n, \quad K_{n-2} * P_2 \text{ and } K_{n-2} * P_3.$$

With the exception of D, all of them are also SI.

Proof. Since K_m are obviously irreducible, it suffices to show that the SFI (resp. SI) which are not $(n-1)$ colourable are exactly K_n, $K_{n-2} * P_2$ and $K_{n-2} * P_3$. Consider such an A and a homomorphism $f: A \to K_n$. Since A is not $(n-1)$-colourable, there are edges between any $f^{-1}(x)$, $f^{-1}(y)$ with $x \neq y$.

Use Lemma 3.2. If f is one-one, we are left with K_n. Let $f^{-1}(x_1) = \{a_1, b_1\}$ and $|f^{-1}(x)| = 1$ for the other x. Let, say, a_1 be not connected with an $a \in f^{-1}(x)$, $x \neq x_1$. Then obviously $[a_1 a]$ is a congruence and we have

$$A = A[a_1 b_1] \wedge \bigwedge \{A[a_1 a] \mid a \in f^{-1}(x), x \neq x_1, a, a_1 \text{ not connected}\}$$

Thus the only candidate for an SFI in this case is K_n with one point redoubled, that is, $K_{n-2} * P_2$. Since there is only one nontrivial congruence, and since it is maximal n-colourable, it is SI.

Now let $f^{-1}(x_i) = \{a_i, b_i\}$ for $i = 1, 2$, and $|f^{-1}(x)| = 1$ otherwise. Again we see that each of the a_i, b_i is connected with any of the remaining points so that it suffices to show these four points form, as an induced subgraph of A, the path P_3.

By 3.2 (c), say, a_1 is connected with a_2. Thus we have to discuss the cases depicted in Fig. 4. In the cases (a), (b) and (f), $A = A[a_1 b_1] \wedge A[a_2 b_2]$, in the case (c), $[a_1 b_2]$ is also a congruence and we have $A = A[a_1 b_1] \wedge A[a_1 b_2]$. The cases (d) and (e) are SFI and both lead to a graph isomorphic to $K_{n-2} * P_3$. This graph, moreover, is meet-irreducible, and the only two congruences, $[a_i b_i]$, are both noncritical. Thus, it is also SI. □

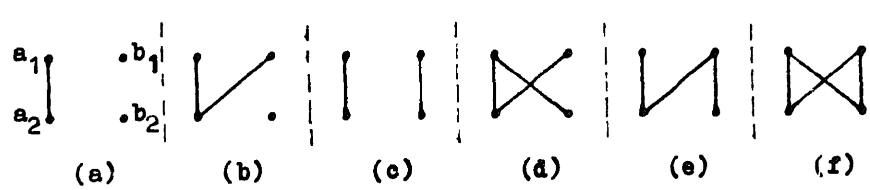

Fig. 4.

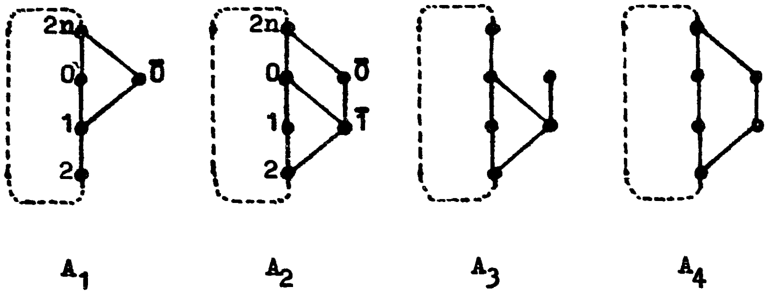

Fig. 5.

Corollary 3.7. *Consequently, each n-partite graph A is an induced subgraph of $(K_{n-2} * P_3)^k$ for a sufficiently large k. Moreover, if A is connected, the embedding μ can be constructed so that each $p_i\mu$ yields a quotient mapping onto $p_i\mu(A)$.*

We will discuss one more simple case of $\to G$.

Theorem 3.8. *In $\to C_{2n+1}$, the SI are P_0, P_1, P_2, P_3, C_{2n+1}, A_1, A_2. There are three more SFI, namely D, A_3 and A_4 (for A_i see Fig. 5).*

Proof. First realize that if a graph from $\to C_{2n+1}$ is not bipartite, its cardinality is at least 5. Consequently, a bipartite graph is irreducible in $\to C_{2n+1}$ iff it is such in $\to K_2$. Thus, it suffices to check the objects A of $\to C_{2n+1}$ which are not bipartite.

Represent C_{2n+1} as the set $\{0, 1, \ldots, 2n\}$ with i connected with $i+1$ (addition mod $2n+1$). Let $f: A \to C_{2n+1}$ be a fixed homomorphism. If A is not bipartite, we obviously have to have $f^{-1}(i) \neq \emptyset$ and $f^{-1}(i)$ interconnected with $f^{-1}(i+1)$.

If $|f^{-1}(i)| = 1$ for all i, we are left with C_{2n+1}. If exactly one of the $f^{-1}(i)$ consists of two points, rename the elements so as to have $f^{-1}(0) = \{0, \bar{0}\}$ and $f^{-1}(i) = \{i\}$ otherwise. Since A is not bipartite, one of 0, $\bar{0}$, say 0, is joined with both $2n$ and 1. But also $\bar{0}$ has to be joined with both of them if A is an SFI. If, say, it is not joined with $2n$, we have also the congruence $[\bar{0}2]$ and

$$A = A[0\bar{0}] \wedge A[\bar{0}2] \quad \text{or} \quad A[0\bar{0}] \wedge A[\bar{0}3]$$

according to whether $\bar{0}$ is joined with 1 or not. Since the only object left, A_1, is maximal and has only one nontrivial congruence, it is SI.

Now let two of the $f^{-1}(i)$ have two elements. If A is irreducible, they have to be neighbours by 3.2(c). Taking into account the fact that A is not bipartite, we easily see that we can rename the elements of A to have $f^{-1}(i) = \{i, \bar{i}\}$ for $i = 0, 1$, $f^{-1}(i) = \{i\}$ otherwise, and that $\{0, 1, \ldots, 2n\}$ induces C_{2n+1} in A.

First, let us deal with the case of $\bar{0}$ not joined with $\bar{1}$. By the same argument as above on the point redoubled, we obtain that if A is irreducible, $\bar{0}$ is joined with $2n$ and 1, and $\bar{1}$ with 0 and 2. This is a graph isomorphic to A_2, obviously meet-irreducible, and with all the possible congruences noncritical. Hence it is an SI.

Let $\bar{0}$ be joined with $\bar{1}$. In the case of both $0\bar{1}$ and $\bar{0}1$ present, $A = A[0\bar{0}] \wedge A[1\bar{1}]$ whatever the situation otherwise is.

Now, let us have $0\bar{1}$ but not $\bar{0}1$. If also $2\bar{1}$ fails, A is easily seen to be reducible. If it is present, we obtain A_3 in the absence and A_4 in the presence of $2n\bar{0}$. Whatever nontrivial congruence ε one takes in any of these cases, $A\varepsilon$ contains $1\bar{0}$. A_3, being meet-reducible, is not SI, though.

Finally let us have neither $0\bar{1}$ nor $1\bar{0}$. We easily check that the absence of any of $2n\bar{0}$ or $\bar{1}2$ allows congruences which make A reducible. If both these edges are present, however, all the congruences add $0\bar{1}$ and $1\bar{0}$. Thus, we obtain the last SFI, namely A_4. It is not SI, being meet-reducible. □

4. The nonlatticed case: How to borrow the $A\varepsilon$ from a larger category

As we have said above, it is often the case that a very reasonable category is not latticed so that 2.4 does not apply. Typically, however, such a category is nicely embedded into a larger one which is already latticed. We will show that this suffices.

Lemma 4.1. *Let $\mathscr{C}$ be an onto-reflective weakly regular subcategory of a weakly regular category $\mathscr{C}'$ (i.e., the embedding preserves the underlying sets and mappings, and the reflection morphisms are onto). Let $\mu : A \to B$ be a subobject in $\mathscr{C}$. Then it is one in $\mathscr{C}'$.*

Proof. Let $\psi : C \to B$ in $\mathscr{C}'$ and $f : U(C) \to U(A)$ be such that $U\mu \circ f = U\psi$. Let $\rho : C \to \bar{C}$ be the reflection morphism and let $\bar{\psi} : \bar{C} \to B$ be the morphism in $\mathscr{C}$ satisfying $\bar{\psi} \circ \rho = \psi$.

If $\rho(x) = \rho(y)$, we have $\psi(x) = \bar{\psi}\rho(x) = \psi(y)$ and hence $\mu(f(x)) = \mu(f(y))$ and consequently $f(x) = f(y)$. Thus, we can define a mapping $g : U(\bar{C}) \to U(A)$ by putting $g(\rho(x)) = f(x)$ and we have $g \circ U\rho = f$. Hence

$$U\mu \circ g \circ U\rho = U\mu \circ f = U\varphi = U\bar{\psi} \circ U\rho$$

and since $U\rho$ is onto, $U\mu \circ g = U\bar{\psi}$ and hence there is a $\gamma : \bar{C} \to A$ such that $U\gamma = g$. Put $\varphi = \gamma\rho$. We have $U\varphi = g \circ U\rho = f$. □

Theorem 4.2. *Let $\mathscr{C}$ be an onto-reflective concrete weakly regular subcategory of a latticed weakly regular category $\mathscr{C}'$ with products. Then a family of quotients in $\mathscr{C}$ $(\varepsilon_i : A \to B_i)_{i \in J}$ induces an SF-representation of A in $\mathscr{C}$ iff*

(1) *$(\varepsilon_i)_i$ is collectionwise monomorphic, and*

(2) *$\bigwedge_i A\varepsilon_i = A$ in $\mathscr{C}'(X)$.*

Proof. We can repeat the proof of 2.4. In part (II) we use 4.1 to conclude that ε' is a subobject in $\mathscr{C}'$. □

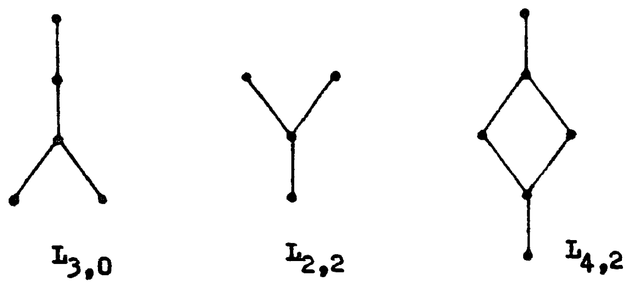

Fig. 6.

Notation 4.3. $L_n = (\{0, 1, \ldots, n\}, <)$, $\bar{L}_n = (\{0, 1, \ldots, n\}, \leqslant)$, P_n is the oriented path $(\{0, 1, \ldots, n\}, \{(i, i+1) \setminus i = 0, \ldots, n-1\})$. $L_{n,k}$ is obtained from L_n by redoubling the point k (see Fig. 6).

4.4. *Reflexive posets*: The maximal objects are exactly the linear orders $\bar{L}_n$ and of those only $\bar{L}_0$ and $\bar{L}_1$ satisfy CT. Each nonmaximal object is meet-reducible so that the

SI are exactly $\bar{L}_0$ and $\bar{L}_1$ (see also [9]).

Now to the SFI: Obviously, if $[ab]$ is a congruence the new couples in $A[ab]$ are exactly the (x, y) with

$$x \leqslant a \;\&\; b \leqslant y \quad \text{or} \quad x \leqslant b \;\&\; a \leqslant y.$$

Further, if a is an immediate successor of b, or if a, b are incomparable, $[ab]$ is a congruence. We easily check that:

(i) if a, b, c are immediate successors then $A = A[ab] \wedge A[bc]$,

(ii) if a, b, c are mutually incomparable then

$$A = A[ab] \wedge A[ac] \wedge A[bc].$$

This leaves us with checking the cases of $|A| \leqslant 4$ which is easily done and yields, besides the SI's, only the 2-antichain.

4.5. *Antireflexive posets:* The maximal objects are exactly the L_n and none of them admits a nontrivial congruence. The nonmaximal objects are meet-reducible and hence we have

SI: exactly the L_n $(n = 0, 1, \ldots)$.

If $[ab]$ is a congruence, the new couples in $A[ab]$ are the (x, y) satisfying

$$(x \leqslant a \;\&\; b \leqslant y \quad \text{or} \quad x \leqslant b \;\&\; a \leqslant y) \;\&\; \{x, y\} \neq \{a, b\}.$$

If a, b are incomparable, $[ab]$ is a congruence and in $A[ab]$ a and b are incomparable again. No congruence identifies comparable elements. We have

$$A = \bigwedge \{A[ab] \mid a, b \text{ distinct incomparable}\}$$

(if there are incomparable couples) and we obtain the list of

$$\text{SFI: } L_n,\ L_{n,k} \quad (n = 0, 1, \ldots ; k \leqslant n).$$

4.6. *Partial unary algebras* (with one unary operation): For $\mathscr{C}'$ take the category of all oriented graphs. Since shrinking one or more components into a single point each is a congruence ε such that in $A\varepsilon$ nothing is added into the remaining components, an irreducible object:

(i) has at most two components,

(ii) if it has two components, at least one of them consists of a single point, and if the other is larger, it cannot contain a loop.

Further, in an irreducible object $(A, \propto)$ one cannot have distinct a, b, c with $\alpha(a) = \alpha(b) = c$. Indeed let this be the case. At least one of a, b, say a, is not of the form $\alpha^k(c)$. Shrinking all the $\alpha^k(b)$ $(k = 0, 1, \ldots)$ is a congruence ε and $[ab]$ is also one. We have $A = A[ab] \wedge A\varepsilon$.

Finally we easily see that a cycle is irreducible iff its length is a power of a prime.

Thus, to the list of SI from [9] (there is an error: instead of 'prime' read 'power of a prime') we add only the obligatory two-element objects.

4.7. *Oriented graphs without cycles:* The maximal objects are exactly the L_n. The meet irreducibles are the $L_n \setminus \{(a, b)\}$ with $a < b$; if a is immediately succeeded by b, $[ab]$ is a critical congruence, otherwise there is no nontrivial congruence at all. Thus, we have the following list of

$$\text{SI: } L_n,\ L_n \setminus \{(a, b)\} \quad (n = 0, 1, \ldots, \exists x, a < x < b).$$

There are plenty new SFI. In particular, any A such that $\vec{P}_n \leqslant A \leqslant L_n$ since no such A admits a nontrivial congruence. On the other hand, an SFI does not contain a 3-antichain, or two 2-antichains which are not connected with each other.

4.8. *Antisymmetric graphs without loops:* One immediately checks that the SI are exactly the tournaments. Again, there are plenty new SFI's, probably very hard to list. For instance any A with $G \leqslant A \leqslant$ tournament is such, where G is the product of L_n and L_2 with lexicographic order.

5. Remarks and problems

5.1. Since other conditions on the $p_i\mu$ in the representation than 'being quotients' (notably, 'being retractions', as pointed out by I. Rival) may be of interest, let us note that the procedure from 2.4 resp. 4.2 can be substantially generalized.

Let $\mathscr{E}$ be a class of epimorphisms in a concrete category $(\mathscr{C}, U)$ such that each $\varepsilon \in \mathscr{E}$ which is also a monomorphism is an isomorphism. Let us say that a subobject $\mu: A \to \prod B_i$ is an $\mathscr{E}$-subdirect representation if all the $p_i\mu$ are in $\mathscr{E}$. Let, moreover, $(\mathscr{C}, U)$ be latticed. For an $\varepsilon: A \to B$ from $\mathscr{E}$ define $A\varepsilon\mathscr{E}$ as the join $\bigvee \{A' \mid \varepsilon = \varepsilon' \circ (A \leq A')\}$ and let $\bar{\varepsilon}\mathscr{E}: A\varepsilon\mathscr{E} \to B$ be the morphism obtained from the colimit condition. Quite analogously to Theorem 4.2 we obtain the following.

Theorem. *Let $\mathscr{C}$ be an onto-reflective concrete weakly regular subcategory of a latticed weakly regular category $\mathscr{C}'$ with products. Let $\mathscr{E}$ be a class of epimorphisms of $\mathscr{C}$ such that each monomorphism in $\mathscr{E}$ is an isomorphism. Then a family $(\varepsilon_i: a \to B_i)_{i \in J}$ of morphisms of $\mathscr{E}$ induces an $\mathscr{E}$-subdirect representation of A in $\mathscr{C}$ iff:*

(1) *$(\varepsilon_i)_i$ is collectionwise isomorphic, and*

(2) *$\bigwedge A\varepsilon_i\mathscr{E} = A$ in $\mathscr{C}'(X)$.*

Corollary. *In the situation above, A is $\mathscr{E}$-subdirectly irreducible iff for each collectionwise monomorphic system $(\varepsilon_i: A \to B_i)_i$ of morphisms from $\mathscr{E}$, $\bigwedge_i A\varepsilon_i\mathscr{E} \neq A$.*

5.2. Recall that for varieties of algebras, CT ≡ SI ≡ SFI. In the general case we obviously have

CT
SFI
SI

and nothing more. Already in the list of 2.9 we find counter-examples to all the other possible implications, and also to SFI ⇒ (SI or CT).

5.3. It should be of interest to explain the situation in some of the examples where the SI coincide with the SFI (symmetric graphs with loops) or almost do so (n-partite graphs, reflexive posets, where we have only the discrete two-point object in the difference, which does not play any role when representing connected objects). It is not as if a subdirect representation in the SI's would be in these cases automatically an SF representation; only, the existence of one—somewhat surprisingly—implies the existence of the other.

5.4. In the SI's the general case differs from the algebraic one in the possible existence of the nonmaximal irreducible objects (with the condition differing from the CT). In several cases (symmetric graphs without loops, reflexive posets, antireflexive posets, antisymmetric graphs) we have observed that the nonmaximal SI's do not appear. None of the mentioned categories is close to categories of algebras in character. It would be of interest to have a characteristics, or nontrivial necessary or sufficient conditions for the lack of nonmaximal SI's.

5.5. Another intriguing question is that of heredity of subobject irreducibility. In all the examples of Section 2 (with the possible exception of the SFI in 2.8, where we do not know the list), or in the categories of posets, subobjects of SI (SFI) are SI (SFI). Not so, however, for instance, in any $\rightarrow K_n$. This deserves a closer examination.

5.6. Observe that, although the classes of SFI often considerably differred from those of the SI, in all our examples, if SI were finitely many, so were also SFI. Is this a fairly general law?

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