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On the Injectivity and Linearization of the Coefficient-to-Solution Mapping for Elliptic Boundary Value Problems

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1. INTRODUCTION

In this paper we study the coefficient-to-solution mapping $a \to u(a)$, where $u(a) = u$ is a solution of

$$-\operatorname{div}(a \operatorname{grad} u) = f \qquad \text{in } \Omega, \tag{1.1}$$

with Ω a bounded domain and f a known function. Conditions will be given that guarantee injectivity of the mapping $a \to u(a)$ at a reference coefficient $\tilde{a}$, i.e., $u(\tilde{a}) = u(a)$ implies $\tilde{a} = a$ provided that $u(\tilde{a})$ or alternatively f satisfies appropriate conditions. Moreover a priori estimates for the coefficient a in terms of u will be derived. We also investigate the linearization T of the coefficient-to-solution mapping, and exhibit topologies for the coefficient—as well as for the solution space such that T is continuously invertible on its range.

Several authors have obtained results on the injectivity of $a \to u(a)$ and we briefly discuss some of them. In [R] Richter considers (1.1) as a hyperbolic equation for a and, using method of characteristics, he derives a

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bound on $|a|_{L^\infty}$ in terms of u and f and from knowledge of a on the "inflow boundary," provided that

$$\inf_{x \in \Omega} [\max |\nabla u(x)|, \Delta u(x)] > 0. \tag{1.2}$$

Chicone and Gerlach in [CG] also use the method of characteristics, to obtain a subset of Ω over which the coefficient a is uniquely determined by knowledge of $u(a)$. In [F] Falk obtains error estimates for Galerkin approximations to the least squares formulation of estimating the coefficient a from the state u, under the assumption that there exists a constant unit vector $\mathbf{v}$ and a constant σ such that $\nabla u(x) \cdot \mathbf{v} \geqslant \sigma > 0$ for all $x \in \Omega$. In [KL], Kohn and Lowe introduce a numerical technique based on a variational formulation for the estimation of a form u and discuss its approximation by means of Galerkin discretization. One of their rate of convergence results assumes condition (1.2). Its proof can be modified to obtain an a priori estimate on a in the L^2-norm in terms of u in the $H^2(\Omega)$-norm, in dimensions 2 and 3. In [K], a priori estimates on the coefficient a in terms of the solution u are obtained, where the coefficient space is endowed with the weighted seminorm $|a \nabla u|_{L^1}$, which realizes the fact that, without further assumptions, a cannot be determined from u over the singular set $S = \{x \in \Omega : \nabla u(x) = 0\}$.

From these results it follows that injectivity of $a \to u(a)$ is an exceptional situation and moreover that a can be bounded by $u(a)$ only if a coarser topology is used for the coefficient than for the solution space. The latter is referred to as ill-posedness of the inverse problem of identifying a from $u(a)$.

In this paper we concentrate on special situations which guarantee injectivity of $a \to u(a)$ at a reference coefficient. This is part of our project of devising optimal input functions f which maximize the robustness of estimating a from $u(a)$ [IK2]. We distinguish between the smooth and the rough case. In the rough case (Sections 2 and 3), $u \in H^1(\Omega)$ and $|\nabla u(x)|$ is assumed to be different from 0 a.e. on Ω, which may typically hold, if f is a sum of point source functions in case $n = 1$ or line sources if $n = 2$. In the smooth case (Section 4), u is assumed to be in $H^3(\Omega)$ and Ω is the countable union of subdomains Ω_i, for each of which a condition of the type (1.2) is assumed to hold. Our analysis uses techniques developed in [KL]. In the rough case we obtain a priori estimates on the L^2-norm of the coefficients involving the $W^{1,p}$-norm of the solutions, whereas in the smooth case, the a priori estimates of the coefficients in the L^2-norm require the $H^2(\Omega)$-norm of the solutions. We also investigate the linearization of the coefficient-to-solution mapping T at a reference coefficient $\tilde{a}$. Besides being of interest in its own right, this analysis is motivated by our study of devising optimal inputs. In [CK], for example, it was shown that for the stability of the least squares formulation of estimating a from data z (corresponding to the solution $u(a^*)$ at the "true" coefficient a^*), given by

$$\min_{a \in \mathscr{A}} |u(a)-z|_Y^2,$$

where $\mathscr{A}$ is the space of admissible parameters, and Y denotes the output norm, it is essential that $T=u'(a^*)$ satisfies a lower bound of the type

$$|Th|_Y \geqslant k\ |h|_X \tag{1.3}$$

for some $k>0$ independent of $h \in X$. Here $(X, |\cdot|_X)$ and $(Y, |\cdot|_Y)$ denote the coefficient and the solution space, respectively, and $u'(a)$ stands for the Fréchet derivative of $a \to u(a)$ at a. From the theory of closed linear operators we recall that (1.3) implies continous invertibility of T on its range $R(T)$, which is known to be closed in this case. In a further study we propose to maximize $\inf_{h \neq 0} |Th|_Y/|h|_X$ be appropriate choice of f. Due to the ill-posed nature of the problem of determining a from u, one cannot take the same topology for X and Y in (1.3). The X-topology necessarily has be coarser than the Y-topology, and it is desirable that the gap between these two is as small as possible. In the rough case conditions will be given which guarantee that modifications of (1.3) hold with $Y=H^1(\Omega)$ and $X=L^2(\Omega)$, and in the smooth case (1.3) can be obtained for $Y=H^2(\Omega)$ and $X=L^2(\Omega)$. These topologies imply that the linearized coefficient-to-solution mapping is continuously invertible. The choice of appropriate topologies for the output (in our case solution) space and for the parameter space as a means of formulating inverse problems in a stable way is also well known in the theory of linear inverse problems. We refer to [L, Chap. 3.6] in this respect. While exhibiting topologies for which the inverse of an ill-posed problem is stable can be of help in understanding the degree of ill-posedness, it is clearly not a technique which itself eliminates, e.g., numerical difficulties that may arise due to ill-posedness.

II. Rough-Case, One Space Dimension

1. *Identifiability, Arbitrary Boundary Conditions*

Let (a, u, f) be functions related via

$$\langle au_x, \varphi_x \rangle = \langle f, \varphi \rangle, \qquad \text{for all} \quad \varphi \in H_0^1. \tag{2.1}$$

Here $\langle \cdot, \rangle$ denotes the inner product in L^2 as well as the duality pairing between H_0^1 and H^{-1}, $f \in H^{-1}$ is fixed throughout and all function spaces are considered over the interval (0, 1). Observe that (2.1) is formally equivalent to

$$-(au_x)_x = f \qquad \text{on } (0, 1);$$

and boundary conditions on u are not specified. A pair $(a, u) \in L^2 \times H^1$ is called a solution of (2.1) if $au_x \in L^2$ and if (2.1) holds. Conditions will be given which, for fixed u, imply uniqueness of the first coordinate in the solution pair (a, u) of (2.1).

We require the hypothesis

$$u_x(x) \neq 0 \qquad \text{a.e. on } (0, 1). \tag{H1}$$

THEOREM 2.1. *Let* (H1) *hold, and let* (a, u), (b, u) *be solutions of* (2.1).

(i) *If* $a-b \in C$, $u \in W^{1,\infty}$, $u \notin C^1$, *then* $a=b$.

(ii) *If there exists* $s \in (0, 1)$ *such that* $\lim_{x \to s} (a-b)(x)$ *exists and* $\lim_{x \to s^+} u_x(x)$ *and* $\lim_{x \to s^-} u_x(x)$ *exist and differ, then* $a=b$ *a.e.*

(iii) *If there exists* $s \in [0, 1]$ *such that* $\lim_{x \to s} (a-b)(x)$ *exists and* $\lim_{x \to s} u_x(x) = \pm\infty$, *or* $\lim_{x \to s} u_x(x) = 0$ *then,* $a=b$ *a.e.*

Proof. Observe that (2.1) implies

$$\langle (a-b)\, u_x, \varphi_x \rangle = 0 \qquad \text{for all} \quad \varphi \in H_0^1$$

and hence

$$(a-b)\, u_x = C, \qquad \text{a.e. on } (0, 1), \tag{2.2}$$

for some constant C [B, p. 122].

To verify (i) assume that there exists $x_0 \in [0, 1]$ such that $a(x_0) \neq b(x_0)$. If $a(x) \neq b(x)$ for all $x \in [0, 1]$, then $u_x = C/(a-b)$, which contradicts $u \notin C^1$. Otherwise there exists an open (relative to $[0, 1]$) interval $I \subset [0, 1]$ such that $x_0 \in I$ and for at least one of the endpoints y of I, $(a-b)(y) = 0$. This contradicts $u_x \in W^{1,\infty}$ and hence $a=b$. The proofs of (ii) and (iii) follow from (2.2).

EXAMPLE 2.2. Let $f = \delta(1/2)$ be the Dirac delta function with weight at 1/2 and let

$$u(x) = \begin{cases} x & \text{on } [0, 1/2] \\ 1-x & \text{on } [1/2, 1]. \end{cases}$$

We further define the family of functions

$$a_\alpha(x) = \begin{cases} 1-\alpha & \text{on } [0, 1/2) \\ 1+\alpha & \text{on } [1/2, 1]. \end{cases}$$

Then (a_α, u) is a solution of (2.1) for every $\alpha \in \mathbb{R}$ and u satisfies the assumptions of Theorem 2.1(i) and (ii). If the additional requirement of continuity of a at $x = 1/2$ is enforced, then a_0 is unique among all such L^2-functions for which (a, u) is a solution of (2.1).

2. *A Priori Estimate*

Let (a, u) and (b, v) denote two solutions of (2.1). In this subsection we derive a priori estimates on $a-b$ in terms of $u-v$. Two additional hypotheses are required.

$$\text{There exists } \kappa > 0 \text{ such that } |u_x(x)| \geqslant \kappa \text{ for a.e. } x \in (0, 1). \tag{H2}$$

$$\text{There exists } w \in L^\infty \text{ such that } \int_0^1 w u_x^{-1}\, dx \neq 0 \text{ and } \int_0^1 (a-b)\, w\, dx = 0. \tag{H3}$$

Remark 2.3. While (H2) is a restrictive assumption, it is satified for a wide class of solutions of two point boundary value problems, if the input f is sufficiently singular. A specific example is given in Example 2.2. More generally one may observe that the weak solution $u \in H_0^1$ of

$$-u_{xx} = \sum_{i=1}^{l} \beta_i\, \delta(x_i)$$
$$u(0) = u(1) = 0,$$

where the x_i are pairwise different elements of $(0, 1)$ and $\beta_i \in \mathbb{R}$, is a piecewise linear function. In this case (H2) holds, if u is not constant on any subinterval of $(0, 1)$. Condition (H3) can be considered as assuming a priori knowledge of $a-b$ on a one dimensional subspace. In view of the one dimensional kernel of the differentiation operator which acts on au_x in the strong form of (2.1) this is a natural assumption.

THEOREM 2.4. *Let* $(a, u) \in L^2 \times W^{1,\infty}$ *and* $(b, v) \in L^2 \times W^{1,\infty}$ *be solutions of* (2.1) *and assume that* (H2), (H3) *hold. Then there exists a constant* K, *independent of* (b, v) *and* a *such that*

$$|a-b|_{L^2} \leqslant K\, |b|_{L^2}\, |u_x - v_x|_{L^\infty}.$$

Proof. Let Δ denote the Laplace operator from H_0^1 to H^{-1}, let D stand for differentiation, and let $P = D\Delta^{-1}D$, considered as an operator on C^∞. It is simple to argue that P has a unique extension as a bounded linear operator on L^2 [IK1], which will be again denoted by P. Moreover, P is an orthogonal projection on L^2 with $\ker P = \{\varphi : \varphi = \text{constant}\}$, $P\varphi = \varphi - \int_0^1 \varphi(s)\, ds$, and $|P\varphi|_{L^2}^2 = |\varphi|_{L^2}^2 - (\int_0^1 \varphi(s)\, ds)^2$. From (2.1) we have

$$\langle (a-b)\, u_x, \varphi_x \rangle + \langle b(u_x - v_x), \varphi_x \rangle = 0, \qquad \text{for all} \quad \varphi \in H_0^1$$

and hence

$$(a-b)\, u_x + b(u_x - v_x) = \text{constant}.$$

and

$$P((a-b)\,u_x) = P(b(v_x - u_x)).$$

From (2.3) we conclude

$$|(a-b)\,u_x|_{L^2}^2 - \left(\int_0^1 (a-b)\,u_x\,dx\right)^2 \leqslant |b|_{L^2}^2\,|u_x - v_x|_{L^\infty}^2. \tag{2.4}$$

Let us put $h = a - b$, and let $\beta \in \mathbb{R}$ be arbitrary. Then using (H3) we have

$$\begin{aligned}\left(\int_0^1 hu_x\,dx\right)^2 &= \left(\int_0^1 h(u_x - \beta w)\,dx\right)^2\\ &\leqslant |hu_x|_{L^2}^2 \int_0^1 \left(1 - \frac{\beta w}{u_x}\right)^2 dx\\ &\leqslant |hu_x|_{L^2}^2 \left[1 - 2\beta \int_0^1 \frac{w}{u_x}\,dx + \beta^2 \int_0^1 \frac{w^2}{u_x^2}\,dx\right].\end{aligned}$$

Let $\beta = \int_0^1 wu_x^{-1}\,dx / \int_0^1 w^2 u_x^{-2}\,dx$, where we observe that $\int_0^1 w^2 u_x^{-2}\,dx \neq 0$ due to (H2) and (H3).

Then for $\lambda := 1 - (\int_0^1 wu_x^{-1}\,dx)^2 / \int_0^1 w^2 u^{-2}\,dx$ we have $\lambda \in (0, 1)$ by (H3) and $(\int_0^1 hu_x\,dx)^2 \leqslant \lambda\,|hu_x|_{L^2}^2$. Inserting this estimate in (2.4) we obtain $(1-\lambda)\,|(a-b)\,u_x|_{L^2}^2 \leqslant |b|_{L^2}^2\,|u_x - v_x|_{L^\infty}^2$. The desired estimate now follows from (H2).

Remark 2.5. The results of this section remain unchanged if the equation contains a convection—and a restoring force term, so that the weak form of the equation is given by

$$\langle au_x, \varphi_x\rangle + \langle a_1 u_x + a_2 u, \varphi\rangle = \langle f, \varphi\rangle, \qquad \text{for all} \quad \varphi \in H_0^1,$$

with a_1, a_2 given.

3. *The Linearization of the Parameter-to-Solution Mapping*

In this subsection we study properties of the linearization of the parameter-to-solution mapping of two point boundary value problems. Besides being of interest in their own right, these properties are essential for the analysis of maximizing the sensitivity of the parameter-to-output mapping by choice of an appropriate inhomogeneity f [IK2]. For ease of presentation we restrict ourselves to homogeneous Dirichlet boundary conditions and consider

$$\begin{aligned} -(au_x)_x &= f \qquad \text{on } (0, 1)\\ u(0) &= u(1) = 0. \end{aligned} \tag{2.5}$$

It is well known that for every $f \in H^{-1}$ and $a \in L^\infty$, with $a(x) \geq \gamma > 0$, for a.e. $x \in (0, 1)$, there exists a unique weak solution $u \in H_0^1$ of (2.5). It can be shown that the mapping $a \to u(a)$ is Fréchet differentiable from L^∞ to H^1 (and a fortiori from L^∞ to L^2), and that the Fréchet derivative $\eta = u'(a)(h)$ of u at a in direction h is characterized by

$$\begin{aligned} -(a\eta_x)_x &= (hu_x(a))_x \\ \eta(0) &= \eta(1) = 0. \end{aligned} \tag{2.6}$$

Let $A(a)\colon H_0^1 \to H^{-1}$ denote the operator given by

$$A(a)\,\varphi = -(a\varphi_x)_x,$$

and observe that $A(a)$ is an isomorphism from H_0^1 onto H^{-1}. Using the operator $A(a)$, the Fréchet derivative can be expressed as

$$u'(a)\,h = A^{-1}(a)\,(hu_x(a))_x.$$

Let us introduce the operator $Th = A^{-1}(a)(hu_x(a))_x$, with dom $T = L^\infty$. If $u \in W^{1,\infty}$, then T has a unique extension as a bounded linear operator from L^2 to H^1 and a fortiori from L^2 into itself, which will again be denoted by T.

THEOREM 2.6. *Assume that* $u \in W^{1,\infty}$.

(i) *The operator T considered from L^2 to L^2 and from H^1 to H_0^1 is a compact linear operator.*

(ii) *The operator $T\colon L^2 \to H_0^1$ is bounded linear operator. If* (H2), (H3), *with $a - b$ replaced by h hold, then there exists $k_1 > 0$ such that*

$$|Th|_{H_0^1} \geq k_1\,|h|_{L^2}$$

for all $h \in L^2$ with $\int_0^1 hw\,dx = 0$; in particular this implies that $R(T)$, the range of T restricted to $\{h \in L^2 : \int_0^1 hw\,dx = 0\}$, is a closed subset of H_0^1, and that T is continuously invertible on $R(T)$.

(iii) *Assume that* (H1) *holds and that there exists $s \in (0, 1)$ such that $\lim_{x \to s^+} u_x(x)$ and $\lim_{x \to s^-} u_x(x)$ exists and differ (or $\lim_{x \to s} u_x(x) = \pm\infty$ or $\lim_{x \to s} u_x(x) = 0$). Then for every $r > 0$ there exists $\tilde{k}_r$ such that*

$$|Th|_{H_0^1} \geq \tilde{k}_r$$

for all $h \in H^1$ with $|h|_{H^1} \leq r$ and $|h|_{L^2} = 1$.

Proof. In view of the fact that $A(a): H_0^1 \to H^{-1}$ is an isomorphism, and due to the compact embedding of H_0^1 into L^2, T is compact from L^2 into L^2. Since (the extension) of T is continuous from L^2 into H^1, compactness of the embedding $H^1 \to L^2$ implies that T is also a compact operator from H^1 into H_0^1. This (i) is verified.

To verify (ii) it suffices to establish the lower bound for T. Observe that

$$|Th|_{H_0^1} = |DA^{-1}(a)\, D(hu_x(a))|_{L^2},$$

where we use $|D\varphi|$ as a norm on H_0^1. It can be shown that there exists a constant $k_2 > 0$ such that $|DA^{-1}(a)\, D(hu_x(a))|_{L^2} \geqslant k_2\, |P(hu_x(a))|_{L^2}$ for all $h \in L^2$. As in the proof of Theorem 2.5 we find $(1-\lambda)\, |hu_x(a)|_{L^2} \leqslant |Phu_x(a)|_{L^2} \leqslant k_2^{-1}\, |DA^{-1}(a)\, D(hu_x(a))|_{L^2} = k_2^{-1}\, |Th|_{H_0^1}$, for all $h \in L^2$ with $\int_0^1 hw\, dx = 0$. This implies the desired estimate in (ii). If (iii) were false, there would exist a sequence h_n in H^1 with $|h_n|_{L^2} = 1$, and $|h_n|_{H^1} \leqslant r$ such that $|Th_n|_{H_0^1} \leqslant 1/n$ for all n. This would imply the existence of a subsequence again denoted by h_n and of an element $h^* \in H^1$ such that $h_n \to h^*$ in L^2, $|h^*|_{L^2} = 1$, and $Th^* = 0$. Since the hypotheses on u_x imply injectivity of T this is a contradiction.

Remark 2.7. Let us interpret Theorem 2.6 from the point of view of well-posedness of the linearized inverse problem. Recall that a linear inverse problem

$$Tx = y \tag{2.7}$$

is well-posed (in the sense of continuous dependence of the least squares solutions of (2.7) on y) if and only if the range of T is closed and, for T compact, if and only if the range of T is finite dimensional. Theorem 2.6(i) gives cases where the linearized inverse problem is not well-posed (unless u_x is trivial), whereas (ii) describes a formulation where the linearized inverse problem is well-posed. The formulations in (ii) and (iii) will be the basis for our study of determining the optimal input to identify a from knowledge of u.

III. Rough-Case, Several Space Dimensions

1. *Three Examples*

We present three examples of generaic situations in which $|\nabla u(x)|$ can be bounded away from zero a.e. uniformly on Ω, and which are used to illustrate the analysis that follows. The examples will be special cases of weak solutions of

$$\begin{aligned} -\nabla \cdot (a \nabla u) &= f \qquad \text{in } \Omega \\ u|_{\partial\Omega} &= 0, \end{aligned} \tag{3.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, with Lipschitz continuous boundary $\partial\Omega$. Unless otherwise specified all function spaces will be considered over Ω.

EXAMPLES 3.1. Let $\Omega \subset \mathbb{R}^2$ be the open unit square with lower left corner at the origin and define $u: \Omega \to \mathbb{R}$ by

$$u(x_1, x_2) = \begin{cases} x_2 & \text{on } \quad I = \{(x_1, x_2): 0 < x_2 \leqslant 1/2, \\ & \quad \text{and } x_2 \leqslant x_1 < 1/2 \text{ or } 1/2 \leqslant x_1 \leqslant 1 - x_2\} \\ 1 - x_1 & \text{on } \quad II = \{(x_1, x_2): 1/2 \leqslant x_1 < 1, \\ & \quad \text{and } 1 - x_1 \leqslant x_2 < 1/2 \text{ or } 1/2 \leqslant x_2 < x_1\} \\ 1 - x_2 & \text{on } \quad III = \{(x_1, x_2): 1/2 \leqslant x_2 < 1, \\ & \quad \text{and } 1 - x_2 \leqslant x_1 < 1/2 \text{ or } 1/2 \leqslant x_1 < x_2\} \\ x_1 & \text{on } \quad IV = \{(x_1, x_2): 0 < x_1 \leqslant 1/2, \\ & \quad \text{and } x_1 \leqslant x_2 < 1/2 \text{ or } 1/2 \leqslant x_2 < 1 - x_1\}. \end{cases}$$

It is simple to check that u is the variational solution in H_0^1 of (3.1) with $a = 1$ and $f \in H^{-1}$ given by

$$\langle \varphi, f \rangle_{H_0^1, H^{-1}} = \sqrt{2} \int_D \varphi \, ds \qquad \text{for all} \quad \varphi \in H_0^1,$$

where $\langle \cdot, \cdot \rangle_{H_0^1, H^{-1}}$ denotes the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ and D is the union of the two diagonals of the unit square. Obviously $|\nabla u(x)| = 1$ for all $x \in \Omega \backslash D$ and $u \in W^{1,\infty}(\Omega)$ in this example.

EXAMPLE 3.2. Let Ω be the open unit disk in $\mathbb{R}^2$ and define

$$f(x_1, x_2) = \frac{4\alpha^2}{(x_1^2 + x_2^2)^{1-\alpha}}, \qquad \text{with} \quad \alpha \in \left(0, \frac{1}{2}\right).$$

If $p(1-\alpha) < 1$ and $p \in [1, \infty]$, then $f \in L^p$. Therefore, for any $\alpha \in (0, 1/2)$ there exists $\varepsilon > 0$ such that $f \in L^{1+\varepsilon}$ and consequently $f \in H^{-1}$ for every $\alpha \in (0, 1/2)$. One check that for this choice of f and Ω the weak solution in H_0^1 of (3.1) with $a = 1$ is given by

$$u(x_1, x_2) = 1 - (x_1^2 + x_2^2)^\alpha$$

and that

$$\nabla u(x_1, x_2) = -2\alpha \operatorname{col}(x_1(x_1^2 + x_2^2)^{\alpha-1}, x_2(x_1^2 + x_2^2)^{\alpha-1}).$$

We find that $|\nabla u(x_1, x_2)|^2 = 4\alpha^2(x_1^2 + x_2^2)^{2\alpha - 1}$ and hence, for any $\alpha \in (0, 1/2)$, there exists $k_\alpha > 0$ such that $|\nabla u(x)| \geqslant k_\alpha$ a.e. on Ω. Moreover if $p < 2/(1 - 2\alpha)$ then $u \in W^{1,p}$ and $u \notin W^{1,\infty}$ for $\alpha \in (0, 1/2)$.

EXAMPLE 3.3. As in the previous example, let Ω be the open unit circle in $\mathbb{R}^2$ and define $f \in H^{-1}$ by

$$\langle \varphi, f \rangle_{H_0^1, H^{-1}} = \int_{-1/2}^{1/2} \varphi(x_1, 0)\, dx_1, \qquad \text{for} \quad \varphi \in H_0^1.$$

Since $F\varphi = \int_{-1/2}^{1/2} \varphi(x_1, 0)\, dx_1$ defines a continuous linear functional on $W^{1,q}$, for any $q \geqslant 1$, see [G], there exists a unique solution u of (3.1) with $a = 1$ and $u \in H_0^1(\Omega) \cap W^{1,p}(\Omega)$ for every $p \in [1, \infty)$. We argue that $|\nabla u(x)|$ can be bounded away from zero a.e. uniformly on Ω. Green's function for the Dirichlet problem on the unit disk is given by [S, p. 518]

$$g(x, \xi) = \frac{1}{2\pi} \log \frac{1}{|x - \xi|} - \frac{1}{2\pi} \log \frac{1}{|x - \xi^*|} + \frac{1}{2\pi} \log |\xi|,$$

where x, ξ, and $\xi^* \in \mathbb{R}^2$ and ξ^* is the inverse point of ξ with respect to the unit circle (i.e., the polar coordinates of ξ^* are $(1/r, \varphi)$ if those of ξ are (r, φ)). The solution of (3.1) is then given by

$$\begin{aligned} u(x_1, x_2) &= \frac{1}{2\pi} \int_{-1/2}^{1/2} \log \frac{1}{\sqrt{(x_1 - \zeta)^2 + x_2^2}}\, d\zeta \\ &\quad - \frac{1}{2\pi} \int_{-1/2}^{1/2} \log \frac{1}{\sqrt{(x_1 - \zeta^{-1})^2 + x_2^2}}\, d\zeta + \frac{1}{2\pi} \int_{-1/2}^{1/2} \log |\zeta|\, d\zeta \\ &=: \tilde{u} + v + \frac{1}{2\pi} \int_{-1/2}^{1/2} \log |\zeta|\, d\zeta, \end{aligned}$$

and v is harmonic on Ω [S]. Let us first consider the free space fundamental solution $\tilde{u}$ and define $J = \{(x_1, 0) : x_1 \in [-1/2, 1/2]\}$. A short calculation shows that for $x = (x_1, x_2) \notin J$

$$\tilde{u}_{x_1}(x) = \frac{1}{4\pi} \log \frac{(x_1 - 1/2)^2 + x_2^2}{(x_1 + 1/2)^2 + x_2^2},$$

and hence for $x \in \Omega \backslash J$

$$\begin{aligned} &\tilde{u}_{x_1}(0, x_2) = 0, \qquad \tilde{u}_{x_1}(x_1, x_2) = -\tilde{u}_{x_1}(-x_1, x_2), \\ &\tilde{u}_{x_1}(x_1, x_2) < 0 \qquad \text{if} \quad x_1 > 0, \\ &\tilde{u}_{x_1}(x_1, x_2) > 0 \qquad \text{if} \quad x_1 < 0, \\ &\lim_{x_1 \to (1/2)^+} \tilde{u}_{x_1}(x_1, 0) = -\infty, \qquad \lim_{x_1 \to -(1/2)} \tilde{u}_{x_1}(x_1, 0) = \infty. \end{aligned} \tag{3.2}$$

Moreover, for every $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that

$$|\tilde{u}_{x_1}(x)| \geqslant M_\varepsilon \qquad \text{for all} \quad x \in \Omega_\varepsilon = \{x \in \Omega : |x_1| \geqslant \varepsilon\}. \tag{3.3}$$

Similarly one finds for $x \in \Omega \backslash J$ that

$$\begin{aligned} |\tilde{u}_{x_2}(x)| &= -\frac{1}{2\pi} \int_{-1/2}^{1/2} \frac{x_2}{\sqrt{(x_1 - \zeta)^2 + x_2^2}} \, d\zeta \\ &= \frac{1}{2\pi} \left(\arc\tan \frac{x_1 - 1/2}{x_2} - \arc\tan \frac{x_1 - 1/2}{x_2} \right), \end{aligned}$$

and consequently for $x_0 \in \{(x_1, 0) : x_1 \in (-1/2, 1/2)\}$,

$$\lim_{\substack{x \to x_0 \\ x_2 > 0}} \tilde{u}_{x_2}(x) = -1/2, \qquad \lim_{\substack{x \to x_0 \\ x_2 < 0}} \tilde{u}_{x_2}(x) = 1/2$$

$$\tilde{u}_{x_2}(x_1, x_2) = -\tilde{u}_{x_2}(-x_1, x_2)$$

and

$$\begin{aligned} &\tilde{u}_{x_2}(x) < 0 \qquad \text{if} \quad x_2 > 0, \\ &\tilde{u}_{x_2}(x) > 0 \qquad \text{if} \quad x_2 < 0. \end{aligned} \tag{3.5}$$

For v one finds

$$\begin{aligned} v_{x_1} &= \frac{1}{\pi} \int_2^\infty \frac{x_1 - z}{(x_1 - z)^2 + x_2^2} \frac{dz}{z^2} + \frac{1}{\pi} \int_{-\infty}^{-2} \frac{x_1 - z}{(x_1 - z)^2 + x_2^2} \frac{dz}{z^2}, \\ v_{x_2} &= \frac{1}{2\pi} \int_{-1/2}^{1/2} \frac{x_2}{(x_1 - z^{-1})^2 + x_2^2} \, dx, \end{aligned} \tag{3.6}$$

and consequently

$$\begin{aligned} &v_{x_1}(0, x_2) = v_{x_2}(x_1, 0) = 0, \qquad v_{x_i}(-x_1, x_2) = -v_{x_i}(x_1, x_2), \quad \text{for} \quad i = 1, 2, \\ &v_{x_1}(x) < 0 \qquad \text{if} \quad x_1 > 0 \\ &v_{x_1}(x) > 0 \qquad \text{if} \quad x_1 < 0. \end{aligned} \tag{3.7}$$

Using (3.4), (3.5), and (3.6) one can argue that there exists $k_1 > 0$ such that

$$|\tilde{u}_{x_2}(x)| - |v_{x_2}(x)| \geqslant k_1 > 0 \qquad \text{on} \quad \Omega - \Omega_{1/4}. \tag{3.8}$$

Combining (3.3) and (3.8) we find that there exists a constant $k > 0$ such that $|\nabla u(x)| \geqslant k$ a.e. on Ω.

Remark 3.4. The solution u of Examples 3.1–3.3 satisfies $u(x) \geqslant 0$ everywhere on Ω.

Above we gave three examples for different choices of f so that the weak solution $u \in H_0^1$ of (3.1) satisfies

(i) there exists $k > 0$ such that $|\nabla u(x)| \geqslant k > 0$ a.e. in Ω,

(ii) $u \in W^{1,p}$ for some $p > n$,

(iii) $u(x) \geqslant 0$ everywhere in Ω.

In the above examples properties (i)–(iii) were established for a fixed value for the coefficient, $a = 1$. We show next that properties (i) and (ii) are stable under appropriate perturbations of the coefficients a if $\partial\Omega$ is smooth. Property (iii) is stable with respect to L^∞ perturbations of a if, e.g., f is such that $\langle f, \varphi \rangle \geqslant 0$ for all $\varphi \in H_0^1$. This follows from the weak maximum principle.

PROPOSITION 3.5. *Let $\partial\Omega$ be $C^{1,1}$ smooth, $f \in H^{-1}$, $p > n$, $\bar{a} \in W^{2,p}$, with $\bar{a}(x) \geqslant \nu > 0$ and let $u = u(\bar{a})$ be the weak solution of* (3.1) *satisfying properties* (i) *and* (ii). *Then there exists a neighborhood N of $\bar{a}$ in $W^{2,p}$ such that for each $a \in N$ the weak solution $u(a)$ of* (3.1) *satisfies* (i) *and* (ii). *The constant $k = k_N$ in* (i) *can be chosen uniformly for all $a \in N$.*

Proof. Since $p > n$ we have $W^{i+1,p} \subset C^i$ for all $i = 0, 1, \ldots$. There exists a neighborhood N_∞ of $\bar{a}$ in L^∞ (and hence in $W^{2,p}$) such that $a(x) \geqslant \nu/2$ for all $a \in N_\infty$. For $a \in N_\infty$ let $v = u(a)$ be the weak solution of (3.1). We introduce $w = v - (\bar{a}/a)\, u$ and observe that

$$\langle \bar{a} \nabla w, \nabla \varphi \rangle_{L^2} = \langle \hat{f}, \varphi \rangle_{L^2} \qquad \text{for} \quad \varphi \in H_0^1,$$

where

$$\hat{f} = \frac{u}{a}\left(a\, \Delta \bar{a} - \bar{a}\, \Delta a - \nabla a \cdot \nabla \bar{a} + \frac{\bar{a}}{a}\, |\nabla a|^2\right) + \frac{\nabla u}{a}\,(a \nabla \bar{a} - \bar{a} \nabla a).$$

Since $u \in W^{1,p}$, $\bar{a} \in W^{2,p}$, and $a \in W^{2,p}$ it follows that $\hat{f} \in L^p$ and $\hat{f} \to 0$ in L^p for $a \to \bar{a}$ in $W^{2,p}$. This implies that $w \in W^{2,p} \cap H_0^1$ and that $w \to 0$ in $W^{2,p}$ if $a \to \bar{a}$ in $W^{2,p}$ [T, p. 180].

Since $u(a) = w + (\bar{a}/a)\,u$, it follows that $u(a)$ satisfies (i) and (ii) whenever a is in a sufficiently small neighborhood N of $\bar{a}$ in $W^{2,p}$.

We close this section with an example illustrating the fact that knowledge of $u = u(a)$ satisfying properties (i)–(iii) is not sufficient for identifiability of a from $u(a)$.

EXAMPLE 3.6. Let Ω and f be as defined in Example 3.1 and let u be the weak solution of (3.1) with $a = 1$. Its explicit form is given in Example 3.1. It can be shown that (b, u) also satisfies

$$\langle b\,\nabla u, \nabla\varphi\rangle = \langle f, \varphi\rangle \qquad \text{for all} \quad \varphi \in H_0^1$$

provided that

$$(a-b)(x, y) = \begin{cases} h_1(x) & \text{on } I \\ h_3(x) & \text{on } III \\ -h_1(1-y) & \text{on } II \quad \text{for} \quad y \leqslant 1/2 \\ -h_3(y) & \text{on } II \quad \text{for} \quad y > 1/2 \\ -h_3(1-y) & \text{on } IV \quad \text{for} \quad y > 1/2 \\ -h_1(y) & \text{on } IV \quad \text{for} \quad y \leqslant 1/2, \end{cases}$$

and $h_1 \in L^2(0, 1)$, $h_3 \in L^2(0, 1)$. Therefore identifiability of a from u fails within the class of L^2 coefficients.

This example is also of interest for the linearization of the parameter-to-solution mapping $a \to u(a)$. The Fréchet derivative of $a \to u(a)$ at $a = 1$ is given by $u'(1)(h) = (-\Delta)^{-1}\,\nabla\cdot(h\,\nabla u)$; for details see Subsection 3.2. Clearly $u'(1)$ is well defined from L^2 to H_0^1 and the choice $h = a - b$ shows that the kernel of $u'(1)$ is infinite dimensional.

2. *Estimates Based on a Variational Procedure*

In this subsection we derive an identifiability result, an a priori estimate, and a lower bound on the linearization of the parameter-to-solution mapping by choice of a proper test function in the variational formulation of the boundary value problem. The results are directly applicable to Examples 3.1–3.3. Moreover, they can also be used in situations where u is smooth. Let Ω be a bounded domain in $\mathbb{R}^n$, $n = 2$ *or* 3, with $C^{1,1}$-smooth boundary $\partial\Omega$ or Ω convex, and $f \in H^{-1}$. We consider

$$\langle a\,\nabla u, \nabla\varphi\rangle = \langle f, \varphi\rangle \qquad \text{for all} \quad \varphi \in H_0^1, \tag{3.9}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in L^2 as well as the duality pairing between H_0^1 and H^{-1}, and we refer to (a, u) as a solution of (3.9) if $(a, u) \in L^\infty \times H_0^1$ and (3.9) holds.

THEOREM 3.7 (Identifiability). *Let* (a, u) *and* (b, v) *be solutions of* (3.9) *in* $W^{1,\infty} \times H_0^1$, *and* $a(x) \geq \nu > 0$. *If*

$$\langle \varphi^2, a\,|\nabla u|^2 + uf \rangle > 0 \qquad \text{for every} \quad \varphi \neq 0 \quad \text{with} \quad \varphi u \in H_0^1 \tag{3.10}$$

then $u = v$ *implies* $a = b$.

THEOREM 3.8. (A Priori Estimate). *Let* (a, u) *and* (b, v) *be solutions of* (3.9) *in* $W^{1,\infty} \times (H_0^1 \cap W^{1,6})$ *and let* $a(x) \geq \nu > 0$. *If there exist* $\hat{\nu} > 0$ *such that*

$$\hat{\nu}\, |\varphi|_{L^2}^2 \leq \langle \varphi^2, a\,|\nabla u|^2 \rangle + \langle \varphi^2 u, f \rangle \quad \text{for every} \quad \varphi \neq 0 \quad \text{with} \quad \varphi u \in H_0^1, \tag{3.11}$$

then there exists a constant $K = K(\nu, \hat{\nu}, |a|_{H^1}, |u|_{W^{1,4}})$ *such that*

$$|a - b|_{L^2}^2 \leq K\, |a|_{L^\infty}^2\, |u - v|_{W^{1,6}}.$$

If $n = 2$, *then* $W^{1,6}$ *can be replaced by* $W^{1,2+\varepsilon}$ *for any* $\varepsilon > 0$.

The constants K in Theorem 3.8 and Theorem 3.10 below can be chosen such that they depend continuously on their arguments.

Proof of Theorem 3.7. Let us first assume that $f \in L^2$. Due to the assumptions on $\partial\Omega$ and a, b it follows that $u \in H_0^1 \cap H^2$ and $v \in H_0^1 \cap H^2$ [G]. Let us put $h = a - b$.

A little algebra shows that

$$A^{-1}(a)\nabla \cdot (h\nabla u) = A^{-1}(a)\nabla \cdot (b\nabla(u - v)) \qquad \text{in } H_0^1(\Omega), \tag{3.12}$$

where $A(a)\colon H_0^1(\Omega) \to H^{-1}(\Omega)$ is defined by $A(a)\varphi = -\nabla \cdot (a\nabla\varphi)$. This equality implies

$$\left\langle A^{-1}(a)\nabla \cdot (h\nabla u), \frac{hu}{a} \right\rangle_{1,a} = \left\langle A^{-1}(a)\nabla \cdot (b\nabla(u - v)), \frac{hu}{a} \right\rangle_{1,a}, \tag{3.13}$$

where for $\varphi, \psi \in H_0^1$ we put $\langle \varphi, \psi \rangle_{1,a} = \langle a\nabla\varphi, \nabla\psi \rangle$.

Hence we have

$$\left\langle a\nabla A^{-1}(a)\nabla \cdot (h\nabla u), \nabla\left(\frac{hu}{a}\right) \right\rangle_{L^2} = \left\langle \nabla A^{-1}(a)\nabla \cdot (b\nabla(u - v)), a\nabla\left(\frac{hu}{a}\right) \right\rangle_{L^2} \tag{3.14}$$

and

$$\left\langle \nabla(h\,\nabla u), \frac{hu}{a} \right\rangle = \left\langle \nabla(b(u-v)), \frac{hu}{a} \right\rangle.$$

This implies

$$\left\langle \nabla\left(\frac{hu}{a}\right), h\,\nabla u \right\rangle = \left\langle \nabla\left(\frac{hu}{a}\right), b\,\nabla(u-v) \right\rangle. \tag{3.15}$$

Next we manipulate the term on the left hand side of (3.15):

$$\begin{aligned}
\left\langle \nabla\left(\frac{hu}{a}\right), h\,\nabla u \right\rangle_{L^2} &= \left\langle \frac{u}{a}\nabla h + \frac{h}{a}\nabla u - \frac{hu}{a^2}\nabla a, h\,\nabla u \right\rangle_{L^2} \\
&= \frac{1}{2}\left\langle \nabla h^2, \frac{u}{a}\nabla u \right\rangle + \left\langle \frac{h^2}{a}, |\nabla u|^2 \right\rangle - \left\langle \frac{h^2}{a^2}\nabla u, u\,\nabla a \right\rangle \\
&= -\frac{1}{2}\left\langle \frac{h^2}{a^2}, a\,|\nabla u|^2 + au\,\Delta u - u\,\nabla u\cdot\nabla a \right\rangle \\
&\quad + \left\langle \frac{h^2}{a}, |\nabla u|^2 \right\rangle - \left\langle \frac{h^2}{a^2}\nabla u, u\,\nabla a \right\rangle \\
&= \frac{1}{2}\left\langle \frac{h^2}{a}, |\nabla u|^2 \right\rangle_{L^2} - \frac{1}{2}\left\langle \frac{h^2}{a^2}, u\,\nabla u\cdot\nabla a \right\rangle_{L^2} \\
&\quad - \frac{1}{2}\left\langle \frac{h^2}{a}u, \Delta u \right\rangle \frac{1}{2}\left\langle \frac{h^2}{a}, |\nabla u|^2 \right\rangle - \left\langle \frac{h^2}{a^2}u, \nabla\cdot(a\,\nabla u) \right\rangle \\
&= \frac{1}{2}\left\langle \frac{h^2}{a^2}, a\,|\nabla u|^2 + uf \right\rangle.
\end{aligned}$$

We summarize these equalities in

$$\left\langle \nabla\left(\frac{hu}{a}\right), h\,\nabla u \right\rangle = \frac{1}{2}\left\langle \frac{h^2}{a^2}, a\,|\nabla u|^2 + uf \right\rangle. \tag{3.16}$$

Density of L^2 in H^{-1} and continuity of $f \to u(f)$ from H^{-1} to H_0^1 imply the validity of (3.16) also for $f \in H^{-1}$.

If $u = v$ then (3.15) and (3.16) imply

$$0 = \left\langle \frac{h^2}{a^2}, a\,|\nabla u|^2 + uf \right\rangle$$

from which it follows that $a = b$, by (3.10).

Proof of Theorem 3.8. This is an extension of the proof of the previous theorem. We estimate the right hand side of (3.15) from above:

$$\left\langle \nabla\left(\frac{hu}{a}\right), b\,\nabla(u-v)\right\rangle_{L^2}$$

$$= \left\langle \frac{1}{a^2}\,(ah\,\nabla u + au\,\nabla h - hu\,\nabla a), b\,\nabla(u, v)\right\rangle_{L^2}$$

$$\leqslant \frac{1}{v}\,|h\,\nabla u|_{L^2}\,|b\,\nabla(u-v)|_{L^2} + \frac{1}{v}\,|u|_{L^\infty}\,|\nabla h|_{L^2}\,|\,b\,\nabla(u-v)|_{L^2}$$

$$+ \frac{1}{v^2}\,|u|_{L^\infty}\,|\nabla a|_{L^2}\,|bh\,\nabla(u-v)|_{L^2}.$$

We now consider the case $n=3$. If $n=2$ the estimate is quite similar and it is left to the reader. We find

$$\left\langle \nabla\left(\frac{hu}{a}\right), b\,\nabla(u-v)\right\rangle_{L^2} \leqslant \frac{1}{\min(v^2, v)}\,(|h|_{L^4}\,|\nabla u|_{L^4}\,|b|_{L^4}\,|\nabla(u-v)|_{L^4})$$

$$+ |u|_{L^\infty}\,|\nabla h|_{L^2}\,|b|_{L^4}\,|\nabla(u-v)|_{L^4}$$

$$+ (|u|_{L^\infty}\,|\nabla a|_{L^2}\,|b|_{L^6}\,|h|_{L^6}\,|\nabla(u-v)|_{L^6})$$

$$\leqslant \kappa\,|u|_{W^{1,4}}\,(|a|_{H^1}\,|b|_{H^1} + |b|_{H^1})$$

$$\times\,|a-b|_{H^1}\,|u-v|_{W^{1,6}}, \tag{3.17}$$

where κ depends on v and embedding constants. Combining (3.11), (3.15), (3.16), and (3.17) we find

$$|a-b|_{L^2}^2 \leqslant 2\kappa\hat{v}\,|a|_{L^\infty}^2\,(|a|_{H^1} + |b|_{H^1})\,|a-b|_{H^1}\,|u-v|_{W^{1,6}}.$$

This estimate implies the desired result.

Remark 3.9. Condition (3.11) is satisfied for the specific cases of Examples 3.1–3.3.

We turn to an estimate of the linearization of the parameter-to-output mapping. It is simple to argue that $L^2(|\nabla u|) \supset L^6$ if $n=3$ and $u \in W^{1,3}$, and that $L^2(|\nabla u|) \supset L^q$ with $q = 2(2+\varepsilon)/\varepsilon$, $\varepsilon > 0$, if $n=2$ and $u \in W^{1,2+\varepsilon}$. In either case $L^2(|\nabla u|) \supset H^1$ and T is well defined on H^1.

THEOREM 3.10. *Let* $(a, u) \in W^{1,4} \times (H_0^1 \cap W^{1,4})$ *with* $a \geq \nu > 0$, *be a solution of* (3.9) *and let* (3.11) *hold. Then the linearization* T *of* $a \to u(a)$ *at* a *satisfies*

$$|h|_{L^2}^2 \leq K\,|Th|_{H_0^1} \qquad \text{for all} \quad h \in H^1,$$

where $K = K(\nu, |a|_{L^\infty}, |a|_{H^1}, |u|_{W^{1,4}}, |h|_{H^1})$. *If* $n = 2$ *then* $W^{1,4}$ *can be replaced by* $W^{1,2+\varepsilon}$, *for any* $\varepsilon > 0$.

Proof. Let $h \in W^{1,4}$, so that $h\,\nabla u \in L_n^2$ and observe that as in the proof of Theorem 3.7 (see (3.13)–(3.15)) we find

$$\left\langle Th, \frac{hu}{a} \right\rangle_{1,a} = \left\langle \nabla\left(\frac{hu}{a}\right), h\,\nabla u \right\rangle_{L^2}.$$

Using (3.16) we find

$$\begin{aligned}
\frac{1}{2}\left\langle \frac{h^2}{a^2}, a\,|\nabla u|^2 + uf \right\rangle &= \left\langle Th, \frac{hu}{a} \right\rangle_{1,a} \leq |Th|_{H_0^1} \left| a\,\nabla\left(\frac{hu}{a}\right) \right|_{L^2} \\
&\leq |Th|_{H_0^1} \left| \frac{1}{a}\,(ah\,\nabla u + au\,\nabla h - hu\,\nabla a) \right|_{L^2} \\
&\leq \min\left(1, \frac{1}{\nu}\right) |Th|_{H_0^1} (|h\,\nabla u|_{L^2} + |u\,\nabla h|_{L^2} + |hu\,\nabla a|_{L^2}).
\end{aligned}$$

Again we give the details for $n = 3$ and leave the case $n = 2$ to the reader. We find

$$\begin{aligned}
\frac{1}{2}\left\langle \frac{h^2}{a^2}, a\,|\nabla u|^2 + uf \right\rangle &\leq \min\left(1, \frac{1}{\nu}\right) |Th|_{H_0^1} \\
&\quad \times (|h|_{L^4}\,|\nabla u|_{L^4} + |u|_{L^\infty}\,|\nabla h|_{L^2} + |u|_{L^\infty}\,|h|_{L^4}\,|\nabla a|_{L^4})
\end{aligned}$$

and by (3.11)

$$\begin{aligned}
|h|_{L^2}^2 &\leq \frac{2\,|a|_{L^\infty}^2}{\hat{\nu}} \min\left(1, \frac{1}{\nu}\right) (|h|_{L^4}\,|\nabla u|_{L^4} + |u|_{L^\infty}\,|\nabla h|_{L^2} \\
&\quad + |u|_{L^\infty}\,|h|_{L^4}\,|\nabla a|_{L^4}). \qquad (3.18)
\end{aligned}$$

This implies the claim for $h \in W^{1,4}$. The assertion of the theorem follows from density of $W^{1,4}$ in H^1.

Remark 3.11. If the assumption on the regularity of u in Theorem 3.7 is increased to $u \in H_0^1 \cap L^\infty$, then hypothesis (3.10), in Theorem 3.7, can be relaxed to

There exists $\gamma \in \mathbb{R}$ such that $\langle \varphi^2, a(1+\gamma u)\, |\nabla u|^2 \rangle + \langle \varphi^2 u, f \rangle > 0$ for every $\varphi \neq 0$ with $\varphi u \in H_0^1$. (3.10′)

Similarly (3.11) in Theorems 3.8 and 3.10 can be relaxed to

There exists $\gamma \in \mathbb{R}$ and $\hat{v} > 0$ such that $\hat{v}\, |\varphi|_{L^2}^2 \leqslant \langle \varphi^2, a(1+\gamma u)\, |\nabla u|^2 + \langle \varphi^2 u, f \rangle$ for all $\varphi \neq 0$ with $\varphi u \in H_0^1$. (3.11′)

This can be verified by making the transformation $\alpha = he^{(-\gamma/2)u}$ in the proofs and using $a^{-1}he^{(\gamma/2)u}$ as a test function.

IV. Smooth Case

1. *A Priori Estimate*

In this section an priori estimate on the diffusion coefficient in terms of the solution of an elliptic boundary value problem is obtained. The result will imply in particular the identifiability of the coefficient from knowledge of the solution. The reference solution of the boundary value problem will be assumed to be smooth. For fixed $f \in L^2$ we consider

$$-\nabla \cdot (a \nabla u) = f \qquad \text{in } \Omega, \tag{4.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n = 1, 2$, or 3. If $n = 1$ then we put $\Omega = (0, 1)$. We refer to $(a, u) \in W^{1,4} \times H^2$ as a solution of (4.1) if (4.1) is satisfied in the L^2-sense. For two solutions (a, u) and (b, v) of (4.1) an a priori estimate of $a - b$ in terms of $u - v$ will be obtained. The following hypotheses are required:

(A1) Ω is bounded domain in $\mathbb{R}^n$, $n = 1, 2$, or 3 with $C^{1,1}$-smooth boundary $\partial\Omega$, if $n = 2$ or 3.

(A2) $(a, u) \in W^{1,4} \times (H^2 \cap C^1(\bar{\Omega}))$, $(b, v) \in W^{1,4} \times (H^2 \cap C^1(\bar{\Omega}))$.

(A3) There is an at most countable family of disjoint connected subdomains $\Omega_i \subset \Omega$, with Lipschitz continuous boundaries $\partial\Omega_i$ such that $\bigcup_{i=1}^{\infty} \bar{\Omega}_i = \bar{\Omega}$. Moreover, there are constants $k > 0$ and $\lambda_i \in [-l, l]$, where $l > 0$, such that for each i

(i) $\lambda_i\, |\nabla u(x)|^2 + \frac{1}{2} \Delta u(x)$ has an a.e. uniform sign on Ω_i for all i,

(ii) $|\lambda_i\, |\nabla u(x)^2| + \frac{1}{2} \Delta u(x)| \geqslant k$ for a.e. $x \in \Omega_i$, and

(iii) $a=b$ on Γ_i for all i, where

$$\Gamma_i=\begin{cases}\{x\in\partial\Omega_i: \nabla u(x)\cdot n_i(x)<0\} \\ \quad \text{if } \lambda_i\,|\nabla u(x)|^2+\frac{1}{2}\Delta u(x)\geqslant k \quad \text{a.e. on } \Omega_i \\ \{x\in\partial\Omega_i: \nabla u(x)\cdot n_i(x)>0\} \\ \quad \text{if } \lambda_i\,|\nabla u(x)|^2+\frac{1}{2}\Delta u(x)\leqslant -k \quad \text{a.e. on } \Omega_i.\end{cases}$$

and n_i denote the unit outer normal to Ω_i.

In the one dimensional case, the requirement $a, b\in W^{1,4}$ in (A.2) can be replaced by $a, b\in H^1$.

An example illustrating (A3) is given by the function

$$u(x, y)=(x^2+y^2)(x^2+y^2-1) \qquad \text{on} \quad \Omega=\{(x, y): x^2+y^2<1\}\subset\mathbb{R}^2. \tag{4.2}$$

We find $\Delta u=4(2r-1)(2r+1)$, and $|\nabla u|^2=4r^2(2r^2-1)^2$, where $r^2=x^2+y^2$, and hence possible choices for the subdivision of Ω are given by

$$\Omega_1=\{(x, y): x^2+y^2<\alpha\}$$

and

$$\Omega_2\{(x, y): \alpha<\alpha^2+y^2<1\}, \qquad \text{with} \quad \alpha\in\left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right).$$

Let us briefly discuss (A3) for the case $n=1$. Assume that there exists an at most countable family of disjoint open intervals $\Omega_i=(l_i, r_i)$ such that $\bar{\Omega}=[0, 1]=\bigcup_i\bar{\Omega}_i$. Moreover, assume that there are constants $k>0$ and $\lambda_i\in[-l, l]$ for some $l>0$, such that for each i

(i*) $\operatorname{sgn} u_x(r_i)=-\operatorname{sgn} u_x(l_i)$, if $\Omega_j=(0, r_j)$ then either $u_x(0)=0$ and $u_x(r_j)\neq 0$, or $\operatorname{sgn} u_x(0)=-\operatorname{sgn} u_x(r_j)$, if $\Omega_k=(l_k, 1)$ then either $u_x(1)=0$ and $u_x(l_k)\neq 0$, or $\operatorname{sgn} u_x(l_k)-\operatorname{sgn} u_x(1)$,

(ii*) $|\lambda_i\, u_x^2(x)+\frac{1}{2}u_{xx}(x)|\geqslant k$ for a.e. $x\in\Omega_i$,

(iii*) $\operatorname{sgn} u_x(r_i)=\operatorname{sgn}(\lambda_i\, u_x(x)^2+\frac{1}{2}u_{xx}(x))$ for a.e. $x\in\Omega_i$, except if $\Omega_k=(l_k, 1]$ and $u_x(1)=0$, in which case $\operatorname{sgn} u_x(l_k)=-\operatorname{sgn}(\lambda_k u_x(x)^2+\frac{1}{2}u_{xx}(x))$, for a.e. $x\in\Omega_k$.

If (i*)–(iii*) hold, then the family Ω_i satisfies (A3). These conditions imply that on each of the subintervals Ω_i, which does not contain a boundary point, u_x has exactly one extremum. Moreover, u cannot have a turning point ($u_x(s)=u_{xx}(s)=0$) with (ii*) holding. Conditions (i*)–(iii*) do not cover the case that u_x has a uniform sign on $(0, 1)$. If, for example, $u \geqslant 0$ on $(0, 1)$ then (A3) can be replaced by

$u_x(0)\,(a(0) - b(0)) = 0$ and if there are constants λ and $k > 0$ such that $\lambda u_x(x)^2 + \frac{1}{2} u_{xx}(x) \geqslant k$ a.e. on $(0, 1)$.

The condition (A3) with only one domain Ω_i was considered in [R] where an estimate on $|a|_{L^\infty(\Omega)}$ in terms of $|f|_{L^\infty(\Omega)}$ and $|a|_{L^\infty(\partial\Omega)}$ is obtained. The technique of proof in [R] is completely different from the one that we use.

THEOREM 4.1. *Let (a, u) and (b, v) be solutions of (4.1) and assume that (A1)–(A3) hold. Then there exists a constant K depending on k, l, and embedding constants, but independent of (b, v), such that*

$$|a - b|_{L^2} \leqslant K\, |b|_{W^{1,4}}\, |u - v|_{H^2}.$$

Proof. From (4.1) we have

$$\nabla \cdot ((a - b)\nabla u) = \nabla \cdot (b\,\nabla(v - u)). \tag{4.3}$$

The right hand side of (4.3) can be estimated as

$$\begin{aligned} |\nabla \cdot (b\,\nabla(v - u))|_{L^2} &\leqslant |b|_{L^\infty}\, |\Delta(u - v)|_{L^2} + |b|_{W^{1,4}}\, |u - v|_{W^{1,4}} \\ &\leqslant M\, |b|_{W^{1,4}}\, |u - v|_{H^2}, \end{aligned} \tag{4.4}$$

where M is an embedding constant. This estimate depends on the fact that $n \leqslant 3$. Next we bound the left hand side of (4.3) from below. Let us set $h = a - b$ and $\mu = \exp(-l \max_{x \in \Omega} |u(x)|)$. Then we find on each Ω_i,

$$\begin{aligned} &|\nabla \cdot ((a - b)\,\nabla u)|_{L^2(\Omega_i)} \cdot |a - b|_{L^2(\Omega_i)} \\ &\quad\geqslant \mu^2\, |\nabla \cdot (h\,\nabla u)\, e^{-\lambda_i u}|_{L^2(\Omega_i)}\, |h e^{-\lambda_i u}|_{L^2(\Omega_i)} \\ &\quad\geqslant \mu^2 |\langle e^{-\lambda_i u}(h\,\Delta u + \nabla h \cdot \nabla u),\, e^{-\lambda_i u} h\rangle_{L^2(\Omega_i)}|, \end{aligned}$$

and substituting $v = e^{-\lambda_i u} h$

$$\begin{aligned} &|\nabla \cdot (h\,\nabla u)|_{L^2(\Omega_i)}\, |h|_{L^2(\Omega_i)} \\ &\quad\geqslant \mu^2\, |\langle v\,\Delta u + \nabla u \cdot \nabla v + \lambda_i v\, |\nabla u|^2,\, v\rangle_{L^2(\Omega_i)}| \\ &\quad= \mu^2 \left| \langle \Delta u + \lambda_i\, |\nabla u|^2,\, v^2\rangle_{L^2(\Omega_i)} + \frac{1}{2}\, \langle \nabla u, \nabla v^2\rangle_{L^2(\Omega_i)} \right| \\ &\quad= \mu^2 \left| \left\langle \frac{1}{2}\, \Delta u + \lambda_i\, |\nabla u|^2,\, v^2 \right\rangle_{L^2(\Omega_i)} + \frac{1}{2} \left\langle \frac{\partial u}{\partial n},\, v^2 \right\rangle_{L^2(\partial\Omega_i)} \right|, \end{aligned}$$

where we used Green's formula. By (A3) the last estimate implies

$$|\nabla \cdot (h \nabla u)|_{L^2(\Omega_i)} |h|_{L^2(\Omega_i)} \geq \mu^2 k \, |e^{-\lambda_i u}|^2_{L^2(\Omega_i)} \geq \mu^4 k \, |h|^2_{L^2(\Omega_i)},$$

and therefore

$$\mu^4 k \, |a-b|_{L^2(\Omega)} \leq |\nabla \cdot ((a-b) \nabla u)|_{L^2(\Omega)}. \tag{4.5}$$

Combining (4.3)–(4.5) we obtain the desired result.

2. The Linearization of the Parameter-to-Solution Mapping

This section is devoted to a study of the linearization of the parameter-to-solution mapping. We restrict ourselves to Dirichlet boundary conditions, i.e., we consider

$$\begin{aligned} -\nabla \cdot (a \nabla u) &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{4.6}$$

For $f \in L^2$, the mapping $a \to u(a)$ is well defined from H^2 to H^2 if $n = 2$ or 3, and from H^1 to H^2 if $n = 1$, whenever $a > 0$. For such a, the Fréchet derivative of $u(a)$ at a in direction $h \in H^2$ ($h \in H^1$ if $n = 1$) is given by

$$u'(a) \, h = A^{-1}(a) \nabla \cdot (h \nabla u(a)),$$

where

$$A(a) \, \varphi = \nabla \cdot (a \nabla \varphi)$$

with domain $D(A(a)) = H^2 \cap H^1_0$ [CK]. Let us henceforth denote by T the operator that characterizes the action of $u'(a)$, i.e.,

$$Th = A^{-1}(a) \nabla \cdot (h \nabla u).$$

We observe that T can be considered as a compact mapping from $H^1(\Omega)$ to itself, it provided that $u(a) \in W^{1,\infty}$. Moreover, T has a bounded extension as a linear operator from L^2 to H^1. Our specific interest, however, lies in the choice of topologies for the domain and the range of T, for which T becomes continuously invertible on its range. For sufficiently smooth u, this is achieved, if T is considered as a mapping from a subset of L^2 to H^2. In this case T becomes an unbounded operator and some analysis is required to interpret T as a closed operator from L^2 to H^2.

Some additional notation and assumptions are introduced next. The solution is assumed to satisfy

(A4) $u = u(a) \in W^{3,4}$.

Here and throughout $a > 0$ and $a \in H^2$ if $n = 2$ or 3 and $a \in H^1(\Omega)$ if $n = 1$. Let U denote that $n \times n$ matrix with elements $U_{ij} = u_{x_i x_j}$ and let λ_M be the maximum modulus of the eigenvalues of U. We define the inflow boundary

$$\Gamma = \mathrm{cl}\{x \in \partial\Omega : \nabla u(a)(x) \cdot n(x) < 0\},$$

where cl denotes closure.

Note that the inflow boundary is not invariant under change of sign of f (or equivalently of $u(a)$). It will be necessary to assume that the closure of the inflow boundary and its complement do not meet, more precisely,

(A5) $\partial\Omega - \Gamma$ is closed relative to $\partial\Omega$.

We do not exclude the case where $\Gamma = \varnothing$. The following two technical lemmas will be needed.

LEMMA 4.2. *Let* (A1), (A4), *and* (A5) *hold. The operator* $B : D(B) \subset L^2(\Omega) \to L^2(\Omega)$ *given by*

$$D(B) = \{\varphi \in H^1 : \varphi \mid \Gamma = 0\} \qquad \text{and} \qquad B\varphi = \nabla \cdot (\varphi \nabla u)$$

is closable.

Here $\varphi|_\Gamma = 0$ is interpreted in $H^{1/2}(\Gamma)$.

We show that $\mu + B$ is closable for some sufficiently large μ. This will imply the claim. The following auxiliary lemma is needed.

LEMMA 4.3. *Let* (A1), (A4), *and* (A5) *hold, and let*

$$\eta > 2k\, |\Delta u|_{W^{1,4}} + \lambda_M,$$

where $k \geqslant 1$ *is a common embedding constant of* H^1 *into* L^4 *and* $W^{1,4}$ *into* L^∞. *Then for every* $b \in H^1$ *with* $g|_\Gamma = 0$ *there exists a unique solution* $y \in H^1$ *of*

$$\begin{aligned} \eta y + \nabla u \cdot \nabla y + \Delta u y &= g \\ y|_\Gamma &= 0, \end{aligned} \tag{4.7}$$

with

$$(\eta - 2k\, |\Delta u|_{W^{1,4}} - \lambda_M)\, |y|_{H^1} \leqslant \sqrt{2}\, |g|_{H^1},$$

and

$$(\eta - \tfrac{1}{2} |\Delta u|_{L^\infty})\, |y|_{L^2} \leqslant |g|_{L^2}.$$

The proofs of Lemmas 4.2 and 4.3 will be given at the end of this section.

Remark 4.4. The assumption $u \in W^{3,4}$ is used in the proof of Lemma 4.3. In the one dimensional case, (A4) can be replaced by the requirement that $u \in H^3$ only. Then we can assert that

$$\eta > \tfrac{3}{2}\,|\Delta u|_{L^\infty} + \tfrac{3}{2}\,k\,|u''|_{L^2},$$

with k the embedding constant from H^1 into L^∞. This implies the existence of a unique solution $y \in H^1$ of (4.7) which satisfies

$$(\eta - \tfrac{3}{2}\,|\Delta u|_{L^\infty} - \tfrac{3}{2}\,k\,|u''|_{L^2})\,|y|_{H^1} \leq \sqrt{2}\,|g|_{H^1},$$

and

$$(\eta - \tfrac{1}{2}\,|\Delta u|_{L^\infty})\,|y|_{L^2} \leq |g|_{L^2}.$$

We now specify T together with its domain so that it becomes closable as an operator from $L^2(\Omega)$ to $H^2(\Omega)$:

$$\begin{aligned} D(T) &= \{h \in H^1 : h \mid \Gamma = 0\}, \\ Th &= A(a)^{-1}\,\nabla \cdot (h\,\nabla u). \end{aligned} \tag{4.8}$$

THEOREM 4.5. *Let* (A1) *and* (A3)–(A5) *hold. Then T is closable operator from $L^2(\Omega)$ to $H^2(\Omega)$ and its closure $\bar{T}$ satifies*

$$|\bar{T}h|_{H^2} \geq K\,|h|_{L^2}$$

for some constant $K > 0$ independent of $h \in D(\bar{T})$. In particular this implies that the range of $\bar{T}$, $R(\bar{T})$, is closed in $H^2(\Omega)$ and that $\bar{T}$ is continuously invertible on $R(\bar{T})$.

Proof. Since $A(a)^{-1}$ is a homeomorphism from L^2 to H^2, T is closable by Lemma 4.3. From the proof of Theorem 4.1, see (4.5), it follows that

$$|Th|_{H^2} = |A^{-1}(a)\,\nabla \cdot (h\,\nabla u)|_{L^2} \geq K\,|h|_{L^2},$$

for some constant $K > 0$ independent of $h \in D(T)$.

Taking the closure of the graph of T one concludes that

$$|\bar{T}h|_{H^2} \geq K\,|h|_{L^2}$$

for all $h \in D(\bar{T})$. This estimate ends the proof.

This section is conclude with the proofs of Lemmas 4.2 and 4.3.

Proof of Lemma 4.3. We follow a parabolic regularization technique used previously in [B, OR]. For $\varepsilon > 0$ and $g \in H^1(\Omega)$ with $g|_\Gamma = 0$ consider

$$
\begin{aligned}
-\varepsilon\,\Delta y+\eta y+\nabla u\cdot\nabla y+\Delta u y &= g \qquad \text{in } \Omega \\
y &= 0 \qquad \text{on } \Gamma \\
\partial y/\partial n &= 0 \qquad \text{on } \quad \partial\Omega-\Gamma.
\end{aligned}
\tag{4.9}
$$

We use the Lax–Milgram theorem to establish existence of a unique solution of (4.9). Taking the inner product in L^2 of (4.9) with $w\in H^1$, $w|_\Gamma=0$, we obtain the weak form of (4.9):

$$
\varepsilon\langle\nabla y,\nabla w\rangle+\eta\langle y,w\rangle+\langle\nabla u\cdot\nabla y,w\rangle+\langle\Delta u y,w\rangle=\langle g,w\rangle. \tag{4.10}
$$

The left hand side of (4.10) with $w=y\in H^1$, $y|_\Gamma=0$ can be bounded from below:

$$
\begin{aligned}
&\varepsilon\,|\nabla y|^2_{L^2}+\eta\,|y|^2_{L^2}+\tfrac{1}{2}\langle\nabla u\cdot n,y^2\rangle_{L^2(\partial\Omega)}+\tfrac{1}{2}\langle\Delta u,y^2\rangle\\
&\quad\geqslant\varepsilon\,|\nabla y|^2_{L^2}+(\eta-\tfrac{1}{2}\,|\Delta u|_{L^\infty})\,|y^2|_{L^2}.
\end{aligned}
$$

Hence there exists a unique variational solution $y_\varepsilon\in H^1(\Omega)$ of (4.9). The regularity assumption (A4) on u and (A5) imply further that $y_\varepsilon\in H^2(\Omega)$ and that y_ε satisfies the boundary conditions in the strong sense, see, e.g., [T, p. 132]. We next take the scalar product in $L^2(\Omega)$ of (4.9) with $-\Delta y_\varepsilon$:

$$
\varepsilon\,|\Delta y_\varepsilon|^2_{L^2}+\eta\,|\nabla y_\varepsilon|^2_{L^2}-\langle\nabla u\cdot\nabla y_\varepsilon,\Delta y_\varepsilon\rangle-\langle\Delta u y_\varepsilon,\Delta y_\varepsilon\rangle=-\langle g,\Delta y_\varepsilon\rangle.
$$

Using the formula

$$
\operatorname{div}((\nabla u\cdot\nabla y_\varepsilon)\,\nabla y_\varepsilon-\tfrac{1}{2}\nabla u\,|\nabla y_\varepsilon|^2)=(\nabla u\cdot\nabla y_\varepsilon)\,\Delta y_\varepsilon+\nabla y_\varepsilon^T u\nabla y_\varepsilon-\tfrac{1}{2}\,\Delta u\,|\nabla y_\varepsilon|^2
$$

We obtain

$$
\begin{aligned}
&\varepsilon\,|\Delta y_\varepsilon|^2_{L^2}+\eta\,|\nabla y_\varepsilon|^2_{L^2}+\langle\nabla y_\varepsilon, u\nabla y_\varepsilon\rangle-\tfrac{1}{2}\langle\Delta u,|\nabla y_\varepsilon|^2\rangle\\
&\quad+\tfrac{1}{2}\langle n\cdot\nabla u,|\nabla y_\varepsilon|^2\rangle_{L^2(\partial\Omega)}-\langle\nabla u\cdot\nabla y_\varepsilon,n\cdot\nabla y_\varepsilon\rangle_{L^2(\partial\Omega)}+\langle\nabla(\Delta u y_\varepsilon),\nabla y_\varepsilon\rangle\\
&=\langle\nabla g,\nabla y_\varepsilon\rangle.
\end{aligned}
$$

Here the boundary conditions for g on Γ and for $\partial y_\varepsilon/\partial n$ on $\partial\Omega-\Gamma$ were used.

In the following calculations we consider $n=3$. The dimensions $n=1$ and $n=2$ can easily be seen to be special cases of $n=3$. On Γ we find

$$
\begin{aligned}
&\tfrac{1}{2}(n\cdot\nabla u)\,|\nabla y_\varepsilon|^2-(\nabla u\cdot\nabla y_\varepsilon)(n\cdot\nabla y_\varepsilon)\\
&\quad=-(n\cdot\nabla u)\,|\nabla y_\varepsilon|^2+\left[\begin{pmatrix}0\\-n_3\\n_2\end{pmatrix}\cdot\nabla y_\varepsilon\right]\nabla u\cdot\begin{pmatrix}0\\-y_{\varepsilon,x_3}\\y_{\varepsilon,x_2}\end{pmatrix}
\end{aligned}
$$

$$+\left[\begin{pmatrix} n_3 \\ 0 \\ -n_1 \end{pmatrix}\cdot\nabla y_\varepsilon\right]\nabla u\cdot\begin{pmatrix} y_{\varepsilon,x_3} \\ 0 \\ -y_{\varepsilon,x_1} \end{pmatrix}$$

$$+\left[\begin{pmatrix} -n_2 \\ n_1 \\ 0 \end{pmatrix}\cdot\nabla y_\varepsilon\right]\nabla u\cdot\begin{pmatrix} -y_{\varepsilon,x_2} \\ y_{\varepsilon,x_1} \\ 0 \end{pmatrix} = -\frac{1}{2}(n\cdot\nabla u)\,|\nabla y_\varepsilon|^2,$$

where we used the fact that $y_\varepsilon = 0$ on Γ. Moreover, on $\partial\Omega - \Gamma$ we have $\partial y_\varepsilon/\partial n = 0$ and hence

$$\tfrac{1}{2}(n\cdot\nabla u)\,|\nabla y_\varepsilon|^2 - (\nabla u\cdot\nabla y_\varepsilon)(n\cdot\nabla y_\varepsilon) = \tfrac{1}{2}(n\cdot\nabla u)\,|\nabla y_\varepsilon|^2.$$

Thus we find

$$\varepsilon\,|\Delta y_\varepsilon|^2_{L^2} + \eta\,|\nabla y_\varepsilon|^2_{L^2} + \left\langle \nabla y_\varepsilon, u\nabla y_\varepsilon\right\rangle + \tfrac{1}{2}\langle \Delta u, |\nabla y_\varepsilon|^2\rangle$$
$$+\tfrac{1}{2}\langle |n\cdot\nabla u|, |\nabla y_\varepsilon|^2\rangle_{L^2(\partial\Omega)} + \langle \nabla(\Delta u y_\varepsilon), \nabla y_\varepsilon\rangle = \langle \nabla g, \nabla y_\varepsilon\rangle.$$

This allows the estimate

$$\varepsilon\,|\Delta y_\varepsilon|^2_{L^2} + (\eta - \tfrac{1}{2}|\Delta u|_{L^\infty} - \lambda_M)\,|\nabla y_\varepsilon|^2_{L^2} - |\nabla\,\Delta u|_{L^4}\,|y_\varepsilon|_{L^4}\,|\nabla y_\varepsilon|_{L^2}$$
$$\leqslant |\nabla g|_{L^2}\,|\nabla y_\varepsilon|_{L^2}. \tag{4.11}$$

Let $k > 1$ be a constant for which $|\varphi|_{L^4} \leqslant k\,|\varphi|_{H^1}$ for all $\varphi \in H^1(\Omega)$ and $|\varphi|_{L^\infty} \leqslant k\,|\varphi|_{W^{1,4}}$ for all $\varphi \in W^{1,4}(\Omega)$. Then we find

$$\varepsilon\,|\Delta y_\varepsilon|^2_{L^2} + (\eta - 2k\,|\Delta u|_{W^{1,4}} - \lambda_M)\,|\nabla y_\varepsilon|^2_{L^2} - \tfrac{1}{2}k\,|\Delta u|_{W^{1,4}}\,|y_\varepsilon|^2_{L^2}$$
$$\leqslant |\nabla g|_{L^2}\,|\nabla y_\varepsilon|_{L^2}. \tag{4.12}$$

Taking the scalar product of (4.9) with y_ε gives in view of (4.10)

$$\varepsilon\,|\nabla y_\varepsilon|^2_{L^2} + (\eta - \tfrac{1}{2}|\Delta u|_{L^\infty})\,|y_\varepsilon|^2_{L^2} \leqslant |g|_{L^2}\,|y_\varepsilon|_{L^2}. \tag{4.13}$$

Combining (4.12) and (4.13) we arrive at

$$\varepsilon(|\nabla y_\varepsilon|^2_{L^2} + |\Delta y_\varepsilon|^2_{L^2}) + (\eta - 3k\,|\Delta u|_{W^{1,4}})(|y_\varepsilon|^2_{L^2} + |\nabla y_\varepsilon|^2_{L^2})$$
$$\leqslant |g|_{L^2}\,|y_\varepsilon|_{L^2} + |\nabla g|_{L^2}\,|\nabla y_\varepsilon|_{L^2}$$
$$\leqslant \sqrt{2}\,|g|_{H^1}\,|y_\varepsilon|_{H^1}. \tag{4.14}$$

From (4.14) we deduce that

$$(\eta - 3k\,|\Delta u|_{W^{1,4}})\,|y_\varepsilon|_{H^1} \leqslant \sqrt{2}\,|g|_{H^1}. \tag{4.15}$$

Since by assumption $\eta > (5/2)\, k\, |\Delta u|_{W^{1,4}}$, there exists a subsequence $\{y_{\varepsilon_k}\}_{k=1}^{\infty}$ of $\{y_\varepsilon\}_{\varepsilon>0}$ and an element $y \in H^1$ such that $w-\lim_k y_{\varepsilon_k} = y$ in H^1. Moreover we have $y|\Gamma = 0$ and $\varepsilon\, |\Delta y_\varepsilon|_{L^2} \to 0$ as $\varepsilon \to 0$. Thus it follows that y is a solution of (4.7). The a priori estimates of Lemma 4.3 on y in terms of g follow from (4.13) and (4.15).

In the case $n = 1$, estimate (4.11) is replaced by

$$\varepsilon\, |\Delta y_\varepsilon|_{L^2}^2 + (\eta - \tfrac{3}{2}\, |\Delta u|_{L^\infty})\, |\nabla y_\varepsilon|_{L^2}^2 + |u''|_{L^2}\, |y_\varepsilon|_{L^\infty}\, |\nabla y_\varepsilon|_{L^2} \leqslant |\nabla g|_{L^2}\, |\nabla y_\varepsilon|_{L^2}, \tag{4.11'}$$

from which it follows that

$$\varepsilon\, |\Delta y_\varepsilon|_{L^2}^2 + \left(\eta - \frac{3}{2}\, |\Delta u|_{L^\infty} - \frac{3}{2}\, k\, |u''|_{L^2}\right) |\nabla y_\varepsilon|_{L^2}^2 - \frac{k}{2}\, |u''|_{L^2}\, |y_\varepsilon|_{L^2}^2 \leqslant |\nabla g|_{L^2}\, |\nabla y_\varepsilon|_{L^2}. \tag{4.12'}$$

Together with (4.13), (4.12') implies

$$\varepsilon\, |\Delta y_\varepsilon|_{L^2}^2 + (\eta - \tfrac{3}{2}\, |\Delta u|_{L^\infty} - \tfrac{3}{2}\, k\, |u''|_{L^2})(|\nabla y_\varepsilon|_{L^2}^2 + |y_\varepsilon|_{L^2}^2) \leqslant \sqrt{2}\, |g|_{H^1}\, |y_\varepsilon|_{H^1}.$$

This implies the assertion in Remark 4.4

Proof of Lemma 4.2. Let C denote the injective mapping of Lemma 4.3 that assigns to every $g \in H^1(\Omega)$ with $g|_\Gamma = 0$ in $H^{1/2}(\Gamma)$ the unique solution y of (4.7). By Lemma 4.3, C can be extended to a bounded linear operator on $L^2(\Omega)$, which will be denoted by $\bar{C}$. Its inverse, $\bar{C}^{-1}$, is well defined on $R(\bar{C})$ and closed. Below it will be argued that $D(\bar{C}^{-1}) = R(\bar{C}) \supset D(B)$. This will imply that $\bar{C}^{-1}$ is a closed extension of $\mu I + B$. Hence $\mu I + B$ and consequently B are closable.

To argue that $R(\bar{C}) \supset D(B)$ let $S = \{y \in H^1(\Omega) : y|_\Gamma = 0,\ \nabla \cdot (h \nabla u) \in H^1(\Omega),\ \nabla \cdot (h \nabla u)|_\Gamma = 0\}$. Since the space $D(B) = \{\varphi \in H^1(\Omega) : \varphi | \Gamma = 0$ in $H^{1/2}(\Gamma)\}$ can be equivalently defined as the closure in $H^1(\Omega)$ of $C_c^\infty(\Omega \cup (\partial\Omega - \Gamma))$ (the set of C^∞-functions with compact support in $\Omega \cup (\partial\Omega - \Gamma)$) it follows that S is dense in $D(B)$ in the $H^1(\Omega)$ topology [T, pp. 67, 75]. Hence there exists for every $y \in D(B)$ a sequence $y_n \in S$ converging to y in $H^1(\Omega)$. Let us define $g_n =: \eta y_n + \nabla \cdot (y_n \nabla u)$. Then $C g_n = y_n$ and g_n is a Cauchy sequence in $L^2(\Omega)$. Hence there exists an element $g \in L^2(\Omega)$ with $\lim_n g_n = g$ in $L^2(\Omega)$. Moreover $\bar{C} g = y$ and therefore $y \in R(\bar{C})$ as desired. This ends the proof.

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