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The Umbral Calculus of Symmetric Functions

Miguel A. Méndez

IVIC, Departamento de Matemática, Caracas, Venezuela; and
UCV, Departamento de Matemática, Facultad de Ciencias, Caracas, Venezuela

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1. INTRODUCTION

In this article we develop an umbral calculus for the symmetric functions in an infinite number of variables. This umbral calculus is analogous to Roman–Rota umbral calculus for polynomials in one variable [45].

Though not explicit in the Roman–Rota treatment of umbral calculus, it is apparent that the underlying notion is the Hopf-algebra structure of the polynomials in one variable, given by the counit ε, comultiplication E^y, and antipode θ

$$\langle \varepsilon \mid p(x) \rangle = p(0)$$

$$E^y p(x) = p(x+y)$$

$$\theta p(x) = p(-x).$$

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The algebraic dual, endowed with the multiplication

$$\langle L * M \mid p(x) \rangle = \langle L^x \cdot M^y \mid p(x+y) \rangle$$

and with a suitable topology, becomes a topological algebra. This topological algebra is called the umbral algebra. It is isomorphic to the algebra of exponential formal power series.

The Hopf-algebra structure of the set of symmetric functions $\Lambda(X)$ over the alphabet $X = \{x_1, x_2, x_3, ...\}$ has been known since the work of Geissinger [20]. $\Lambda(X)$ is reflexive in the following sense: the graded dual of $\Lambda(X)$ is isomorphic to $\Lambda(X)$.

To make the analogy with the one-variable case work, and to use the arsenal of umbral calculus, we have to get rid of the notion of a graded dual. We consider instead the whole dual $\Lambda(X)^*$. The Hopf algebra structure of $\Lambda(X)$ provides the product on $\Lambda(X)^*$,

$$\langle LM \mid R(x) \rangle = \langle L^X M^Y \mid R(X+Y) \rangle.$$

Endowing $\Lambda(X)^*$with the appropriate topology, it becomes a topological algebra. $\Lambda(X)^*$ is isomorphic to the algebra of symmetric series.

The algebra, coalgebra, and bialgebra maps of $\Lambda(X)$ are then classified in terms of families of symmetric functions of binomial type and admissible systems of functionals. Umbral substitution of basic families of binomial type, i.e., composition of umbral maps, has as dual operation the substitution of admissible systems of functionals Many examples of families of binomial type are studied: the power symmetric functions, the monomial symmetric functions, the plethystic exponential polynomials, and the plethystic analogues of the decreasing factorial, the plethystic analogues of the increasing factorial, etc. The classical definition of Sheffer sequences of polynomials is extended to this context, and analogues of Bernoulli and Euler polynomials are introduced.

Chen's compositional calculus [9] is criptormorphic to the present calculus by identifying x_n with $p_n(X)$, $p_n(X)$ being the power sum symmetric function. The underlying combinatorics in Chen's compositional calculus is the counting of colored structures in an infinite number of colors. In the present approach the underlying combinatorics is related to the counting of structures kept fixed by permutations and to Bergeron's S-species [3].

In the Roman–Rota paper [45], the classical recursive formula (Rodrigues formula) for the generation of sequences of binomial type is generalized by the use of umbral shifts as adjoint operators of continuous derivations in the umbral algebra. This procedure was later extended by Roman to polynomials in many variables [44]. We extend this duality to the present situation by associating to each invertible system of lineal functionals an infinite system of derivations in the umbral algebra. Its adjoint operators form an infinite system of umbral shifts. We obtain in this way two recursive formulas for basic families of symmetric functions of binomial type.

We go one step forward in this direction. A family of symmetric functions of Schur type is defined as the set of images by an umbral map of the classical Schur functions. The dual notion of a family of symmetric functions of Schur type is that of a family of functionals of Schur type. A family of symmetric functions of Schur type satisfies all the coalgebraic properties of the Schur functions. The functionals of Schur type satisfy all the algebra properties of the Schur functions (in particular, analogous of the Jacobi–Trudi identity). Taking exterior powers of umbral shifts we generalize the recursive formulas for the connecting coefficients between symmetric functions of binomial type. When the umbral map is also an algebra map, this generalization is the bridge between the Jacobi–Trudi formulas and the quotient of alternant formulas: a general formula for the symmetric functions of Schur type as a quotient of alternants is proved.

A generalized Schur function in the variables $x_1, x_2, ..., x_n$ is defined as the quotient of alternants

$$\frac{|t_{\lambda_r + r - 1}(x_s)|_{r, s=1}^{n}}{|x_s^{r-1}|_{r, s=1}^{n}}$$

where $t_n(x)$ is any polynomial sequence. They form a basis of $\Lambda(x_1, x_2, ..., x_n)$. By using the exterior power of umbral shifts we obtain a generalization of the Jacobi–Trudi formula for the dual pseudobasis. The factorial symmetric functions [4, 5, 22, 32] and the Macdonald 6th variation on Schur functions [32] are special cases of the generalized Schur function. We obtain a necessary and sufficient condition over the polynomials $t_n(x)$ for the existence of the inverse limit of the corresponding generalized Schur functions. We define a Sheffer–Schur family as the image under an invertible shift invariant operator $N(D)$ of a family of functions of Schur type. When the functional N is multiplicative, and the family of functions of Schur type are the classical Schur functions we get back the class of symmetric functions that can be obtained as inverse limits of generalized symmetric functions. We think that this fact has particular relevance in the calculus of Witt vectors.

We introduce here a notion of determinant for certain kinds of infinite jacobian matrices. With this notion, and using one of the recursive formulas, we generalize Joni's transfer formula [24, 25] to this context. A very general Lagrange's inversion formula is obtained as the dual form of the transfer formula.

2. PRELIMINARIES

Let S be a set. We define the power sum symmetric functions over the set S as

$$p_0(S) = 1$$

$$p_n(S) = \sum_{s \in S} s^n, \qquad n > 0.$$

A *partition* of a positive integer n is a collection of positive integers whose sum is equal to n. Since the order in which the integers are written is immaterial, we may represent a partition as a nondecreasing sequence $\lambda = (\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_l)$ of positive integers. We call it the standard representation of the partition λ. We say that $\mu \sqsubseteq \lambda$ if $\mu_i \leqslant \lambda_i$ for every i. For $\lambda \sqsubseteq \lambda$ we denote by $S_{\lambda/\mu}$ the skew Schur function.

We also use the multiset notation $\lambda = (1^{\alpha_1} 2^{\alpha_2} 3^{\alpha_3} \cdots)$. It means that exactly α_i parts in the partition are equal to i. In this article we frequently identify a partition λ with the vector of multiplicities of its parts $\vec{\alpha}(\lambda) = \vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3, ...)$. In symbols,

$$\vec{\alpha} = \sum_i \vec{e}_{\lambda_i} \tag{1}$$

where $\vec{e}_r$ is the rth unit vector.

The expression $\vec{\alpha} \vdash n$ (resp., $\lambda \vdash n$) means that $\vec{\alpha}$ (resp. λ) is a partition of n, i.e., $\sum_i i\alpha_i = n$ (resp., $\sum_i \lambda_i = n$). The expression $|\vec{\alpha}|$ (resp. $|\lambda|$) denotes the sum of the parts of the partition $\vec{\alpha}$ (resp. λ), and $l(\vec{\alpha})$ (resp. $l(\lambda)$) denotes the number of parts of it, i.e.,

$$|\vec{\alpha}| = \sum_i i\alpha_i = \sum_{i=1}^{l} \lambda_i$$

$$l(\vec{\alpha}) = \sum_i \alpha_i = l = l(\lambda).$$

We say that $\vec{\beta} \leqslant \vec{\alpha}$ if every component of $\vec{\beta}$ is less than or equal to the corresponding component of $\vec{\alpha}$. We denote by $\mathfrak{P}$ the set of all partitions.

A permutation $\sigma: E \to E$ of a finite set E is said to be of class $\vec{\alpha}$ if σ has α_i cycles of size i. The number of permutations commuting with a permutation of class $\vec{\alpha}$ is given by the formula

$$z_{\vec{\alpha}} = \prod_i i^{\alpha_i} \alpha_i!.$$

For three partitions $\vec{\alpha}$, $\vec{\beta}$, and $\vec{\gamma}$ satisfying $\vec{\beta}+\vec{\gamma}=\vec{\alpha}$ we define the binomial coefficient

$$\binom{\vec{\alpha}}{\vec{\beta},\vec{\gamma}}=\frac{z_{\vec{\alpha}}}{z_{\vec{\beta}}z_{\vec{\gamma}}}=\frac{\vec{\alpha}!}{\vec{\beta}!\,\vec{\gamma}!}, \tag{2}$$

where $\vec{\alpha}! := \prod_i \alpha_i!$

Let $X=\{x_1, x_2, x_3, ...\}$ be an infinite set of variables. The algebra $\Lambda(X)$ of symmetric functions over X is the $\mathbb{C}$-algebra freely generated by the symmetric functions $p_n(X)$, i.e.,

$$\Lambda(X)=\mathbb{C}[p_1(X), p_2(X), p_3(X), ...].$$

Clearly, the family of power symmetric functions $\{p_{\vec{\alpha}}(X)\}_{\vec{\alpha}}$,

$$p_{\vec{\alpha}}(X)=p_1(X)^{\alpha_1}\,p_2(X)^{\alpha_2}\,p_3(X)^{\alpha_3}\cdots,$$

form a basis of $\Lambda(X)$. Throughout this article we will follow Macdonald's notation (see [33]) for the rest of the classical basis of $\Lambda(X)$.

3. THE UMBRAL ALGEBRA

Consider another totally ordered set $Y=\{y_1, y_2, y_3, ...\}$. Denote by $X+Y$ the set of variables $\{x_1, x_2, ..., y_1, y_2, ...\}$. Clearly $\Lambda(X)$ is isomorphic to $\Lambda(Y)$. Let

$$\Lambda(X, Y)=\mathbb{C}[\{p_i(X)\,p_j(Y)\}_{i,\,j=1,2,...}]$$

be the algebra of $\mathbb{C}$-polynomials, symmetric in X and Y. $\Lambda(X, Y)$ is isomorphic to $\Lambda(X)\otimes\Lambda(X)$. For a symmetric function $R(X)=\sum_{\vec{\alpha}} a_{\vec{\alpha}}\,p_{\vec{\alpha}}(X)$, $R(XY)$ will denote the element of $\Lambda(X, Y)$ defined by

$$R(XY)=\sum_{\vec{\alpha}} a_{\vec{\alpha}}\,p_{\vec{\alpha}}(X)\,p_{\vec{\alpha}}(Y).$$

DEFINITION 3.1. The translation operator E^Y is defined by

$$E^Y\colon \Lambda(X)\to\Lambda(X, Y)\sim\Lambda(X)\otimes\Lambda(X)$$

$$E^Y R(X)=R(X+Y).$$

The algebra of symmetric functions becomes a Hopf algebra with comultiplication E^Y, the evaluation at zero of $\varepsilon: \Lambda(X) \to \mathbb{C}$ as the counit, and the involution

$$\theta(p_{\vec{\alpha}}(X)) = (-1)^{l(\vec{\alpha})} p_{\vec{\alpha}}(X)$$

as the antipode (see [20]).

Since $\Lambda(X)$ is a Hopf algebra, for a $\mathbb{C}$-algebra $\mathfrak{A}$, the vector space of linear homomorphisms $\operatorname{Hom}(\Lambda(X), \mathfrak{A})$ is an algebra.

Recall that a directed set $(\mathfrak{J}, \leqslant)$ is a partially ordered set satisfying the condition that, for every pair $i, j \in \mathfrak{J}$, there exists an element $k \in \mathfrak{J}$ such that $k \geqslant i$ and $j \geqslant i$. Consider the discrete topology over the algebras $\Lambda(X)$ and $\mathfrak{A}$. All the homomorphisms $\operatorname{Hom}(\Lambda(X), \mathfrak{A})$ are continuous, and the discrete topology over $\mathfrak{A}$ induces a topology over $\operatorname{Hom}(\Lambda(X), \mathfrak{A})$ described as follows: A sequence of elements of $\operatorname{Hom}(\Lambda(X), \mathfrak{A})$, $\{M_j\}_{j \in \mathfrak{J}}$ converges to M if for any symmetric function $R(X)$ there exists some $j_0 \in \mathfrak{J}$ depending on $R(X)$ such that

$$\langle M_j \mid R(X) \rangle = \langle M \mid R(X) \rangle \qquad \text{for} \quad j \geqslant j_0.$$

A series $\sum_{k \in K} M_k$ is convergent if the sequence of partial sums $S_F = \sum_{k \in F} M_k$, F ranging over the directed set of finite parts of K, is convergent. Clearly, a series of the form $\sum_{k=1}^{\infty} L_k$ is convergent if and only if $L_k \to 0$.

Since $p_n(X+Y) = p_n(X) + p_n(Y)$, it is easy to check the identity

$$E^Y p_{\vec{\alpha}}(X) = p_{\vec{\alpha}}(X+Y) = \sum_{\vec{\beta}+\vec{\gamma}=\vec{\alpha}} \binom{\vec{\alpha}}{\vec{\beta}, \vec{\gamma}} p_{\vec{\beta}}(X)\, p_{\vec{\gamma}}(Y). \tag{3}$$

Moreover, if $X_1, X_2, ..., X_k$ is a sequence of sets of variables, we have the identity

$$p_{\vec{\alpha}}(X_1 + X_2 + \cdots + X_k) = \sum \binom{\vec{\alpha}}{\vec{\alpha}^{(1)}, \vec{\alpha}^{(2)}, ..., \vec{\alpha}^{(k)}} \prod_{i=1}^{k} p_{\vec{\alpha}^{(i)}}(X_i) \tag{4}$$

where the sum of the above ranges over the tuples of partitions $(\vec{\alpha}^{(1)}, \vec{\alpha}^{(2)}, ..., \vec{\alpha}^{(k)})$ satisfying $\sum_{i=1}^{k} \vec{\alpha}^{(i)} = \vec{\alpha}$ and the coefficient $\binom{\vec{\alpha}}{\vec{\alpha}^{(1)}, \vec{\alpha}^{(2)}, ..., \vec{\alpha}^{(k)}}$ is defined by

$$\binom{\vec{\alpha}}{\vec{\alpha}^{(1)}, \vec{\alpha}^{(2)}, ..., \vec{\alpha}^{(k)}} = \frac{\vec{\alpha}!}{\vec{\alpha}^{(1)}!\, \vec{\alpha}^{(2)}! \cdots \vec{\alpha}^{(k)}!}. \tag{5}$$

Let M and N be two elements of $\operatorname{Hom}(\Lambda(X), \mathfrak{A})$. By formula (3), the product $M \cdot N$ is given explicitly by

$$\begin{aligned}\langle M \cdot N \mid p_{\vec{\alpha}}\rangle &= \langle M^X N^Y \mid p_{\vec{\alpha}}(X+Y)\rangle \\ &= \sum_{\vec{\beta}+\vec{\gamma}=\vec{\alpha}} \binom{\vec{\alpha}}{\vec{\beta}, \vec{\gamma}} \langle M \mid p_{\vec{\beta}}(X)\rangle \langle N \mid p_{\vec{\gamma}}(X)\rangle. \qquad (6)\end{aligned}$$

Consider now a sequence $M_1, M_2, ..., M_k$ of elements in $\operatorname{Hom}(\Lambda(X), \mathfrak{A})$. Using formula (4) we get

$$\langle M_1 \cdot M_2 \cdots M_k \mid p_{\vec{\alpha}}(X)\rangle = \sum \binom{\vec{\alpha}}{\vec{\alpha}^{(1)}, \vec{\alpha}^{(2)}, ..., \vec{\alpha}^{(k)}} \prod_{i=1}^{k} \langle M_i \mid p_{\vec{\alpha}^{(i)}}(X)\rangle. \qquad (7)$$

Proposition 3.1. *Let M be an element of* $\operatorname{Hom}(\Lambda(X), \mathfrak{A})$. *If* $\langle M \mid 1\rangle = 0$, *then* $M^n \to 0$.

Proof. By Eq. (7), if $k > l(\vec{\alpha})$, then $\langle M^k \mid p_{\vec{\alpha}}(X)\rangle = 0$. ∎

Proposition 3.2. *Let M be an element of* $\operatorname{Hom}(\Lambda(X), \mathfrak{A})$. *$M$ is invertible if and only if $\langle M \mid 1\rangle$ is invertible in $\mathfrak{A}$.*

Proof. Since $\langle MM' \mid A\rangle = \langle M \mid 1\rangle \langle M' \mid 1\rangle$ for any pair M, M' in $\operatorname{Hom}(\Lambda(X), \mathfrak{A})$, if M is invertible then $\langle M \mid 1\rangle$ is invertible. Conversely, if $\langle M \mid 1\rangle$ is invertible we will define M' in a recursive way. Let $\langle M' \mid 1\rangle = (\langle M \mid 1\rangle)^{-1}$. Assuming that we have defined $\langle M' \mid p_{\vec{\beta}}(X)\rangle$ for every $\vec{\beta} < \vec{\alpha}$, define

$$\langle M' \mid p_{\vec{\alpha}}(X)\rangle = \Bigg(- \sum_{\substack{\vec{\beta}+\vec{\gamma}=\vec{\alpha} \\ \vec{\beta}<\vec{\alpha}}} \binom{\vec{\alpha}}{\vec{\beta}, \vec{\gamma}} \langle M' \mid p_{\vec{\beta}}(X)\rangle \langle M \mid p_{\vec{\gamma}}(X)\rangle \Bigg) \langle M \mid 1\rangle^{-1}.$$

It is easy to check that M' is the left inverse of M. In a similar way we define the right inverse of M. ∎

The family of sets of the form

$$\mathcal{N}(M, \vec{\alpha}) = \{L \in \operatorname{Hom}(\Lambda(X), \mathfrak{A}) \colon \langle L \mid p_{\vec{\beta}}(X)\rangle = \langle M \mid p_{\vec{\beta}}(X)\rangle, \vec{\beta} \leqslant \vec{\alpha}\},$$
$$M \in \operatorname{Hom}(\Lambda(X), \mathfrak{A}), \quad \vec{\alpha} \in \mathfrak{P}$$

is a fundamental set of neighborhoods for the topology defined on $\operatorname{Hom}(\Lambda(X), \mathfrak{A})$.

The proof of the following proposition is straightforward from the definition of product and the proof of the previous proposition.

PROPOSITION 3.3. *Let L_1, L_2, M, and N be arbitrary elements of the algebra* $\mathrm{Hom}(\Lambda(X), \mathfrak{A})$. *We have*

1. *If $L_1 \in \mathcal{N}(M, \vec{\alpha})$ and $L_2 \in \mathcal{N}(N, \vec{\alpha})$, then $cL_1 \in \mathcal{N}(cM, \vec{\alpha})$ for every $c \in \mathbb{C}$, $L_1 + L_2 \in \mathcal{N}(M+N, \vec{\alpha})$, and $L_1 \cdot L_2 \in \mathcal{N}(M \cdot N, \vec{\alpha})$.*

2. *Assume that M is invertible. If $N \in \mathcal{N}(M, \vec{\alpha})$, then N is invertible and $N^{-1} \in \mathcal{N}(M^{-1}, \vec{\alpha})$.*

From the previous proposition we get that $\mathrm{Hom}(\Lambda(X), \mathfrak{A})$, with the topology defined above, is a topological algebra.

The map $\diamondsuit : R(X) \mapsto R(XY)$ defines another coproduct on $\Lambda(X)$. The *inner plethysm* is the corresponding product on $\mathrm{Hom}(\Lambda(X), \mathfrak{A})$, defined by

$$\langle L \odot M \mid R(X) \rangle = \langle L^X M^Y \mid \diamondsuit R(X) \rangle = \langle L^X M^Y \mid R(XY) \rangle. \tag{8}$$

DEFINITION 3.2. As a particular instance of $\mathrm{Hom}(\Lambda(X), \mathfrak{A})$, $\Lambda(X)^*$, the linear dual of $\Lambda(X)$ is a topological algebra. Following the terminology in [45], we call it the *Umbral Algebra.*

EXAMPLE 3.1. Let $\mathfrak{M} = \{a_1^{n_1}, a_2^{n_2}, ..., a_k^{n_k}\}$ be a multiset of complex numbers of cardinality $n_1 + n_2 + \cdots + n_k = n$. The evaluation $\varepsilon_{\mathfrak{M}} \in \Lambda(X)^*$ is the linear functional defined by

$$\langle \varepsilon_{\mathfrak{M}} \mid R(X) \rangle = R(\overbrace{a_1, ..., a_1}^{n_1}, \overbrace{a_2, ..., a_2}^{n_2}, ..., \overbrace{a_k, ..., a_k}^{n_k}, 0, 0, ...) = \rho_n(R)(\mathfrak{M}), \tag{9}$$

where ρ_n is the projection of $\Lambda(X)$ over the algebra $\Lambda(x_1, x_2, ..., x_n)$ of symmetric polynomials in the variables $x_1, x_2, ..., x_n$. By abuse of notation we write $R(\mathfrak{M}) = \langle \varepsilon_{\mathfrak{M}} \mid R(X) \rangle$. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be two multisets. For any symmetric function $R(X)$ we have

$$\langle \varepsilon_{\mathfrak{M}_1} \varepsilon_{\mathfrak{M}_2} \mid R(X) \rangle = R(\mathfrak{M}_1 + \mathfrak{M}_2) = \langle \varepsilon_{\mathfrak{M}_1 + \mathfrak{M}_2} \mid R(X) \rangle,$$

where $\mathfrak{M}_1 + \mathfrak{M}_2$ is the disjoint union of the multisets $\mathfrak{M}_1$ and $\mathfrak{M}_2$. Then $\varepsilon_{\mathfrak{M}_1} \varepsilon_{\mathfrak{M}_2} = \varepsilon_{\mathfrak{M}_1 + \mathfrak{M}_2}$.

We define the functional $\varepsilon_{(-)\mathfrak{M}}$ by

$$\langle \varepsilon_{(-)\mathfrak{M}} \mid p_{\vec{\alpha}}(X) \rangle = (-1)^{l(\vec{\alpha})} \, p_{\vec{\alpha}}(\mathfrak{M}) = \langle \varepsilon_{\mathfrak{M}} \mid \theta p_{\vec{\alpha}}(X) \rangle.$$

Since $\varepsilon_{\mathfrak{M}}$ is multiplicative, $\varepsilon_{(-)\mathfrak{M}}$ is its inverse, $\varepsilon_{\mathfrak{M}} \varepsilon_{(-)\mathfrak{M}} = \varepsilon_{\varnothing} = \varepsilon$.

PROPOSITION 3.4. *Let $R(X)$ and $S(X)$ be two symmetric functions satisfying, for every multiset $\mathfrak{M}$,*

$$\langle \varepsilon_{\mathfrak{M}} \mid R(X) \rangle = \langle \varepsilon_{\mathfrak{M}} \mid S(X) \rangle.$$

Then $R(X) = S(X)$.

Proof. Since $\rho_n(R(\mathfrak{M})) = \rho_n S(\mathfrak{M})$, for every finite multiset $\mathfrak{M}$ we have $\rho_n(R(X)) = \rho_n(S(X))$, $n \geqslant 0$. Then, $R(X) = S(X)$. ∎

For a sequence of elements in the umbral algebra $\{L_{\vec{\alpha}}\}_{\vec{\alpha}}$, indexed by partitions, $L_{\vec{\alpha}} \to L$ means that for every symmetric function $R(X)$, $\langle L_{\vec{\alpha}} \mid R(X) \rangle = \langle L \mid R(X) \rangle$ for $|\vec{\alpha}|$ big enough.

The following proposition is easy to prove:

PROPOSITION 3.5. *The series $\sum_{\vec{\alpha} \in \mathfrak{P}} L_{\vec{\alpha}}$ of functionals is convergent if and only if $L_{\vec{\alpha}}$ converges to zero.*

DEFINITION 3.3. We define the functional A_n by the relation

$$\langle A_n \mid p_{\vec{\alpha}}(X) \rangle = n\delta(\vec{e}_n, \vec{\alpha}), \qquad n > 0,$$

where δ is Kronecker's. For a partition $\vec{\alpha}$, define $A^{\vec{\beta}} = \prod_i A_i^{\beta_i}$.

PROPOSITION 3.6. *We have the biorthogonality relations*

$$\langle A^{\vec{\beta}} \mid p_{\vec{\alpha}}(X) \rangle = z_{\vec{\beta}} \delta(\vec{\alpha}, \vec{\beta}).$$

Proof. Assume first that $\vec{\beta} = k\vec{e}_n$ for some integers n and k. Applying Eq. (7) we have

$$\langle A^{\vec{\beta}} \mid p_{\vec{\alpha}}(X) \rangle = \langle A_n^k \mid p_{\vec{\alpha}}(X) \rangle = \sum \binom{\vec{\alpha}}{\vec{\alpha}^{(1)}, \vec{\alpha}^{(2)}, \ldots, \vec{\alpha}^{(k)}} \prod_{i=1}^{k} n\delta\,(\vec{e}_n, \vec{\alpha}^{(i)})$$

$$= \binom{\vec{\alpha}}{\vec{e}_n, \vec{e}_n, \ldots, \vec{e}_n} n^k \delta(k\vec{e}_n, \vec{\alpha}) = n^k k!\ \delta(k\vec{e}_n, \vec{\alpha}) = z_{k\vec{e}_n} \delta(k\vec{e}_n, \vec{\alpha}).$$

For an arbitrary partition $\vec{\beta} = (\beta_1, \beta_2, \ldots, \beta_r, 0, 0, \ldots)$ we have

$$\langle A^{\vec{\beta}} \mid p_{\vec{\alpha}}(X) \rangle = \sum \binom{\vec{\alpha}}{\vec{\alpha}^{(1)}, \vec{\alpha}^{(2)}, \ldots, \vec{\alpha}^{(r)}} \prod_{i=1}^{r} \langle A_i^{\beta_i} \mid p_{\vec{\alpha}^{(i)}}(X) \rangle$$

$$= \sum \binom{\vec{\alpha}}{\vec{\alpha}^{(1)}, \vec{\alpha}^{(2)}, \ldots, \vec{\alpha}^{(r)}} \prod_{i=1}^{r} \delta(\beta_i e_i, \vec{\alpha}_i)\, z_{\vec{\beta}^{(i)}} = z_{\vec{\beta}} \delta(\vec{\alpha}, \vec{\beta}). \quad ∎$$

By Proposition 3.6 we have

PROPOSITION 3.7. *The sequence of functionals* $\{A_{\vec{\alpha}}\}_{\vec{\alpha} \in \mathfrak{P}}$ *converges to zero.*

PROPOSITION 3.8. *Let* $M \in \Lambda(X)^*$ *be a linear functional. Then we have the expansion*

$$M = \sum_{\vec{\alpha} \in \mathfrak{P}} \langle M \mid p_{\vec{\alpha}}(X) \rangle \frac{A^{\vec{\alpha}}}{z_{\vec{\alpha}}}. \tag{10}$$

Proof. Since $A^{\vec{\alpha}} \to 0$, the series $\sum_{\vec{\alpha} \in \mathfrak{P}} \langle M \mid p_{\vec{\alpha}}(X) \rangle A^{\vec{\alpha}}/z_{\vec{\alpha}}$ converges to some functional L. We have the identity

$$\langle L \mid p_{\vec{\beta}}(X) \rangle = \sum \langle M \mid p_{\vec{\alpha}}(X) \rangle \frac{\langle A^{\vec{\alpha}} \mid p_{\vec{\beta}}(X) \rangle}{z_{\vec{\alpha}}} = \langle M \mid p_{\vec{\beta}}(X) \rangle.$$

Since the $p_{\vec{\alpha}}$'s form a basis, we obtain the result. ∎

Associating to each functional M the symmetric series

$$\sum_{\vec{\alpha} \in \mathfrak{P}} \langle M \mid p_{\vec{\alpha}}(X) \rangle \frac{p_{\vec{\alpha}}(X)}{z_{\vec{\alpha}}},$$

we get an isomorphism

$$\Lambda(X)^* \to \Lambda((X)) = \mathbb{C}[[p_1(X), p_2(X), ...]] \tag{11}$$

between the umbral algebra and the algebra $\Lambda((X))$ of symmetric series. The topology of $\Lambda(X)^*$ is in this way transported to $\Lambda((X))$.

We call the series

$$\mathscr{I}_M(X) = \sum_{\vec{\alpha} \in \mathfrak{P}} \langle M \mid p_{\vec{\alpha}}(X) \rangle \frac{p_{\vec{\alpha}}(X)}{z_{\vec{\alpha}}}$$

the indicator of M. We frequently write $M(X)$ instead of $\mathscr{I}_M(X)$. For a symmetric series $S(X)$, $S(A)$ will denote the linear functional whose indicator is $S(X)$.

A functional M is called *polynomial* if $\langle M \mid p_{\vec{\alpha}}(X) \rangle = 0$ for almost every $\vec{\alpha}$, or equivalently, if $M(X) \in \Lambda(X)$. If M is polynomial and $R(X)$ is any symmetric function the evaluation of M in R, $\langle M \mid R(X) \rangle$, coincides with the Hall inner product $(M(X), R(X))_H$.

Consider now the sequence of infinite sets of variables $X^{(1)}, X^{(2)}, ..., X^{(k)}$. Since $\Lambda(X^{(1)}, X^{(2)}, ..., X^{(k)})$ is isomorphic to the tensor product

$$\Lambda(X^{(1)}) \otimes \Lambda(X^{(2)}) \otimes \cdots \otimes \Lambda(X^{(k)}),$$

every functional $L \in \Lambda(X^{(1)}, ..., X^{(k)})^*$ has an expansion of the form

$$L = \sum_{\vec{\alpha}^{(1)}, ..., \vec{\alpha}^{(k)}} \prod_{i=1}^{k} \langle L \mid p_{\vec{\alpha}^{(i)}}(X^{(i)}) \rangle \prod_{i=1}^{k} \frac{(A^{\vec{\alpha}^{(i)}})^{X^{(i)}}}{z_{\vec{\alpha}^{(i)}}}, \tag{12}$$

where for a functional N in $\Lambda(X)^*$, $N^{X^{(i)}}$ denotes the image of N by the injection

$$\Lambda(X)^* \cong \Lambda(X^{(i)})^* \hookrightarrow \Lambda(X^{(1)}, ..., X^{(k)})^*.$$

Corollary 3.1. *We have the expansions*

$$\varepsilon_{\mathfrak{M}} = \sum_{\vec{\alpha} \in \mathfrak{P}} p_{\vec{\alpha}}(\mathfrak{M}) \frac{A^{\vec{\alpha}}}{z_{\alpha}} = e^{(\sum_{n \geqslant 1} p_n(\mathfrak{M})\, A_n/n)} \tag{13}$$

$$\varepsilon_{(-)\,\mathfrak{M}} = \sum_{\vec{\alpha} \in \mathfrak{P}} (-1)^{l(\vec{\alpha})}\, p_{\vec{\alpha}}(\mathfrak{M}) \frac{A^{\vec{\alpha}}}{z_{\alpha}} = e^{(-\sum_{n \geqslant 1} p_n(\mathfrak{M})\, A_n/n)}. \tag{14}$$

Remark. The indicators of $\varepsilon_{\mathfrak{M}}$ and $\varepsilon_{(-)\,\mathfrak{M}}$ are, respectively,

$$\mathscr{I}_{\varepsilon_{\mathfrak{M}}}(X) = \sum_{n \geqslant 0} h_n(X\mathfrak{M}) =: h(X\mathfrak{M}).$$

$$\mathscr{I}_{\varepsilon_{(-)\,\mathfrak{M}}}(X) = \sum_{n \geqslant 0} (-1)^n\, e_n(X\mathfrak{M})$$

where $h_n(X)$ and $e_n(X)$ are the nth complete homogeneous and elementary symmetric functions, respectively.

In particular, for $\mathfrak{M} = \{1\}$ we have

$$\mathscr{I}_{\varepsilon_{\{1\}}}(X) = \sum_{n \geqslant 0} h_n(X) = h(X). \tag{15}$$

Corollary 3.2. *If $R(X)$ is a symmetric function, then*

$$R(X) = \sum_{\vec{\alpha}} \langle A^{\vec{\alpha}} \mid R(X) \rangle \frac{p_{\vec{\alpha}}(X)}{z_{\vec{\alpha}}}. \tag{16}$$

Proof. Let us call $S(X)$ the symmetric function defined by the right hand side of Eq. (16). By formula (13), for every multiset $\mathfrak{M}$, $R(\mathfrak{M}) = S(\mathfrak{M})$. The result follows from Proposition 3.4. ▮

Example 3.2 (The Hammond Functionals). For a positive integer n, we define the Hammond functionals

$$\langle \varepsilon_1^{(n)} \mid R(X) \rangle = R^{(n)}(1),$$

where $R^{(n)}(X)$ denotes the component of homogeneous degree n of R. We have the expansion

$$\varepsilon_1^{(n)} = \sum_{\vec{\alpha} \vdash n} \frac{A^{\vec{\alpha}}}{z_\alpha} = h_n(A).$$

4. CLASSIFICATION OF THE ALGEBRA, COALGEBRA, AND HOPF ALGEBRA MAPS

Let V be an arbitrary linear space, and V^* be its linear dual. Using a procedure analogous to that used to define a topology on $\Lambda(X)^*$, we define a topology on V^*. We have the following proposition.

PROPOSITION 4.1. *Let $U: V \to \Lambda(X)$ be a linear operator. Then the adjoint T of U, $T: \Lambda(X)^* \to V^*$ is a continuous linear operator. Conversely, if $T: \Lambda(X)^* \to V^*$ is a continuous linear operator, then there exists a linear operator $U: V \to \Lambda(X)$ such that $T = U^*$.*

Proof. The proof of the first part of the proposition is straightforward. Assume that T is continuous. Choose a basis $\{v_i\}_i$ of V. Since $L_{\vec{\alpha}} = T(A^{\vec{\alpha}}) \to 0$, the symmetric function

$$q_i(X) = \sum_{\vec{\beta} \in \mathfrak{P}} \frac{\langle TA^{\beta} \mid v_i \rangle}{z_{\vec{\beta}}} \, p_{\vec{\beta}}(X)$$

is well defined.

Clearly $\langle TA^{\vec{\beta}} \mid v_i \rangle = \langle A^{\vec{\beta}} \mid q_i(X) \rangle$, for every partition $\vec{\alpha}$ and every index i.

Define the operator

$$U: V \to \Lambda(X), \qquad Uv_i = q_i(X).$$

We have

$$\langle U^* A^{\vec{\beta}} \mid v_i \rangle = \langle A^{\vec{\beta}} \mid q_i(X) \rangle = \langle TA^{\vec{\beta}} \mid v_i \rangle, \qquad \text{for every } \alpha \text{ and } i.$$

Since T is continuous, by Proposition 3.8 we have $T = U^*$. ∎

As an easy consequence of the previous proposition we obtain the following

COROLLARY 4.1. *A linear operator $T: \Lambda(X)^* \to V^*$ is continuous if and only if $L_{\vec{\alpha}} = TA^{\alpha}$ converges to zero.*

DEFINITION 4.1. A family of functionals $\{L_{\vec{\alpha}}\}_{\vec{\alpha} \in \mathfrak{P}}$ in $\Lambda(X)^*$ is called a pseudobasis if

1. $L_{\vec{\alpha}} \to 0$ when $|\vec{\alpha}| \to \infty$
2. Every functional M has an unique expansion of the form

$$M = \sum_{\vec{\alpha} \in \mathfrak{P}} a_{\vec{\alpha}} L_{\vec{\alpha}}.$$

For every basis $\{R_{\vec{\alpha}}\}_{\vec{\alpha} \in \mathfrak{P}}$ of $\Lambda(X)$, the family of functionals $\{L_{\vec{\alpha}}\}_{\vec{\alpha} \in \mathfrak{P}}$ defined by

$$\langle L_{\vec{\alpha}} \mid R_{\vec{\beta}}(X) \rangle = \delta_{\vec{\alpha}, \vec{\beta}}$$

is easily seen to be a pseudobasis. The converse is also true.

PROPOSITION 4.1. *Given a pseudobasis $\{L_{\vec{\alpha}}\}_{\vec{\alpha} \in \mathfrak{P}}$, there exists a sequence of symmetric functions $\{R_{\vec{\alpha}}(X)\}_{\vec{\alpha} \in \mathfrak{P}}$ satisfying $\langle L_{\vec{\alpha}} \mid R_{\vec{\beta}} \rangle = \delta_{\vec{\alpha}, \vec{\beta}}$. Moreover, we have the expansions*

$$M = \sum_{\vec{\alpha}} \langle M \mid R_{\vec{\alpha}}(X) \rangle \, L_{\vec{\alpha}}, \qquad M \in \Lambda(X)^* \tag{17}$$

$$R(X) = \sum_{\vec{\alpha}} \langle L_{\vec{\alpha}} \mid R(X) \rangle \, R_{\vec{\alpha}}(X), \qquad R(X) \in \Lambda(X), \tag{18}$$

and hence, $\{R_{\vec{\alpha}}(X)\}_{\vec{\alpha} \in \mathfrak{P}}$ is a basis.

Proof. Consider the map $T: \Lambda(X)^* \to \Lambda(X)^*$, $TM = \sum_{\vec{\alpha}} \langle M \mid p_{\vec{\alpha}}(X) \rangle$ $L_{\vec{\alpha}}/z_{\vec{\alpha}}$. Since $TA^{\vec{\alpha}} = L_{\vec{\alpha}} \to 0$, by Corollary 4.1 T is continuous. Since $L^{\vec{\alpha}}$ is a pseudobasis, T is an isomorphism. Define $R_{\vec{\alpha}}(X) = U^{-1}(p_{\vec{\alpha}}(X)/z_{\vec{\alpha}})$, where U is the map satisfying $U^* = T$. It is straightforward to verify that $\{R_{\vec{\alpha}}(X)\}_{\vec{\alpha} \in \mathfrak{M}}$ is the required pseudobasis. The proof of the expansions is standard. ∎

By Proposition 4.1, the adjoint μ^* of the multiplication $\mu: \Lambda(X, Y) \to \Lambda(X)$, the adjoint θ^* of the antipode $\theta: \Lambda(X) \to \Lambda(X)$, the adjoint e^* of the identity $e: \Lambda \to \mathbb{C}$, and the adjoint $\odot$ of $\diamond: \Lambda(X) \to \Lambda(X, Y)$, $\diamond(R(X)) = R(XY)$, are continuous maps:

$$\mu^*: \Lambda(X)^* \to \Lambda(X, Y)^*$$

$$\theta^*: \Lambda(X)^* \to \Lambda(X)^*$$

$$e^*: \mathbb{C} \to \Lambda(X)^*$$

$$\diamond: \Lambda(X)^* \to \Lambda(X, Y)^*.$$

μ^*, θ^* and e^* are explicitly defined by

$$\mu^* A^{\vec{\alpha}} = \sum_{\vec{\beta}+\vec{\gamma}=\vec{\alpha}} \binom{\vec{\alpha}}{\vec{\beta}, \vec{\gamma}} (A^{\vec{\beta}})^X (A^{\vec{\gamma}})^Y$$
$$\theta^* A^{\vec{\alpha}} = (-1)^{l(\vec{\alpha})} A^{\vec{\alpha}}$$
$$e^* A^{\vec{\alpha}} = \langle A\vec{\alpha} \,|\, 1 \rangle.$$

Consider the discrete topology on $\mathbb{C}$. The topological bidual $\Lambda(X)^{**}$ is a topological algebra with the product

$$\langle \mathscr{L}_1 \cdot \mathscr{L}_2 \,|\, A^{\vec{\alpha}} \rangle = \sum_{\vec{\beta}+\vec{\gamma}=\vec{\alpha}} \binom{\vec{\alpha}}{\vec{\beta}, \vec{\gamma}} \langle \mathscr{L}_1 \,|\, A^{\vec{\beta}} \rangle \langle \mathscr{L}_2 \,|\, A^{\vec{\gamma}} \rangle$$

and identity e^*. By Proposition 4.1 the map $\Lambda(X)^{**} \to \Lambda(X)$, $\mathscr{L} \mapsto \sum_{\vec{\alpha}} \langle \mathscr{L} \,|\, A^{\vec{\alpha}} \rangle\, p_{\vec{\alpha}}(X)/z_{\vec{\alpha}}$ defines an algebra isomorphism.

Definition 4.2. Let T be a continuous operator $T\colon \Lambda(X)^* \to \Lambda(X)^*$. T is called a μ^*-map if it commutes with e^* and

$$\mu^* T = T_X T_Y \mu^*,$$

where $T_X T_Y$ is the continuous operator of $\Lambda(X, Y)^*$ uniquely defined by

$$T_X T_Y((A^{\vec{\alpha}})^X (A^{\vec{\beta}})^Y) = T(A^{\vec{\alpha}})\, T(A^{\vec{\beta}}).$$

Proposition 4.2. *Let $U\colon \Lambda(X) \to \Lambda(X)$ be a linear operator, and $T = U^*$ be its adjoint. Then we have*

1. *U is a coalgebra map if and only if T is a continuous algebra map.*
2. *U is an algebra map if and only if T is a μ^*-map.*
3. *U is a Hopf algebra map if and only if T is simultaneously a μ^*-map and an algebra map.*

Definition 4.3. A sequence of symmetric functions $\{q_{\vec{\alpha}}\}_{\vec{\alpha} \in \mathfrak{P}}$ is of binomial type if

1. $q_{\vec{0}}(X) = 1$ and $q_{\vec{\alpha}}(0) = 0$, $\vec{\alpha} \neq \vec{0}$.
2. For every $\vec{\alpha} \in \mathfrak{P}$,

$$E^Y q_{\vec{\alpha}}(X) = q_{\vec{\alpha}}(X + Y) = \sum_{\vec{\beta}+\vec{\gamma}=\vec{\alpha}} \binom{\vec{\alpha}}{\vec{\beta}, \vec{\gamma}} q_{\vec{\beta}}(X)\, q_{\vec{\gamma}}(Y).$$

A basic sequence of binomial type is called a binomial basis.

Analogously, a sequence of functionals $\{L_{\vec{\alpha}}\}_{\vec{\alpha}\in\mathfrak{P}}$ is of binomial type if

1. $L_{\vec{\alpha}} \to 0$ and $\langle L_{\vec{0}} \mid 1\rangle = 1$.
2. For every $\vec{\alpha}\in\mathfrak{P}$,

$$\mu^* L_{\vec{\alpha}} = \sum_{\vec{\beta}+\vec{\gamma}=\vec{\alpha}} \binom{\vec{\alpha}}{\vec{\beta}, \vec{\gamma}} (L^{\vec{\beta}})^X (L^{\vec{\gamma}})^Y.$$

Definition 4.4. An admissible system of linear functionals is a sequence $\vec{L} = (L_1, L_2, L_3, \ldots)$ of functionals satisfying

1. $\langle L_k \mid 1\rangle = 0$ for $k = 1, 2, \ldots$.
2. $L_k \to 0$, $k \to \infty$.

Proposition 4.3. *A system $\vec{L}$ is admissible if and only if $L^{\vec{\beta}} = \prod_i L_i^{\beta_i} \to 0$.*

Proof. Assume that $L^{\vec{\beta}} \to 0$. Then clearly

$$\lim_{k\to\infty} L_k = 0$$

and

$$\lim_{j\to\infty} L_i^j = 0, \qquad \text{for} \quad i = 1, 2, \ldots.$$

Since $\langle L_i^j \mid 1\rangle = \langle L_i \mid 1\rangle^j$ we obtain that $\langle L_i \mid 1\rangle = 0$.

Assume now that $\vec{L}$ is admissible. By an expansion argument, we have only to prove that for every partition $\vec{\alpha}$, $\langle L_{\vec{\beta}} \mid p_{\vec{\alpha}}(X)\rangle \to 0$. Assuming that $l = l(\vec{\beta}) > l(\vec{\alpha})$, for any sequence of partitions $(\vec{\alpha}^{(1)}, \vec{\alpha}^{(2)}, \ldots, \vec{\alpha}^{(l)})$, adding up to $\vec{\alpha}$, at least one of them is equal to zero. Let $(\tau_1 \geqslant \tau_2 \geqslant \cdots \geqslant \tau_l)$ be the partition $\vec{\beta}$ in standard notation. Since $\langle L_{\tau_i} \mid 1\rangle = 0$ for $i = 1, 2, \ldots, l$, by formula (7) we obtain that $\langle L^{\vec{\beta}} \mid p_{\vec{\alpha}}(X)\rangle = \langle L_{\tau_1} L_{\tau_2} \cdots L_{\tau_l} \mid p_{\vec{\alpha}}(X)\rangle = 0$.

We can assume now that $l(\vec{\beta}) \leqslant l(\vec{\alpha})$. Since $L_k \to 0$ there exists an integer k_0 for which we have $\langle L_k \mid p_{\vec{\gamma}}(X)\rangle = 0$, $k \geqslant k_0$, and $\gamma_i \leqslant \alpha_i$ $(i = 1, \ldots, l(\vec{\alpha}))$.

If $|\vec{\beta}| \geqslant l(\vec{\alpha})\, k_0$, since $l(\vec{\beta}) \leqslant l(\vec{\alpha})$ there exists at least one part of $\vec{\beta}$ which is greater than k_0. Using formula (7) we obtain the result. ∎

Proposition 4.4. *Let $T\colon \Lambda(X)^* \to \Lambda(X)^*$ be a linear map. Then T is a continuous algebra map if and only if the system $\vec{L}$, $L_i = TA_i$, is admissible and $TA^{\vec{\alpha}} = K^{\vec{\alpha}} = \prod_i L_i^{\alpha_i}$ for every $\vec{\alpha}\in\mathfrak{P}$.*

Proof. By Proposition 4.3 if $\vec{L}$ is admissible $L^{\vec{\alpha}} \to 0$, and by Corollary 4.1 T is continuous. Since $TA^{\vec{\alpha}}A^{\vec{\beta}} = TA^{\vec{\alpha}+\vec{\beta}} = L^{\vec{\alpha}+\vec{\beta}} = L^{\vec{\alpha}}L^{\vec{\beta}} = TA^{\vec{\alpha}}TA^{\vec{\beta}}$, by an expansion argument, and because T is continuous, T is an algebra map. Conversely if T is a continuous algebra map, $L^{\vec{\alpha}} \to 0$. By Proposition 4.3, $\vec{L}$ is admissible. ∎

PROPOSITION 4.5. *Let $U: \Lambda(X) \to \Lambda(X)$ be a linear map. U is a coalgebra map if and only if the sequence of symmetric functions $q_{\vec{\alpha}}(X) = Up_{\vec{\alpha}}(X)$ is of binomial type.*

Proof. If $q_{\vec{\alpha}}(X)$ is of binomial type, using an expansion argument is easy to prove that U is a coalgebra map. If U is a coalgebra map we have

$$q_{\vec{\alpha}}(0) = \langle \varepsilon \mid Up_{\vec{\alpha}}(X) \rangle = \langle \varepsilon \mid p_{\vec{\alpha}}(X) \rangle = p_{\vec{\alpha}}(0).$$

Since $p_{\vec{\alpha}}(X)$ satisfies the binomial identity, obviously $q_{\vec{\alpha}}(X)$ also satisfies it. It remains to show that $q_{\vec{0}}(X) = 1$. By Proposition 4.2 the adjoint U^* of U is a continuous algebra map, and then the system $\vec{L} = (U^*A_1, U^*A_2, ...)$ is admissible. For any partition $\vec{\beta}$ we have

$$\langle A^{\vec{\beta}} \mid q_{\vec{0}}(X) \rangle = \langle A^{\vec{\beta}} \mid U1 \rangle = \langle U^*A^{\vec{\beta}} \mid 1 \rangle = \langle L^{\vec{\beta}} \mid 1 \rangle.$$

Since $\vec{L}$ is admissible $\langle L^{\vec{\beta}} \mid 1 \rangle = \delta(\vec{\beta}, \vec{0})$. By the Corollary 3.2 we get the result. ∎

COROLLARY 4.2. *If $\vec{L}$ is an admissible system, then the sequence of symmetric functions*

$$q_{\vec{\alpha}}(X) = \sum_{\vec{\beta}} \langle L^{\vec{\beta}} \mid p_{\vec{\alpha}}(X) \rangle \frac{p_{\vec{\beta}}(X)}{z_{\vec{\beta}}} \tag{19}$$

is of binomial type.

In conclusion, for each coalgebra map U there exists a unique sequence $q_{\vec{\alpha}}(X) = Up_{\vec{\alpha}}(X)$ of symmetric functions of binomial type, and a unique admissible system satisfying $\vec{L} = U^*\vec{A}$. The binomial sequence $q_{\vec{\alpha}}(X)$ and the map $U = U_{\vec{L}}$ are respectively called the *conjugate sequence* and the *conjugate coalgebra map* of $\vec{L}$. For any symmetric function $R(X)$ we have $U_{\vec{L}}R(X) = R(\mathbf{q}(X))$, where $R(\mathbf{q}(x))$ is the symmetric function obtained by substituting $p_{\vec{\alpha}}(X)$ by $q_{\vec{\alpha}}(X)$ in the expansion of $R(X)$ in terms of the $p_{\vec{\alpha}}$'s.

The following propositions are the dual versions of Propositions 4.4 and 4.5, respectively.

PROPOSITION 4.6. *A linear map $U: \Lambda(X) \to \Lambda(X)$ is an algebra map if and only if the sequence of symmetric functions $q_{\vec{\alpha}}(X) = Up_{\vec{\alpha}}(X)$ is multiplicative, i.e.,*

$$q_{\vec{\alpha}}(X) = \prod_n q_{\vec{e}_n}(X)^{\alpha_n}, \qquad \alpha \in \mathfrak{P}.$$

PROPOSITION 4.7. *Let $T: \Lambda(X)^* \to \Lambda(X)^*$ be a linear map. Then T is a μ^*-map if and only if the sequence of functionals $L_{\vec{\alpha}} = TA^{\vec{\alpha}}$ is of binomial type.*

We have the dual form of Corollary 4.2.

COROLLARY 4.3. *If $q_{\vec{\alpha}}(X)$ is a multiplicative sequence of symmetric functions, then the sequence of functionals defined by*

$$L_{\vec{\beta}} = \sum_{\vec{\alpha}} \langle A^{\vec{\beta}} | q_{\vec{\alpha}}(X) \rangle \frac{A^{\vec{\alpha}}}{z_{\vec{\alpha}}}$$

is of binomial type.

Using the fact that A_n satisfies the identity $\mu^* A_n = (A_n)^X + (A_n)^Y$, the two previous propositions combine into

PROPOSITION 4.8. *Let U and T be an adjoint pair. Then T is simultaneously an algebra and a μ^*-map (equivalently, U is a Hopf algebra map), if and only if $L_{\vec{\alpha}} = TA^{\vec{\alpha}}$ is of the form $L_{\vec{\alpha}} = \prod_i L_i^{\alpha_i}$, $\vec{L}$ being an admissible system, and each L_k satisfies the identity*

$$\mu^* L_k = (L_k)^X + (L_k)^Y. \tag{20}$$

U is a Hopf-algebra map (equivalently, T is simultaneously an algebra and a μ^*-map) if and only if the polynomials $q_{\vec{\alpha}}(X) = Up_{\vec{\alpha}}(X)$ are multiplicative and every $q_{\vec{e}_n}(X)$ is a primitive element of $\Lambda(X)$, i.e.,

$$q_{\vec{e}_n}(X + Y) = q_{\vec{e}_n}(X) + q_{\vec{e}_n}(Y).$$

4.1. *Frobenius and Verschiebung Operators*

For a positive integer n, let F_n be the continuous algebra map on $\Lambda(X)^*$ defined by $F_n A_k = A_{nk}$ ($k = 1, 2, ...$). Since each $L_k = A_{nk}$ satisfies Eq. (20), F_n is a μ^*-map. We have

$$F_n A^{\vec{\alpha}} = A^{\vec{\alpha}\{n\}},$$

where

$$\vec{\alpha}^{\{n\}} = (\overbrace{0, 0, ..., \alpha_1}^{n}, \overbrace{0, ..., \alpha_2}^{n}, \overbrace{0, ..., \alpha_3}^{n}, ...).$$

The Verschiebung operator $V_n = U_{(A_n, A_{2n}, ...)}$ is the linear operator on $\Lambda(X)$, satisfying $V_n^* = F_n$. V_n is a Hopf algebra map.

The Frobenius operator on $\Lambda(X)$ is defined by

$$F_n p_{\vec{\alpha}}(X) = p_{\vec{\alpha}^{\{n\}}}(X) = \prod_k p_{nk}(X)^{\alpha_k}.$$

The Verschiebung operator V_n, acting over the umbral algebra, is defined as the adjoint of the Frobenius operator over $\Lambda(X)$.

Since $F_n p_k(X) = p_{nk}(X)$ is a primitive element of $\Lambda(X)$, F_n is a Hopf algebra map. F_n is the Adam's operator in the usual λ-ring structure of $\Lambda(X)$. We prefer the previous notation because of its significance in the theory of Witt vectors (see [39]).

PROPOSITION 4.9. *The action of the operator V_n over the symmetric power functions is given by the formula*

$$V_n p_{\vec{\alpha}}(X) = \begin{cases} n^{l(\vec{\alpha})} p_{\vec{\tau}}(X) & \text{if } \vec{\alpha} \text{ is of the form } \vec{\alpha} = \vec{\tau}^{\{n\}} \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

Proof. It is easy to check that $z_{\vec{\tau}^{\{n\}}} = n^{l(\vec{\tau})} z_{\vec{\tau}}$. Then we have

$$\begin{aligned} \langle F_n A^{\vec{\tau}} \mid p_{\vec{\alpha}}(X) \rangle &= \langle A^{\vec{\tau}^{\{n\}}} \mid p_{\vec{\alpha}}(X) \rangle = n^{l(\vec{\tau})} z_{\vec{\tau}}\, \delta(\vec{\tau}^{\{n\}}, \vec{\alpha}) \\ &= n^{l(\vec{\alpha})} z_{\vec{\tau}}\, \delta(\vec{\tau}^{\{n\}}, \vec{\alpha}) = \langle A^{\vec{\tau}} \mid V_n p_{\vec{\alpha}}(X) \rangle. \quad \blacksquare \end{aligned}$$

DEFINITION 4.5. A coalgebra operator $U_{\vec{L}}$ that commutes with the Verschiebung operator V_n, for every n, is called a *plethystic* coalgebra operator. The conjugate sequence of $\vec{L}$, $q_{\vec{\alpha}}(X) = U_{\vec{L}} p_{\vec{\alpha}}(X)$, is called a sequence of *plethystic type*.

We easily prove the proposition:

PROPOSITION 4.10. *Let $q_{\vec{\alpha}}(X)$ be the conjugate sequence of the admissible system $\vec{L}$. The following statements are equivalent.*

1. $\{q_{\vec{\alpha}}(X)\}_{\vec{\alpha} \in \mathfrak{P}}$ *is of plethystic type.*
2. $V_n q_{\vec{\alpha}}(X) = \begin{cases} n^{l(\vec{\alpha})} q_{\vec{\alpha}}(X) & \text{if } \vec{\alpha} \text{ is of the form } \vec{\alpha} = \tau^{\{n\}} \\ 0 & \text{otherwise.} \end{cases}$
3. $L_n = F_n L_1$, $n = 1, 2, 3, ...$

PROPOSITION 4.11. *We have the identities*

1. $F_n \varepsilon_a = \varepsilon_{r\Upsilon_n}$
2. $V_n \varepsilon_a = \varepsilon_{a^n}$
3. $F_n \varepsilon_a^{(m)} = \varepsilon_{r\Upsilon}^{(mn)}$
4. $F_n V_n \varepsilon_a = \varepsilon_{a\Upsilon_n}$
5. $V_n F_n \varepsilon_a = \varepsilon_{\underbrace{\{a, ..., a\}}_{n}}$
6. $F_n V_n \varepsilon_a^{(m)} = \varepsilon_{a\Upsilon_n}^{(m)}$
7. $F_n \varepsilon_1 = \varepsilon_{\Upsilon_n}$
8. $F_n \varepsilon_1^{(m)} = \varepsilon_{\Upsilon_n}^{(mn)}$

where $r\Upsilon_n$ is the set of the nth complex roots of a, r is any nth root of a, and $\Upsilon_n = \{1, \iota, \iota^2, ..., \iota^{n-1}\}$, ι being any primitive nth root of the unity.

Proof. Let us prove the first identity. Since for every symmetric function $R(X)$,

$$\langle F_n \varepsilon_a \mid R(X) \rangle = \langle \varepsilon_a \mid V_n R(X) \rangle,$$

and because F_n and V_n are both algebra maps, it is enough to prove that for every m

$$p_m(r\Upsilon_n) = V_n p_m(a). \tag{22}$$

The left-hand side of Eq. (22) is equal to

$$p_m(r, r\iota, ..., r\iota^{n-1}) = r^m \sum_{k=0}^{n-1} (\iota^k)^m = \begin{cases} a^{m/n} n & \text{if} \quad n \mid m \\ r^m \dfrac{1-(\iota^m)^n}{1-\iota^m} = 0 & \text{otherwise} \end{cases}$$

By Proposition 4.9 we easily check that this is equal to $V_n p_m(a)$.

The proof of the rest of the identities is similar. ∎

COROLLARY 4.4. *For any multiset $\mathfrak{M}$,*

1. $F_n \varepsilon_{\mathfrak{M}} = \varepsilon_{\mathfrak{M}^{1/n}}$
2. $V_n \varepsilon_{\mathfrak{M}} = \varepsilon_{\mathfrak{M}^n}$
3. $F_n \varepsilon_{\mathfrak{M}}^{(m)} = \varepsilon_{\mathfrak{M}^{1/n}}^{(mn)}$
4. $V_n F_n \varepsilon_{\mathfrak{M}} = \varepsilon_{\Upsilon_n \mathfrak{M}}$
5. $V_n F_n \varepsilon_{\mathfrak{M}}^{(m)} = \varepsilon_{\Upsilon_n \mathfrak{M}}^{(m)}$,

where $\mathfrak{M}^{1/n}$ ($\mathfrak{M}^n$) is the multiset whose elements are the nth complex roots (powers) of each of the elements of $\mathfrak{M}$ (multiplicities counted).

Proof. The first, second, and fourth identities follow directly from the fact that $\varepsilon_{\mathfrak{M}} = \prod_{a \in \mathfrak{M}} \varepsilon_a$.

It is easy to see that if $\mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2$, then

$$\varepsilon_{\mathfrak{M}}^{(m)} = \sum_{k=0}^{m} \varepsilon_{\mathfrak{M}_1}^{(m-k)} \varepsilon_{\mathfrak{M}_2}^{(k)}.$$

Since F_n and V_n are algebra maps, by induction on the cardinal of $\mathfrak{M}$ the third and fifth identities follow. ∎

COROLLARY 4.5. *For any symmetric function $R(X)$ we have*

$$F_n V_n R(X) = R(X\Upsilon_n).$$

Proof. For any multiset $\mathfrak{M}$,

$$R(\Upsilon_n \mathfrak{M}) = \langle F_n V_n \varepsilon_{\mathfrak{M}} \mid R(X) \rangle = \langle \varepsilon_{\mathfrak{M}} \mid F_n V_n R(X) \rangle = (F_n V_n R)(\mathfrak{M}).$$

By Proposition 3.4 the result follows. ∎

PROPOSITION 4.12. *The action of the Verschiebung operator over the homogeneous and elementary symmetric functions is given by the formulas*

$$F_n h_m(X) = h_{m/n}(X) \tag{23}$$

$$V_n e_m(X) = \begin{cases} e_{m/n}(X) & \textit{if } m/n \cong m \ (\textit{module } 2) \\ -e_{m/n}(X) & \textit{otherwise}, \end{cases} \tag{24}$$

where $h_{m/n}(X)$ and $e_{m/n}(X)$ are defined to be zero if n does not divide m.

Proof. Using the expansion of $h_m(X)$ in terms of the $p_{\vec{\alpha}}$'s, if n divides m we have

$$V_n h_m(X) = \sum_{\vec{\alpha} \vdash m} V_n \frac{p_{\vec{\alpha}}(X)}{z_{\vec{\alpha}}} = \sum_{\vec{\tau} \vdash m/n} n^{l(\vec{\tau})} \frac{p_{\vec{\tau}}(X)}{z_{\vec{\tau}\{n\}}} = \sum_{\vec{\tau} \vdash m/n} \frac{p_{\vec{\tau}}(X)}{z_{\vec{\tau}}} = h_{m/n}(X).$$

Clearly, if n does not divide m, $V_n h_m(X) = 0$. In an analogous way we prove identity (24). ∎

In [28] an algorithm to express $V_n S_{\lambda/\mu}$ as a product of skew Schur functions is given. See also [7, 47].

5. SUBSTITUTION OF ADMISSIBLE SYSTEMS, COMPOSITION OF COALGEBRA MAPS, AND PLETHYSM

DEFINITION 5.1 (Substitution of Admissible Systems and Plethysm). Let L be a functional and $\vec{M}$ an admissible system. We define the substitution $L(\vec{M})$ as the functional obtained by substituting $A^{\vec{\alpha}}$ by $M^{\vec{\alpha}}$ in the expansion of L, i.e.,

$$L(\vec{M}) = \sum_{\vec{\alpha} \in \mathfrak{P}} \langle L \mid p_{\vec{\alpha}}(X) \rangle \frac{M^{\vec{\alpha}}}{z_{\vec{\alpha}}}.$$

Let

$$f(X) = \sum_{\vec{\alpha} \in \mathfrak{P}} a_{\vec{\alpha}} \frac{p_{\vec{\alpha}}(X)}{z_{\vec{\alpha}}}$$

by the indicator of L and $G_i(X)$ be the indicator of M_i $(i = 1, 2, ...)$. Observe that the indicator of $L(\vec{M})$ is the symmetric series $F(\vec{G})$ defined by

$$F(\vec{G})(X) = \sum_{\vec{\alpha} \in \mathfrak{P}} a_{\vec{\alpha}} \frac{G^{\vec{\alpha}}(X)}{z_{\vec{\alpha}}},$$

where $G^{\vec{\alpha}} = \prod_i G_i(X)^{\alpha_i}$. When $\vec{M}$ is of the form

$$\vec{M} = (M, F_2 M, F_3 M, ...),$$

M being any functional satisfying $\langle M \mid 1 \rangle = 0$, we denote the substitution $L(\vec{M})$ by $L[M]$. In this case the indicator of $L[M]$ is the classical Littlewood plethysm $F[G](X)$ of the symmetric series $F(X)$ with $G(X) = G_1(X)$.

Let L be a multivariated functional $L \in \Lambda(X)^*$ with expansion

$$L = \sum_{\vec{\alpha}^{(1)}, ..., \vec{\alpha}^{(k)}} a_{\vec{\alpha}^{(1)}, \vec{\alpha}^{(2)}, ..., \vec{\alpha}^{(k)}} \prod_{i=1}^{k} \frac{(A^{\vec{\alpha}^{(i)}})^{X_i}}{z_{\vec{\alpha}^{(i)}}}. \tag{25}$$

Definition (5.1) generalizes in an obvious way to allow the substitution $L(\vec{M}^{(1)}, \vec{M}^{(2)}, ..., \vec{M}^{(k)})$ of a sequence of admissible systems $(\vec{M}^{(1)}, ..., \vec{M}^{(k)})$ into the multivariate functional L.

THEOREM 5.1. *Let $\vec{L}$ be an admissible system. Then*

$$\exp\left\{\sum_{n \geq 1} \frac{(L_n)^X (A_n)^Y}{n}\right\} = \sum_{\vec{\alpha} \in \mathfrak{P}} \frac{(A^{\vec{\alpha}})^X (q_{\vec{\alpha}}(A))^Y}{z_{\vec{\alpha}}}. \tag{26}$$

Proof. By formula (15) the left-hand side of Eq. (26) is the substitution of the admissible system $(\vec{L}^X, \vec{A}^Y)$ into the bivariate functional $\Diamond\varepsilon_1(\vec{A}^X, \vec{A}^Y)$. For any pair of partitions $\vec{\beta}$ and $\vec{\gamma}$,

$$\begin{aligned}\langle \Diamond\varepsilon_1(\vec{L}^X, \vec{A}^Y) \,|\, p_{\vec{\beta}}(X)\, p_{\vec{\gamma}}(Y)\rangle &= \langle \Diamond\varepsilon_1 \,|\, U_{\vec{L}}\, p_{\vec{\beta}}(X)\, p_{\vec{\gamma}}(Y)\rangle \\ &= \langle \Diamond\varepsilon_1 \,|\, q_{\vec{\beta}}(X)\, p_{\vec{\gamma}}(Y)\rangle \\ &= \langle \varepsilon_1 \,|\, q_{\vec{\beta}}(X) \odot p_{\vec{\gamma}}(X)\rangle = \langle q_{\vec{\beta}}(A) \,|\, p_{\vec{\gamma}}(X)\rangle \\ &= \left\langle \sum_{\vec{\alpha}\in\mathfrak{P}} \frac{(A^{\vec{\alpha}})^X\, (q_{\vec{\alpha}}(A))^Y}{z_{\vec{\alpha}}} \,\middle|\, p_{\vec{\beta}}(X)\, p_{\vec{\gamma}}(Y)\right\rangle. \quad \blacksquare\end{aligned}$$

Denote by $L_n(X)$ the indicator of the functional L_n. Taking indicators in both sides of Eq. (26) we obtain

COROLLARY 5.1 (The Exponential Formula). *Let $\vec{L}$ and $q_{\vec{\alpha}}(X)$ as above, then*

$$\exp\left\{\sum_{n\geq 1} \frac{L_n(X)\, p_n(Y)}{n}\right\} = \sum_{\vec{\alpha}\in\mathfrak{P}} \frac{p_{\vec{\alpha}}(X)\, q_{\vec{\alpha}}(Y)}{z_{\vec{\alpha}}}. \tag{27}$$

COROLLARY 5.2. *If $\vec{L}$ and $q_{\vec{\alpha}}(X)$ are as above, we have the identity*

$$\sum_{\vec{\alpha}\in\mathfrak{P}} S_\lambda(\vec{L})(X)\, S_\lambda(Y) = \sum_{\vec{\alpha}\in\mathfrak{P}} S_\lambda(X)\, S_\lambda(\mathbf{q})(Y). \tag{28}$$

Proof. From the Cauchy identity we have

$$\sum_{\lambda\in\mathfrak{P}} S_\lambda(X)\, S_\lambda(Y) = \exp\left\{\sum_{n\geq 1} \frac{p_n(X)\, p_n(Y)}{n}\right\} = \sum_{\vec{\alpha}\in\mathfrak{P}} \frac{p_{\vec{\alpha}}(X)\, p_{\vec{\alpha}}(Y)}{z_{\vec{\alpha}}}. \tag{29}$$

Then, the left-hand side of Eq. (27) is equal to the left-hand side of Eq. (28), and the right-hand side of Eq. (27) is equal to the right-hand side of Eq. (28). $\blacksquare$

Applying identity (28) with $\vec{L} = (A_n, A_{2n}, A_{3n}, \ldots)$, we obtain the equation

$$\sum_{\lambda\in\mathfrak{P}} S_\lambda[p_n](X)\, S_\lambda(Y) = \sum_{\lambda\in\mathfrak{P}} S_\lambda(X)\, V_n S_\lambda(Y). \tag{30}$$

PROPOSITION 5.1. *Let $\vec{L}$ and $\vec{M}$ be two admissible systems. Let $U_{\vec{L}}$ and $U_{\vec{M}}$ be the respective conjugate coalgebra maps. Then the system $\vec{L}(\vec{M}) = (L_1(\vec{M}), L_2(\vec{M}), ...)$ is admissible and $U_{\vec{L}} U_{\vec{M}} = U_{\vec{L}(\vec{M})}$.*

Proof. Let $T_1 = (U_{\vec{L}})^*$ and $T_2 = (U_{\vec{M}})^*$. We have

$$(U_{\vec{L}} U_{\vec{M}})^* = (U_{\vec{M}})^* \, (U_{\vec{L}})^* = T_2 T_1.$$

Since $T_2 T_1$ is continuous, and $T_2 T_1 A_n = T_2(L_n) = L_n(\vec{M})$, by Proposition 4.4 the system $\vec{L}(\vec{M})$ is admissible and clearly $U_{\vec{L}} U_{\vec{M}} = U_{\vec{L}(\vec{M})}$. ∎

COROLLARY 5.3. *Let $\{r_{\vec{\alpha}}(X)\}$ and $\{q_{\vec{\alpha}}\}$ be the conjugate sequences of $\vec{L}$ and $\vec{M}$ respectively. Then the sequence $\{q_{\vec{\alpha}}(\mathbf{r})(X)\}$ is of binomial type and is the conjugate sequence of $\vec{L}(\vec{M})$.*

Proof.

$$U_{\vec{L}(\vec{M})} p_{\vec{\alpha}}(X) = U_{\vec{L}} q_{\vec{\alpha}}(X) = q_{\vec{\alpha}}(\mathbf{r})(X). \quad ∎$$

Remark. The admissible systems with the operation of substitution form a semigroup with identity $\vec{A} = (A_1, A_2, ...)$. Then, from Proposition 5.1 the correspondence $\vec{L} \mapsto U_{\vec{L}}$ establishes an isomorphism between the semigroup of admissible systems and the semigroup of coalgebra maps of $\Lambda(X)$.

5.1. *Umbral Operators and Delta-Systems*

DEFINITION 5.2. An *umbral operator* is an invertible coalgebra operator on $\Lambda(X)$. An admissible system $\vec{L}$ is called a delta-system if it is invertible with respect to the operation of substitution.

THEOREM 5.2. *The coalgebra map $U_{\vec{L}}$ is an umbral map if and only if $\vec{L}$ is a delta system. Furthermore, the inverse of $U_{\vec{L}}$ is also an umbral map with*

$$(U_{\vec{L}})^{-1} = U_{\vec{L}^{\langle -1 \rangle}},$$

where $\vec{L}^{\langle -1 \rangle}$ is the substitutional inverse of $\vec{L}$.

Proof. Assume that $\vec{L}$ is a delta system. By Proposition 5.1 it is straightforward to verify that $U_{\vec{L}^{\langle -1 \rangle}} = (U_{\vec{L}})^{-1}$.

Assume now that $U_{\vec{L}}$ is an umbral map. Let $q_{\vec{\alpha}}(X)$ be the conjugate binomial sequence of $\vec{L}$. Since $\{q_{\vec{\alpha}}(X)\}_{\vec{\alpha} \in \mathfrak{M}}$ is a binomial basis we can define an admissible system $\vec{M} = (M_1, M_2, ...)$ by the relations

$$\langle M_n \,|\, q_{\vec{\alpha}}(X) \rangle = n\delta(\vec{\alpha}, \vec{e}_n), \qquad \text{for all } n. \tag{31}$$

Then we have

$$\langle M_n(\vec{L}) \mid p_{\vec{\alpha}}(X)\rangle = n\delta(\vec{\alpha}, \vec{e}_n), \qquad \text{for all } n,$$

which is equivalent to saying that $\vec{M}(\vec{L}) = \vec{A}$. By Proposition 5.1 $U_{\vec{M}}$ is the right inverse of $U_{\vec{L}}$. Since $U_{\vec{L}}$ is invertible, $U_{\vec{M}}$ is also the left inverse of $U_{\vec{L}}$ and we have $\vec{L}(\vec{M}) = \vec{A}$. ∎

A binomial basis $\{q_{\vec{\alpha}}\}_{\vec{\alpha} \in \mathfrak{P}}$ satisfying the relations (31) is called the binomial basis *associated* to the delta system $\vec{M}$. The binomial basis $r_{\vec{\alpha}}(X)$ associated to the delta system $\vec{L} = \vec{M}^{\langle -1 \rangle}$ is called the umbral inverse sequence of $q_{\vec{\alpha}}(X)$. It is easy to verify that

$$r_{\vec{\alpha}}(\mathbf{q})(W) = q_{\vec{\alpha}}(\mathbf{r})(X) = p_{\vec{\alpha}}(X).$$

Using a procedure similar to that used to prove Proposition 3.6 we obtain

PROPOSITION 5.2. *The sequence of binomial type $\{q_{\vec{\alpha}}(X)\}_{\vec{\alpha} \in \mathfrak{P}}$ is the associated sequence of the delta system $\vec{M}$ if and only if*

$$\langle \vec{M}^{\vec{\beta}} \mid q_{\vec{\alpha}}(X)\rangle = z_{\vec{\alpha}}\delta(\vec{\alpha}, \vec{\beta}). \tag{32}$$

From the biorthogonality relations (32) we easily obtain the following proposition and corollaries.

PROPOSITION 5.3. *Let N be a linear functional. If $\vec{M}$ is a delta system With associated sequence $q_{\vec{\alpha}}(X)$, we have the expansion*

$$N = \sum_{\vec{\alpha} \in \mathfrak{M}} \langle N \mid q_{\vec{\alpha}}(X)\rangle \frac{\vec{M}^{\vec{\alpha}}}{z_{\vec{\alpha}}}. \tag{33}$$

Remark. Proposition 5.3 is still valid if we assume that N is an arbitrary element of the topological algebra $\operatorname{Hom}(\Lambda(X), \mathfrak{A})$. An expression of the form aL, $a \in \mathfrak{A}$, and $L \in \Lambda(X)$, is interpreted as the element of $\operatorname{Hom}(\Lambda(X), \mathfrak{A})$ that maps a symmetric function $R(X)$ to $\langle L \mid R(X)\rangle\, a$.

COROLLARY 5.4. *We have the expansion*

$$\varepsilon_{\mathfrak{M}} = \sum_{\vec{\alpha} \in \mathfrak{M}} q_{\vec{\alpha}}(\mathfrak{M}) \frac{\vec{M}^{\vec{\alpha}}}{z_{\vec{\alpha}}}. \tag{34}$$

From the previous corollary, and using Proposition 3.4, we obtain the following formula:

COROLLARY 5.5. *For any pair of finite or infinite alphabets, X and Y, we have*

$$\sum_{\vec{\alpha} \in \mathfrak{P}} q_{\vec{\alpha}}(X) \frac{\vec{M}^{\vec{\alpha}}(Y)}{z_{\vec{\alpha}}} = h(XY) = \prod_{\substack{x \in X \\ y \in Y}} \frac{1}{1-xy}. \tag{35}$$

COROLLARY 5.6 (Taylor Expansion). *If $R(X)$ is a symmetric function, then*

$$R(X) = \sum_{\vec{\alpha} \in \mathfrak{P}} \langle \vec{M}^{\vec{\alpha}} \mid R(X) \rangle \frac{q_{\vec{\alpha}}(X)}{z_{\vec{\alpha}}}. \tag{36}$$

5.2. *Examples of Binomial Basis*

EXAMPLE 5.1 (Monomial Symmetric Functions). Consider the monomial symmetric functions $m_{\vec{\alpha}}(X)$. We have the identity (see [20])

$$m_{\vec{\alpha}}(X+Y) = \sum_{\vec{\beta}+\vec{\gamma}=\vec{\alpha}} m_{\vec{\beta}}(X)\, m_{\vec{\gamma}}(Y). \tag{37}$$

Then the families of symmetric functions

$$\tilde{m}_{\vec{\alpha}}(X) = z_{\vec{\alpha}} m_{\vec{\alpha}}(X)$$

$$\hat{m}_{\vec{\alpha}}(X) = \vec{\alpha}!\, m_{\vec{\alpha}}(X)$$

are binomial bases.

Consider the admissible systems

$$\vec{H} = (\varepsilon_1^{(1)}, \varepsilon_1^{(2)}, \varepsilon_1^{(3)}, ...) = (h_1(A), h_2(A), h_3(A), ...) \tag{38}$$

$$\vec{K} = (\varepsilon_1^{(1)}, 2\varepsilon_1^{(2)}, 3\varepsilon_1^{(3)}, ...). \tag{39}$$

We easily obtain

$$\varepsilon_1^{(n)} \tilde{m}_{\vec{\alpha}}(X) = \delta(|\vec{\alpha}|, n)\, \tilde{m}_{\vec{\alpha}}(1) = n\delta(\vec{\alpha}, \vec{e}_n) \tag{40}$$

$$n\varepsilon_1^{(n)} \hat{m}_{\vec{\alpha}}(X) = n\delta(|\vec{\alpha}|, n)\, \hat{m}_{\vec{\alpha}}(1) = n\delta(\vec{\alpha}, \vec{e}_n). \tag{41}$$

Then by Proposition 5.2 $\tilde{m}_{\vec{\alpha}}(X)$ and $\hat{m}_{\vec{\alpha}}(X)$ are the associated sequences of $\vec{H}$ and $\vec{K}$, respectively.

The umbral inverse of $\tilde{m}_{\vec{\alpha}}(X)$ is the binomial family

$$\tilde{n}_{\vec{\alpha}}(X) = \sum_{\vec{\alpha} \in \mathfrak{P}} \varphi_{\vec{\alpha}}^{\lambda} \frac{p_\lambda(X)}{z_\lambda}, \tag{42}$$

the coefficients $\varphi_{\vec{\alpha}}^{\lambda}$ given by

$$\begin{aligned}
\varphi_{\vec{\alpha}}^{\lambda} &= \langle \vec{H}_\lambda \mid p_{\vec{\alpha}}(X) \rangle \\
&= \left\langle \prod_i H_{\lambda_i} \mid p_{\vec{\alpha}}(X) \right\rangle \\
&= \sum_{\vec{\alpha}^{(1)}+\vec{\alpha}^{(2)}+\cdots+\vec{\alpha}^{(k)}=\vec{\alpha}} \binom{\vec{\alpha}}{\vec{\alpha}^{(1)}, \vec{\alpha}^{(2)}, ..., \vec{\alpha}^{(k)}} \prod_i \langle \varepsilon_1^{(\lambda_i)} \mid p_{\vec{\alpha}^{(i)}}(X) \rangle \\
&= \sum_{\substack{|\vec{\alpha}^{(i)}|=\lambda_i \\ i=1,2,...,k}} \binom{\vec{\alpha}}{\vec{\alpha}^{(1)}, \vec{\alpha}^{(2)}, ..., \vec{\alpha}^{(k)}}. \qquad (43)
\end{aligned}$$

Example 5.2 (The Forgotten Symmetric Functions). The classical involution defined by $\omega p_{\vec{\alpha}}(X) = \operatorname{sig}(\vec{\alpha})\, p_{\vec{\alpha}}(X)$, $\operatorname{sig}(\vec{\alpha}) = (-1)^{\sum (i-1)\vec{\alpha}_i}$, is an umbral map. It is easy to see that $\omega^* A_n = (-1)^{n-1} A_n$. Then ω is a Hopf algebra map. Define the delta system $\vec{E} = \omega \vec{H}$. Then

$$E_n = \omega H_n = \sum_{\vec{\alpha} \in \mathfrak{P}} = \operatorname{sig}(\vec{\alpha}) \frac{A^{\vec{\alpha}}}{z_{\vec{\alpha}}} = \vec{e}_n(A),$$

where $e_n(X)$ is the elementary symmetric function. The associated binomial sequence of $\vec{E}$ is

$$\tilde{f}_{\vec{\alpha}}(X) = \omega \tilde{m}_{\vec{\alpha}}(X).$$

We have $\tilde{f}_{\vec{\alpha}}(X) = z_{\vec{\alpha}} f_{\vec{\alpha}}(X)$, $f_{\vec{\alpha}}(X)$ being the so-called "forgotten symmetric functions", studied by Doubilet (see [13]).

Example 5.3 (The Plethystic Exponential Polynomials and Their Umbral Inverse). The *plethystic forward difference system* is the delta system

$$\vec{G} = (\varepsilon_1 - \varepsilon, F_2(\varepsilon_1 - \varepsilon), F_3(\varepsilon_1 - \varepsilon), ...) = (\varepsilon_{\Upsilon_1} - \varepsilon, \varepsilon_{\Upsilon_2} - \varepsilon, \varepsilon_{\Upsilon_3} - \varepsilon, ...). \quad (44)$$

For a partition $\vec{\beta}$, the functional $\vec{G}^{\vec{\beta}}$ has the expansion

$$\vec{G}^{\vec{\beta}} = \prod_i (\varepsilon_{\Upsilon_i} - \varepsilon)^{\beta_i} = \sum_{\vec{\tau}+\vec{\gamma}=\vec{\beta}} \binom{\vec{\beta}}{\vec{\tau}, \vec{\gamma}} (-1)^{l(\vec{\gamma})} \prod_i \varepsilon_{\Upsilon_i^{[\tau_i]}}, \quad (45)$$

where $\Upsilon_i^{[\tau_i]}$ is the multiset

$$\Upsilon_i^{[\tau_i]} = \{\overbrace{1, 1, ..., 1}^{r_i}, \overbrace{\iota, \iota, ..., \iota}^{r_i}, \overbrace{\iota^2, \iota^2, ..., \iota^2}^{r_i}, ..., \overbrace{\iota^{(i-1)}, \iota^{(i-1)}, ..., \iota^{(i-1)}}^{r_i}\}. \quad (46)$$

Defining the multiset $\Upsilon_{\vec{\tau}}$ as the disjoint union

$$\Upsilon_{\vec{\tau}} = \sum_i \Upsilon_i^{[\tau_i]},$$

we obtain

$$\vec{G}^{\vec{\beta}} = \sum_{\vec{\tau}+\vec{\gamma}=\vec{\alpha}} \binom{\vec{\beta}}{\vec{\tau}, \vec{\gamma}} (-1)^{l(\vec{\tau})} \varepsilon_{\Upsilon_{\vec{\tau}}}. \tag{47}$$

In order to find the conjugate sequence of $\vec{G}$, we have to complete $\langle \varepsilon_{\Upsilon_{\vec{\tau}}} | p_{\vec{\alpha}}(X) \rangle$.

$$p_n(\Upsilon_{\vec{\tau}}) = \sum_i p_n(\Upsilon_i^{[\tau_i]}) = \sum_i \tau_i p_n(\Upsilon_i) = \sum_{d|n} d\tau_d,$$

$$\langle \varepsilon_{\Upsilon_{\vec{\tau}}} | p_{\vec{\alpha}}(X) \rangle = \prod_n (p_n(\Upsilon_{\vec{\tau}}))^{\alpha_n} = \prod_n \left(\sum_{d|n} d\tau_d \right)^{\alpha_n}$$

Then the conjugate family of $\vec{G}$ are the plethystic exponential polynomials $\phi_{\vec{\alpha}}(X)$ introduces in [36],

$$\phi_{\vec{\alpha}}(X) = \sum_{\vec{\beta}} z_{\vec{\beta}} S(\vec{\alpha}, \vec{\beta}) \frac{p_{\vec{\beta}}(X)}{z_{\vec{\beta}}}, \tag{48}$$

where $S(\vec{\alpha}, \vec{\beta})$ is the plethystic number of the second kind, given by the formula

$$z_{\vec{\beta}} S(\vec{\alpha}, \vec{\beta}) = \langle \vec{G}^{\vec{\beta}} | p_{\vec{\alpha}}(X) \rangle = \sum_{\vec{\tau}+\vec{\gamma}=\vec{\beta}} \binom{\vec{\beta}}{\vec{\tau}, \vec{\gamma}} (-1)^{l(\vec{\gamma})} \prod_n \left(\sum_{d|n} d\tau_d \right)^{\alpha_n}.$$

From the exponential formula we obtain the following generating function:

$$\sum_{\vec{\alpha} \in \mathfrak{P}} \frac{p_{\vec{\alpha}}(X)\, \phi_{\vec{\alpha}}(Y)}{z_{\vec{\alpha}}} = \exp \left\{ \sum_{n \geqslant 0} \frac{F_n(h(X)-1)\, p_n(Y)}{n} \right\}.$$

The umbral inverse of $\{\phi_{\vec{\alpha}}(X)\}_{\vec{\alpha}}$ is the family $\{\psi_{\vec{\alpha}}(X)\}_{\vec{\alpha}}$ (associated to the system $\vec{G}$)

$$\psi_{\vec{\alpha}}(X) = \sum_{\vec{\beta}} s(\vec{\alpha}, \vec{\beta}) \frac{p_{\vec{\beta}}(X)}{z_{\vec{\beta}}},$$

where $s(\vec{\alpha}, \vec{\beta})$ is the plethystic Stirling number of the first kind (see [36], and [41]).

The family $\{\phi_{\vec{\alpha}}(X)\}_{\vec{\alpha}}$ is the conjugate sequence of the system $\vec{G}^{\langle -1\rangle}$,

$$\vec{G}_n^{\langle -1\rangle} = \log \prod_{k\geqslant 1} (\varepsilon + A_{nk})^{\mu(k)/k},$$

where μ is the classical Möbius function.

By Proposition 5.3 every functional N has the expansion

$$N = \sum_{\vec{\alpha}} \langle N \mid \psi_{\vec{\alpha}}(X)\rangle \frac{\vec{G}^{\alpha}}{z_{\vec{\alpha}}}.$$

By the Taylor formula, every symmetric function R has an expansion of the form

$$R(X) = \sum_{\vec{\alpha}} \left(\sum_{\vec{\beta}+\vec{\gamma}=\vec{\alpha}} \binom{\vec{\alpha}}{\vec{\beta}, \vec{\gamma}} (-1)^{l(\vec{\gamma})} R(\Upsilon_{\vec{\beta}}) \right) \frac{\psi_{\vec{\alpha}}(X)}{z_{\vec{\alpha}}}. \tag{49}$$

Equation (49) can be rewritten as

$$R(X) = h^{-1}(\boldsymbol{\psi})(X) \sum_{\vec{\beta}} R(\Upsilon_{\vec{\beta}}) \frac{\psi_{\vec{\beta}}(X)}{z_{\vec{\beta}}}, \tag{50}$$

where $h^{-1}(X) = \exp(-\sum_n (p_n(X)/n)$ is the multiplicative inverse of $h(X)$.

Applying the umbral operator $U_{\vec{G}}$ to both sides of (50) we get

$$R(\boldsymbol{\phi}) = \exp\left(-\sum_n \frac{p_n(X)}{n} \right) \sum_{\vec{\beta}} R(\Upsilon_{\vec{\beta}}) \frac{p_{\vec{\beta}}(X)}{z_{\vec{\beta}}}.$$

In particular, for $R(X) = p_{\vec{\alpha}}(X)$ we obtain the plethystic version of the classical Dobinski identity

$$\phi_{\vec{\alpha}}(X) = \exp\left(-\sum_n \frac{p_n(X)}{n} \right) \sum_{\vec{\beta}} \prod_n \left(\sum_{d\mid n} d\beta_d \right)^{\alpha_n} \frac{p_{\vec{\beta}}(X)}{z_{\vec{\beta}}}.$$

EXAMPLE 5.4 (A Variation on the Exponential Plethystic Polynomials). For a multiset $\mathfrak{M}$ define the functional $\varepsilon_{\mathfrak{M},(n)}$ by

$$\langle \varepsilon_{\mathfrak{M},(n)} \mid p_{\vec{\beta}\alpha}(X)\rangle = \begin{cases} p_{\vec{\alpha}}(\mathfrak{M}) & \text{if } \vec{\alpha} \text{ is of the form } \vec{\alpha} = k\vec{e}_n \\ 0 & \text{otherwise.} \end{cases} \tag{51}$$

For two multisets $\mathfrak{M}_1$, $\mathfrak{M}_2$, we have the formula $\varepsilon_{\mathfrak{M}_1,(n)} \varepsilon_{\mathfrak{M}_2,(n)} = \varepsilon_{\mathfrak{M}_1+\mathfrak{M}_2,(n)}$.

Consider the admissible system

$$\vec{\tilde{G}} = (\varepsilon_{1,(1)} - \varepsilon, \varepsilon_{1,(2)} - \varepsilon, \varepsilon_{1,(3)} - \varepsilon, \ldots).$$

Since $\varepsilon_{1,(n)}$ has the expansion

$$\varepsilon_{1,(n)} = e^{A_n/n},$$

$\vec{\tilde{G}}$ is a delta system with inverse

$$\vec{\tilde{G}}_n^{\langle -1 \rangle} = n \log(A_n + 1).$$

Clearly

$$\vec{\tilde{G}}^{\vec{\beta}} = \sum_{\vec{\tau}+\vec{\gamma}=\vec{\beta}} \binom{\vec{\beta}}{\vec{\tau}, \vec{\gamma}} (-1)^{l(\vec{\gamma})} \prod_i \varepsilon_{1,(i)}^{\tau_i} = \sum_{\vec{\tau}+\vec{\gamma}=\vec{\beta}} \binom{\vec{\beta}}{\vec{\tau}, \vec{\gamma}} (-1)^{l(\vec{\gamma})} \prod_i \varepsilon_{1^{\tau_i},(i)}.$$

Then, the conjugate sequence of $\vec{\tilde{G}}$ is the polynomials $\tilde{\phi}_{\vec{\alpha}}(X)$,

$$\begin{aligned}
\tilde{\phi}_{\vec{\alpha}}(X) &= \sum_{\vec{\beta}} \tilde{S}(\vec{\alpha}, \vec{\beta})\, p_{\vec{\beta}}(X), \\
\tilde{S}(\vec{\alpha}, \vec{\beta}) &= \sum_{\vec{\tau}+\vec{\gamma}=\vec{\beta}} \binom{\vec{\beta}}{\vec{\tau}, \vec{\gamma}} (-1)^{l(\vec{\gamma})} \prod_i (\tau_i)^{\alpha_i} \\
&= = \prod_i \left(\sum_{\tau_i=0}^{\beta_i} \binom{\beta_i}{\tau_i} (-1)^{(l(\gamma_i)} (\tau_i)^{\alpha_i} \right) = \prod_i \tilde{S}(\alpha_i, \beta_i),
\end{aligned} \tag{52}$$

the coefficients $\tilde{S}(m, n)$ being the ordinary Stirling numbers of the second kind.

It is easy to check that the polynomials

$$\tilde{\psi}_{\vec{\alpha}}(X) = \prod_n n^{\alpha_n} (p_n(X))_{\alpha_n},$$

where

$$(p_n(X))_{\alpha_n} = \prod_{k=0}^{\alpha_n - 1} (p_n(X) - k),$$

are the associated sequence of $\vec{\tilde{G}}$.

For any functional N and any symmetric function $R(X)$ we have the expansions

$$\begin{aligned}
N &= \sum_{\vec{\alpha}} \langle N \mid \tilde{\psi}_{\vec{\alpha}}(X) \rangle \frac{\prod_i (\varepsilon_{1,(i)} - \varepsilon)^{\alpha_i}}{z_{\vec{\alpha}}} \\
R(X) &= \sum_{\vec{\alpha}} \left(\sum_{\vec{\beta}+\vec{\gamma}=\vec{\alpha}} \binom{\vec{\alpha}}{\vec{\beta}, \vec{\gamma}} (-1)^{l(\vec{\gamma})} R(\vec{\beta}) \right) \frac{\prod_n n^{\alpha_n} (p_n(X))_{\alpha_n}}{z_{\vec{\alpha}}},
\end{aligned} \tag{53}$$

where $R(\vec{\beta})$ represents the number obtained by replacing each $p_{\vec{\alpha}}(X)$ in the expansion of $R(X)$ by the number $\prod_n \beta_n^{\alpha_n}$. As before, we have

$$R(\tilde{\boldsymbol{\phi}}) = \exp\left(-\sum_n \frac{p_n(X)}{n}\right) \sum_{\vec{\beta}} R(\vec{\boldsymbol{\beta}}) \frac{\tilde{\psi}_{\vec{\beta}}(X)}{z_{\vec{\alpha}}}$$

and

$$\tilde{\phi}_{\vec{\alpha}}(X) = \exp\left(-\sum_n \frac{p_n(X)}{n}\right) \sum_{\vec{\beta}} \prod_i \beta_i^{\alpha_i} \frac{p_{\vec{\beta}}(X)}{z_{\vec{\alpha}}}.$$

Example 5.5 (Plethystic Morphisms and Necklace Polynomials). Consider the admissible system $\vec{M}$.

$$M_n = n \sum_{k \geqslant 1} \frac{A_{nk}}{k}.$$

Since $\mu^* M_n = M_n^X + M_n^Y$, $U_{\vec{M}}$ is a Hopf algebra operator. Then, $\operatorname{mor}_{\vec{\alpha}}(X) = U_{\vec{M}} p_{\vec{\alpha}}(X)$, the conjugate sequence of $\vec{M}$, is multiplicative. We have

$$\operatorname{mor}_n = \sum_k \langle M_k \mid p_n(X) \rangle \frac{p_k(X)}{k} = \sum_{d \mid n} d p_d(X)$$

$$\operatorname{mor}_{\vec{\alpha}} = \prod_n \left(\sum_{d \mid n} d p_d(X)\right)^{\alpha_n}.$$

We can easily check that the inverse system of $\vec{M}$ is $\vec{L}$,

$$L_k = \sum_n \frac{\mu(n)}{n^2 k} A_{nk}.$$

The associated sequence of $\vec{M}$ is the necklace polynomials

$$\operatorname{nec}_{\vec{\alpha}}(X) = \prod_n \left(\frac{1}{n} \sum_{d \mid n} \mu(n/d)\, p_d(x)\right)^{\alpha_n}.$$

The exponential formula in Corollary 5.5. becomes in this case the following version of the cyclotomic identity:

$$h(XY) = \exp\left(\sum_{n \geqslant 1} \frac{L_n(Y) \operatorname{nec}_n(X)}{n}\right) = \prod_{n \geqslant 1} \exp\left\{\left(\sum_{k \geqslant 1} \frac{p_{nk}(Y)}{k}\right) \operatorname{nec}_n(X)\right\}$$

$$= \prod_{n \geqslant 1} \prod_{i \geqslant 1} \left(\frac{1}{1 - y_i^n}\right)^{\operatorname{nec}_n(X)}. \tag{54}$$

Substituting the system $\vec{M}$ into $\tilde{\vec{G}}$, we obtain

$$\tilde{G}_n(\vec{M}) = e^{M_n/n} - \varepsilon = \exp\left(\sum_{k \geqslant 1} \frac{A_{nk}}{k}\right) - \varepsilon = G_n.$$

By Proposition 5.1 and its corollary we obtain

$$\phi_{\vec{\alpha}}(X) = \mathrm{mor}_{\vec{\alpha}}(\tilde{\boldsymbol{\phi}})(X).$$

Then we have the identity (in umbral notation),

$$\phi_{\vec{\alpha}} = \prod_n \left(\sum_{d \mid n} d\,\tilde{\boldsymbol{\phi}}_{\mathrm{d}}\right)^{\alpha_n}.$$

Taking umbral inverses, we obtain

$$\psi_{\vec{\alpha}}(X) = \prod_n n^{\alpha_n} \left(\frac{1}{n} \sum_{d \mid n} \mu(n/d)\, p_d(X)\right)_{\alpha_n}. \tag{55}$$

EXAMPLE 5.6 (Witt Polynomials). Define the Witt system $\vec{\mathfrak{W}}$ as the inverse of the delta system $(\sum_{d \mid n} A_d^{n/d})_{n=1}^{\infty}$. The indicators $\mathfrak{W}_n(X)$ are the symmetric functions introduced in [43]. The Witt sequence $\{\omega_{\vec{\alpha}}(X)\}_{\vec{\alpha}}$ is defined as the binomial basis associated to $\vec{\mathfrak{W}}^{\langle -1 \rangle} = (\sum_{d \mid n} A_d^{n/d})_{n=1}^{\infty}$.

6. SHIFT-INVARIANT OPERATORS

Using an procedure analogous to that used to define a topology on $\Lambda(X)^*$, we define a topology on the algebra of all linear operators on $\Lambda(X)$.

For every linear functional $L \in \Lambda(X)^*$ we define the continuous operator

$$m_L^*\colon \Lambda(X)^* \to \Lambda(X)^*$$

$$m_L^*(N) := LN, \qquad N \in \Lambda(X)^*.$$

Then, m_L^* is the adjoint of some operator $m_L\colon \Lambda(X) \to \Lambda(X)$.

PROPOSITION 6.1. *The operator m_L is explicitly given by*

$$m_L R(X) = \langle L^Y \mid R(X+Y)\rangle, \qquad R(X) \in \Lambda(X).$$

Proof. For every multiset $\mathfrak{M}$,

$$\begin{aligned} m_L R(\mathfrak{M}) &= \langle \varepsilon_{\mathfrak{M}} \mid m_L R(X)\rangle = \langle L\varepsilon_{\mathfrak{M}} \mid R(X)\rangle = \langle \varepsilon_{\mathfrak{M}}^X L^Y \mid R(X+Y)\rangle \\ &= \langle L^Y \mid R(\mathfrak{M}+Y)\rangle. \end{aligned}$$

By Proposition 3.4, we obtain the result. ∎

The operator $E^{\mathfrak{M}} := m_{\varepsilon_{\mathfrak{M}}}$ is explicitly defined by

$$E^{\mathfrak{M}} R(X) = R(X + \mathfrak{M}).$$

For $\mathfrak{M} = a$, $E^a R(X) = R(X+a)$. For $\mathfrak{M} = \varnothing$, $\varepsilon_{\varnothing} = \varepsilon$, then $m_\varepsilon = E^{\varnothing} = I$. $E^{\mathfrak{M}}$ is called a translation operator.

The action of the operator $E^{\mathfrak{M}, (k)} = m_{\varepsilon^{(k)}_{\mathfrak{M}}}$ is given by

$$E^{\mathfrak{M}, (k)} p_{\vec{\alpha}}(X) = \sum_{|\vec{\beta}| = k} \binom{\vec{\alpha}}{\vec{\beta}, \vec{\gamma}} p_{\vec{\beta}}(\mathfrak{M})\ p_{\vec{\gamma}}(X).$$

For a partition $\lambda = \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3, ...,$ define $E^{1, (\lambda)} = \prod_i E^{1(\lambda_i)} = m_{\prod_i \varepsilon_1^{(\lambda_i)}}$. By Eq. (43),

$$E^{1, (\lambda)} p_{\vec{\alpha}}(X) = \sum_{\vec{\beta} + \vec{\gamma} = \vec{\alpha}} \binom{\vec{\alpha}}{\vec{\beta}, \vec{\gamma}} \varphi^{\lambda}_{\vec{\beta}} p_{\vec{\gamma}}(X). \tag{56}$$

The operator $D_n := m_{An}$ is the derivation $D_n = n\partial_{p_n}$, where ∂_{p_n} is the partial derivative with respect to the symmetric function $p_n(X)$.

DEFINITION 6.1. A linear operator $T: \Lambda(X) \to \Lambda(X)$ is called shift invariant if it commutes with all the translation operators of the form E^a, $a \in \mathbf{C}$.

Since $E^{\mathfrak{M}} = \prod_{a \in \mathfrak{M}} E^a$, any shift-invariant operator also commutes with all the translations of the form $E^{\mathfrak{M}}$. By a standard argument we obtain

PROPOSITION 6.2. *The operator T is shift-invariant if and only if*

$$E^Y T = T E^Y.$$

A sequence of shift invariant operators T_j, $j \in \mathfrak{I}$, is said to be convergent to a shift invariant operator T, if for every symmetric function $R(X)$ there exists j_0 such that $T_j R(X) = TR(X)$ for $j \geqslant j_0$. The shift invariant operators form a topological algebra that we denote by Σ.

PROPOSITION 6.3. *The map $\sigma: L \mapsto m_L$ is a bicontinuous algebra isomorphism between the umbral algebra $\Lambda(X)^*$ and the topological algebra Σ of shift-invariant operators on $\Lambda(X)$.*

Proof. It is easy to prove that σ is one-to-one. By Proposition 6.1 we obtain that σ is an algebra map.

Assume that N_j, $j \in \mathfrak{I}$, is a sequence of linear functionals converging to zero. Given a partition $\vec{\alpha}$, choose $j_0 \in \mathfrak{I}$ such that $j > j_0$ then $N_j \in \mathcal{N}(0, \vec{\alpha})$. Then

$$\sigma(N_j)\, p_{\vec{\alpha}}(X) = \sum_{\vec{\beta}+\vec{\gamma}=\vec{\alpha}} \binom{\vec{\alpha}}{\vec{\beta}, \vec{\gamma}} \langle N_j \mid p_{\vec{\beta}}(X)\rangle\, p_{\vec{\gamma}}(X) = 0, \qquad j > j_0.$$

Given an operator $T \in \Sigma$, $\sigma^{-1}T$ is the linear functional $R(X) \mapsto \langle \varepsilon \mid TR(X)\rangle = TR(\varnothing)$. Clearly if T_j, $j \in \mathfrak{I}$, is a sequence of operators converging to zero, then $\langle \varepsilon \mid T_j R(X)\rangle = 0$ for j big enough. ▮

For any functional $L \in \Lambda(X)^*$ we denote the operator $\sigma(L)$ by $L(D)$, that is, by identifying the functional L with its $\vec{A}$-indicator.

Every continuous operator $T\colon \Sigma \to \Sigma$ is of the form $T = \sigma P \sigma^{-1}$, for some continuous operator $P\colon \Lambda(X)^* \to \Lambda(X)^*$. We will denote T by $\hat{P}$.

Definition 6.2. An admissible system (respectively, a delta-system) of operators is a collection of operators of the form $(Q_1, Q_2, Q_3, ...) = (M_1(D), M_2(D), M_3(D), ...)$, where $(M_1, M_2, M_3, ...)$ is an admissible (respectively, a delta system) of functionals. The conjugate sequence of an admissible system of operators $\vec{Q} = \vec{M}(D)$ is the conjugate sequence of the respective admissible system of functionals $\vec{M}$. The associate sequence of a delta system of operators $\vec{Q} = \vec{M}(D)$ is the associated sequence of the respective delta system of linear functionals $\vec{M}$.

We omit the proof of the following proposition.

Proposition 6.4. *The binomial basis $q_{\vec{\alpha}}(X)$ is the associated basis of the delta system of operators $\vec{Q}$ if and only if*

$$Q_n q_{\vec{\alpha}}(X) = n\alpha_n q_{\vec{\alpha}-\vec{e}_n}(X).$$

Proposition 5.3 and Corollaries 5.4 and 5.6 lead to the following proposition and corollaries.

Proposition 6.5. *Let $\vec{Q}$ be a delta system of operators with associated sequence $\{q_{\vec{\alpha}}(X)\}_{\vec{\alpha}}$, and T be a shift-invariant operator. Then*

$$T = \sum_{\vec{\alpha}} \frac{\langle \varepsilon \mid Tq_{\vec{\alpha}}(X)\rangle}{z_{\vec{\alpha}}} \vec{Q}^{\vec{\alpha}}.$$

Corollary 6.1. *For any multiset $\mathfrak{M}$ of complex numbers, we have the expansion*

$$E^{\mathfrak{M}} = \sum_{\vec{\alpha} \in \mathfrak{P}} \frac{q_{\vec{\alpha}}(\mathfrak{M})}{z_{\vec{\alpha}}} \vec{Q}^{\vec{\alpha}}.$$

COROLLARY 6.2 (Taylor Expansion). *If R is any symmetric function, then*

$$R(X+Y)=\sum_{\vec{\alpha}\in\mathfrak{P}}\vec{Q}^{\vec{\alpha}}R(X)\frac{q_{\vec{\alpha}}(Y)}{z_{\vec{\alpha}}}.$$

6.1. *Sheffer Families*

Let $\{q_{\vec{\alpha}}(X)\}_{\vec{\alpha}\in\mathfrak{P}}$ be a binomial basis as above. A *Sheffer* family relative to $\{q_{\vec{\alpha}}(X)\}_{\vec{\alpha}\in\mathfrak{P}}$ is any family of polynomials of the form

$$g_{\vec{\alpha}}(X)=m^*(N)\,q_{\vec{\alpha}}(X)=N(D)\,q_{\vec{\alpha}}(X),$$

where N is any invertible functional. A Sheffer family relative to $p_{\vec{\alpha}}(X)$ is called an *Appel* family. Clearly any Sheffer family $\{g_{\vec{\alpha}}\}_{\vec{\alpha}}$ relative to $\{q_{\vec{\alpha}}(X)\}_{\vec{\alpha}\in\mathfrak{P}}$ satisfies the properties

1. $g_{\vec{\alpha}}(X+Y)=\sum_{\vec{\gamma}+\vec{\beta}=\vec{\alpha}}\binom{\vec{\alpha}}{\vec{\beta},\vec{\gamma}}\,g_{\vec{\beta}}(X)\,q_{\vec{\gamma}}(Y)$
2. $Q_n g_{\vec{\alpha}}(X)=g_{\vec{\alpha}-\vec{e}_n}(X)$
3. $\langle N^{-1}M_n\,|\,g_{\vec{\alpha}}(X)\rangle=\delta_{\vec{e}_n,\vec{\alpha}}$.

We have the following expansions:

PROPOSITION 6.6. *Let $\vec{Q}$ be a delta system of operators with associated sequence $\{q_{\vec{\alpha}}(X)\}_{\vec{\alpha}}$, and let T be a shift-invariant operator. Then*

$$T=\sum_{\vec{\alpha}}\frac{\langle\varepsilon\,|\,Tg_{\vec{\alpha}}(X)\rangle}{z_{\vec{\alpha}}}N^{-1}(D)\,\vec{Q}^{\vec{\alpha}}.$$

COROLLARY 6.3. *For any multiset $\mathfrak{M}$ of complex numbers, we have the expansion*

$$E^{\mathfrak{M}}=\sum_{\vec{\alpha}\in\mathfrak{P}}\frac{g_{\vec{\alpha}}(\mathfrak{M})}{z_{\vec{\alpha}}}N^{-1}(D)\,\vec{Q}^{\vec{\alpha}}.$$

COROLLARY 6.4 (Taylor Expansion). *If R is any symmetric function, then*

$$R(X+Y)=\sum_{\vec{\alpha}\in\mathfrak{P}}N^{-1}(D)\,\vec{Q}^{\vec{\alpha}}R(X)\frac{g_{\vec{\alpha}}(Y)}{z_{\vec{\alpha}}}.$$

EXAMPLE 6.1. Let N be the invertible functional $(\sum_{n=1}^{\infty}A_n/n)/(\varepsilon_1-\varepsilon)$. The Bernoulli family is the Appel family

$$B_{\vec{\alpha}}(X)=N(D)\,p_{\vec{\alpha}}(X)=\frac{\sum_{n=1}^{\infty}D_n/n}{E^1-I}\,p_{\vec{\alpha}}(X).$$

EXAMPLE 6.2. The Euler family is the Appel family corresponding to the functional $2/(\varepsilon_1+\varepsilon)$,

$$\mathfrak{e}_{\vec{\alpha}}(X)=\frac{2}{E^1+I}\,p_{\vec{\alpha}}.$$

7. HAMMOND DERIVATIVES AND UMBRAL SHIFTS

In this section we generalize to an infinite number of variables the theory of the duality between umbral shifts and derivations developed in [45] for one variable and in [44] for a finite number of variables. We obtain two recursive formulas for families of binomial type. The second recursive formula involves the inverse of an infinite matrix (the jacobian matrix of the associated delta system). We prove a third formula for the particular case when the delta system is diagonal. This formula involves the inverse of a finite sub-matrix of the jacobian matrix.

DEFINITION 7.1. Consider a delta system $\vec{M}$. The Hammond derivative δ_{M_n} is the unique continuous derivation of the umbral algebra satisfying

$$\delta_{M_n}M_m=n\delta(m,n).$$

Clearly, $\delta_{M_n}=n\partial_{M_n}$. The Hammond derivative δ_{A_n} is denoted by δ_n. We denote by

$$\hat{\delta}_{Q_n}=\hat{\delta}_{M_n(D)}=\sigma\delta_{M_n}\sigma^{-1}$$

the operator on Σ corresponding to δ_{M_n}.

Let $\{q_{\vec{\alpha}}(X)\}_{\vec{\alpha}}$ be the associated sequence of $\vec{M}$. The umbral shift $\vartheta_{M_n}\colon \Lambda(X)\to\Lambda(X)$ is defined by

$$\vartheta_{M_n}q_{\vec{\alpha}}(X)=q_{\vec{\alpha}+\vec{e}_n}(X).$$

The symbol $\vartheta_{\vec{M}}$ denotes the vector of umbral shifts $(\vartheta_{M_1},\vartheta_{M_2},\vartheta_{M_3},\ldots)$. It is trivial to check that $\vartheta^*_{M_n}=\delta_{M_n}$. We denote by ϑ_n the umbral shift ϑ_{A_n}.

Consider the delta system of operators $\vec{Q}=\vec{M}(D)$. It is easy to check the identity

$$\hat{\delta}_{Q_n}T=T\vartheta_{\vec{M}}^{\vec{e}_n}-\vartheta_{\vec{M}}^{\vec{e}_n}T. \tag{57}$$

For a functional L, $\delta_{\vec{M}}L$ denotes the column vector

$$\delta_{\vec{M}}L=(\delta_{M_1}L,\delta_{M_2}L,\delta_{M_3}L,\ldots)^t.$$

For an admissible system $\vec{L}$ define the infinite matrix $\delta_{\vec{M}}\vec{L}$ by

$$\delta_{\vec{M}}\vec{L}=\begin{pmatrix} \delta_{M_1}L_1 & \dfrac{\delta_{M_1}L_2}{2} & \dfrac{\delta_{M_1}L_3}{3} & \cdots \\ \delta_{M_2}L_1 & \dfrac{\delta_{M_2}L_2}{2} & \dfrac{\delta_{M_2}L_3}{3} & \cdots \\ \delta_{M_3}L_1 & \dfrac{\delta_{M_3}L_2}{2} & \dfrac{\delta_{M_3}L_3}{3} & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{pmatrix}=\left(\frac{\delta_{M_i}L_j}{j}\right)_{i,\,j=1}^{\infty}. \tag{58}$$

Since δ_{M_n} is continuous, $\lim_{j\to\infty}\delta_{M_n}L_j=0$ for every n. We call $\delta_{\vec{A}}\vec{L}$ the *jacobian matrix* of $\vec{L}$.

It is easy to verify that for every sequence of functionals $\{N_i\}_{i=1}^{\infty}$, $\lim_{j\to\infty}(\delta_{M_n}L_j)\,N_j$ is also zero. Then the series

$$\sum_{j=1}^{\infty}\frac{\delta_{M_i}L_j}{j}\cdot N_j$$

is convergent, and the product

$$(\delta_{\vec{M}}\vec{L})(N_1, N_2, N_3, ...)^t$$

is well defined.

In a similar way we define $\hat{\delta}_{\vec{Q}}\vec{P}$, $\vec{Q}$ being any delta system of shift invariant operators and $\vec{P}$ being any system of shift-invariant operators.

Consider the set $\mathscr{R}_f$ of infinite matrices of the form

$$B=(b_{i,\,j})_{i,\,j=1}^{\infty}\ b_{i,\,j}\in\mathbb{C},$$

where in every row there are only a finite number of non-zero entries (row finite matrices). With the usual operations, the set $\mathscr{R}_f$ is an algebra. The set $\mathscr{R}_f(\Lambda(X)^*)$ of matrices of the form $\mathscr{B}=(B_{i,\,j})_{i,\,j=1}^{\infty}$, $B_{i,\,j}\in\Lambda(X)^*$, satisfying $\lim_{j\to\infty}B_{i,\,j}=0$ for every j, can be identified with the topological algebra $\operatorname{Hom}(\Lambda(X), \mathscr{R}_f)$.

We say that a matrix $\mathscr{B}\in\mathscr{R}_f(\Lambda(X)^*)$ is invertible if it has a two sided inverse as an element of $\operatorname{Hom}(\Lambda(X), \mathscr{R}_f)$.

PROPOSITION 7.1 (Chain Rule). *Let $\vec{M}$ be a delta system and $\vec{L}$ be an admissible system. For any functional N we have*

$$\delta_{\vec{M}}N(\vec{L})=(\delta_{\vec{M}}\vec{L})\cdot(\delta_{\vec{A}}N)(\vec{L}). \tag{59}$$

Proof. It is easy to prove the chain rule for the partial derivatives ∂_{M_n},

$$\partial_{M_n} N(\vec{L}) = \sum_{k=1}^{\infty} \partial_{M_n}(L_k)(\partial_k N)(\vec{L})$$

$$n\partial_{M_n} N(\vec{L}) = \sum_{k=1}^{\infty} \frac{n\partial_{M_n}(L_k)}{k} k(\partial_k M)(\vec{L}),$$

equivalently

$$\delta_{M_n} N(\vec{L}) = \sum_{k=1}^{\infty} \frac{\delta_{M_n} L_k}{k} (\delta_k N)(\vec{L}). \quad \blacksquare$$

From Proposition 7.1 we easily obtain the corollary

COROLLARY 7.1. *If $\vec{N}$ is an admissible system then*

$$\delta_{\vec{M}} \vec{N}(\vec{L}) = (\delta_{\vec{M}} \vec{L}) \cdot (\delta_{\vec{A}} \vec{N})(\vec{L}). \tag{60}$$

COROLLARY 7.2. *If $\vec{L}$ is a delta system, then*

$$\delta_{\vec{M}} = (\delta_{\vec{M}} \vec{L}) \cdot \delta_{\vec{L}}, \tag{61}$$

where the symbol $\delta_{\vec{L}}$ represents the transformation $\vec{W} \mapsto \delta_{\vec{L}} \vec{W}$, mapping any admissible system $\vec{W}$ of linear functionals to the matrix $\delta_{\vec{L}} \vec{W}$.

Proof. since $\vec{L}$ is a delta system, every system of functionals $\vec{W}$ is of the form $\vec{W} = \vec{N}(\vec{L})$, $\vec{N}(X)$ being the system of $\vec{L}$-indicators of $\vec{W}$. For every j, $(\delta_k N_j)(\vec{L}) = \delta_{L_K} W_j$. From Eq. (61) we obtain the result. $\blacksquare$

THEOREM 7.1 (First Recursive Formula). *Let $q_{\vec{\alpha}}(X)$ be the conjugate sequence of the admissible system $\vec{L}$. Then we have the formula*

$$q_{\vec{\alpha}+\vec{e}_n}(X) = \sum_{k=1}^{\infty} \frac{p_k(X)}{k} \sum_{\vec{\beta}+\vec{\gamma}=\vec{\alpha}} \binom{\vec{\alpha}}{\vec{\beta}, \vec{\gamma}} \langle L_k \mid p_{\vec{\beta}+\vec{e}_n}(X) \rangle \, q_{\vec{\gamma}}(X). \tag{62}$$

Proof. Using the chain rule with $\vec{M} = \vec{A}$, we obtain that for every functional N

$$\delta_n N(\vec{L}) = \sum_{k=1}^{\infty} \frac{\delta_n L_k}{k} (\delta_k N)(\vec{L}). \tag{63}$$

Equivalently,

$$\delta_n U^*_{\vec{L}} = \sum_{k=1}^{\infty} \frac{\delta_n L_k}{k} U^*_{\vec{L}} \delta_k.$$

Taking adjoints,

$$U_{\vec{L}}\vartheta_n = \sum_{k=1}^{\infty} \frac{\vartheta_k}{k}\, U_{\vec{L}}\delta_n L_k(D). \tag{64}$$

By Proposition 6.1,

$$\delta_n L_k(D)\, p_{\vec{\alpha}}(X) = \sum_{\vec{\beta}+\vec{\gamma}=\vec{\alpha}} \binom{\vec{\alpha}}{\vec{\beta},\, \vec{\gamma}} \langle L_k \,|\, p_{\vec{\beta}+\vec{e}_n}(X)\rangle\, p_{\vec{\gamma}}(X).$$

Applying both sides of (64) to $p_{\vec{\alpha}}(X)$, we obtain the result. ∎

Corollary 7.3. *If $q_{\vec{\alpha}}(X)$ is of plethystic type with $L_n = F_n L$ for some functional L, then we have*

$$q_{\vec{\alpha}+\vec{e}_n}(X) = \sum_{d\,|\,n} p_d(X) \sum_{\vec{\tau}\{d\}+\vec{\gamma}=\vec{\alpha}} \binom{\vec{\alpha}}{\vec{\tau}^{\{d\}},\, \vec{\gamma}} d^{l(\tau)} \langle L \,|\, p_{\vec{\tau}+\vec{e}_{n/d}}(X)\rangle\, q_{\vec{\gamma}}(X).$$

Proposition 7.2. *Let $\vec{M}$ and $\vec{L}$ be any pair of delta systems of functionals. Then the infinite matrix $\delta_{\vec{M}}\vec{L}$ is invertible, $(\delta_{\vec{M}}\vec{L})^{-1} = \delta_{\vec{L}}\vec{M}$, and we have*

$$\delta_{\vec{L}} = (\delta_{\vec{M}}\vec{L})^{-1}\, \delta_{\vec{M}}. \tag{65}$$

Proof. Applying both sides of (61) to $\vec{M}$ we obtain $\varepsilon Id = (\delta_{\vec{M}}\vec{L})(\delta_{\vec{L}}\vec{M})$. Then $\delta_{\vec{L}}\vec{M}$ is the right inverse of $\delta_{\vec{M}}\vec{L}$. Similarly, we prove that $\delta_{\vec{L}}\vec{M}$ is the left inverse of $\delta_{\vec{M}}\vec{L}$. Equation (65) now follows directly from Eq. (61). ∎

We easily obtain the dual version of the previous proposition.

Corollary 7.4. *Defining the systems $\vec{Q} := \vec{M}(D)$ and $\vec{R} := \vec{L}(D)$ we have*

$$\vartheta_{\vec{L}} = \vartheta_{\vec{M}} (\hat{\delta}_{\vec{Q}}\vec{R})^{-1}. \tag{66}$$

The following corollary is obtained by making $\vec{M} = \vec{A}$ in Eq. (66).

Corollary 7.5 (Second Recursive Formula). *Let $r_{\vec{\alpha}}(X)$ be the binomial basis associated to $\vec{R}$. We have the recursion*

$$r_{\vec{\alpha}+\vec{e}_n}(X) = \sum_{k=1}^{\infty} p_k(X)\, C_{n,k}(D)\, r_{\vec{\alpha}}(X), \tag{67}$$

where

$$\mathscr{C}(D)=(C_{n,k}(D))_{n,k=1}^{\infty}=(\delta_{\vec{A}}\vec{L}(D))^{-1}=(\hat{\delta}_{\vec{D}}\vec{R})^{-1}.$$

PROPOSITION 7.3. *If $\vec{L}$ is an admissible system such that the matrix $\delta_{\vec{A}}\vec{L}$ is invertible, then $\vec{L}$ is a delta system.*

Proof. Define the admissible system of operators $\vec{R}:=\vec{L}(D)$. We define the polynomial sequence $r_{\vec{\alpha}}(X)$ by setting $r_0(X)=1$ and using the recursion (67). It is not difficult to check that $r_{\vec{\alpha}}(X)$ is of binomial type and $\langle L_k \mid r_{\vec{\alpha}}(X)\rangle = k\delta(\vec{\alpha}, \vec{e}_k)$ for every k and $\vec{\alpha}$. Let $\vec{M}$ be the admissible system of functionals satisfying

$$U_{\vec{M}}\, p_{\vec{\alpha}}(X)=r_{\vec{\alpha}}(X).$$

We have

$$k\delta(\vec{\alpha}, \vec{e}_k)=\langle L_k \mid r_{\vec{\alpha}}(X)\rangle=\langle L_k(\vec{M}) \mid p_{\vec{\alpha}}(X)\rangle.$$

Then $L_k(\vec{M})=A_k$, or equivalently, $\vec{M}$ is the right inverse of $\vec{L}$. By the chain rule (59), and since $\delta_{\vec{A}}\vec{L}$ is invertible, $\delta_{\vec{A}}\vec{M}$ is invertible. Using the previous argument, $\vec{M}$ has a right inverse $\vec{L}_1$. By associativity $\vec{L}_1=\vec{L}$. ∎

By Proposition 3.2 we obtain the corollary

COROLLARY 7.6. *An admissible system $\vec{L}$ is a delta system if and only if the matrix $\langle \delta_{\vec{\alpha}}\vec{L} \mid 1\rangle$ is invertible.*

7.1. *Applications of the First Recursive Formula*

EXAMPLE 7.1. Let $u_{\vec{\alpha}}(X)$ be the conjugate family of the system $\vec{L}$, $L_k=F_k\varepsilon_1^{(m)}=F_k h_m(A)$. We have the recursion

$$u_{\vec{\alpha}+\vec{e}_n}(X)=\sum_{d\mid n} p_d(X) \sum_{\substack{\tau^{\{d\}}+\vec{\gamma}=\vec{\alpha}\\ |\vec{\tau}|=m-m/d}} \binom{\vec{\alpha}}{\vec{\tau}^{\{d\}},\ \vec{\gamma}} d^{l(\tau)} u_{\vec{\gamma}}(X). \tag{68}$$

For every functional N, $U_{\vec{L}}N=N[h_n(A)]$. Denoting by $u_{\vec{\alpha}}^{(r)}(X)$ the component of degree r of $u_{\vec{\alpha}}(X)$, we have that

$$u_{\vec{\alpha}}^{(r)}(1)=\langle h_r(A) \mid u_{\vec{\alpha}}(X)\rangle=\langle h_r[h_m] \mid p_{\vec{\alpha}}(X)\rangle=(h_r[h_m](X), p_{\vec{\alpha}}(X))_H.$$

The symmetric function $h_r[h_m](X)$ is the Frobenius character of the permutation representation given by the action of the symmetric group $\mathfrak{S}_{mr}$ over the set $\Pi_{(m)}[mr]$ of partitions of $\{1, 2, ..., mr\}$, with blocks having m

elements. Then $\varsigma_{\vec{\alpha}}^{(r[m])} := u_{\vec{\alpha}}^{(r)}(1)$ is the character of that representation evaluated at a permutation of type $\vec{\alpha}$. From Eq. (68) we obtain the recursion

$$\varsigma_{\vec{\alpha}+\vec{e}_n}^{(r[m])} = \sum_{d|n} \sum_{\substack{\vec{\tau}^{\{n\}}+\vec{\gamma}=\vec{\alpha} \\ |\vec{\tau}|=m-n/d}} \binom{\vec{\alpha}}{\vec{\tau}^{\{d\}},\, \vec{\gamma}} d^{l(\tau)} \varsigma_{\vec{\gamma}}^{((r-d)[m])}.$$

EXAMPLE 7.2. The exponential polynomials $\phi_{\vec{\alpha}}(X)$ satisfy the recursion

$$\phi_{\vec{\alpha}+\vec{e}_n}(X) = \sum_{d|n} p_d(X) \sum_{\vec{\tau}^{\{d\}}+\vec{\gamma}=\vec{\alpha}} \binom{\vec{\alpha}}{\vec{\tau}^{\{d\}},\, \vec{\gamma}} d^{l(\tau)} \phi_{\vec{\gamma}}(X). \tag{69}$$

Equating the coefficients of $p_{\vec{\beta}}(X)$ in Eq. (69), we obtain the following recursion for the Stirling numbers of the second kind

$$S(\vec{\alpha}+\vec{e}_n, \vec{\beta}) = \sum_{d|n} \sum_{\vec{\tau}^{\{d\}}+\vec{\gamma}=\vec{\alpha}} \binom{\vec{\alpha}}{\vec{\tau}^{\{d\}},\, \vec{\gamma}} d^{l(\tau)} S(\vec{\gamma}, \vec{\beta}-e_d).$$

EXAMPLE 7.3. The conjugate sequence of the system of Hammond functionals (Example 5.1) satisfies the recursion

$$\tilde{n}_{\vec{\alpha}+\vec{e}_n}(X) = \sum_{k\geqslant n} p_k(X) \sum_{\substack{\vec{\beta}+\vec{\gamma}=\vec{\alpha} \\ |\vec{\beta}|=k-n}} \binom{\vec{\alpha}}{\vec{\beta},\, \vec{\gamma}} \tilde{n}_{\vec{\gamma}}(X).$$

7.2. *Applications of the Second Recursive Formula*

EXAMPLE 7.4. Consider the delta system $\vec{K}$, Example 5.1. We have

$$\delta_{\vec{A}}\vec{K} = \begin{pmatrix} \varepsilon & \varepsilon_1^{(1)} & \varepsilon_1^{(2)} & \varepsilon_1^{(3)} & \cdots \\ 0 & \varepsilon & \varepsilon_1^{(1)} & \varepsilon_1^{(2)} & \cdots \\ 0 & 0 & \varepsilon & \varepsilon_1^{(1)} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix} = (h_{k-n}(A))_{n,k=1}^{\infty},$$

where $h_k(X)$ is assumed to be zero for $k<0$. By the Newton identity (see [33]), the inverse of $\delta_{\vec{A}}\vec{K}$ is

$$(\delta_{\vec{A}}\vec{K})^{-1} = ((-1)^{k-n} e_{k-n}(A))_{n,k=1}^{\infty},$$

where $e_n(X)$ is the nth elementary symmetric function.

We have the recursive formula

$$\hat{m}_{\vec{\alpha}+\vec{e}_n}(X) = \sum_{k=n}^{|\vec{\alpha}|+n} p_k(X)(-1)^{k-n}\, e_{k-n}(D)\, \hat{m}_{\vec{\alpha}}(X).$$

Then

$$\begin{aligned}(\alpha_n+1)\, m_{\vec{\alpha}+\vec{e}_n}(X) &= \sum_{k=n}^{|\vec{\alpha}|+n} p_k(X)(-1)^{k-n}\, e_{k-n}(D)\, m_{\vec{\alpha}}(X)\\ &= \sum_{k=n}^{|\vec{\alpha}|+n} p_k(X) \sum_{\vec{\beta}+\eta\gamma} = \vec{\alpha}\, m_{\vec{\beta}}(X)\langle \theta^*\varepsilon_1^{(k-1)}(A) \,|\, m_\gamma(X)\rangle\\ &= \sum_{k=n}^{|\vec{\alpha}|+n} p_k(X) \sum_{\vec{\beta}+\vec{\gamma}=\vec{\alpha}} m_{\vec{\beta}}(X)\langle \varepsilon_1^{(k-n)} \,|\, \theta m_\gamma(X)\rangle\\ &= \sum_{k=n}^{|\vec{\alpha}|+n} p_k(X) \sum_{\substack{\vec{\beta}+\vec{\gamma}=\vec{\alpha}\\ |\vec{\gamma}|=k-n}} m_{\vec{\beta}}(X)\, \theta^* m_\gamma(1).\end{aligned} \tag{70}$$

From the above formula we obtain the following recursion for the coefficients $c_{\vec{\alpha},\vec{\tau}}$, connecting the $p_{\vec{\tau}}$'s with the $m_{\vec{\alpha}}$'s.

$$c_{\vec{\alpha}+\vec{e}_n,\tau} = \frac{1}{\alpha_n+1} \sum_{k=n}^{|\alpha|+n} \sum_{\substack{\vec{\beta}+\vec{\gamma}=\vec{\alpha}\\ \vec{\gamma}=k-n}} c_{\vec{\beta},\vec{\tau}-\vec{e}_k} u_{\vec{\gamma}} \tag{71}$$

where $u_{\vec{\gamma}} = \theta m_{\vec{\gamma}}(1) = \sum_{|\vec{\mu}|=k-n} (-1)^{l(\vec{\mu})}\, c_{\vec{\gamma},\vec{\mu}}$.

EXAMPLE 7.5. Consider the exponential polynomials $\phi_{\vec{\alpha}}$ of Example 5.3 and its umbral inverses $\psi_{\vec{\alpha}}$. The polynomials $\psi_{\vec{\alpha}}$ are the associate family of the delta system $F_k \Delta = F_k E^1 - I = E^{\Upsilon_k} - I$. Then,

$$(E^{\Upsilon_n} - I)\, \psi_{\vec{\alpha}}(X) = n\alpha_n \psi_{\vec{\alpha}-\vec{e}_n}(X). \tag{72}$$

The exponential polynomials are the associate sequence of the delta system $Q_k = \sum_{n=1}^{\infty} (\mu(n)/n) \log(1 + D_{nk})$.

Then,

$$\frac{\tilde{\delta}_n Q_k}{k} = \begin{cases} \dfrac{\mu(n/k)}{I+D_n} & \text{if } k \,|\, n\\ 0 & \text{otherwise.}\end{cases}$$

The inverse of the $\tilde{\delta}_D \vec{Q}$ is the matrix $\mathscr{C}$,

$$C_{n,k} = \begin{cases} I + D_k & \text{if } k \mid n \\ 0 & \text{otherwise.} \end{cases}$$

We get the recursive formula

$$\phi_{\vec{\alpha}+\vec{e}_n}(X) = \sum_{d \mid n} p_d(X)(\phi_{\vec{\alpha}}(X) + D_d \phi_{\vec{\alpha}}(X)). \tag{73}$$

Equating the coefficients of $p_{\vec{\beta}}(X)$ on both sides of (73), we obtain the following recursion for the plethystic Stirling numbers of the second kind:

$$S(\vec{\alpha}+\vec{e}_n, \vec{\beta}) = \sum_{d \mid n} S(\vec{\alpha}, \vec{\beta}-\vec{e}_d) + d\beta_d S(\vec{\alpha}, \vec{\beta}). \tag{74}$$

Computing the jacobian matrix $\tilde{\delta}_{\vec{D}} \vec{\Delta}$ we obtain

$$\frac{\tilde{\delta}_n F_k \Delta}{k} = \begin{cases} E^{\Upsilon_k} & \text{if } k \mid n \\ 0 & \text{otherwise.} \end{cases}$$

We easily check that the inverse $\mathscr{L} = (L_{n,k})_{n,k=1}^{\infty}$ of $\tilde{\delta}_{\vec{D}} \vec{\Delta}$ is given by

$$L_{n,k} = \begin{cases} \mu(n/k)\, E^{(-)\,\Upsilon_n} & \text{if } k \mid n \\ 0 & \text{otherwise.} \end{cases}$$

We get the recursion

$$\psi_{\vec{\alpha}+\vec{e}_n}(X) = \sum_{d \mid n} p_d(X)\, \mu(n/d)\, E^{(-)\,\Upsilon_n} \psi_{\vec{\alpha}}(X). \tag{75}$$

From Eq. (72) we obtain $\psi_{\vec{\alpha}}(X) = \psi_{\vec{\alpha}}(X + \Upsilon_k) - n\alpha_n \psi_{\vec{\alpha}-\vec{e}_n}(X)$. From Eq. (55) we get

$$\psi_{\vec{\alpha}-\vec{e}_n}(X) = \frac{\psi_{\vec{\alpha}}(X+\Upsilon_n)}{\sum_{d \mid n} \mu(n/d)\, p_d(X+\Upsilon_n)}.$$

Then

$$E^{(-)\,\Upsilon_n} \psi_{\vec{\alpha}}(X) = \psi_{\vec{\alpha}}(X) - n\alpha_n \frac{\psi_{\vec{\alpha}}(X)}{\sum_{d \mid n} \mu(n/d)\, p_d(X)}.$$

Substituting in Eq. (75) we get

$$\psi_{\vec{\alpha}+\vec{e}_n}(X) = \left(\sum_{d \mid n} p_d(X)\, \mu(n/d)\, \psi_{\vec{\alpha}}(X) \right) - n\alpha_n \psi_{\vec{\alpha}}(X). \tag{76}$$

The plethystic Stirling numbers of the second kind $s(\vec{\alpha}, \vec{\beta})$ are defined as the coefficients connecting the $p_{\vec{\beta}}$'s with the $\psi_{\vec{\alpha}}$'s. Consider the lattice

$$\Pi[S, \sigma] = \{\pi \mid \sigma(B) \in \pi, \forall B \in \pi\}$$

of partitions on an n-set S compatible with some permutation $\sigma: S \to S$ of class $\vec{\alpha}$. For $\pi \in \Pi[S, \sigma]$, σ induces a permutation σ/π on the blocks of π. The plethystic Stirling numbers of the first kind have the combinatorial interpretation (see [36] and [41])

$$s(\vec{\alpha}, \vec{\beta}) = \sum_{\sigma,\, cl(\sigma/\pi) = \vec{\beta}} \mu_\sigma(\hat{0}, \pi),$$

where μ_σ is the Möbius function of $\Pi[S, \sigma]$.

From Eq. (76) we obtain the recursion

$$s(\vec{\alpha} + \vec{e}_n, \vec{\beta}) = \left(\sum_{d \mid n} \mu(n/d)\, s(\vec{\alpha}, \vec{\beta} - \vec{e}_d) \right) - n\alpha_n s(\vec{\alpha}, \vec{\beta}). \tag{77}$$

EXAMPLE 7.6. The backward difference delta system of operators $F_n \nabla$, $\nabla = I - E^{(-)1}$, has as an associated sequence the family

$$\kappa_{\vec{\alpha}}(X) = \prod_n n^{\alpha_n} \left\langle \frac{1}{n} \sum_{d \mid n} \mu(n/d)\, p_d(X) \right\rangle_{\alpha_n},$$

where $\langle x \rangle_n$ denotes the increasing factorial $\langle x \rangle_n = x(x+1)(x+2) \cdots (x+n-1)$. We have the recursive formula

$$\begin{aligned} \kappa_{\vec{\alpha} + \vec{e}_n}(X) &= \sum_{d \mid n} p_d(X)\, \mu(n/d)\, E^{\Upsilon_n} \kappa_{\vec{\alpha}}(X) \\ &= \left(\sum_{d \mid n} p_d(X)\, \mu(n/d)\, \kappa_{\vec{\alpha}}(X) \right) + n\alpha_n \kappa_{\vec{\alpha}}(X). \end{aligned}$$

The sequence $\kappa_{\vec{\alpha}}(X)$ is the conjugate sequence of the delta system of operators

$$W_k = F_k \log \prod_n \left(\frac{1}{I - D_n} \right)^{\mu(n)/n}.$$

$W_1(X)$ is the Frobenius character of the free Lie algebra considered as an analytic functor (see [27]).

7.3. *Diagonal Systems*

DEFINITION 7.2. A diagonal system is a delta system $\vec{M}$ where each component M_k is of the form $M_k = A_k P_k$. The matrix of $\langle \delta_{\vec{A}} \vec{M} | 1 \rangle$ is the diagonal $\mathrm{diag}(\langle P_1 | 1 \rangle, \langle P_2 | 1 \rangle, \langle P_3 | 1 \rangle, ...)$. Then, $\langle P_k | 1 \rangle \neq 0$ and each functional P_k is invertible. Since each component $M_k^{\langle -1 \rangle}$ of the compositional inverse $\vec{M}^{\langle -1 \rangle}$ satisfies the functional equation

$$M_k^{\langle -1 \rangle} = A_k P_k^{-1}(\vec{M}^{\langle -1 \rangle}),$$

$\vec{M}^{\langle -1 \rangle}$ is also a diagonal system.

For a partition $\vec{\alpha}$ denote by $\mathrm{supp}(\vec{\alpha})$ the finite set $\{i: \alpha_i \neq 0\}$.

PROPOSITION 7.4. *A binomial basis* $\{q_{\vec{\alpha}}(X)\}_{\vec{\alpha} \in \mathfrak{P}}$ *is the associated sequence of a diagonal system if and only if for every partition* $\vec{\alpha}$ *we have*

$$\langle A^{\vec{\beta}} | q_{\vec{\alpha}}(X) \rangle = 0 \qquad \textit{for every } \vec{\beta} \nleq \vec{\alpha}. \tag{78}$$

Proof. Assume that $\{q_{\vec{\alpha}}(X)\}$ is the associated sequence of a diagonal system $\vec{M}$. The compositional inverse $\vec{L}$ of $\vec{M}$ is also diagonal. Since each L_k is of the form $L_k = A_k R_k$, for $\vec{\beta} \nleq \vec{\alpha}$,

$$\langle A^{\vec{\beta}} | q_{\vec{\alpha}}(X) \rangle = \langle \vec{L}^{\vec{\beta}} | p_{\vec{\alpha}}(X) \rangle = \langle A^{\vec{\beta}} \vec{R}^{\vec{\beta}} | p_{\vec{\alpha}}(X) \rangle = \langle \vec{R}^{\vec{\beta}} | D^{\vec{\beta}} p_{\vec{\alpha}}(X) \rangle = 0.$$

Assuming Eq. (78), true is for every k and $\vec{\alpha}$ we have $\langle A_k | q_{\vec{\alpha}}(X) \rangle = \langle L_k | p_{\vec{\alpha}}(X) \rangle = 0$ if k is not in $\mathrm{supp}(\vec{\alpha})$. Then $\vec{L}$ is diagonal, and $\vec{M} = \vec{L}^{\langle -1 \rangle}$ is also diagonal. ∎

Let S be any finite set of positive integers. For an admissible system $\vec{M}$, $\delta_{\vec{A}} \vec{M}^S$ will denote the finite matrix $(\delta_n M_k / k)_{n, k \in S}$.

THEOREM 7.2 (Recursive Formula for Diagonal Systems). *Assume that* $\{q_{\vec{\tau}}(X)\}_{\vec{\tau} \in \mathfrak{P}}$ *is the associated sequence of a diagonal system* $\vec{M}$. *For a fixed partition* $\vec{\alpha}$ *let* $S = \mathrm{supp}(\vec{\alpha})$ *and* n *be any element of* S. *Then we have the recursive formula*

$$q_{\vec{\alpha} + \vec{e}_n}(X) = \sum_{k \in S} p_k(X)\, \tilde{C}_{n,k}^{(S)}(D)\, q_{\vec{\alpha}}(X), \tag{79}$$

where $\tilde{\mathscr{C}}^S$ *is the inverse of the finite matrix* $\mathscr{B}^S = \delta_{\vec{A}} \vec{M}^S = (\delta_{A_i} M_j / j)_{i, j \in S}$.

Proof. Define the infinite matrix $\mathscr{B}^{[S]}$ by

$$B_{n,k}^{[S]} = \begin{cases} 0 & n \in S,\ k \notin S \\ \dfrac{\delta_n M_k}{k} = B_{n,k} & \text{otherwise.} \end{cases}$$

We claim that $\mathscr{B}^{[S]}$ is in the neighborhood $\mathscr{N}(\mathscr{B}, \vec{\alpha})$. By the definition of $\mathscr{B}^{[S]}$ we have only to check that for $n \in S$, $k \notin S$, and every $\vec{\tau} \leqslant \vec{\alpha}$, $\langle \delta_n M_k | p_{\vec{\tau}}(X) \rangle = 0$. But this is equal to $\langle M_k | p_{\vec{\tau}+\vec{e}_n}(X) \rangle$ which is equal to zero because $\vec{M}$ is a diagonal system and $k \notin \operatorname{supp}(\vec{\tau}+\vec{e}_n) \subseteq S$.

Then, by Proposition 3.3 $(\mathscr{B}^{[S]})^{-1} \in \mathscr{N}(\mathscr{B}^{-1}, \vec{\alpha})$. It means that for every partition $\vec{\tau} \leqslant \vec{\alpha}$, $(\mathscr{B}^{[s]})^{-1}(D)\, p_{\vec{\tau}}(X) = \mathscr{B}^{-1}(D)\, p_{\vec{\tau}}(X)$. Since in the expansion of $q_{\vec{\alpha}}(X)$ there only appear power functions of the form $p_{\vec{\tau}}(X)$, with $\vec{\tau} \leqslant \vec{\alpha}$, we have

$$(\mathscr{B}^{[S]})^{-1}(D)\, q_{\vec{\alpha}}(X) = \mathscr{B}^{-1}(D)\, q_{\vec{\alpha}}(X).$$

By the second recursive formula

$$q_{\vec{\alpha}+\vec{e}_n}(X) = \sum_k p_k(X)(\mathscr{B}^{[S]})^{-1}_{n,k}\, q_{\vec{\alpha}}(X). \tag{80}$$

Since the matrix $\mathscr{B}^{[S]}$ has the block structure

$$\mathscr{B}^{[S]} = \begin{pmatrix} \mathscr{B}^S & 0 \\ \mathscr{F} & \mathscr{G} \end{pmatrix},$$

the inverse $(\mathscr{B}^{[S]})^{-1}$ is of the form

$$\begin{pmatrix} \widetilde{\mathscr{C}}^S & 0 \\ -\mathscr{G}^{-1}\mathscr{F}\widetilde{\mathscr{C}}^S & \mathscr{G}^{-1} \end{pmatrix}.$$

Then for $n \in S$,

$$(\mathscr{B}^{[S]})^{-1}_{n,k} = \begin{cases} \widetilde{C}^S_{n,k} & k \in S \\ 0 & k \notin S. \end{cases}$$

Substituting in (80) we obtain the result. ∎

7.4. *Exterior Powers of Umbral Shifts*

Denote by $\mathbb{P}^{[n]}$ the set of n-subsets of positive integers. For I and J in $\mathbb{P}^{[n]}$ we say that $I \preccurlyeq J$ if $|I| < |J|$ or if $|I| = |J|$ and I precedes J in the reverse lexicographic order. $\mathbb{P}^{[n]} = \{I_1 \prec I_2 \prec I_3 \cdots\}$ is a totally ordered set. For examples,

$$\mathbb{P}^{[2]} = \{\{1, 2\} \prec \{1, 3\} \prec \{2, 3\} \prec \{1, 4\} \prec \cdots\}.$$

Let $\mathscr{M} = (M_{i,j})_{i,j=1}^{\infty}$ be an infinite matrix in $\operatorname{Hom}(\Lambda(X), \mathscr{R}_f)$. For I and J in $\mathbb{P}^{[n]}$, denote by $\mathscr{M}(I, J)$ the determinant $|M_{i,j}|_{i \in I, j \in J}$. We denote by $\mathscr{M}^{(n)}$ the compound matrix of the minors $(\mathscr{M}(I, J))_{I, J \in \mathbb{P}^{[n]}}$. $\mathscr{M}^{(n)}$ can be thought of as an element of $\operatorname{Hom}(\Lambda(X), \mathscr{R}_f)$ by defining the entry $\mathscr{M}^{(n)}_{r,s}$ to be the minor $\mathscr{M}(I_r, I_s)$, I_r and I_s being the rth and sth elements of $\mathbb{P}^{[n]}$ respectively. It is not difficult to verify that $\mathscr{M}_{r,s} \to 0$ when $s \to \infty$.

For a set $I = \{i_1, i_2, ..., i_n\}$ in $\mathbb{P}^{[n]}$ let $\delta^I_{\vec{L}}$ be the continuous map

$$\delta^I_{\vec{L}}\colon \Lambda(X^{(1)}, X^{(2)}, ..., X^{(n)})^* \to \Lambda(X)^*$$

that assigns to a multivariate decomposed functional of the form $\prod_{s=1}^n T_s^{X^{(s)}}$ the determinant $|\delta_{L_{ir}} T_s|_{r,s=1}^n$ in $\Lambda(X)^*$.

Proposition 7.1. *The transformation*

$$\vartheta^I_{\vec{L}}\colon \Lambda(X) \to \Lambda(X^{(1)}, X^{(2)}, ..., X^{(n)})$$

whose adjoint is $\delta^I_{\vec{L}}$ *is given by*

$$\vartheta^I_{\vec{L}} R(X) = |\vartheta^{X^{(s)}}_{L_{ir}}|^n_{r,s=1} R(X^{(1)} + X^{(2)} + \cdots + X^{(n)}). \tag{81}$$

Proof. The adjoint of $E^{\sum_{i=2}^n X^{(i)}}$ is the n-multiplication

$$\mu^{(n-1)}\colon \Lambda(X^{(1)}, X^{(2)}, ..., X^{(n)})^* \to \Lambda(X)^*,$$

$$\mu^{(n-1)} \prod_{s=1}^n T_s^{X^{(s)}} = \prod_{s=1}^n T_s.$$

Using this notation we have that

$$\vartheta^I_{\vec{L}} = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sign}(\sigma) \prod_{r=1}^n \vartheta^{X^{(\sigma(r))}}_{L_{i_r}} \circ E^{X^{(2)} + X^{(3)} + \cdots + X^{(n)}}.$$

Taking adjoints we get

$$(\vartheta^I_{\vec{M}})^* = \mu^{(n-1)} \circ \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sign}(\sigma) \prod_{r=1}^n \delta^{X^{(\sigma(r))}}_{L_{ir}} = \delta^I_{\vec{L}}. \quad \blacksquare$$

We denote by $\vartheta^{(n)}_{\vec{L}}$ the infinite vector $(\vartheta^{I_1}_{\vec{L}}, \vartheta^{I_2}_{\vec{L}}, ...)$ and by $\delta^{(n)}_{\vec{L}}$ the vector $(\delta^{I_1}_{\vec{L}}, \delta^{I_2}_{\vec{L}}, ...)^t$.

Theorem 7.3. *Let $\vec{M}$ and $\vec{N}$ be two delta systems. We have the identities*

$$\delta^{(n)}_{\vec{L}} = {}^m\{(\delta_{\vec{L}} \vec{M})^{(n)}\}\, \delta^{(n)}_{\vec{M}} \tag{82}$$

$$\vartheta^{(n)}_{\vec{L}} = \vartheta^{(n)}_{\vec{M}} (\delta_{\vec{L}} \vec{M})^{(n)\,t}\,(D). \tag{83}$$

Proof. Equation (83) is the dual form of (82). To prove (82) we only have to prove that for every multivariate functional of the form $T_1^{X(1)} T_2^{X(2)} \cdots T_n^{X(n)}$ and any set $I \in \mathbb{P}^{[n]}$,

$$\delta_{\vec{L}}^{I} \prod_{s=1}^{n} T_s^{X(s)} = \sum_{J \in \mathbb{P}^{[n]}} (\delta_{\vec{L}} \vec{M})^{(n)} (I, J)\, \delta_{\vec{M}}^{J} \prod_{s=1}^{n} T_n^{X(s)}. \tag{84}$$

Consider the system $\vec{T} = (T_1, T_2, ..., T_n, , 0, ...)$. By Eq. (61) we have

$$\delta_{\vec{L}} \vec{T} = (\delta_{\vec{L}} \vec{M})(\delta_{\vec{M}} \vec{T}).$$

Taking compound matrices and using the Cauchy–Binet identity we get

$$(\delta_{\vec{L}} \vec{T})^{(n)} = (\delta_{\vec{L}} \vec{M})^{(n)} (\delta_{\vec{M}} \vec{T})^{(n)}. \tag{82}$$

We obtain Eq. (84) by computing the entry $(I, \{1, 2, ..., n\})$ on both sides of Eq. (85). ∎

8. SYMMETRIC FUNCTIONS OF SCHUR TYPE

Definition 8.1. A family of symmetric functions of the form $\{t_{\lambda/\mu}(X)\}_{\mu \sqsubseteq \lambda}$ said to be of Schur type if they are the images of the skew Schur function $S_{\lambda/\mu}(X)$ by an umbral operator. Similarly, a family of functionals $\{L_{\lambda/\mu}\}_{\lambda \sqsubseteq \mu}$ is called of Schur type if they are the images of the Schur functionals $S_{\lambda/\mu}(A)$ by an algebra automorphism of $\Lambda(X)^*$.

If U is an umbral operator, we denote by $S_{\lambda/\mu}^{U}(X)$ the Schur-type symmetric function $US_{\lambda/\mu}(X)$. Similarly, we denote by $H_{\lambda/\mu}^{U}$ the Schur-type functional $(U^{-1})^* S_{\lambda/\mu}(A)$. From now on, L^U will denote the functional $(U^{-1})^* L$.

It is clear that $\langle H_\lambda^U | S_\tau^U(X) \rangle = \delta_{\lambda, \mu}$. In other words, $\{H_\lambda^U\}_\lambda$ is the dual pseudobasis of $\{S_\lambda^U(X)\}_\lambda$.

Any family $\{S_{\lambda/\mu}^U(X)\}$ satisfies the coalgebra properties of the Schur functions

$$S_{\lambda/\mu}^U(0) = 0, \qquad \text{when} \quad \mu \neq \lambda$$

$$S_{\lambda/\mu}^U(X + Y) = \sum_{\mu \sqsubseteq \tau \sqsubseteq \lambda} S_{\tau/\mu}^U(X)\, S_{\lambda/\tau}^U(Y).$$

Given a set $I = \{i_1, i_2, ..., i_n\}$ in $\mathbb{P}^{[n]}$, the nonzero elements of the sequence $i_1 - 1, i_2 - 2, ..., i_n - n$ form a partition of length less than or equal to n. We denote it by $\pi_n(I)$.

Denote by $\mathfrak{P}_n$ the set of all the partitions whose length is less than or equal to n. A partition λ in $\mathfrak{P}_n$ can be represented as a vector of integers $(\lambda_1, \lambda_2, ..., \lambda_n)$ by placing zeroes in the $n-k$ first positions when $l(\lambda)=k<n$. We denote by $\varsigma_n(\lambda)$ the set $\{\lambda_1+1, \lambda_2+2, ..., \lambda_n+n\}$ in $\mathbb{P}^{[n]}$, where ς_n is a bijection from $\mathfrak{P}_n$ to $\mathbb{P}^{[n]}$ with inverse π_n.

Given two partitions $\mu \sqsubseteq \lambda$ in $\mathfrak{P}_n$, the entry $(\varsigma_n(\mu), \varsigma_n(\lambda))$ of the matrix $(\delta_{\vec{A}}\vec{K})^{(n)}$ is the determinant $|\varepsilon_1^{(\lambda_j+j-\mu_i-i)}|_{i,j=1}^n = |h_{\lambda_j-\mu_i+j-i}(A)|_{i,j=1}^n$. By the Jacoby–Trudi identity, $S_{\lambda/\mu}(A) = (\delta_{\vec{A}}\vec{K})^{(n)}(\varsigma_n(\mu), \varsigma_n(\lambda))$. Since the algebra properties are preserved by the operator $(U^{-1})^*$, $H^U_{\lambda/\mu}$ also satisfies the Jacobi–Trudy identity:

$$H^U_{\lambda/\mu} = (\delta_{\vec{A}^U}\vec{K}^U)^{(n)}(\varsigma_n(\mu), \varsigma_n(\lambda)) = |\varepsilon_1^{(\lambda_j-\mu_i+j-i),\,U}|_{i,j=1}^n.$$

Assume now that U is a bialgebra map. Since $Up_n(X+Y) = Up_n(X) + Up_n(Y)$ we have that $q_n(X) = Up_n(X)$ is of the form

$$q_n(X) = \sum_{k>0} c_{n,k}\, p_k(X) = \sum_{s=1}^{\infty} \sum_{k>0} c_{n,k}\, x_s = \sum_{r=1}^{\infty} q_n(x_r),$$

where $\{q_n(x)\}$ is a basis of $\mathbb{C}[x]$. Since U is an algebra map, $Up_{\vec{\alpha}}(X) = q_{\vec{\alpha}}(X) = \prod_n q_n^{\alpha_n}(X)$. Our goal is to generalize the quotient of alternants formula for the Schur function $S_\lambda(x_1, x_2, ..., x_n)$ to $S^U_\lambda(x_1, x_2, ..., x_n)$, U being a bialgebra map.

THEOREM 8.1. *Let U be a bialgebra map as above. For any sequence of alphabets $X^{(1)}, X^{(2)}, ..., X^{(n)}$, $S^U_\lambda(X^{(1)}+\cdots+X^{(n)})$ satisfies the identity*

$$\begin{aligned} S^U_\lambda(X^{(1)}+\cdots+X^{(n)})\, &|q_r(X^{(s)})|_{r,s=1}^n \\ &= \sum_{\mu\in\mathfrak{P}_n} \sum |\vartheta_{K^U_{\mu_r+r}} S^U_{\lambda^{(s)}/\lambda^{(s-1)}}(X^{(s)})|_{r,s=1}^n, \end{aligned} \tag{86}$$

where the second sum is over the sequences of partitions $\mu = \lambda^{(0)} \sqsubseteq \lambda^{(1)} \cdots \sqsubseteq \lambda^{(n)} = \lambda$.

Proof. By Eq. (83), for $I \in \mathbb{P}^{[n]}$ we have

$$\vartheta^I_{\vec{A}^U} = \sum_{J\in\mathbb{P}^{[n]}} \vartheta^J_{\vec{K}^U} (\delta_{\vec{A}^U} K^U)^{(n)}(I, J)(D).$$

Since $(\delta_{\vec{A}^U} K^U)^{(n)}(I, J)(D) = H_{\mu/\tau}(D)$, $\mu = \pi_n(J)$, $\tau = \pi_n(I)$, we have

$$\vartheta^{\varsigma_n(\tau)}_{\vec{A}^U} = \sum_{\mu\in\mathfrak{P}_n} \vartheta^{\varsigma_n(\mu)}_{\vec{K}^U} H^U_{\mu/\tau}(D). \tag{87}$$

Applying both sides of Eq. (87), with $\tau = 0$, to $S_\lambda^U(X)$, we obtain

$$S_\lambda^U(X^{(1)} + X^{(2)} + \cdots + X^{(n)}) \, |q_r(X^{(s)})|_{r,\, s=1}^n = \sum_{\mu \in \mathfrak{P}_n} |\vartheta_{K^U_{\mu_r + r}}^{X^{(s)}}|_{r,\, s=1}^n \, S_{\lambda/\mu}^U(X^{(1)} + \cdots + X^{(n)}). \tag{88}$$

The theorem follows, since $S_{\lambda/\mu}(X)$ have the same coalgebraic properties as $S_{\lambda/\mu}(X)$. ▮

Corollary 8.1. *If $q_n(x)$ is a polynomial sequence* $(\deg(q_n(x)) = n)$, *then $S_\lambda^U(x_1, x_2, ..., x_n)$ can be expressed as a sum of quotients of alternants as*

$$S_\lambda^U(x_1, x_2, ..., x_n) = \sum_{\mu \in \mathfrak{P}_n} \frac{1}{|q_r(x_s)|_{r,\, s=1}^n} \sum |(\vartheta_{K^U_{\mu_r + r}} S^U_{\lambda^{(s)}/\lambda^{(s-1)}})(x_s)|_{r,\, s=1}^n, \tag{89}$$

where the second sum ranges over all chains of partitions $\mu = \lambda^{(0)} \sqsubseteq \lambda^{(1)} \cdots \sqsubseteq \lambda^{(n)} = \lambda$.

Proof. Taking one element alphabets $X^{(s)} = \{x_s\}$ in Eq. (86) we obtain

$$S_\lambda^U(x_1, x_2, ..., x_n) \, |q_r(x_s)|_{r,\, s=1}^n = \sum_{\mu \in \mathfrak{P}_n} \sum |(\vartheta_{K^U_{\mu_r + r}} S^U_{\lambda^{(s)}/\lambda^{(s-1)}})(x_s)|_{r,\, s=1}^n.$$

If $q_n(x)$ is a polynomial sequence, the determinant $|q_r(x_s)|_{r,\, s=1}^n$ is of the form $c_n \, |x_r^s|_{r,\, s=1}^n$, for some sequence of complex numbers c_n. Since the Vandermonde determinant divides any alternant polynomial, it divides $(\vartheta_{K^U}^{\varsigma_n(\mu)} S^U_{\lambda/\mu})(x_1, x_2, ..., x_n)$ for every pair of partitions λ, μ. Then, we obtain the corollary. ▮

Observe that if the umbral shifts $\vartheta_{K_n^U}$ have the property

$$(\vartheta_{K_n^U} R)(x) = 0, \qquad \text{if} \quad R(0) = 0, \tag{90}$$

then

$$(\vartheta_{K^U_{\mu_r + r}} S^U_{\lambda^{(s)}/\lambda^{(s-1)}})(x) = 0$$

unless $\lambda^{(s)} = \lambda^{(s-1)}$.

Hence, Eq. (89) would reduce to

$$S_\lambda^U(x_1, x_2, ..., x_n) = \frac{1}{|q_s(x_r)|_{r,\, s=1}^n} \, |(U m_{\vec{e}_{\lambda_r + r}})(x_s)|_{r,\, s=1}^n.$$

Equation (90) is satisfied when U is the identity (and we obtain the classical formula for the Schur functions). We shall construct in the next section a general class of umbral operators U that satisfy (90).

8.1. *Generalized Schur Functions*

Let $q_0(x) \equiv 1$ and $q_n(x) = x t_{n-1}(x)$, $t_n(x)$, $n \geqslant 0$, being a polynomial sequence ($\deg(t_n(x)) = n$).

Define the map

$$U_q m_{\vec{\alpha}}(X) = \sum_{S_1, S_2, \dots, S_m} \prod_{i=1}^{m} \prod_{s \in S_i} q_i(x_s),$$

where m is the number of nonzero components of $\vec{\alpha}$ and the sum is taken over the m-tuples of pairwise disjoint sets of positive integers $(S_1, S_2, \dots, S_m)$, satisfying $|S_i| = \alpha_i$, $i = 1, 2, \dots, m$. The symmetric functions $m_{\vec{\alpha}}^{U_q}(X)$ can be written in the form $c_{\vec{\alpha}} m_{\vec{\alpha}}(X) +$ terms of lower degree. Then $m_{\vec{\alpha}}^{U_q}(X)$ is a basis. Moreover, since $\hat{m}_{\vec{\alpha}}^{U_q}(X) = \vec{\alpha}!\, m_{\vec{\alpha}}^{U_q}(X)$ is of binomial type, U_q is an umbral operator. Equation (90) is satisfied by U_q since this is equivalent to saying that $\hat{m}_{\vec{\alpha}}^{U_q}(x) = 0$ when $\vec{\alpha}$ has more than one non-zero component.

Let ρ_n be the projection $\Lambda(X) \to \Lambda(x_1, x_2, \dots, x_n)$ and ρ_n^*: $\Lambda(x_1, x_2, \dots, x_n) \to \Lambda(X)^*$ be its adjoint. For a sequence of functionals $\{\tilde{L}_i\}_{i=1}^{n}$ of $\mathbb{C}[x]^* = \Lambda(x)^*$, define the functional $\tilde{L}$ in $\Lambda(x_1, x_2, \dots, x_n)$ by

$$\langle \tilde{L} \mid m_{\vec{\alpha}}(x_1, x_2, \dots, x_n) \rangle = \sum_{\vec{\alpha}^{(1)} + \cdots + \vec{\alpha}^{(n)} = \vec{\alpha}} \prod_{i=1}^{n} \langle \tilde{L}_i \mid m_{\vec{\alpha}^{(i)}}(x) \rangle.$$

It is easy to check that $\rho_n^* \tilde{L} = \prod_{i=1}^{n} \rho_1^* \tilde{L}_i$.

Define the evaluation $\tilde{\varepsilon}_a^{(n)} \in \mathbb{C}[x]^*$ by $\langle \tilde{\varepsilon}_a^{(n)} \mid x^m \rangle = \delta_{m,n} a^m$. Let $c_{n,k}$ be the coefficients connecting $q_k(X)$ with x^n:

$$x^n = \sum_{k=1}^{n} c_{n,k} q_k(x).$$

If

$$\tilde{L}_j := \sum_{n \geqslant j} c_{n,j}\, \tilde{\varepsilon}_1^{(n)},$$

then

$$\langle \tilde{L}_j \mid q_i(x) \rangle = \sum_{n \geqslant j} c_{n,j} \hat{c}_{i,n} = \delta_{i,j}.$$

Let $L_j = \rho_1^* \tilde{L}_j$. If $|\vec{\alpha}| = n$,

$$\langle L_j \mid \hat{m}_{\vec{\alpha}}^{U_q}(X) \rangle = \langle \tilde{L}_j \mid \hat{m}_{\vec{\alpha}}^{U_q}(x) \rangle = \langle \tilde{L}_j \mid \delta_{\vec{\alpha}, \vec{e}_n} q_n(x) \rangle = \delta_{\vec{\alpha}, \vec{e}_j}.$$

We have obtained that $jL_j = K_j^{U_q} = j\varepsilon_1^{(j),\, U_q}$ is the delta system associated to $\hat{m}_{\vec{\alpha}}^{U_q}(X)$. A simple computation will give us

$$\frac{\delta_{A_i} K_j^{U_q}}{j} = \sum_{k \geqslant j-i} c_{k+i,\, j} \varepsilon_1^{(k)}.$$

DEFINITION 8.2. Define $L_{\lambda,\, n}$, $\lambda \in \mathfrak{P}_n \in \Lambda(X)^*$, and $\tilde{L}_{\lambda,\, n} \in \Lambda(x_1, x_2, ..., x_n)^*$ to be the functionals

$$L_{\lambda,\, n} = (\delta_{\vec{A}} K^{U_q})(\varsigma_n(0), \varsigma_n(\lambda)) = \left| \sum_{k \geqslant \lambda_r + r - s} c_{k+s,\, \lambda_r + r} \varepsilon_1^{(k)} \right|_{r,\, s=1}^{n}$$

$$\tilde{L}_{\lambda,\, n} = \rho_n^* L_{\lambda,\, n} = \left| \sum_{k \geqslant \lambda_r + r - s} c_{k+s,\, \lambda_r + r} \tilde{\varepsilon}_1^{(k)} \right|_{r,\, s=1}^{n}.$$

We define the *generalized Schur functions*

$$R_\lambda^q(x_1, x_2, x_3, ..., x_n) = \frac{|t_{\lambda_r + r - 1}(x_s)|_{r,\, s=1}^n}{|x_x^{r-1}|_{r,\, s=1}^n}.$$

THEOREM 8.2. *The symmetric functions $R_\lambda^q(x_1, x_2, ..., x_n)$ form a basis of $\Lambda(x_1, x_2, ..., x_n)$. For every $Q(x_1, x_2, ..., x_n)$ in $\Lambda(x_1, x_2, ..., x_n)$ we have the expansion*

$$Q(x_1, x_2, ..., x_n) = \sum_{\lambda \in \mathfrak{P}_n} \langle \tilde{L}_{\lambda,\, n} \,|\, Q(x_1, x_2, ..., x_n) \rangle \, R_\lambda^q(x_1, x_2, ..., x_n). \tag{91}$$

Proof. By Eq. (83)

$$\vartheta_{\vec{A}}^{\varsigma_n(0)} = \sum_{\lambda \in \mathfrak{P}_n} \vartheta_{\vec{K}^{U_q}}^{\varsigma_n(\lambda)} L_{\lambda,\, n}(D). \tag{92}$$

Applying both sides of (92) to $p_{\vec{\alpha}}(X)$, we get

$$\begin{aligned} &p_{\vec{\alpha}}(X^{(1)} + \cdots + X^{(n)}) \,|\, p_r(X^{(s)})|_{r,\, s=1}^n \\ &\quad = \sum_{\lambda \in \mathfrak{P}_n} \sum_{\vec{\beta} + \vec{\gamma} = \vec{\alpha}} \binom{\vec{\alpha}}{\vec{\beta}, \vec{\gamma}} \langle L_\lambda \,|\, p_{\vec{\beta}}(X) \rangle \, |\vartheta_{K_{\lambda_r + r}^{U_q}}^{X^{(s)}}|_{r,\, s=1}^n \, p_{\vec{\gamma}}(X^{(1)} + \cdots + X^{(n)}). \end{aligned} \tag{93}$$

Since U_q satisfies Eq. (90), for $\vec{\gamma} \neq 0$ we have

$$\begin{aligned} &(|\vartheta_{K_{\lambda_r + r}^{U_q}}^{X^{(s)}}|_{r,\, s=1}^n \, p_{\vec{\gamma}})(x_1, ..., x_n) \\ &\quad = \sum_{\vec{\gamma}^{(1)} + \cdots + \vec{\gamma}^{(n)} = \vec{\gamma}} \binom{\vec{\gamma}}{\vec{\gamma}^{(1)}, ..., \vec{\gamma}^{(n)}} |(\vartheta_{K_{\lambda_r + r}^{U_q}}^{X^{(s)}} p_{\vec{\gamma}^{(s)}})(x_s)|_{r,\, s=1}^n = 0. \end{aligned}$$

Then taking one element alphabets in (93), we obtain

$$
\begin{aligned}
p_{\vec{\alpha}}(x_1, x_2, ..., x_n) &= \sum_{\lambda \in \mathfrak{P}_n} \langle L_{\lambda, n} | p_{\vec{\alpha}}(X) \rangle \frac{|\hat{m}^{U_q}_{\vec{e}_{\lambda_r + r}}(x_s)|^n_{r, s=1}}{|x^r_s|^n_{r, s=1}} \\
&= \sum_{\lambda \in \mathfrak{P}_n} \langle L_{\lambda, n} | p_{\vec{\alpha}}(X) \rangle \frac{|x_s t_{\lambda_r + r - 1}(x_s)|^n_{r, s=1}}{|x^r_s|^n_{r, s=1}} \\
&= \sum_{\lambda \in \mathfrak{P}_n} \langle L_{\lambda, n} | p_{\vec{\alpha}}(X) \rangle R^q_\lambda(x_1, x_2, ..., x_n). \qquad (94)
\end{aligned}
$$

By linearity, for any symmetric function $Q(X)$,

$$
\begin{aligned}
Q(x_1, x_2, ..., x_n) &= \sum_{\lambda \in \mathfrak{P}_n} \langle L_{\lambda, n} | Q(X) \rangle R^q_\lambda(x_1, x_2, ..., x_n) \\
&= \sum_{\lambda \in \mathfrak{P}_n} \langle \tilde{L}_{\lambda, n} | Q(x_1, x_2, ..., x_n) \rangle R^q_\lambda(x_1, x_2, ..., x_n). \qquad (95)
\end{aligned}
$$

The result follows since the map $\rho_n Q(X) = Q(x_1, x_2, ..., x_n)$ is surjective. ∎

PROPOSITION 8.1. *Let C be the infinite lower triangular matrix $C = (c_{i,j})^\infty_{i,j=1}$ of the coefficients connecting the family $q_n(x)$ with the powers x^n, and $\hat{C}$ its inverse. Then, we have the expansions*

$$L_{\lambda, n} = \sum_{\mu \sqsupseteq \lambda} C^{(n)}_{\mu, \lambda} H_\mu \qquad (96)$$

$$H_\lambda = \sum_{\mu \sqsupseteq \lambda} \hat{C}^{(n)}_{\mu, \lambda} L_{\lambda, n} \qquad (97)$$

$$R^q_\mu = \sum_{\lambda \sqsupseteq \mu} \hat{C}_{\mu, \lambda} S_\lambda(x_1, x_2, ..., x_n), \qquad (98)$$

where $C^{(n)}_{\mu, \lambda} = C^{(n)}(\zeta_n(\mu), \zeta_n(\lambda)) = |c_{\mu_r + r, \lambda_s + s}|^n_{r, s=1}$.

Proof. By the chain rule we have $\delta_{\vec{A}} \vec{K}^{U_q} = (\delta_{\vec{A}} \vec{K})(\delta_{\vec{K}} \vec{K}^{U_q}) = (\delta_{\vec{A}} \vec{K})\, C$. We also have $\delta_{\vec{A}} \vec{K} = (\delta_{\vec{A}} \vec{K}^{U_q})\, \hat{C}$. By Cauchy–Binet we obtain (96) and (97). Equation (98) is the dual form of (97). ∎

EXAMPLE 8.1 (Factorial Symmetric Function). For $q_n(x) = x(x)_{n-1}$, the corresponding generalized Schur function

$$R^q_\lambda(x_1, x_2, ..., x_n) = \frac{|(x_s)_{\lambda_r + r - 1}|^n_{r, s=1}}{|x^{r-1}_s|^n_{r, s=1}}$$

is the factorial symmetric function [4, 5, 11, 22, 32]. Since $\sum_{k=1}^{n} \tilde{S}(n,k)(x)_k$,

$$x^n = \sum_{k=1}^{n} \tilde{S}(n-1, k-1)\, q_k(x),$$

where $\tilde{S}(n,k)$ is the ordinary Stirling number of the second kind. The dual pseudobasis is given by

$$\tilde{L}_{\lambda,n} = \left| \sum_{k \geqslant \lambda_r + r - s} \tilde{S}(k+s-1, \lambda_r + r - 1)\, \tilde{\varepsilon}_1^{(k)} \right|_{r,s=1}^{n}.$$

EXAMPLE 8.2 (Increasing Factorial Schur Functions). When $\{q_n(x)\}_{n \geqslant 0}$ is an (ordinary) family of binomial type, $\tilde{L}_j = \tilde{L}^{*j}$ for some fixed functional $\tilde{L} \in \mathbb{C}[x]^*$ satisfying $\langle L \mid 1 \rangle = 0$, $\langle L \mid x \rangle \neq 0$, where the product $*$ is the classical product on the umbral algebra [45]

$$\langle L * M \mid p(x) \rangle = \langle L^x M^y \mid p(x+y) \rangle.$$

For example, if $q_n(x) = \langle x \rangle_n = x(x+1) \cdots (x+n-1)$, $L = \tilde{\varepsilon} - \tilde{\varepsilon}_1$ and $L_j = \sum_{k=0}^{j} \binom{j}{k} (-1)^k \varepsilon_k$. The functional $L_{\lambda,n}$ is given by

$$\tilde{L}_{\lambda,n} = \left| \sum_{k=0}^{\lambda_r + r} \binom{\lambda_r + r}{k} (-1)^k k^s \tilde{\varepsilon}_k \right|_{r,s=1}^{n}.$$

They are the dual pseudobasis of the increasing factorial Schur functions

$$R_\lambda^q = \frac{|\langle x_s - 1 \rangle_{\lambda_r + r - 1}|_{r,s=1}^{n}}{|x_s^{r-1}|_{r,s=1}^{n}}.$$

EXAMPLE 8.3 ((MacDonald 6th Variation on Schur Functions). Let $(a_n)_{n=1}^{\infty}$ be any sequence of integers. Defining $(x \mid a)^r = \prod_{i=1}^{r} (x + a_i)$ and $q_n(x) = x(x \mid a)^{n-1} = \sum_{k=1}^{n} e_{n-k}(a_1, a_2, ..., a_{n-1}) x^k$ we obtain

$$R_\lambda^q(x_1, x_2, ..., x_n) = \frac{|(x_s \mid a)^{\lambda_r + r - 1}|_{r,s=1}^{n}}{|x_s^{r-1}|_{r,s=1}^{n}}$$

which is Macdonald's 6th variation of Schur functions [32]. It generalizes the factorial Schur functions and the α-paired Schur functions [11]. The functionals L_j are obtained by computing the inverse of the triangular matrix $(e_{n-k}(a_1, a_2, ..., a_{n-1}))_{n,k=1}^{\infty}$.

The next proposition is easy to prove, and is left as an exercise to the reader.

PROPOSITION 8.2. *The symmetric functions $R^q_\lambda(x_1, x_2, ..., x_n)$ satisfy the condition*

$$R^q_\lambda(x_1, x_2, ..., x_n, 0) = R^q_\lambda(x_1, x_2, ..., x_n) \qquad \text{for every } \lambda \text{ and } n,$$

if and only if the polynomials $t_n(x)$ are monic and satisfy the divided difference equation

$$\frac{t_n(x) - t_n(0)}{x} = t_{n-1}(x), \qquad n \geqslant 1, \tag{99}$$

EXAMPLE 8.4. The polynomial sequences satisfying (99) are easy to classify. They are determined by the sequence $(a_n)_{n \geqslant 1}$, $a_n = t_n(0)$:

$$t_n(x) = x^n + a_1 x^{n-1} + \cdots + a_n.$$

Then, $q_n(x) = t_n(x) - t_n(0)$. The matrix $\hat{C} = (a_{i-j})_{i,j=1}^{\infty}$ connecting the powers x^n with $q_n(x)$ is triangular Toeplitz. Then

$$R^q_\mu(x_1, x_2, ..., x_n) = \sum_{\lambda \sqsubseteq \mu} |a_{\mu_r - \lambda_s + r - s}|_{r,s=1}^{n} S_\lambda(X).$$

The matrix C is also Toeplitz $C = (b_{i-j})_{i,j=1}^{\infty}$,

$$x^n = \sum_{k=1}^{n} b_{n-k} q_k(x),$$

where

$$b_n = (-1)^n \begin{vmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_2 & a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & 1 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{vmatrix}.$$

Since $\tilde{L}_j = \sum_{k \geqslant 0} b_k \tilde{\varepsilon}_1^{(j+k)}$ and $\delta_{A_i} L_j = L_{j-i}$, the jacobian $\delta_{\vec{A}} \vec{K}^{U_q}$ is the Toeplitz matrix

$$\delta_{\vec{A}} \vec{K}^{U_q} = \begin{pmatrix} L_0 & L_1 & L_2 & \cdots \\ L_{-1} & L_0 & L_1 & \cdots \\ L_{-2} & L_{-1} & L_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The dual pseudobasis is

$$\tilde{L}_{\lambda, n} = \left| \sum_k b_k \tilde{\varepsilon}_1^{(k+\lambda_r+r-s)} \right|_{r, s=1}^{n} = \sum_{\mu \sqsupseteq \lambda} |b_{\mu_r - \lambda_s + r - s}|_{r, s=1}^{n} H_\mu.$$

The inverse limit $R^q_\lambda(X)$ exists for every λ. By Theorem 8.2, they form a basis of $\Lambda(X)$. By Eq. (96) the functionals $L_{\lambda, n}$ do not depend on n, and $L_\lambda = L_{\lambda, n}$ are the dual pseudobasis of R^q_λ.

8.2. *Sheffer–Schur Symmetric Functions*

Let $\{S^U_{\lambda/\mu}(X)\}$ be a family of Schur type. A *Sheffer–Schur* family relative to $\{S_{\lambda/\mu}(X)\}$ is any family of the form

$$R^{N, U}_{\lambda/\mu}(X) = N(D)\, S^U_{\lambda/\mu}(X),$$

where N is any invertible functional. An *Appel–Schur* family is a Sheffer–Schur family relative to the Schur functions.

Obviously, $R^{N, U}_{\lambda/\mu}(X)$ satisfies

$$R^{N, U}_{\lambda/\mu}(X+Y) = \sum_{\mu \sqsubseteq \tau \sqsubseteq \lambda} R^{N, U}_{\tau/\mu}(X)\, S^U_{\lambda/\tau}(X).$$

The symmetric functions $\{R^{N, U}_\lambda(X)\}_{\lambda \in \mathfrak{P}}$ form a basis of $\Lambda(X)$. The dual pseudobasis is $\{N^{-1} H^U_\lambda\}_{\lambda \in \mathfrak{P}}$.

Example 8.5 (The Bernouilli–Schur Functions). The Bernoulli–Schur functions are the elements of the Appel family corresponding to the functional $N = (\sum_{n=1}^\infty A_n/n)/(\varepsilon_1 - \varepsilon)$,

$$S^B_{\lambda/\mu} = \frac{\sum_{n=1}^\infty D_n/n}{E^1 - I} S_{\lambda/\mu}(X).$$

Example 8.6. Assume that N is a multiplicative functional with $\langle N | h_n(X) \rangle = a_n$, $n \geqslant 1$. Consider the Appel sequence $R^N_{\lambda/\mu}(X) = N(D)\, S_{\lambda/\mu}(X)$. The inverse functional N^{-1} satisfies $\langle N^{-1} | h_n \rangle = b_n$, where b_n is as in Example 8.4. We have

$$\begin{aligned}
&\langle N^{-1} H_\lambda | S_\mu(X) \rangle \\
&\quad = \langle H_\lambda | N^{-1}(D)\, S_\mu(X) \rangle \\
&\quad = \sum_{\tau \sqsubseteq \mu} \langle N^{-1} | S_{\mu/\tau}(x) \rangle \langle H_\lambda | S_\tau(X) \rangle = \langle N^{-1} | S_{\mu/\lambda}(x) \rangle \\
&\quad = |b_{\mu_r - \lambda_s + r - s}|_{r, s=1}^{n}.
\end{aligned}$$

Then, $N^{-1}H_\lambda = L_{\lambda, n}$, $L_{\lambda, n}$ being as in Example 8.4. The Appel–Schur functions R_λ^M are exactly the generalized Schur functions of Example 8.4 corresponding to the polynomials

$$t_n(x) = x^n + \langle N \mid h_1(X) \rangle\, x^{n-1} + \langle N \mid h_2(X) \rangle\, x^{n-1} + \cdots + \langle N \mid h_n(X) \rangle.$$

9. TRANSFER FORMULA AND LAGRANGE INVERSION FORMULAS

We introduce in this section the notion of a jacobian determinant for certain kinds of infinite jacobian matrices. Based upon this notion, and using the second recursive formula, we generalize Joni's general transfer formula [24] to an infinite number of variables. By a standard procedure a very general Lagrange inversion formula is obtained as the dual form of the transfer formula. Particular cases of this Lagrange inversion are the plethystic formulas obtained in [29]. By using the recursive formula for diagonal systems we present here a new proof of the inversion formula for an infinite number of variables obtained in [16, Theorem 4].

9.1. *The Jacobian Determinant*

For a jacobian matrix $\delta_{\vec{A}} \vec{M}$, the minor $\delta_{\vec{A}} \vec{M}(S, S)$ is denoted by $\partial \vec{M}^S$.

PROPOSITION 9.1. *For any admissible system* $\vec{M}$ *and any finite set* $S = \{i_1, i_2, ..., i_{k-1}\}$, *we have the identity*

$$\begin{vmatrix} \delta_{i_1} & \delta_{i_1} \dfrac{M_{i_1}}{i_1} & \delta_{i_1} \dfrac{M_{i_2}}{i_2} & \cdots & \delta_{i_1} \dfrac{M_{i_{k-1}}}{i_{k-1}} \\ \delta_{i_2} & \delta_{i_2} \dfrac{M_{i_1}}{i_1} & \delta_{i_2} \dfrac{M_{i_2}}{i_2} & \cdots & \delta_{i_2} \dfrac{M_{i_{k-1}}}{i_{k-1}} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \delta_{i_k} & \delta_{i_k} \dfrac{M_{i_1}}{i_1} & \delta_{i_k} \dfrac{M_{i_2}}{i_2} & \cdots & \delta_{i_k} \dfrac{M_{i_{k-1}}}{i_{k-1}} \end{vmatrix} = 0. \tag{100}$$

Proof. Expanding the determinant by the second column, and using induction on the cardinal of S, we easily get the result. ∎

PROPOSITION 9.2. *Let $\vec{M}$ be a delta system. For a finite set S let $\tilde{\mathscr{C}}^S$ be the inverse of the matrix $\mathscr{B}^S = \delta_{\vec{A}} \vec{M}^S$. Then we have the identity*

$$\sum_{j \in S} \delta_j (\tilde{C}^S_{n,\,j} \partial \vec{M}^S) = 0 \tag{101}$$

for every n in S.

Proof. Let $S = \{i_1, i_2, i_3, ..., i_k\}$ and $n = i_r$, for some $1 \leqslant r \leqslant k$. The left-hand side of (101) is equal to the symbolic determinant

$$\begin{vmatrix} \frac{\delta_{i_1} M_{i_1}}{i_1} & \cdots & \frac{\delta_{i_1} M_{i_{r-1}}}{i_{r-1}} & \delta_{i_1} & \frac{\delta_{i_1} M_{i_{r+1}}}{i_{r+1}} & \cdots & \frac{\delta_{i_1} M_{i_k}}{i_k} \\ \frac{\delta_{i_2} M_{i_1}}{i_1} & \cdots & \frac{\delta_{i_2} M_{i_{r-1}}}{i_{r-1}} & \delta_{i_2} & \frac{\delta_{i_2} M_{i_{r+1}}}{i_{r+1}} & \cdots & \frac{\delta_{i_2} M_{i_k}}{i_k} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \frac{\delta_{i_k} M_{i_1}}{i_1} & \cdots & \frac{\delta_{i_k} M_{i_{r-1}}}{i_{r-1}} & \delta_{i_k} & \frac{\delta_{i_k} M_{i_{r+1}}}{i_{r+1}} & \cdots & \frac{\delta_{i_k} M_{i_k}}{i_k} \end{vmatrix}$$

$$= (-1)^{r-1}$$

$$\times \begin{vmatrix} \delta_{i_1} & \frac{\delta_{i_1} M_{i_1}}{i_1} & \cdots & \frac{\delta_{i_1} M_{i_{r-1}}}{i_{r-1}} & \frac{\delta_{i_1} M_{i_{r+1}}}{i_{r+1}} & \cdots & \frac{\delta_{i_1} M_{i_k}}{i_k} \\ \delta_{i_2} & \frac{\delta_{i_2} M_{i_1}}{i_1} & \cdots & \frac{\delta_{i_2} M_{i_{r-1}}}{i_{r-1}} & \frac{\delta_{i_2} M_{i_{r+1}}}{i_{r+1}} & \cdots & \frac{\delta_{i_2} M_{i_k}}{i_k} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \delta_{i_k} & \frac{\delta_{i_k} M_{i_1}}{i_1} & \cdots & \frac{\delta_{i_k} M_{i_{r-1}}}{i_{r-1}} & \frac{\delta_{i_k} M_{i_{r+1}}}{i_{r+1}} & \cdots & \frac{\delta_{i_k} M_{i_k}}{i_k} \end{vmatrix} = 0. \quad \blacksquare \tag{102}$$

DEFINITION 9.1. An admissible system $\vec{M}$ is called a Schröder system if it is of the form $\vec{M} = \vec{A} - \vec{G}$, where $\langle G_n | 1 \rangle = 0$ and $\langle G_n | p_m(X) \rangle = 0$ for every m and n.

Since $\langle \delta_{\vec{A}} \vec{M} | 1 \rangle = \mathrm{Id}$, by Corollary 7.6 we have that every Schröder system is a delta system.

DEFINITION 9.2. Let $\mathscr{B}$ be a matrix of the form $\mathscr{B} = \delta_{\vec{A}} \vec{M}$, $\vec{M}$ being a Schröder system satisfying

$$\lim_{k \to \infty} \delta_k C_{n,\,k} = 0 \tag{103}$$

where $\mathscr{C}$ is the inverse of $\mathscr{B}$. The determinant of $\mathscr{B}$, $\partial\vec{M}$, is the element of $\Lambda(X)^*$ recursively defined by

$$\begin{aligned}&\langle \partial\vec{M} \mid 1\rangle = 1,\\ &\langle \partial\vec{M} \mid p_{\vec{\alpha}+\vec{e}_n}(X)\rangle = \left\langle \partial\vec{M} \,\middle|\, \sum_k B_{n,k}(D)\left(-\sum_j \hat{\delta}_j C_{k,j}(D)\right) p_{\vec{\alpha}}(X)\right\rangle.\end{aligned} \tag{104}$$

PROPOSITION 9.3. *Let M be a Schröder system as in Definition* 9.2. *Then, the determinant $\partial\vec{M}$ is the unique functional satisfying*

$$\begin{aligned}\langle \partial\vec{M} \mid 1\rangle &= 1\\ \sum_{k=1}^{\infty} \delta_k(C_{n,k}\partial\vec{M}) &= 0.\end{aligned} \tag{105}$$

Proof. We shall prove that Eq. (105) is equivalent to the recursion (104).

From Eq. (105) we obtain

$$\sum_{k=1}^{\infty} (\delta_k C_{n,k})\, \partial\vec{M} + \sum_{k=1}^{\infty} C_{n,k}\delta_k \partial\vec{M} = 0. \tag{106}$$

Then, we have the identity

$$\mathscr{C}(\delta_1\partial\vec{M}, \delta_2\partial\vec{M}, \delta_3\partial\vec{M}, ...)^t = \partial\vec{M}\vec{v}$$

where

$$v_n = -\sum_{k=1}^{\infty} \delta_k C_{n,k},$$

and then

$$\delta_n \partial\vec{M} = \left(\sum_{k=1}^{\infty} B_{n,k} v_k\right)\partial\vec{M}.$$

For every partition $\vec{\alpha}$ we have

$$\langle \delta_n \partial\vec{M} \mid p_{\vec{\alpha}}(X)\rangle = \left\langle \sum_{k=1}^{\infty} B_{n,k} v_k \partial\vec{M} \mid p_{\vec{\alpha}}(X)\right\rangle,$$

which is equivalent to the recursion (104). Reversing the above steps we obtain (105) from (104). ∎

PROPOSITION 9.4. *Let $\vec{M}$ be a Schröder system as in Definition 9.2. If $\delta_{\vec{A}}\vec{M}$ is upper triangular and the product $\prod_{j=1}^{\infty}(\varepsilon-\delta_j G_j)$ is convergent, then $\partial\vec{M}=\prod_{j=1}^{\infty}(\varepsilon-\delta_j G_j)$.*

Proof. By Proposition 9.3 we have only to prove that

$$\sum_k \delta_k\left(C_{n,k}\prod_{j=1}^{\infty}(\varepsilon-\delta_j G_j)\right)=0.$$

Since $\delta_k(C_{n,k}\prod_{j=1}^{\infty}(\varepsilon-\delta_j G_j))\to 0$, for every $\vec{\alpha}$ there exists an integer $K(\vec{\alpha})$ such that

$$\sum_{k=1}^{\infty}\left\langle \delta_k\left(C_{n,k}\prod_{j=1}^{\infty}(\varepsilon(\delta_j G_j)\right)\middle| p_{\vec{\alpha}}(X)\right\rangle$$
$$=\sum_{k=1}^{K(\vec{\alpha})}\left\langle \delta_k\left(C_{n,k}\prod_{j=1}^{\infty}(\varepsilon-\delta_j G_j)\right)\middle| p_{\vec{\alpha}}(X)\right\rangle.$$

Since $\partial\vec{M}^{(r)}=\prod_{j=1}^{r}(\varepsilon-\delta_j G_j)\to\prod_{j=1}^{\infty}(\varepsilon-\delta_j G_j)$, there exists an integer r_0 such that if $r>r_0$,

$$\langle\partial\vec{M}^{(r_0)}\mid p_{\vec{\beta}+\vec{e}_k}(X)\rangle$$
$$=\left\langle\prod_{j=1}^{\infty}(\varepsilon-\delta_j G_j)\mid p_{\vec{\beta}+\vec{e}_k}(X)\right\rangle,\qquad k=1,2,...,K(\vec{\alpha}),\quad \vec{\beta}\leqslant\vec{\alpha}.$$

Taking $l=\max\{r_0,K(\vec{\alpha})\}$ we have

$$\sum_{k=1}^{\infty}\left\langle \delta_k\left(C_{n,k}\prod_{j=1}^{\infty}(\varepsilon-\delta_j G_j)\right)\middle| p_{\vec{\alpha}}(X)\right\rangle$$
$$=\sum_{k=1}^{l}\left\langle \delta_k\left(C_{n,k}\prod_{j=1}^{l}(\varepsilon-\delta_j G_j)\right)\middle| p_{\vec{\alpha}}(X)\right\rangle. \tag{107}$$

Since $\delta_{\vec{A}}\vec{M}$ is upper triangular, the matrix $\tilde{\mathscr{C}}^{(l)}=(\mathscr{B}^{(l)})^{-1}$ is equal to $\mathscr{C}^{(l)}=(\mathscr{B}^{-1})^{(l)}$. Then, by Proposition 9.2, the right-hand side of (107) is equal to zero. ∎

In the same way we prove

PROPOSITION 9.5. *Let $\vec{M}$ be a Schröder system. If $\delta_{\vec{A}}\vec{M}$ is lower triangular then $\prod_{j=1}^{\infty}(\varepsilon-\delta_j G_j)$ is convergent and $\partial\vec{M}=\prod_{j=1}^{\infty}(\varepsilon-\delta_j G_j)$.*

THEOREM 9.1 (Transfer Formula). *Let $\partial\vec{M}=\vec{A}-\vec{G}$ be a Schröder system as in Definition 9.2. Then the umbral operator $U_{\vec{M}\langle -1\rangle}$ is given by the formula*

$$U_{\vec{M}\langle -1\rangle}=\sum_{\vec{\alpha}}\partial\vec{M}(D)\,\frac{\vec{G}^{\vec{\alpha}}(D)}{z_{\vec{\alpha}}}\,\vartheta_{\vec{A}}^{\vec{\alpha}}. \tag{108}$$

Proof. Denote by T the right-hand side of Eq. (108). Let $q_{\vec{\alpha}}(X)$ be the binomial family associated to $\vec{M}$. We have to prove that $U_{\vec{M}\langle -1\rangle}p_{\vec{\alpha}}(X)=q_{\vec{\alpha}}(X)$ for every $\vec{\alpha}$. By an inductive argument we have only to prove that

$$T1=1$$

and

$$\vartheta_{\vec{M}}^{\vec{e}_n}T=T\vartheta_n.$$

Since $\vec{G}^{\vec{\beta}}(D)\,p_{\vec{\beta}}(X)=0$, $\vec{\beta}\neq\vec{0}$, and $\partial\vec{M}(D)\,1=1$, we obtain $T1=1$. Using the second recursive formula (66)

$$\begin{aligned}\vartheta_{\vec{M}}^{\vec{e}_n}T&=\sum_{k,\vec{\alpha}}\vartheta_n C_{n,k}(D)\,\frac{\partial\vec{M}}{z_{\vec{\alpha}}}\,\vec{G}^{\vec{\alpha}}(D)\,\vartheta_{\vec{A}}^{\vec{\alpha}}\\&=\sum_{k,\vec{\alpha}}\left[C_{n,k}(D)\,\frac{\partial\vec{M}}{z_{\vec{\alpha}}}\,\vec{G}^{\vec{\alpha}}(D)\,\vartheta_{\vec{A}}^{\vec{\alpha}+\vec{e}_k}\right.\\&\qquad\left.-\frac{\hat{\delta}_k(C_{n,k}(D)\,\partial\vec{M}(D)\,\vec{G}^{\vec{\alpha}}(D))}{z_{\vec{\alpha}}}\,\vartheta_{\vec{M}}^{\vec{\alpha}}\right],\end{aligned} \tag{109}$$

the last identity follows by applying formula (57).

Using Eq. (105) we obtain

$$\begin{aligned}&\sum_{k,\vec{\alpha}}\hat{\delta}_k\left(C_{n,k}(D)\,\frac{\partial\vec{M}}{z_{\vec{\alpha}}}\,\vec{G}^{\vec{\alpha}}(D)\right)\vartheta_{\vec{A}}^{\vec{\alpha}}\\&\quad=\sum_{k,\vec{\alpha}}\left[\hat{\delta}_k\left(C_{n,k}(D)\,\frac{\partial\vec{M}}{z_{\vec{\alpha}}}\right)\vec{G}^{\vec{\alpha}}(D)\,\vartheta_{\vec{A}}^{\vec{\alpha}}+\frac{C_{n,k}(D)\,\partial\vec{M}(D)}{z_{\vec{\alpha}}}\right.\\&\qquad\left.\times\sum_i i\alpha_i\vec{G}^{\vec{\alpha}-\vec{e}_i}(D)\,\hat{\delta}_iG_k(D)\,\vartheta_{\vec{A}}^{\vec{\alpha}}\right]\\&\quad=\sum_{k,\vec{\alpha},i}\left(C_{n,k}(D)\,\frac{\partial\vec{M}}{z_{\vec{\alpha}}}\,\vec{G}^{\vec{\alpha}}(D)\,\hat{\delta}_kG_i(D)\right)\vartheta_{\vec{A}}^{\vec{\alpha}+\vec{e}_i}.\end{aligned}$$

Substituting in (109) and rearranging the sum we get

$$\begin{aligned}\vartheta_{\vec{M}}^{\vec{e}_n} T &= \sum_{\vec{\alpha},\, i} \frac{\partial \vec{M}}{z_{\vec{\alpha}}} \vec{G}^{\vec{\alpha}}(D) \sum_{k} C_{n,k}(D)\, (\delta(k, i) \\ &= \hat{\delta}_k G_i(D))\, \vartheta_{\vec{A}}^{\vec{\alpha}+\vec{e}_i} \\ &= \sum_{\vec{\alpha}} \frac{\partial \vec{M}(D)}{z_{\vec{\alpha}}} \vec{G}^{\vec{\alpha}}(D)\, \vartheta_{\vec{M}}^{\vec{\alpha}+\vec{e}_n} = T\vartheta_n. \qquad \blacksquare\end{aligned}$$

Corollary 9.1 (Lagrange Inversion Formula). *If $\vec{M}(X)$ is a system of symmetric series such that $\vec{M}(A)$ is as in Definition 9.2 and $L(X)$ is any symmetric series, then*

$$L(\vec{M}^{\langle -1\rangle})(X) = \sum_{\vec{\alpha}} \frac{1}{\vec{\alpha}!} \left(\prod_{k} \partial_{p_i}^{\alpha_i} \right) (\partial \vec{M}(X)\, \vec{G}^{\vec{\alpha}}(X)\, L(X)). \tag{110}$$

Proof. Denote by $\vec{M}$ the system $\vec{M}(A)$. Taking adjoints in formula (108) we get

$$U^{*}_{\vec{M}^{\langle -1\rangle}} = \sum_{\vec{\alpha}} \delta_{\vec{A}}^{\vec{\alpha}} \cdot m_{\{\partial \vec{M} \vec{G}^{\vec{\alpha}}/z_{\vec{\alpha}}\}}.$$

Applying $U^{*}_{\vec{M}^{\langle -1\rangle}}$ to the functional $L = L(A)$ we obtain

$$U_{\vec{M}^{\langle -1\rangle}} L = L(\vec{M}^{\langle -1\rangle}) = \sum_{\vec{\alpha}} \left(\delta_{\vec{A}}^{\vec{\alpha}} \cdot m_{\{\partial \vec{M} \vec{G}^{\vec{\alpha}}/z_{\vec{\alpha}}\}} \right) (L) = \sum_{\vec{\alpha}} \delta_{\vec{A}}^{\vec{\alpha}} \left(\frac{L \partial \vec{M} \vec{G}^{\vec{\alpha}}}{z_{\vec{\alpha}}} \right).$$

Taking indicators we get the result. $\blacksquare$

The jacobian matrix of a delta system of the form $(M, F_2 M, F_3 M, \ldots)$ is lower triangular. Then, by Proposition (9.5) formula (110) generalizes Theorem B in [29].

Proposition 9.6 (Transfer Formula for Diagonal Systems). *Assume that $\{q_{\vec{\tau}}(X)\}_{\vec{\tau}}$ is the associated sequence of a diagonal system $M_k = A_k P_k$, each P_k being of the form $P_k = \varepsilon - \tilde{P}_k$. Then, for every partition $\vec{\alpha}$,*

$$q_{\vec{\alpha}}(X) = \partial \vec{M}^{S}(D)\, \vec{P}^{((\vec{\alpha}+\vec{e}_S))}(D)\, p_{\vec{\alpha}}(X) \tag{111}$$

where $S = \mathrm{supp}(\vec{\alpha})$ and $\vec{e}_S = \sum_{s \in S} \vec{e}_s$.

Proof. Using the recursive formula for diagonal systems (7.2) and Proposition 9.2, by a procedure similar to that use in the Proof of Proposition 9.1 we obtain

$$q_{\vec{\alpha}} = \sum_{l(\vec{\beta}) < l(\vec{\alpha})} \partial \vec{M}^S(D) \frac{D_{\vec{\beta}} \vec{P}^{\vec{\beta}}(D)}{z_{\vec{\beta}}} p_{\vec{\beta}+\vec{\alpha}}(X). \tag{112}$$

The rest of the proof is as given in [24, Theorem 3.1.] ∎

We easily obtain the dual form of the previous proposition.

COROLLARY 9.2 (Lagrange Inversion Formula for Diagonal Systems). *If $\vec{M}(X)$ is a system of symmetric series such that $\vec{M}(A)$ is as in Proposition* 9.6 *and $L(X)$ is any symmetric series, then*

$$L(\vec{M}^{\langle -1 \rangle})(X)|_{p_{\vec{\alpha}}(X)} = L(X)\, \partial \vec{M}^S(X)\, \vec{P}^{-(\vec{\alpha}+\vec{e}_S)}(X)|_{p_{\vec{\alpha}}(X)}. \tag{113}$$

EXAMPLE 9.1. The plethystic Abel polynomials $\mathscr{A}_{\vec{\alpha}}(X, -1)$ are defined as the associated sequence of the system $F_k D_1 E^{(-)1} = D_k E^{(-)\Upsilon_k}$. According to the transfer formula for diagonal systems we have

$$\mathscr{A}_{\vec{\alpha}}(X, -1) = \left(\prod_{s \in S} E^{(-)\Upsilon_s}(I - D_s) \right) E^{\Upsilon_{\vec{\alpha}+\vec{e}_S}} p_{\vec{\alpha}}(X),$$

where $\Upsilon_{\vec{\alpha}}$ is the multiset defined in Example 5.3. Then

$$\begin{aligned}
\mathscr{A}_{\vec{\alpha}}(X, (-)1) &= \prod_{s \in S} (I - D_s)\, E^{\Upsilon_{\vec{\alpha}}} p_{\vec{\alpha}}(X) \\
&= \sum_{\varnothing \subseteq J \subseteq S} (-1)^{|J|}\, D^{\vec{e}_J} p_{\vec{\alpha}}(X + \Upsilon_{\vec{\alpha}}) \\
&= \sum_{\varnothing \subseteq J \subseteq S} (-1)^{|J|} \left(\prod_{j \in J} j\alpha_j \right) p_{\vec{\alpha}-\vec{e}_J}(X + \Upsilon_{\vec{\alpha}}) \\
&= p_{\vec{\alpha}-\vec{e}_S}(X + \Upsilon_{\vec{\alpha}}) \sum_{\varnothing \subseteq J \subseteq S} (-1)^{|J|} \left(\prod_{j \in J} j\alpha_j \right) p_{\vec{e}_{S-J}}(X + \Upsilon_{\vec{\alpha}}) \\
&= p_{\vec{\alpha}-\vec{e}_S}(X + \Upsilon_{\vec{\alpha}}) \prod_{s \in S} (p_s(X + \Upsilon_{\vec{\alpha}}) - s\alpha_s).
\end{aligned}$$

In [9], Chen defined a family of plethystic Abel polynomials. His construction is analogous to ours. However, the combinatorial meaning of the present plethystic Abel polynomials is different from that in the Chen construction. In [9], the Abel polynomials are the generating function of forests of plethystic trees. Here, the coefficient connecting the Abel polynomial $\mathscr{A}_{\vec{\alpha}}(X, (-)\, 1)$ with $p_{\vec{\beta}}(X)$ is the number of rooted forests kept fixed by a permutation of type $\vec{\alpha}$, where the type of the permutation induced on the partition associated to the forest is $\vec{\beta}$. Such a coefficient is equal to

$$\begin{aligned}
&\langle A^{\vec{\beta}} \mid \mathscr{A}_{\vec{\alpha}}(X) \rangle / z_{\vec{\beta}} \\
&\quad = D^{\vec{\beta}}(\mathscr{A}_{\vec{\alpha}})(0)/z_{\vec{\beta}} \\
&\quad = \sum_{\substack{\varnothing \subseteq J \subseteq \operatorname{supp}(\vec{\beta}) \\ \vec{e}_{S-J} \leqslant \vec{\alpha} - \vec{\beta}}} \binom{\vec{\alpha} - \vec{e}_S}{\vec{\beta} - \vec{e}_J, \vec{\alpha} - \vec{\beta} - \vec{e}_{S-J}} \\
&\qquad \times \frac{\prod_{s \in S} (\sum_{d \mid s} d\alpha_d)^{\alpha_s - \beta_s} \prod_{s \in S-J} (\sum_{d \mid s,\, d \neq s} d\alpha_d)}{\prod_{s \in S-J} (\sum_{d \mid s} d\alpha_d)}.
\end{aligned}$$

The number of forests kept fixed by a permutation of type $\vec{\alpha}$ is

$$\begin{aligned}
\mathscr{A}_{\vec{\alpha}}(1) &= p_{\vec{\alpha} - \vec{e}_S}(1 + \Upsilon_{\vec{\alpha}}) \prod_{s \in S} \left(1 + \sum_{d \mid s} s\alpha_s\right) \\
&= \sum_{\vec{e}_S \leqslant \vec{\beta} \leqslant \vec{\alpha}} \binom{\vec{\alpha} - \vec{e}_S}{\vec{\beta} - \vec{e}_S, \vec{\alpha} - \vec{\beta}} \prod_{s \in S} \left(\sum_{d \mid s} d\alpha_d\right)^{\vec{\alpha}_s - \vec{\beta}_s} \left(1 + \sum_{d \mid s,\, d \neq s} d\alpha_d\right).
\end{aligned} \tag{114}$$

See [12] for an equivalent form of formula (114).

The Abel plethystic family $\mathscr{A}_{\vec{\alpha}}(X, -\mathfrak{M})$ is defined as the binomial sequence associated to the delta system $F_k D_1 E^{(-)\,\mathfrak{M}}$. From Theorem 4.11 we obtain that for any multiset $\mathfrak{M}$, $F_k E^{\mathfrak{M}} = E^{\mathfrak{M}^{1/n}}$, where $\mathfrak{M}^{1/n}$ is the multiset whose elements are the nth complex roots of the elements of $\mathfrak{M}$. Denote by $\mathfrak{M}_{\vec{\alpha}}$ the multiset defined as the disjoint union of $\vec{\alpha}_1$ copies of $\mathfrak{M}^{1/1}$, α_2 copies of $\mathfrak{M}^{1/2}$, etc. Following the same procedure as above, we obtain

$$\begin{aligned}
\mathscr{A}_{\vec{\alpha}}(X, (-)\, \mathfrak{M}) &= \prod_{s \in S} (I - p_1(\mathfrak{M})\, D_s)\, E^{\mathfrak{M}_{\vec{\alpha}}} p_{\vec{\alpha}}(X) \\
&= p_{\vec{\alpha} - \vec{e}_S}(X + \mathfrak{M}_{\vec{\alpha}}) \prod_{s \in S} (p_s(X + \mathfrak{M}_{\vec{\alpha}}) - s\alpha_s p_1(\mathfrak{M})).
\end{aligned}$$

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