On Jordan left-$I$-centralizers of prime and semiprime gamma rings with involution

Kalyan Kumar Dey $^{a,*}$, Akhil Chandra Paul $^a$, Bijan Davvaz $^b$

$^a$ Department of Mathematics, Rajshahi University, Rajshahi 6205, Bangladesh

$^b$ Department of Mathematics, Yazd University, Yazd, Iran

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Abstract Let $M$ be a 2-torsion free $I$-ring with involution $I$ satisfying the condition $xzy\beta z = x\beta yxz$ for all $x, y, z \in M$ and $x, \beta \in I$. The object of our paper is to show that every Jordan left-$I$-centralizer on a semiprime $I$-ring with involution $I$, is a reverse left-$I$-centralizer. We solve some functional equations in prime and semiprime $I$-rings with involution $I$ by means of the above results. Moreover, we discuss some more related results.

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1. Introduction

The notion of a $I$-ring was first introduced by Nobusawa [6] as a generalization of a ring and then Barnes [2] generalized the definition of Nobusawa’s $I$-ring in a more general nature. For the $I$-rings we refer to Barnes [2]. Let $M$ and $I$ be additive abelian groups. If there is a mapping $M \times I \times M \rightarrow M$ (sending $(a, x, b) \rightarrow axb$) which satisfies the conditions (1) $(a + b)xc = axc + bxc, ax(x + \beta) = axb + a\beta b, ax(b + c) = axb + axc$; (2) $ax(b)\beta c = ax(b\beta c)$; for all $a, b, c \in M$ and $x, \beta \in I$. Then, $M$ is a $I$-ring in the sense of Barnes [2]. Let $M$ be a $I$-ring. A mapping $I: M \rightarrow M$ is called an involution if (1) $I(a + b) = I(a) + I(b)$; (2) $I(ab) = I(b)\alpha I(a)$; (3) $I^2(a) = a$ for all $a, b \in M$ and $x \in I$. In the ring theory, Bai, Dar and Vukman [1] proved that every Jordan left $\ast$-centralizer on a semiprime ring with involution, of char $R$ different from 2 is a reverse left $\ast$-centralizer. They used this result to make it possible to solve some fundamental equations in prime and semiprime rings. Also, see [3,4,7]. This paper deals with the study of Jordan left-$I$-centralizers of prime and semiprime $I$-rings with involution $I$, and was motivated by the work of [1]. Throughout, $M$ will represent a $I$-ring with center $Z(M)$. We shall denote by $C(M)$ the extended centroid of a prime $I$-ring $M$. For the explanation of $C(M)$ we refer to the reader in the paper of Soyturk [8]. Given an integer $n \geq 2$, a $I$-ring $M$ is said to be $n$-torsion free, if for $x \in M, nx = 0$ implies $x = 0$. As usual $[x, y]_n$ and $\langle x, y \rangle_n$ will denote the commutator $x^{2n} - y^{2n}$ and anti-commutator $xy + yx$, respectively for all $x, y \in M$ and $x \in I$. An additive mapping $T: M \rightarrow M$ is called a left centralizer in case $T(xyz) = T(x)yz$ holds for all $x, y, z \in M$. A mapping $I: M \rightarrow M$ is called an involution if (1) $I(a + b) = I(a) + I(b)$; (2) $I(ab) = I(b)\alpha I(a)$; (3) $I^2(a) = a$ for all $a, b \in M$ and $x \in I$. In the ring theory, Bai, Dar and Vukman [1] proved that every Jordan left $\ast$-centralizer on a semiprime ring with involution, of char $R$ different from 2 is a reverse left $\ast$-centralizer. They used this result to make it possible to solve some fundamental equations in prime and semiprime rings. Also, see [3,4,7]. This paper deals with the study of Jordan left-$I$-centralizers of prime and semiprime $I$-rings with involution $I$, and was motivated by the work of [1]. Throughout, $M$ will represent a $I$-ring with center $Z(M)$. We shall denote by $C(M)$ the extended centroid of a prime $I$-ring $M$. For the explanation of $C(M)$ we refer to the reader in the paper of Soyturk [8]. Given an integer $n \geq 2$, a $I$-ring $M$ is said to be $n$-torsion free, if for $x \in M, nx = 0$ implies $x = 0$. As usual $[x, y]_n$ and $\langle x, y \rangle_n$ will denote the commutator $x^{2n} - y^{2n}$ and anti-commutator $xy + yx$, respectively for all $x, y \in M$ and $x \in I$. An additive mapping $T: M \rightarrow M$ is called a left centralizer in case $T(xyz) = T(x)yz$ holds for all $x, y, z \in M$. A mapping $I: M \rightarrow M$ is called an involution if (1) $I(a + b) = I(a) + I(b)$; (2) $I(ab) = I(b)\alpha I(a)$; (3) $I^2(a) = a$ for all $a, b \in M$ and $x \in I$. In the ring theory, Bai, Dar and Vukman [1] proved that every Jordan left $\ast$-centralizer on a semiprime ring with involution, of char $R$ different from 2 is a reverse left $\ast$-centralizer. They used this result to make it possible to solve some fundamental equations in prime and semiprime rings. Also, see [3,4,7]. This paper deals with the study of Jordan left-$I$-centralizers of prime and semiprime $I$-rings with involution $I$, and was motivated by the work of [1]. Throughout, $M$ will represent a $I$-ring with center $Z(M)$. We shall denote by $C(M)$ the extended centroid of a prime $I$-ring $M$. For the explanation of $C(M)$ we refer to the reader in the paper of Soyturk [8]. Given an integer $n \geq 2$, a $I$-ring $M$ is said to be $n$-torsion free, if for $x \in M, nx = 0$ implies $x = 0$. As usual $[x, y]_n$ and $\langle x, y \rangle_n$ will denote the commutator $x^{2n} - y^{2n}$ and anti-commutator $xy + yx$, respectively for all $x, y \in M$ and $x \in I$. An additive mapping $T: M \rightarrow M$ is called a left centralizer in case $T(xyz) = T(x)yz$ holds for all
$x, y \in M$ and $\alpha \in \Gamma$. The definition of a right centralizer should be self-explanatory. An additive mapping $T$ is called a two-sided centralizer in case $T: M \to M$ is a left and a right centralizer. An additive mapping $T: M \to M$ is called a Jordan left centralizer if $T(xyx) = T(x)yx$ holds for all $x \in M$ and $y \in \Gamma$. Let $M$ be a $T$-ring with involution $I$. An additive mapping $T: M \to M$ is said to be a left-I-centralizer (resp. reverse left-I-centralizer) if $T(xy) = T(x)I(y)$ (resp. $T(xy) = T(y)I(x)$) holds for all $x, y \in M$ and $z \in \Gamma$. An additive mapping $T: M \to M$ is called a Jordan left-I-centralizer in case $T(xyx) = T(x)yxI(y)$ holds for all $x, y \in M$ and $z \in \Gamma$. The definition of a right-I-centralizer and a Jordan right-I-centralizer should be self-explanatory. For some fixed element $a \in M$, the map $M \to M$ defined by $x \mapsto axI(x)$ is a Jordan left-I-centralizer and the map $x \mapsto I(x)ax$ is a Jordan right-I-centralizer on $M$. Clearly, every left-I-centralizer on a $T$-ring $M$ is a Jordan left-I-centralizer. Further, we establish a result concerning additive mapping $T: M \to M$ satisfying the relation $T(xyxz) = T(x)zxI(x)$ for all $x \in M$ and $z \in \Gamma$. We showed that if $M$ is a 2-torsion free semiprime $T$-ring and $S: T \to M$ are left centralizers such that $[S(x), T(y)] + S(y)I[S(x), T(y)]$ for all $x, y \in M$ and $z \in \Gamma$. In case $M$ is a prime $T$-ring and $T \neq 0$ (i.e., $T(0) = 0$), then there exists $p, q \in C(M)$ such that $T = pS + (qT)$. We shall restrict our attention on Jordan left-I-centralizers, since all results presented in this article are also true for Jordan right-I-centralizers because of left-right symmetry. Throughout this paper, $M$ will represent a $T$-ring and $Z(M)$ will be its center. A $T$-ring $M$ is prime if $x\neq y\neq 0$ implies that $x \neq 0$ or $y \neq 0$, and is semi-prime if $x^2M \neq 0$ implies $x \neq 0$. Let $x, y \in M$ and $z \in \Gamma$, the commutator $x - y$ will be denoted by $[x, y]$. We know that $[x, y]z = [x, z]y + [y, z]x$ for all $x, y, z \in M$. Using this assumption, consider the above identities reduce to $[x, y]z = [x, z]y + [y, z]x$. Also, $[x, y]z = [x, y]z - [x, z]y + [y, z]x$ for all $x, y, z \in M$. Applying the assumption, we have $[x, y]z = [x, z]y + [y, z]x$, which are used extensively in our results.

### 2. Basic results

**Lemma 2.1.** Let $M$ be a prime $T$-ring with the central closure $C(M)$. Suppose that the elements $a, b \in C(M)$ satisfying the condition $\sum_{i=1}^{n}a_iT(x)I(y) = 0$ for all $x \in M$ and $\alpha \in \Gamma$. If $b_i = 0$ for some $i$, then $a_i$ are $C(M)$-independent.

**Proof.** We show that $a_i$ are linearly independent over $C(M)$. If not, there are a minimal $n$ elements $a_1, a_2, \ldots, a_n \in M$ linearly independent over $C(M)$ such that $\sum_{i=1}^{n}a_iI(x)I(y) = 0$ for all $x \in M$ and $\alpha \in \Gamma$, where $b_i$ are non-zero elements of $M$. Since $M$ is prime, $n > 1$. Suppose that $x_i, y_i \in M$ are such that $\sum_{i=1}^{n}x_iI(y_i) = 0$. If $r \in M$, then $\sum_{i=1}^{n}a_iI(x)I(y) = \sum_{i=1}^{n}a_iI(x)I(y)I(x) = \sum_{i=1}^{n}a_iI(x)I(y)I(x)$ for all $r \in M$.Define $b_i = 0$, we obtain a shorter relation than $n$, we have that $\sum_{i=1}^{n}x_iI(y) = 0$ for all $i$. Hence, the map $\psi: MTb, GM \to M$ defined by $\psi(\sum_{i=1}^{n}u_iI(y)I(x)) = \sum_{i=1}^{n}u_iI(y)I(x) = 0$ is well defined. It is trivial that $\psi_i$ is an additive map of the ideal $MTb, GM \to M$. Hence, $\psi_i$ gives an element $b_i$ such that $\psi_i(b_i) \in C(M)$. Moreover, by definition $\psi_i(b_i) = b_i$. Thus, $\sum_{i=1}^{n}a_iI(x)I(y) = \sum_{i=1}^{n}a_iI(x)I(y)I(x) = \sum_{i=1}^{n}a_iI(x)I(y)I(x)$. Since the primeness of $M$, we get that $\sum_{i=1}^{n}a_iI(x)I(y) = 0$, since $a_i$ are linearly independent over $C(M)$, we must have $\psi_i = 0$. But then by definition of $\psi_i, MTb, GM = 0$, gives a contradiction $b_i = 0$. □

**Lemma 2.2.** Let $M$ be a prime $T$-ring with involution $I$ satisfying the condition $xyI(x)I(x) = xyI(x)I(x)$ for all $x, y \in M$, $\alpha \beta \in \Gamma$ and let $T: M \to M$ be a Jordan left-I-centralizer on $M$. If $T(x) = T(x)I(x)I(x)$ for all $x \in M$, then $T = 0$.

**Proof.** By the assumption we have $[T(x), y] = 0$ for all $x, y \in M$ and $\alpha \beta \in \Gamma$. Substituting $xI(x)$ for $x$ in the above relation, then we obtain $0 = [T(xI(x)), y] = [T(x), y]I(xI(x))$ for all $x, y \in M$ and $\alpha \beta \in \Gamma$. In view of our hypothesis, the last expression yields $T(xI(x))I(x) = 0$ for all $x, y \in M$ and $\alpha \beta \in \Gamma$. Because the center of a prime $T$-ring is free from zero divisors, every $T(x) = 0$ or $[T(x), y] = 0$ for all $x, y \in M$ and $\alpha \beta \in \Gamma$. Let $A = \{x \in M \mid T(x) = 0\}$ and $B = \{x \in M \mid [T(x), y] = 0\}$ for all $y \in M$ and $\alpha \beta \in \Gamma$. It can be easily seen that $A$ and $B$ are two additive subgroups of $M$ whose union is $M$ and hence by Brauer’s trick, we get $A = M = B = M$. If $B = M$, then $M$ is commutative, which gives a contradiction. Thus, the only possibility remains that $A = M$. That is, $T(x) = 0$ for all $x \in M$. This completes the proof. □

**Theorem 2.3.** Let $M$ be a semiprime $T$-ring with involution $I$ satisfying the condition $xyI(x)I(x) = xyI(x)I(x)$ for all $x, y \in M$, $\alpha \beta \in \Gamma$, of characteristic different from two and $T: M \to M$ an additive mapping which satisfies $T(xyxz) = T(x)yxI(x)$ for all $x \in M$ and $\alpha \beta \in \Gamma$. Then, $T$ is a reverse left-I-centralizer, that is, $T(xy) = T(y)xI(x)$ for all $x, y \in M$, $\alpha \beta \in \Gamma$.

**Proof.** We have $T(xyxz) = T(x)yxI(x)$ for all $x, y \in M$, $\alpha \beta \in \Gamma$. Applying involution $I$ both sides to the above expression, we obtain $I(T(xyxz)) = xI(y)xI(x)I(x)$ for all $x, y \in M$ and $\alpha \beta \in \Gamma$. Define a new map $S: M \to M$ such that $S(x) = I(T(x))$ for all $x \in M$ and $\alpha \beta \in \Gamma$. Then, we see that $S(xy) = I(T(xy)) = I(xI(y)xI(x)) = xI(y)xI(x)$ for all $x, y \in M$ and $\alpha \beta \in \Gamma$. Hence, we obtain $S(xyxz) = xI(y)xI(x)$ for all $x, y \in M$ and $\alpha \beta \in \Gamma$. Thus, $S$ is a Jordan right-centralizer on $M$. In view of [5], $S$ is a right-centralizer that is, $S(xy) = xI(y)xI(x)$ for all $x, y \in M$, $\alpha \beta \in \Gamma$. This implies that $I(T(xyxz)) = xI(y)xI(x)I(x)$ for all $x, y \in M$ and $\alpha \beta \in \Gamma$. By applying involution to the both sides of the last relation, we find that $T(xy) = T(y)xI(x)$ for all $x, y \in M$ and $\alpha \beta \in \Gamma$. This completes the proof. □

**Lemma 2.4.** Let $M$ be a prime $T$-ring with involution $I$ satisfying the condition $xyI(x)I(x) = xyI(x)I(x)$ for all $x, y, z \in M$, $\alpha \beta \in \Gamma$, of characteristic different from two and $T: M \to M$ an additive mapping which satisfies $T(xyxz) = I(xI(y)xI(x))I(x)$ for all $x, y \in M$ and $\alpha \beta \in \Gamma$. Then, $T(xy) = T(y)xI(x)$ for all $x, y, z \in M$, $\alpha \beta \in \Gamma$. Applying involution $I$ on both sides to the above expression, we get $I(T(xyxz)) = xI(y)xI(x)$ for all $x, y \in M$, $\alpha \beta \in \Gamma$. Define
a new map \( S : M \to M \) such that \( S(x) = I(T(x)) \) for all \( x \in M \). Then, we see that \( S(xxzx) = I(I(T(xxzx))) = I(I(x)zT(x)xI(x)) = xxI(T(x))x = xxS(x)xz \) for all \( x \in M \) and \( z \in \Gamma \). Hence, we conclude that \( S(xxzx) = xxS(x)xz \) for all \( x \in M \) and \( z \in \Gamma \). Thus, \( S \) is an additive mapping such that \( S(xxzx) = xxS(x)xz \). In view of [5], we are forced to conclude that \( S \) is a two-sided centralizer that is, \( S(xyzy) = S(yxzy) \) for all \( x, y \in M \) and \( z \in \Gamma \). This implies that \( I(T(xyzy)) = I(T(yxzy)) = I(T(y)zT(x)xI(x)) = (I(y)zT(x))I(x) = I(y)zT(x) \) for all \( x, y \in M \) and \( z \in \Gamma \). Again applying involutions both sides to the last relation, we find that \( T(xyzy) = T(yxzy)I(x) = I(y)zT(x) \) for all \( x, y \in M \) and \( z \in \Gamma \). □

**Lemma 2.5.** Let \( M \) be a prime \( \Gamma \)-ring with involution \( I \) satisfying the condition \( xxyyz = xlyyly \) for all \( x, y, z \in M \) and \( x, y \in \Gamma \), and let \( S, T : M \to M \) be Jordan left-\( I \)-centralizers. Suppose that \( \langle S(x), T(x) \rangle_x = 0 \) holds for all \( x \in M \) and \( x \in \Gamma \). If \( T \neq 0 \), then there exists \( p \in C(M) \) such that \( S = pT \).

**Proof.** By Theorem 2.3, we conclude that \( S \) and \( T \) are reverse left-\( I \)-centralizers on \( M \). In view of the hypothesis, we have

\[
[S(x), T(x)]_z = 0, \quad (1)
\]

for all \( x \in M \) and \( z \in \Gamma \). Linearizing (1) and using it, we get

\[
[S(x), T(y)]_z + [S(y), T(x)]_z = 0, \quad (2)
\]

for all \( x, y \in M \) and \( z \in \Gamma \). Representing \( x \) by \( yz \) in (2), we obtain

\[
[S(x), T(y)]_z + [S(y), T(x)]_z = 0, \quad (3)
\]

for all \( x, y \in M \) and \( z \in \Gamma \). Replacing \( x \) by \( zyx \) in (3), we get

\[
[S(x), T(y)]_z + [S(y), T(x)]_z = 0, \quad (4)
\]

Replacing \( x \) by \( wzy \) in (4), we get

\[
[S(x), T(y)]_z + [S(y), T(x)]_z = 0, \quad (5)
\]

Replacing \( w \) by \( I(w) \) and \( z \) by \( I(z) \) in (5), we obtain

\[
[S(x), T(y)]_z + [S(y), T(x)]_z = 0. \quad (6)
\]

It follows from Lemma 2.2 that \( \langle S(x), T(x) \rangle_x = 0 \) holds for all \( x, y, z \in M \) and \( \beta, \gamma \in \Gamma \), and \( \langle S(x), T(x) \rangle_x \neq 0 \) holds for all \( x, y, z \in M \). It follows from Lemma 2.1 and from relation (6) that \( \langle S(x), T(x) \rangle_x \neq 0 \) holds for all \( x, y, z \in M \). Hence, by the relation (6) forces that for some \( p, q \in C(M) \), \( 0 = pT(x)bwT(y)z - T(x)bwT(y)z = pT(x)bwT(y)z - T(x)bwT(y)z = (p - q)T(x)bwT(y)z \) for all \( x, y, z \). Since \( M \) is a prime \( \Gamma \)-ring, the above expression yields that either \( (p - q)T(x) = 0 \) or \( [T(x), z]_z = 0 \). Since \( [T(x), z]_z \neq 0 \), we have \( (p - q)T(x) = 0 \) for all \( x \in M \) and \( z \in \Gamma \). This implies that \( pT(x) = qT(x) \) for all \( x \in M \). This gives \( S(x) = pT(x) \) for all \( x \in M \) as desired. If we replace the commutator by anti-commutator in Lemma 2.5, the corresponding result also holds. □

**Lemma 2.6.** Let \( M \) be a prime \( \Gamma \)-ring with involution \( I \) satisfying the condition \( xxyyz = xlyyly \) for all \( x, y, z \in M \) and \( x, y, z \in \Gamma \), and let \( S, T : M \to M \) be Jordan left-\( I \)-centralizers. Suppose that \( \langle S(x), T(x) \rangle_x = 0 \) holds for all \( x \in M \) and \( x \in \Gamma \). If \( T \neq 0 \), then there exists \( p \in C(M) \) such that \( S = pT \).

**Proof.** By the assumption, we have

\[
\langle S(x), T(x) \rangle_x = 0, \quad (7)
\]

for all \( x \in M \) and \( x \in \Gamma \). Replacing \( x \) by \( x + y \) in (7), we obtain

\[
\langle S(x), T(x) \rangle_x + \langle S(y), T(x) \rangle_x = \langle S(y), T(x) \rangle_x
\]

\[
= 0. \quad (8)
\]

Using (7) in (8), we get

\[
\langle S(x), T(x) \rangle_x + \langle S(y), T(x) \rangle_x = 0. \quad (9)
\]

Substituting \( z \beta y \) for \( y \) in (9) and using the fact that \( S \) and \( T \) are reverse left-\( I \)-centralizers, we find that \( 0 = \langle S(x), T(z \beta y) \rangle_x + \langle S(y), T(z \beta y) \rangle_x + \langle T(x), S(y) \beta z \rangle_x = \langle S(x), T(y) \beta z \rangle_x + \langle T(x), S(y) \beta z \rangle_x + \langle T(x), S(y) \beta z \rangle_x \beta z \rangle_x \). Application of (9) yields that

\[
T(y)\beta [S(x), I(z)]_x + S(y)\beta [T(x), I(z)]_x = 0. \quad (10)
\]

Replacing \( y \) by \( wzy \) in (10), we obtain

\[
T(y)\beta [wz \delta I(z), I(z)]_x + S(y)\beta [wz \delta I(T(x), I(z)]_x = 0. \quad (11)
\]

Replacing \( w \) by \( I(w) \) and \( z \) by \( I(z) \) in (11), we get

\[
T(y)\beta [wz \delta I(S(x), I(z)]_x + S(y)\beta [wz \delta I(T(x), I(z)]_x = 0. \quad (12)
\]

Henceforth using similar approach as we have used after Eq. (6) in the proof of Lemma 2.5, we get the required result. This finishes the proof of the lemma. □

**3. Main results**

The main result of the present paper is the following theorem which is inspired by [1].

**Theorem 3.1.** Let \( M \) be a 2-torsion free semiprime \( \Gamma \)-ring with involution \( I \) satisfying the condition \( xxyyz = xlyyly \) for all \( x, y, z \in M \) and \( x, y, z \in \Gamma \), and let \( S, T : M \to M \) be Jordan left-\( I \)-centralizers. Suppose that \( \langle S(x), T(x) \rangle_x \beta S(x) - S(x) \beta [S(x), T(x)]_x = 0 \) holds for all \( x \in M \) and \( x, \beta \in \Gamma \). Then, \( [S(x), T(x)]_x = 0 \) for all \( x \in M \) and \( x, \beta \in \Gamma \). Moreover, if \( M \) is a prime \( \Gamma \)-ring and \( S \neq 0 \), then there exists \( p \in C(M) \) such that \( T = pS \). (\( S = qT \), \( q \in C(M) \)).

**Proof.** In view of Lemma 2.3, we conclude that \( S \) and \( T \) are reverse left-\( I \)-centralizers. By the hypothesis, we have

\[
\langle S(x), T(x) \rangle_x \beta S(x) - S(x) \beta [S(x), T(x)]_x = 0, \quad (13)
\]

for all \( x \in M \) and \( x, \beta \in \Gamma \). Linearization of the relation (13) yields that...
Replacing \( \gamma I(T(x)) \) in (20), we obtain
\[
S(x) \delta I(T(x)) I(y) \beta [S(x), T(x)]_{y} \mu S(x) - S(x) \beta S(x) \delta T(y) I(y) \mu [S(x), T(x)]_{y} = 0.
\] (21)

Left multiplying (20) by \( T(x) \) gives
\[
T(x) \gamma S(x) \delta I(T(x)) I(y) \beta [S(x), T(x)]_{y} \mu S(x) - T(x) \gamma S(x) \beta S(x) \delta I(y) \mu [T(x), S(x)]_{y} = 0.
\] (22)

On combining (21) and (22), we obtain
\[
[S(x), T(x)]_{y} \gamma I(y) \delta [S(x), T(x)]_{y} \mu S(x) - [S(x) S(x), T(x)]_{y} \gamma I(y) \mu [S(x), T(x)]_{y} = 0.
\] (23)

By our hypothesis, we have
\[
0 = [S(x), T(x)]_{y} S(x) - S(x) \beta S(x), T(x)]_{y} S(x) + T(x) \gamma S(x) \beta S(x) - S(x) \beta S(x) \gamma T(x) = S(x) \beta S(x) \gamma T(x).
\] The above expression can be further written as
\[
[S(x) \beta S(x), T(x)]_{y} = 0.
\] (24)

Using (24) in (23), we get
\[
[S(x), T(x)]_{y} \gamma I(y) \delta [S(x), T(x)]_{y} \mu S(x) = 0.
\] (25)

Replacing \( y \) by \( y I(S(x)) \) in (25), we obtain
\[
[S(x), T(x)]_{y} \gamma S(x) \mu I(y) \delta [S(x), T(x)]_{y} \mu S(x) = 0.
\] (26)

Since \( M \) is a semiprime \( T \)-ring it follows from relation (26) that
\[
[S(x), T(x)]_{y} \delta I(y) = 0.
\] (27)

In view of relation (24) and (27), we have
\[
[S(x) \mu [S(x), T(x)]_{y} = 0.
\] (28)

Replacing \( x \) by \( x + y \) in (28) and using the same techniques as we used to obtain (16) from (13), we get
\[
S(y) \mu [S(x), T(x)]_{y} + S(x) \mu [S(y), T(x)]_{y} = 0.
\]

Substituting \( y \beta x \) for \( y \) in (29), we obtain
\[
S(x) \beta \mu [S(x), T(x)]_{y} + S(x) \mu [S(x) \beta I(y), T(x)]_{y} + S(x) \mu [S(x), T(x)]_{y} \beta I(y) + S(x) \mu [S(x), T(x)]_{y} \beta I(y) = 0.
\] (30)

This implies that
\[
S(x) \beta I(y) [S(x), T(x)]_{y} + S(x) \mu [S(x) \beta I(y), T(x)]_{y} + S(x) \mu [S(x), T(x)]_{y} \beta I(y) = 0.
\]

Thus we have the relation
\[
S(x) \beta I(y) [S(x), T(x)]_{y} + S(x) \mu [S(x) \beta I(y), T(x)]_{y} + S(x) \mu [S(x), T(x)]_{y} \beta I(y) = 0.
\]

which can be further written in the form
\[
S(x) \beta I(y) [S(x), T(x)]_{y} + S(x) \mu [S(x) \beta I(y), T(x)]_{y} + S(x) \mu [S(x), T(x)]_{y} \beta I(y) = 0.
\] (30)

Application of (28) forces that
Further reduces to

\[ S(x) \beta I(y) \mu [S(x), T(x)]_a + S(x) \mu S(x) \beta I(y) z T(x) - S(x) \mu T(x) \beta I(y) z S(x) = 0. \]  \hspace{2cm} (31)

Left multiplication of (31) by \( T(x) \) gives

\[ T(x) z S(x) \beta I(y) \mu [S(x), T(x)]_a + T(x) z S(x) \mu S(x) \beta I(y) z T(x) - T(x) z S(x) \mu T(x) \beta I(y) z S(x) = 0. \]  \hspace{2cm} (32)

On substituting \( y z I(T(x)) \) for \( y \) in (31), we have

\[ S(x) \mu T(x) \beta I(y) z [S(x), T(x)]_a + S(x) \mu S(x) \beta T(x) \beta I(y) z T(x) - S(x) \mu T(x) \beta S(x) z I(y) z S(x) = 0. \]  \hspace{2cm} (33)

Combining (32) and (33), we obtain

\[ [S(x), T(x)]_a \mu [S(x), T(x)]_a + [S(x) z S(x), T(x)]_a \mu I(T(x)) + T(x) S(x) \beta I(T(x)) z S(x) = 0. \]  \hspace{2cm} (34)

Using (24), the above expression reduces to

\[ [S(x), T(x)]_a \beta I(T(x)) z [S(x), T(x)]_a + [T(x), S(x)]_a \beta T(x) \beta I(T(x)) z S(x) = 0. \]  \hspace{2cm} (35)

Substituting \( z \beta I(T(x)) \) for \( y \) in (35), we get

\[ [S(x), T(x)]_a \beta I(T(x)) z [S(x), T(x)]_a + [T(x), S(x)]_a \beta T(x) \beta I(T(x)) z S(x) = 0. \]  \hspace{2cm} (36)

On the other hand right multiplying to (35) by \( I(z) \delta S(x) \), we get

\[ [S(x), T(x)]_a \beta I(T(x)) z \delta A(x, z) = 0, \]  \hspace{2cm} (38)

where

\[ A(x, z) = [S(x), T(x)]_a \beta I(z) \mu S(x) - S(x) \mu I(z) \beta [S(x), T(x)]_a. \]

Substituting \( y z I(S(x)) \mu z \) for \( y \) in (38) gives

\[ [S(x), T(x)]_a \beta I(z) \delta S(x) \mu [S(x), T(x)]_a = 0. \]  \hspace{2cm} (39)

Left multiplying to (38) by \( S(x) \mu I(z) \), we get

\[ S(x) \beta I(z) \delta A(x, z) = 0. \]  \hspace{2cm} (40)

From (39) and (40), we arrive at \( A(x, z) \mu I(z) = 0 \). That is, \( A(x, z) = \mu I(z) = 0 \). The semiprimeness of \( M \) forces that \( A(x, z) = 0 \). In other words, we have

\[ [S(x), T(x)]_a \beta I(z) \delta S(x) = S(x) \delta I(z) \mu [S(x), T(x)]_a. \]  \hspace{2cm} (41)

Replacing \( z \) by \( y z I(T(x)) \) in (41), we obtain

\[ [S(x), T(x)]_a \beta I(y) \delta I(S(x)) = S(x) \delta T(x) \beta I(y) \mu [S(x), T(x)]_a. \]  \hspace{2cm} (42)

Combining (35) and (42), we obtain \( [S(x), T(x)]_a \beta I(y) \delta I(S(x)) + S(x) \delta T(x) \beta I(y) \mu [S(x), T(x)]_a = 0. \) This further reduces to

\[ T(x) \delta S(x) \beta I(y) \mu [S(x), T(x)]_a = 0. \]  \hspace{2cm} (43)

If we substitute \( y z I(T(x)) \) for \( y \) in (43), we find that

\[ T(x) \delta S(x) \beta T(x) \mu I(S(x), T(x)]_a = 0. \]  \hspace{2cm} (44)

Multiplying (43) from the left side by \( T(x) \), we get

\[ T(x) \delta T(x) \mu S(x) \beta I(y) z [S(x), T(x)]_a = 0. \]  \hspace{2cm} (45)

Subtracting (45) from (44), we get

\[ T(x) \delta [S(x), T(x)]_a \beta I(y) \mu [S(x), T(x)]_a = 0. \]  \hspace{2cm} (46)

Replacing \( I(T(x)) \delta y \) for \( y \) in (46), we obtain

\[ T(x) \delta [S(x), T(x)]_a \beta I(T(x)) \delta T(x) \mu [S(x), T(x)]_a = 0. \]  \hspace{2cm} (47)

That is, \( T(x) \Gamma [S(x), T(x)]_a \Gamma M \Gamma T(x) [S(x), T(x)]_a = 0 \) for all \( x \in M \). The semiprimeness of \( M \) yields that

\[ T(x) \Gamma [S(x), T(x)]_a = 0. \]  \hspace{2cm} (48)

Replacing \( y \) by \( I(T(x)) y z \) in (42) gives, because of (48)

\[ [S(x), T(x)]_a \beta I(T(x)) y z T(x) \mu S(x) = 0. \]  \hspace{2cm} (49)

Substituting \( x + y \) for \( x \) in (27) and using the same approach as we used to obtain (16) from (13), we get

\[ [S(x), T(x)]_a \beta S(y) + [S(x), T(x)]_a \beta S(x) + [S(y), T(x)]_a \beta S(x) = 0. \]  \hspace{2cm} (50)

On substituting \( y \) for \( y \) in (50), we obtain \( [S(x), T(x)]_a \beta S(x) \beta I(y) + T(x) [S(x), I(y)]_a \beta S(x) + [S(x), T(x)]_a \beta I(y) \beta S(x) + [S(x), T(x)]_a \beta I(y) \beta S(x) + [S(x), T(x)]_a \beta I(y) \beta S(x) + [S(y), T(x)]_a \beta S(x) = 0. \) Application of (27) yields that

\[ [S(x), T(x)]_a \beta I(y) \beta S(x) + T(x) \beta [S(x), I(y)]_a \beta S(x) + [S(x), T(x)]_a \beta I(y) \beta S(x) + [S(x), T(x)]_a \beta I(y) \beta S(x) + [S(x), T(x)]_a \beta I(y) \beta S(x) = 0. \]  \hspace{2cm} (51)

This implies that

\[ 2 [S(x), T(x)]_a \beta I(y) \beta S(x) + T(x) \beta [S(x), I(y)]_a \beta S(x) + S(x) \beta I(y), \mu S(x) \beta S(x) = 0. \]  \hspace{2cm} (52)

This can be further written as

\[ 2[S(x), T(x)]_a \beta I(y) \beta S(x) + T(x) \beta S(x) \beta I(y) z S(x) - T(x) z I(y) \beta S(x) + S(x) \beta I(y) \beta T(x) \beta S(x) - S(x) \beta T(x) \beta I(y) \beta S(x) \beta S(x) = 0, \]  \hspace{2cm} (53)

which reduces to

\[ [S(x), T(x)]_a \beta I(y) \beta S(x) + S(x) \beta I(y) \beta S(x) + T(x) = \beta I(y) \beta S(x) \beta S(x) = 0. \]  \hspace{2cm} (54)

Using (41) in (53), we obtain \( 0 = [S(x), T(x)]_a \beta I(y) \beta S(x) + T(x) \beta S(x) \beta I(y) z S(x) - T(x) z I(y) \beta S(x) + S(x) \beta I(y) \beta S(x) \beta S(x) = S(x) \beta I(y) \beta S(x) \beta S(x) + T(x) = \beta I(y) \beta S(x) \beta S(x). \]  \hspace{2cm} (55)

The above expression yields that

\[ S(x) \beta I(y) \beta S(x) \beta T(x) = T(x) \beta I(y) \beta S(x) \beta S(x). \]  \hspace{2cm} (54)
Left multiplication to (54) by $T(x)$ leads to
\[ T(x)\alpha S(x)\beta I(y)\gamma S(x)\delta T(x) = T(x)\beta T(x)\alpha I(y)\gamma S(x)\beta S(x). \]  
(55)

By combining (55) and (56), we arrive at
\[ [S(x), T([x])]_{\beta I(y)}\gamma S(x)\alpha T(x) = 0. \]  
(57)

Using (61) in (62), we conclude that
\[ S(x)\beta I(y)\gamma S(x), T([x])]_{\beta I(y)}\gamma S(x)\alpha T(x) = 0. \]  
(63)

Substituting $y[I(T(x))]$ for $y$ in the above relation, we obtain
\[ S(x)\beta T(x)\beta I(y)\gamma S(x), T([x])]_{\beta I(y)}\gamma S(x)\alpha T(x) = 0. \]  
(64)

On the other hand left multiplication of (63) by $T(x)$ gives
\[ T(x)\delta S(x)\beta I(y)\gamma S(x), T([x])]_{\beta I(y)}\gamma S(x)\alpha T(x) = 0. \]  
(65)

By comparing (64) and (65), we obtain
\[ 0 = [S(x), T([x])]_{\beta I(y)}\gamma S(x), T([x])]_{\beta I(y)}\gamma S(x)\alpha T(x) = 0. \]  
(66)

If we multiply (66) by $S(x)$ from left, we get
\[ S(x)\delta S(x), T([x])]_{\beta I(y)}\gamma S(x), T([x])]_{\beta I(y)}\gamma S(x)\alpha T(x) = 0. \]  
(67)

On the other hand putting $y[S(x), T([x])]_{\gamma y}$ for $y$ in (63), we arrive at
\[ S(x)\beta S(x), T([x])]_{\beta I(y)}\gamma S(x), T([x])]_{\beta I(y)}\gamma S(x)\alpha T(x) = 0. \]  
(68)

By combining (67) and (68), we obtain
\[ S(x)\delta S(x), T([x])]_{\beta I(y)}\gamma S(x), T([x])]_{\beta I(y)}\gamma S(x)\alpha T(x) = 0. \]  
(69)

Using (58) in the above expression, we obtain
\[ S(x)\beta S(x), T([x])]_{\beta I(y)}\gamma S(x), T([x])]_{\beta I(y)}\gamma S(x)\alpha T(x) = 0. \]  
(70)

Since $M$ is semiprime, it follows that $S(x)\beta S(x), T([x])]_{\beta I(y)}\gamma S(x), T([x])]_{\beta I(y)}\gamma S(x)\alpha T(x) = 0$. The last two expressions are same as Eqs. (27) and (28) and hence, by using similar approach as we have used after (27) and (28) in the proof of Theorem 3.1, we get the required result. The theorem is thereby proved. □

Corollary 3.3. Let $M$ be a $2$-torsion free semiprime $\Gamma$-ring with involution $I$ satisfying the condition $xy I\beta z = x\beta y I\gamma z$ for all $x, y, z \in M$ and $\beta, \gamma, \lambda \in \Gamma$, and $T: M \rightarrow M$ a Jordan left-$\Gamma$-centralizer.
(1) Suppose that \(\langle T(x), I(x)\rangle\beta I(x) - I(x)\beta\langle T(x), I(x)\rangle = 0\) holds for all \(x \in M\) and \(\alpha, \beta \in \Gamma\). Then, \(T\) is a reverse \(I\)-centralizer on \(M\).

(2) Suppose that \(\langle T(x), I(x)\rangle\beta T(x) - T(x)\beta\langle T(x), I(x)\rangle = 0\) holds for all \(x \in M\) and \(\alpha, \beta \in \Gamma\). Then, \(T\) is a reverse \(I\)-centralizer on \(M\).

(3) Suppose that \([T(x), I(x)]\beta I(x) - I(x)\beta[T(x), I(x)] = 0\) holds for all \(x \in M\) and \(\alpha, \beta \in \Gamma\). In this case, \(T\) is a reverse \(I\)-centralizer on \(M\).

(4) Suppose that \([T(x), I(x)]\beta T(x) - T(x)\beta[T(x), I(x)] = 0\) holds for all \(x \in M\) and \(\alpha, \beta \in \Gamma\). In this case, \(T\) is a reverse \(I\)-centralizer on \(M\).

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References


