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Journal of Pure and Applied Algebra 202 (2005) 313–324

JOURNAL OF PURE AND APPLIED ALGEBRA

www.elsevier.com/locate/jpaa

Automatic quotients of free groups

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Received 7 May 2003; received in revised form 28 January 2005
Available online 7 April 2005
Communicated by C.A. Weibel

Abstract

Automatic groups admitting prefix-closed automatic structures with uniqueness are characterized as the quotients of free groups by normal subgroups possessing sets of free generators satisfying certain language-theoretic conditions.

MSC: Primary: 20F65; secondary: 68Q45

1. Introduction

It is an open problem whether or not every synchronously automatic group admits a prefix-closed automatic structure with uniqueness [4, Open Problem 2.5.10]. Motivated by this problem, we give new characterizations of synchronous and asynchronous automatic groups which have prefix-closed automatic structures with uniqueness.

Automatic groups are a class of finitely presented groups modeled on the fundamental groups of compact 3-manifolds and defined by the property that group multiplication can be carried out by finite automata. The standard introduction is [4]; [3] is a more recent introduction for those familiar with the theory of finite automata. Accounts of the context in which automatic groups occur are given in [2,9].

[1] The author thanks Imperial College for its hospitality while this paper was being written.

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0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2005.02.004

Theorem 1. *A group has a prefix-closed asynchronous automatic structure with uniqueness if and only if it is isomorphic to the quotient of a finitely generated free group F by a normal subgroup N admitting a linear language L of freely reduced generators with significant letters*.

L is a subset of the free monoid over a set of free generators and their inverses for F. Linear languages lie between the better known classes of regular and context free languages. L has significant letters if each word in L has a distinguished letter such that free reduction of a product of two words from L or their inverses does not affect the distinguished letters except when the product consists of a word times its inverse. Precise definitions are given in the next section.

Theorem 2. *A group has a prefix-closed synchronous automatic structure with uniqueness, if and only if it is isomorphic to the quotient of a finitely generated free group by a normal subgroup admitting a linear language of freely reduced generators with significant letters satisfying either of the following two conditions*:

(1) *For some constant k each significant letter is k-central*.
(2) *The significant letters are o-central*.

A significant letter is k-central if it is within a distance k of the center of its word. A set of words with significant letters has o-central significant letters if either the set is finite or if the distance from significant letters to the center of their words is o of the length of the word. Thus k-central implies o-central.

The generators mentioned in Theorems 1 and 2 are essentially the Schreier generators corresponding to the combing from the automatic structure. The desired properties of these generators are derived in a straightforward way from the combing, but the argument in the other direction is more complicated.

2. Preliminary definitions and results

2.1. Formal languages

An alphabet is a finite nonempty set, Σ. A formal language over Σ is a subset of Σ^*, the free monoid over Σ. Elements of Σ^* are called words. The identity element of Σ^* is the empty word, denoted ε. $|w|$ is the length of a word w. If $w = a_1 \ldots a_n$, the distance from the letter a_i to the center of w is $|i - (n+1)/2|$.

Σ^* is well ordered by the shortlex order, which is defined by $u < v$ if either $|u| < |v|$ or $|u| = |v|$ and u is less than v in the lexicographic order corresponding to some fixed ordering of Σ. The shortlex order has the property that $u < v$ implies $xuy < xvy$ for all $x, y \in \Sigma^*$.

We assume the reader is familiar with the theory of automatic groups including the basic facts about regular languages and finite automata necessary for the development of that theory. We include some additional definitions and results from formal language theory which we shall need. See [7] for a survey of the whole field.

Recall that regular languages are the languages accepted by finite automata. A finite automaton $\mathscr{A}$ over Σ is a finite-directed graph with edge labels from $\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$, a designated initial vertex, and some terminal vertices. A path in $\mathscr{A}$ is called successful if it starts at the initial vertex and ends at a terminal vertex. The language accepted by $\mathscr{A}$ is the collection of labels of successful paths. The label of a path is just the product of its edge labels. The label of a path of length 0 is ε. We assume without loss of generality that every edge and every vertex of an automaton occur in some successful path. Other edges and vertices can simply be deleted. We denote by $|\mathscr{A}|$ the number of vertices in a finite automaton $\mathscr{A}$.

Regular languages are closed under union, product and generation of submonoid. We require some additional properties.

Lemma 2.1. *Every regular language R may be expressed as a finite union $R = \bigcup X_i Y_i$ of products of regular languages X_i, Y_i in such a way that $w = uv \in R$ if and only if $u \in X_i$, $v \in Y_i$ for some i.*

Proof. Let R be accepted by an automaton $\mathscr{A}$ with vertices $p_1, \ldots, p_n$. For each i between 1 and n define an automaton $\mathscr{A}_i$ by altering $\mathscr{A}$ so that p_i is its single terminal vertex. Likewise define $\mathscr{A}'_i$ by making p_i the initial vertex. The languages X_i accepted by $\mathscr{A}_i$ and Y_i accepted by $\mathscr{A}'_i$ are as required. $\square$

Definition 2.2. For any word w, w^r is w written backwards. Likewise for any language L, $L^r = \{w^r \mid w \in L\}$.

Lemma 2.3. *If R is a regular language, so is R^r.*

Proof. Let R be accepted by the automaton $\mathscr{A}$. Reverse the orientation of the edges of $\mathscr{A}$ and make the initial vertex the single terminal. Taking each original terminal vertex in turn as the initial vertex, we obtain a set of finite automata. The union of the languages accepted by these automata is R^r. $\square$

2.2. Transductions

Finite automata over $\Sigma \times \Sigma$ are defined just like finite automata over Σ except that edge labels are from $\Sigma_\varepsilon \times \Sigma_\varepsilon$. The label of a path of length 0 is $(\varepsilon, \varepsilon)$, and the collection of labels of successful paths is a subset of $\Sigma^* \times \Sigma^*$ called a rational transduction over Σ. A rational transduction is a binary relation on Σ^*.

Lemma 2.4. *Every finite binary relation on Σ^* is a rational transduction. Projections of rational transduction onto either coordinate yield regular languages. Rational transductions are closed under union and product. They are also closed under intersection with direct products $R \times S \subset \Sigma^* \times \Sigma^*$ of regular languages R, S over Σ.*

Proof. The first two assertions are immediate from the definition of rational transduction. To show closure under union combine two automata over $\Sigma \times \Sigma$ by adding a new initial vertex together with edges labeled $(\varepsilon, \varepsilon)$ from the new vertex to the initial vertex of each automaton. The new automaton accepts the union of the two rational transductions accepted by the original automata. Closure under product is demonstrated similarly using edges from the terminal vertices of the first automaton to the initial vertex of the second.

To complete the proof of the lemma it suffices to show that if ρ is a rational transduction accepted by an automaton $\mathscr{A}$ over $\Sigma \times \Sigma$ and R is a regular language accepted by the automaton $\mathscr{B}$ over Σ, then $\rho \cap (\Sigma^* \times R)$ and $\rho \cap (R \times \Sigma^*)$ are both rational transductions. The argument is the same in both cases. We will show that $\rho \cap (\Sigma^* \times R)$ is a rational transduction.

First observe that it does no harm to require a loop (an edge from a vertex to itself) with label $(\varepsilon, \varepsilon)$ at each vertex of $\mathscr{A}$ and a loop with label ε at each vertex of $\mathscr{B}$. Now define an automaton $\mathscr{C} = \mathscr{A} \times \mathscr{B}$ over $\Sigma \times \Sigma$ as follows. The set of vertices of $\mathscr{C}$ is the Cartesian product of the vertices of $\mathscr{A}$ with the vertices of $\mathscr{B}$. There is an edge with label (a, b) from (p_1, q_1) to (p_2, q_2) if and only if there is an edge from p_1 to p_2 with label (a, b) in $\mathscr{A}$ and an edge from q_1 to q_2 with label b in $\mathscr{B}$.

It is easy to see that if there is path in $\mathscr{C}$ with label (u, v) from (p_1, q_1) to (p_2, q_2), then there is a path in $\mathscr{A}$ from p_1 to p_2 with label (u, v) and a path in $\mathscr{B}$ from q_1 to q_2 with label v. The converse is also straightforward once we observe that if there are paths in $\mathscr{A}$ from p_1 to p_2 with label (u, v) and in $\mathscr{B}$ from q_1 to q_2 with label v, then by judiciously inserting loops with labels $(\varepsilon, \varepsilon)$ or ε into the two paths we can arrange things so that v is expressed in exactly the same way as a product of elements of Σ_ε along both paths. (This argument is given in greater detail in the proof of [6, Theorem 4.4].)

Take the initial vertex of $\mathscr{C}$ to be (p_0, q_0) where p_0 is the initial vertex of $\mathscr{A}$ and q_0 is the initial vertex of $\mathscr{B}$. Likewise (p, q) is terminal if both p and q are. It follows from the preceding paragraph that (u, v) is the label of a successful path in $\mathscr{C}$ if and only if (u, v) is the label of a successful path in $\mathscr{C}$ and v is the label of a successful path in $\mathscr{B}$. $\square$

Lemma 2.5. *For each regular language R over Σ the binary relation $\rho_R = \{(u, u) \mid u \in R\}$ is a rational transduction.*

Proof. Since $\rho_R = \rho_{\Sigma^*} \cap (R \times R)$, it suffices by Lemma 2.4 to consider the case $R = \Sigma^*$. The automaton with one vertex p (which is both initial and terminal) and edges $p \overset{(a,a)}{\to} p$ for each $a \in \Sigma$ accepts ρ_{Σ^*}. $\square$

2.3. Linear languages

Definition 2.6. A language L over Σ is linear if for some rational transduction ρ over Σ, $L = \{uv^r \mid (u, v) \in \rho\}$.

In other words a linear language consists of all words uv^r such that (u, v) is the label of a successful path in some fixed automaton over $\Sigma \times \Sigma$. Other characterizations are given in [7, Chapter 3, Section 6.1]. Automata over $\Sigma \times \Sigma$ serve as acceptors for both rational transductions and linear languages.

Lemma 2.7. *The union of two linear languages is linear. The intersection of a linear language over Σ and a regular language over Σ is linear.*

Proof. The first assertion is immediate from Lemma 2.4. For the second let L be linear and R regular. L is accepted by an automaton $\mathscr{A}$ which also accepts a rational transduction ρ such that $L = \{uv^r \mid (u, v) \in \rho\}$. Express $R = \bigcup X_i Y_i$ as in Lemma 2.1. By Lemmas 2.3 and 2.4 $\rho' = \bigcup(\rho \cap (X_i \times Y_i^r))$ is a rational transduction. Since $L \cap R = \{uv^r \mid (u, v) \in \rho'\}$, $L \cap R$ is linear. $\square$

2.4. Languages and groups

Consider a group G and a surjective homomorphism $\mu : F \to G$ from a finitely generated free group F. Let N be the kernel of μ. Take Σ to be an alphabet of free generators and their inverses for F and let $\pi : \Sigma^* \to F$ be the projection which sends each word $w \in \Sigma^*$ to the element of F it represents. Notice that Σ^* is equipped with formal inverses in a natural way, and π respects inverses. We will call this configuration a choice of generators for G. From now on Σ stands for an alphabet with formal inverses.

$$\Sigma^* \stackrel{\pi}{\to} F \stackrel{\mu}{\to} G. \tag{1}$$

When we wish to avoid explicit reference to μ and π, we will use $\widehat{x}$ and $\overline{x}$ to denote the image of x in F and G, respectively.

Given a choice of generators (1), we see that for every language L over Σ there is a subgroup $H = \langle \widehat{L} \rangle$ generated by the image of L in F. We call L a language of generators for H.

2.5. Significant letters

Definition 2.8. Let $L \subset \Sigma^*$ be a language of freely reduced words which does not contain the empty word. L has significant letters if every $w \in L$ can be written as a product $w = uav^{-1}$ with $a \in \Sigma$ such that for all $w_1, w_2 \in L$ and $\varepsilon_1, \varepsilon_2 = \pm 1$, free reduction of $(w_1)^{\varepsilon_1}(w_2)^{\varepsilon_2} = (u_1 a_1 v_1^{-1})^{\varepsilon_1}(u_2 a_2 v_2^{-1})^{\varepsilon_2}$ does not affect a_1 or a_2 unless the product reduces to ε.

When considering a word w in a language L with significant letters, $w = uav^{-1}$ will always mean the significant letter decomposition of w. Significant letters need not be uniquely determined, but it is clear from Definition 2.8 that we may assume that if $w, w^{-1} \in L$ and $w = uav^{-1}$, then $w^{-1} = va^{-1}u^{-1}$ is the significant letter decomposition of w. It follows that if L has significant letters, so does $L \cup L^{-1}$. We record this fact along with two immediate consequences of Definition 2.8.

Lemma 2.9. *Let L have significant letters. Then $L \cup L^{-1}$ has significant letters. Consider $w = uav^{-1}, w_1 \in L$. If either ua is a prefix of w_1 or av^{-1} is a suffix, then $w = w_1$. If either va^{-1} is a prefix or $a^{-1}u$ is a suffix, then $w = w_1^{-1}$.*

If $L \subset \Sigma^*$ has significant letters, then $\widehat{L}$ is a set of free generators for the subgroup $\langle \widehat{L} \rangle \subset F$ generated by $\widehat{L}$. Indeed if $w_1^{\varepsilon_1} \dots w_n^{\varepsilon_n}$ is any product of words from L and their inverses such that no $w_i^{\varepsilon_i}$ is followed by its inverse, then free reduction of the product does not affect the significant letter of any w_i. Consequently $\widehat{w_1}^{\varepsilon_1} \dots \widehat{w_n}^{\varepsilon_1} \neq 1$.

2.6. Combings

Definition 2.10. A combing is a language C over Σ such that $\overline{C} = G$. C is prefix closed if every prefix of any $w \in C$ is also in C. C is a combing with uniqueness if C maps bijectively to G. C is regular if it is a regular language.

There are other definitions of combing in the literature.

Lemma 2.11. *If C is a prefix-closed combing with uniqueness, then no nontrivial subword of a word in C defines the identity in G. In particular C consists of freely reduced words.*

Proof. If not, then there is $uxv \in C$ with $\overline{x} = 1$. By closure under prefixes, $u, ux \in C$, contradicting uniqueness. □

Lemma 2.12. *If ρ is a rational transduction over Σ, then $L = \{uv^{-1} \mid (u, v) \in \rho\}$ is a linear language. If L is linear, then so is $L^{-1} = \{w \mid w^{-1} \in L\}$.*

Proof. For the first part replace each edge label (a, b) with (a, b^{-1}). For the second assertion change each label (a, b) to (b^{-1}, a^{-1}). □

In practice we will not replace (a, b) by (a, b^{-1}). Instead we will just read (uv^{-1}) instead of (uv^r) for each path with label (u, v) in an automaton accepting ρ. From now on the linear language corresponding to a transduction τ will be $L = \{(uv^{-1} \mid (u, v) \in \tau\}$.

We will make use of the following possibly infinite automaton.

Definition 2.13. Let G be a group and (1) a choice of generators. The Cayley automaton $\mathscr{A}_G$ has vertices G and edges $g \overset{(a,b)}{\to} h$ for all $g, h \in G$ and $a, b \in \Sigma_\varepsilon$ with $g\overline{b} = \overline{a}h$. The initial state of $\mathscr{A}_G$ is 1, and all states are terminal states.

Lemma 2.14. *There is a path in $\mathscr{A}_G$ with label (u, v) from 1 to h if and only if $h = \overline{u^{-1}v}$. If u and v are asynchronous k-fellow travelers, then the path may be chosen in the ball of radius k around 1.*

Proof. It is straightforward to prove by induction on length that there is a path with label (u, v) from g to h in $\mathscr{A}_G$ if and only if $g\overline{u} = \overline{v}h$. The second assertion follows directly from the definition of asynchronous fellow traveler. □

2.7. Automatic structures

Lemma 2.15. *A combing C for a group G supports a prefix-closed asynchronous automatic structure with uniqueness if and only if C is prefix-closed with uniqueness, and for each $a \in \Sigma$ the binary relation $\rho_a = \{(u, v) \mid u, v \in C, \overline{ua} = \overline{v}\}$ is a rational transduction.*

Proof. Suppose C supports a prefix-closed asynchronous automatic structure with uniqueness in the sense of [4, Definition 7.2.1]. Then C is a prefix-closed combing with uniqueness, and it is not hard to check that the corresponding binary relations are transductions.

For the converse take C and ρ_a, $a \in \Sigma$ as above, and let k be an upper bound for the number of vertices in automata $\mathscr{A}_a$ accepting ρ_a. Suppose γ is a successful path in some $\mathscr{A}_a$. For each vertex p of γ there is a path of length at most k from p to a terminal vertex of $\mathscr{A}_a$. Thus if (u, v) is the label of γ up to p, there are words x, y of length at most k such that $(ux, vy) \in \rho_a$. Consequently, $\overline{uxa} = \overline{vy}$, which implies that the word difference $u^{-1}v$ has the same image in G as some word of length at most $2k + 1$. From this observation together with the fact that ρ_ε is the identity binary relation on C we see that C satisfies the asynchronous fellow traveler property. By Theorems 1 and 2 of [8] some subset of C is a regular combing supporting an asynchronous automatic structure. Since C is a combing with uniqueness, the subset must be C itself. $\square$

An analog of Lemma 2.15 holds for synchronous automatic structures, but the rational transductions are of a special type.

Definition 2.16. A rational transduction $\rho \subset \Sigma^* \times \Sigma^*$ is called synchronized if $(u, v) \in \rho$ implies the lengths $|u|$ and $|v|$ differ by at most k for some constant k. A finite automaton over $\Sigma \times \Sigma$ is synchronized if it is built up from a subautomaton $\mathscr{A}_0$ with edge labels all in $\Sigma \times \Sigma$ by attaching directed paths of length at most k such that the edge labels along each path are either all in $\Sigma \times \{\varepsilon\}$ or all in $\{\varepsilon\} \times \Sigma$. These paths are attached at their initial points only and are otherwise disjoint from each other.

It is clear that any rational transduction accepted by a synchronized automaton is synchronized. The converse follows from [5, Proposition 2.1].

Lemma 2.17. *A rational transduction is synchronized if and only if it is accepted by a synchronized finite automaton.*

Lemma 2.18. *A combing C for a group G supports a prefix-closed synchronous automatic structure with uniqueness if and only if C is prefix-closed with uniqueness and for each $a \in \Sigma$ the binary relation $\rho_a = \{(u, v) \mid u, v \in C, \overline{ua} = \overline{v}\}$ is a synchronized rational transduction.*

Proof. Suppose C supports a prefix-closed synchronous automatic structure with uniqueness in the sense of [4, Definition 2.3.1]. It is clear that the associated binary relations ρ_a are rational transductions. By [4, Lemma 2.3.9] the uniqueness condition on C implies that the ρ_a's are synchronized rational transductions.

For the converse take C and ρ_a, $a \in \Sigma$ as above. Synchronized finite automata accepting the ρ_a's fit the definition of the automata occurring in [4, Definition 2.3.1] once labels (a, ε) and (ε, a) are replaced by labels $(a, \$)$ and $(\$, a)$, respectively. The same conclusion holds for ρ_ε, as it is the identity on C. $\square$

Automatic structures can also be defined in terms of regular combings satisfying fellow traveler conditions. These conditions are defined in terms of the word metric d corresponding to a choice of generators (1). We write $D_a(w, v) \leqslant k$ if two words $w, v \in \Sigma^*$ satisfy the asynchronous k-fellow traveler condition and $D_s(w, v) \leqslant k$ if they satisfy the synchronous k-fellow traveler condition. The following lemma records some well known properties.

Lemma 2.19. *If $D_s(w, v) \leqslant k$, then $D_a(w, v) \leqslant k$. Further,*

(1) *If $D_a(u, v) \leqslant k$ and $D_a(v, v') \leqslant k'$, then $D_a(u, v') \leqslant k + k'$.*
(2) *If $D_s(u, v) \leqslant k$ and $D_s(v, v') \leqslant k'$, then $D_s(u, v') \leqslant k + k'$.*
(3) *$D_s(u, uv) \leqslant |v|$.*

3. Finding generators

In this section we prove Theorems 1 and 2 in one direction by extracting from an automatic structure a language of generators of the required type. The arguments are identical for both types of automatic group except for one paragraph which applies only to the synchronous case.

Let G be automatic of either type. Make a choice of generators (1), and take N to be the kernel of μ. As in Definitions 2.15 and 2.18 C is a combing supporting a prefix-closed automatic structure with uniqueness, and for each $a \in \Sigma$, $\rho_a = \{(u, v) \mid u, v \in C, a \in \Sigma, \overline{ua} = \overline{v}\}$ is a rational transduction. In the synchronous case ρ_a is a synchronized rational transduction. We will show that $L = \{uav^{-1} \mid u, v \in C, a \in \Sigma, \overline{ua} = \overline{v}, uav^{-1} \text{ is freely reduced}\}$ is the desired language of generators.

First we note that by construction L is closed under inverse. Next we show that L is a linear language. By Lemma 2.4 the product $(\rho_a)\{(a, \varepsilon)\} = \{(ua, v) \mid u, v \in C, a \in \Sigma, \overline{ua} = \overline{v}\}$ is a rational transduction. Likewise $\rho = \bigcup_{a \in \Sigma} \rho_a = \{(ua, v) \mid u, v \in C, a \in \Sigma, \overline{ua} = \overline{v}\}$ is also a rational transduction. Hence $L' = \{uav^{-1} \mid u, v \in C, a \in \Sigma, \overline{ua} = \overline{v}\}$ is a linear language. As L is the intersection of L' with the regular language of nontrivial freely reduced words, L is linear by Lemma 2.7.

In the synchronous case ρ is synchronous because each ρ_a is. Thus for some positive integer k, $(ua, v) \in \rho$ implies that $|ua|$ and $|v|$ differ by at most k. We conclude that in the synchronous case the a's are k-central for words in L and hence o-central as well.

It remains to show that in both cases L is a language of generators and the a's are significant letters for L. Observe that prefix closure and uniqueness for C imply that $\widehat{C}$ is a set of prefix-closed coset representatives for N in F. We will interpret this fact geometrically.

Each $w \in \Sigma^*$ may be thought of as a path beginning at 1 in Γ, the Cayley diagram of G with respect to the set of generators Σ. We pick one letter from each pair a, a^{-1} to use as edge labels in Γ. An edge of Γ traversed backwards is construed as a forward edge with the inverse label. C is a spanning tree for Γ, and any word uav^{-1} with $\overline{ua} = \overline{v}$ is a cycle. If a labels an edge of Γ in the spanning tree C, then because of our convention about edge labels, uav^{-1} is a cycle in C and thus freely equal to ε. Otherwise uav^{-1} is freely reduced by inspection. Likewise free reduction of a product of two words in $u_1a_1v_1^{-1}, u_2a_2v_2^{-1} \in L$, cannot involve the a_i's unless they are labels of inverse edges in Γ in which case the product is freely equal to ε. Finally if uav^{-1} is freely reduced, so is $va^{-1}u^{-1}$. It follows that the a's are significant letters for L. Hence L is a language of free generators and their inverses for the subgroup $\langle \widehat{L} \rangle \subset F$.

A word $w \in \Sigma^*$ represents an element of N if and only if w is a cycle in Γ. Thus $\langle \widehat{L} \rangle \subset N$. On the other hand suppose w is a cycle in Γ. A short argument by induction on the number, n, of edges of w not in Γ shows that w is freely equal to a product of words in L. Indeed if $n = 0$, then as above w is a cycle in C and so freely equal to ε, which is the empty product. Otherwise $w = uax$ where a labels the first edge not in C. But then $u \in C$, and there is $uav^{-1} \in L$. Consequently w is freely equal to $(uav^{-1})vx$. But vx is a cycle to which the induction hypothesis applies.

4. Finding automatic structures

We complete the proofs of Theorems 1 and 2 by finding the required automatic structures. Assume that G is a group with choice of generators (1) and that N has a linear language, L, of freely reduced generators with significant letters. By Lemmas 2.9 and 2.7 we may assume that L is closed under inverse.

It suffices to show that Σ^* contains a prefix-closed regular combing with uniqueness which satisfies the appropriate fellow-traveler property. The arguments in the two cases are almost identical. When it is necessary to distinguish between them, we refer to the central and noncentral cases.

Let L be accepted by an automaton $\mathscr{A}$ over $\Sigma_\varepsilon \times \Sigma_\varepsilon$. If possible choose $\mathscr{A}$ to be synchronized. Let τ be the rational transduction accepted by $\mathscr{A}$; $L = \{uv^{-1} \mid (u, v) \in \tau\}$.

If $\mathscr{A}$ is synchronized, there are no edges with label $(\varepsilon, \varepsilon)$. However, in general there may be some. If there is a cycle with label $(\varepsilon, \varepsilon)$, then identifying all the vertices in the cycle, discarding the edges in the cycle, and taking the resulting vertex to be initial or terminal if one of the identified vertices does not change the set of labels of successful paths. Consequently, we assume that there are no such cycles.

Without loss of generality delete edges and vertices of $\mathscr{A}$ not lying on successful paths. If $\mathscr{A}$ was synchronized before this change, it remains so. Choose a constant K greater than the number of vertices and edges in $\mathscr{A}$.

If G is free or finite, there is nothing to prove. Thus we may assume $N \neq 1$ and N has infinite index in F. As finitely generated normal subgroups of free groups have finite index, N is not finitely generated. Thus L is infinite, and consequently $\mathscr{A}$ has at least one cycle.

Define $\mathscr{A}_0$ to be the subgraph of $\mathscr{A}$ consisting of all vertices and edges which are in cycles or in paths leading to cycles. As every edge of $\mathscr{A}$ is on a successful path, the initial vertex of $\mathscr{A}$ must be in $\mathscr{A}_0$. By definition of $\mathscr{A}_0$ there are no cycles in $\mathscr{A} - \mathscr{A}_0$ and no edges from $\mathscr{A} - \mathscr{A}_0$ into $\mathscr{A}_0$. Consequently every path in $\mathscr{A}$ lies in $\mathscr{A}_0$ except for its last j vertices for some $j \leqslant K$.

Lemma 4.1. *An edge of $\mathscr{A}$ whose label holds the significant letter for some successful path does not lie in $\mathscr{A}_0$.*

Proof. Assume otherwise. There is a successful path $\gamma = \gamma_1\gamma_2$ such that γ_1 is in $\mathscr{A}_0$ and the significant letter occurs in an edge label of γ_1. Since γ_1 is in $\mathscr{A}_0$, there are successful paths $\gamma_1\gamma_3\gamma_4\gamma_5$ where γ_4 is a cycle. By Lemma 2.9 all these paths have the same label. But then γ_4 must have label $(\varepsilon, \varepsilon)$ contrary to our assumption about cycles in $\mathscr{A}$. $\square$

Lemma 4.2. *In the central case τ is a synchronized rational transduction, and the edge labels of $\mathscr{A}_0$ lie in $\Sigma \times \Sigma$.*

Proof. We claim that every cycle in $\mathscr{A}$ has label (u, v) with $|u| = |v|$. As every path in $\mathscr{A}$ has at most K edges which do not occur in cycles along the path, it will follow that $||x| - |y|| \leqslant K$ for every $(x, y) \in \tau$. Hence, τ will be synchronized.

To verify our claim suppose (u, v) is the label of a cycle in $\mathscr{A}$ and $|u| \neq |v|$. As all cycles lie in $\mathscr{A}_0$, Lemma 4.1 implies that for fixed words u_0, u_1, v_0, v_1 and all integers $i \geqslant 0$, L contains words $u_0 u^i u_1 v_1^{-1} v^{-i} v_0^{-1}$ whose significant letters occur in the subword $u_1 v_1^{-1}$. It follows by a straightforward argument that the significant letters of L are not o-central and hence not k-central. Thus our claim is valid.

Finally, since τ is synchronized, our choice of $\mathscr{A}$ guarantees that $\mathscr{A}$ is too. It follows from Definition 2.16 that the edge labels of $\mathscr{A}_0$ lie in $\Sigma \times \Sigma$. $\square$

The choice of generators (1) determines a Cayley diagram Γ for G with the corresponding word metric d. Each word in Σ^* is the label of a unique path from 1 in Γ, and we will use w to refer to the path as well as the word. A word represents an element of N if and only if it is a cycle in Γ.

Lemma 4.3. *The language C consisting of all prefixes not including the significant letter of each $w \in L$ is a prefix-closed combing with uniqueness for G. Further, if $\overline{ua} = \overline{v}$ for $u, v \in C$ and $a \in \Sigma$, then either uav^{-1} is freely equal to ε or $uav^{-1} \in L$ with significant letter a.*

Proof. C is obviously prefix closed. For any $g \in G$ there is a simple path w in Γ from 1 to g. Since $N \neq 1$, there is a simple cycle of length at least 1 starting at g. Extend the path w by continuing around this cycle until its first return to w and then following w back to 1. This extension of w is a cycle passing through g with freely reduced label. Consequently w is the free reduction of a product of generators from L. Each of these generators is a cycle, and one of them, say $w = uav^{-1}$, must contain the vertex g. Consequently some prefix of u or of v is a path from 1 to g. As L is closed under taking inverses, that prefix lies in C. Thus C maps onto G.

To prove that C maps injectively to G suppose $\overline{u} = \overline{v}$ for $u, v \in C$ with $u \neq v$. It follows that uv^{-1} is freely equal to a nonempty product of generators from L. By the nature of significant letters, the prefix $u_1 a_1$ from the first generator $u_1 a_1 v_1^{-1}$ in the product and the suffix $a_n v_n^{-1}$ from the last generator are not affected by free reduction of the product. As u and v are both freely reduced, it follows that $u_1 a_1$ is a prefix of u or $a_n v_n^{-1}$ is a suffix of v^{-1}. But by the definition of C together with Lemma 2.9 this is impossible.

The last assertion is proved in the same manner. As u and v are freely reduced, uav^{-1} is either freely reduced or freely equal to $u_1 v_1^{-1}$ for prefixes u_1 of u and v_1 of v. In the latter case $u_1 = v_1$ by injectivity whence uav^{-1} is freely equal to ε. In the former case the argument of the previous paragraph yields either $ua = u_1 a_1$ or $av^{-1} = a_n v_n^{-1}$. It follows that $n = 1$ and $uav^{-1} = u_1 a_1 v_1^{-1}$. $\square$

Lemma 4.4. *In the noncentral case there is a constant k such that C satisfies the asynchronous k-fellow traveler condition. In the central case there is a constant k such that C satisfies the synchronous k-fellow traveler condition.*

Proof. Suppose $u, v \in C$ with $d(\overline{u}, \overline{v}) \leqslant 1$ in Γ. If $d(\overline{u}, \overline{v}) = 0$, then $u = v$ by uniqueness and both fellow traveler conditions are satisfied with $k = 0$. If $d(\overline{u}, \overline{v}) = 1$, then $\overline{ua} = \overline{v}$ for some $a \in \Sigma$. By Lemma 4.3 either ua is freely equal to v or $uav^{-1} \in L$. In the first case both fellow traveler conditions are satisfied with $k = 1$.

Assume the second case holds, and suppose γ is a successful path in $\mathscr{A}$ with label uav^{-1}. γ consists of a prefix γ_0 in $\mathscr{A}_0$ followed by a suffix of length at most K. Let (u_0, v_0) be the label of γ_0. By Lemma 4.1 u_0 includes all but the last j letters of u for some $j \leqslant 2K$, and likewise for v_0. In the central case $|u_0| = |v_0|$ as the edge labels of $\mathscr{A}_0$ are all from $\Sigma \times \Sigma$. By Lemma 2.19 it suffices to prove that u_0 and v_0 are k_0 fellow travelers of the appropriate type for some constant k_0.

Consider any vertex p of γ_0. There is a path of length at most K from p to a terminal vertex of $\mathscr{A}$. Thus $u_0 x(v_0 y)^{-1} \in L$ for some words x, y with $|x|, |y| \leqslant K$. Consequently $\overline{u_0 x} = \overline{v_0 y}$, which implies that the word difference $u_0^{-1} v_0$ has the same image in G as some word of length at most $2K$. In the noncentral case we see immediately that $D_a(u_0, v_0) \leqslant 2K$. In the central case $D_s(u_0, v_0) \leqslant 2K$ because the edge labels of $\mathscr{A}_0$ are all in $\Sigma \times \Sigma$. $\square$

It remains only to show that C is regular, but unfortunately there does not seem to be any reason why this should be so. However, by replacing certain suffixes of length at most $2K$ of words in C with new suffixes of length at most $2K$ we obtain a combing C' which works.

Recall that $\mathscr{A}_0$ is the subgraph of $\mathscr{A}$ supported by all vertices which are in cycles or in paths leading to cycles and that $\mathscr{A}_0$ contains the initial vertex of $\mathscr{A}$. Make $\mathscr{A}_0$ into an automaton $\mathscr{B}_0$ over Σ by replacing each edge label (a, b) with the label a. The initial vertex of $\mathscr{B}_0$ is the initial vertex of $\mathscr{A}$, and all vertices are terminal. $\mathscr{B}_0$ accepts a prefix-closed regular language C_0. By Lemma 4.1 C_0 is a collection of prefixes of C. It follows from the structure of $\mathscr{A}$ that each word in C is obtained by appending a word of length at most $2K$ to a word in C_0. We will define C' by appending other suffixes of at most the same length.

Let X be the set of all words in Σ^* of length at most $2K$. Clearly $C \subset C_0 X$. Define C' as follows. For each $g \in G$ pick the unique $x \in X$ minimum in the shortlex order such that there exists $u_0 \in C_0$ with $\overline{u_0 x} = g$. Since $C \subset C_0 X$, such a u_0 exists. By the uniqueness property of C, there is just one choice for u_0. Also since ε is the minimum element of X in the shortlex order, our construction guarantees $C_0 \subset C'$.

Lemma 4.5. *C' is a prefix-closed combing with uniqueness. For some constant k', C' satisfies the appropriate k'-fellow traveler condition.*

Proof. C' has uniqueness by construction. Likewise Lemma 4.4 and the properties listed in Lemma 2.19 insure that C' satisfies the appropriate fellow traveler condition. To show prefix closure consider a prefix v of $u_0 x \in C'$. If v is a prefix of u_0, then $v \in C_0 \subset C'$. Otherwise $v = u_0 x_1$ for some prefix x_1 of $x = x_1 x_2$. If $v \notin C'$, then there exists $u_1 y \in C'$ with $u_1 \in C_0$, $\overline{u_1 y} = \overline{u_0 x_1}$ and $y < x_1$.But then $y x_2 < x$ and $\overline{u_1 y x_2} = \overline{u_0 x}$ contradicting the construction of C'. $\square$

We must show that C' is a regular language. For each $x \in X$ let $C_x = \{r \mid r \in C_0, rx \in C'\}$. $C' = \bigcup_x C_x x$ is regular if each C_x is. $C_x = C_0 - \bigcup_{y \in X, y < x} C_{x,y}$ where $C_{x,y} = \{r \mid r \in C_0, \overline{rx} = \overline{sy}$ for some $s \in C_0\}$, so it suffices to show that $C_{x,y}$ is regular.

Define a finite automaton over $\Sigma_\varepsilon \times \Sigma_\varepsilon$ from the ball of radius $k + 4K$ around 1 in the Cayley automaton $\mathscr{A}_G$ by taking 1 as the initial vertex and $\overline{xy^{-1}}$ as the single terminal vertex. Let $\tau_{x,y}$ be the rational transduction accepted by this automaton. By Lemma 2.14 $\tau_{x,y}$ is contained in the set of (u, v) such that $\overline{u^{-1}v} = \overline{xy^{-1}}$ and contains all (u, v) such that $\overline{u^{-1}v} = \overline{xy^{-1}}$, and $D_a(u, v) \leqslant k + 4K$.

Suppose $r, s \in C_0$ with $\overline{rx} = \overline{sy}$ for $x, y \in X$. As $C_0 \subset C$, Lemmas 4.4 and 2.19 imply $D_a(r, s) \leqslant k + |x| + |y| \leqslant k + 4K$. Hence $\tau_{x,y} \cap (C_0 \times C_0)$ is a rational transduction whose projection onto the first coordinate is $C_{x,y}$. Thus $C_{x,y}$ is regular.

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