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Extensions of Picard stacks and their homological interpretation

Cristiana Bertolin

Dip. di Matematica, Università di Torino, Via Carlo Alberto 10, 10123 Torino, Italy

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A B S T R A C T

Let **S** be a site. We introduce the notion of extensions of strictly commutative Picard **S**-stacks. We define the pull-back, the push-down, and the sum of such extensions and we compute their homological interpretation: if $\mathcal{P}$ and $\mathcal{Q}$ are two strictly commutative Picard **S**-stacks, the equivalence classes of extensions of $\mathcal{P}$ by $\mathcal{Q}$ are parametrized by the cohomology group $\mathrm{Ext}^1([\mathcal{P}],[\mathcal{Q}])$, where $[\mathcal{P}]$ and $[\mathcal{Q}]$ are the complex associated to $\mathcal{P}$ and $\mathcal{Q}$ respectively.

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E-mail address: cristiana.bertolin@googlemail.com.

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Introduction

Let **S** be a site. Let $\mathcal{P}$ and $\mathcal{Q}$ be two strictly commutative Picard **S**-stacks. We define an extension of $\mathcal{P}$ by $\mathcal{Q}$ as a strictly commutative Picard **S**-stack $\mathcal{E}$, two additive functors $I:\mathcal{Q}\to\mathcal{E}$ and $J:\mathcal{E}\to\mathcal{P}$, and an isomorphism of additive functors $J\circ I\cong 0$, such that the following equivalent conditions are satisfied:

- $\pi_0(J):\pi_0(\mathcal{E})\to\pi_0(\mathcal{P})$ is surjective and I induces an equivalence of strictly commutative Picard **S**-stacks between $\mathcal{Q}$ and $\ker(J)$,
- $\pi_1(I):\pi_1(\mathcal{Q})\to\pi_1(\mathcal{E})$ is injective and J induces an equivalence of strictly commutative Picard **S**-stacks between $\operatorname{coker}(I)$ and $\mathcal{P}$.

By [D73, §1.4] there is an equivalence of categories between the category of strictly commutative Picard **S**-stacks and the derived category $\mathcal{D}^{[-1,0]}(\mathbf{S})$ of complexes K of abelian sheaves on **S** such that $\mathrm{H}^i(K)=0$ for $i\neq -1$ or 0. Via this equivalence, the above notion of extension of strictly commutative Picard **S**-stacks furnishes a notion of extension for complexes of abelian sheaves over **S** concentrated in degrees -1 and 0. Let K and L be two complexes of abelian sheaves on **S** concentrated in degrees -1 and 0. In this paper we prove that, as for extensions of abelian sheaves on **S**, the extensions of K by L are parametrized by the cohomology group $\mathrm{Ext}^1(K,L)$.

More precisely, the extensions of $\mathcal{P}$ by $\mathcal{Q}$ form a 2-category $\mathcal{E}xt(\mathcal{P},\mathcal{Q})$ where

- the objects are extensions of $\mathcal{P}$ by $\mathcal{Q}$,
- the 1-arrows are additive functors between extensions,
- the 2-arrows are morphisms of additive functors.

Equivalence classes of extensions of strictly commutative Picard **S**-stacks are endowed with a group law. We denote by $\mathcal{E}xt^1(\mathcal{P},\mathcal{Q})$ the group of equivalence classes of objects of $\mathcal{E}xt(\mathcal{P},\mathcal{Q})$, by $\mathcal{E}xt^0(\mathcal{P},\mathcal{Q})$ the group of isomorphism classes of arrows from an object of $\mathcal{E}xt(\mathcal{P},\mathcal{Q})$ to itself, and by $\mathcal{E}xt^{-1}(\mathcal{P},\mathcal{Q})$ the group of automorphisms of an arrow from an object of $\mathcal{E}xt(\mathcal{P},\mathcal{Q})$ to itself. With these notation our main theorem is:

Theorem 0.1. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two strictly commutative Picard* **S***-stacks. Then we have the following isomorphisms of groups*

(a) $\mathcal{E}xt^1(\mathcal{P},\mathcal{Q})\cong \mathrm{Ext}^1([\mathcal{P}],[\mathcal{Q}])=\mathrm{Hom}_{\mathcal{D}(\mathbf{S})}([\mathcal{P}],[\mathcal{Q}][1])$,
(b) $\mathcal{E}xt^0(\mathcal{P},\mathcal{Q})\cong \mathrm{Ext}^0([\mathcal{P}],[\mathcal{Q}])=\mathrm{Hom}_{\mathcal{D}(\mathbf{S})}([\mathcal{P}],[\mathcal{Q}])$,
(c) $\mathcal{E}xt^{-1}(\mathcal{P},\mathcal{Q})\cong \mathrm{Ext}^{-1}([\mathcal{P}],[\mathcal{Q}])=\mathrm{Hom}_{\mathcal{D}(\mathbf{S})}([\mathcal{P}],[\mathcal{Q}][-1])$,

where $[\mathcal{P}]$ and $[\mathcal{Q}]$ denote the complex of $\mathcal{D}^{[-1,0]}(\mathbf{S})$ corresponding to $\mathcal{P}$ and $\mathcal{Q}$ respectively.

This paper is organized as follows: in Section 1 we recall some basic results on strictly commutative Picard **S**-stacks. In Section 2 we introduce the notions of fibered product and fibered sum of strictly commutative Picard **S**-stacks. In Section 3 we define extensions of strictly commutative Picard **S**-stacks and morphisms between such extensions. The results of Section 2 will allow us to define a group law for equivalence classes of extensions of strictly commutative Picard **S**-stacks (Section 4). Finally in Section 5 we prove the main Theorem 0.1.

The most relevant ancestor of this paper is [BV02] where extensions of symmetric Picard categories (named symmetric categorical groups), together with their pull-back, push-down and sum, are studied. Moreover, the non-abelian analogue of §3 has been developed by Breen in [B90,B92], by Rousseau in [R03], by Aldrovandi and Noohi in [AN09] and by Yekutieli in [Y10].

Notation

Let **S** be a site. Denote by $\mathcal{K}(\mathbf{S})$ the category of complexes of abelian sheaves on the site **S**: all complexes that we consider in this paper are cochain complexes. Let $\mathcal{K}^{[-1,0]}(\mathbf{S})$ be the subcategory of $\mathcal{K}(\mathbf{S})$ consisting of complexes $K=(K^i)_i$ such that $K^i=0$ for $i\neq -1$ or 0. The good truncation $\tau_{\leqslant n}K$ of a complex K of $\mathcal{K}(\mathbf{S})$ is the following complex: $(\tau_{\leqslant n}K)^i=K^i$ for $i<n$, $(\tau_{\leqslant n}K)^n=\ker(d^n)$ and $(\tau_{\leqslant n}K)_i=0$ for $i>n$. For any $i\in\mathbb{Z}$, the shift functor $[i]:\mathcal{K}(\mathbf{S})\to\mathcal{K}(\mathbf{S})$ acts on a complex $K=(K^n)_n$ as $(K[i])^n=K^{i+n}$ and $d^n_{K[i]}=(-1)^i d^{n+i}_K$.

Denote by $\mathcal{D}(\mathbf{S})$ the derived category of the category of abelian sheaves on $\mathbf{S}$, and let $\mathcal{D}^{[-1,0]}(\mathbf{S})$ be the subcategory of $\mathcal{D}(\mathbf{S})$ consisting of complexes K such that $\mathrm{H}^i(K)=0$ for $i\neq -1$ or 0. If K and K' are complexes of $\mathcal{D}(\mathbf{S})$, the group $\mathrm{Ext}^i(K,K')$ is by definition $\mathrm{Hom}_{\mathcal{D}(\mathbf{S})}(K,K'[i])$ for any $i\in\mathbb{Z}$. Let $\mathrm{RHom}(-,-)$ be the derived functor of the bifunctor $\mathrm{Hom}(-,-)$. The cohomology groups $\mathrm{H}^i(\mathrm{RHom}(K,K'))$ of $\mathrm{RHom}(K,K')$ are isomorphic to $\mathrm{Hom}_{\mathcal{D}(\mathbf{S})}(K,K'[i])$.

A 2-*category* $\mathcal{A}=(A,C(a,b),K_{a,b,c},U_a)_{a,b,c\in A}$ is given by the following data:

- a set A of objects $a,b,c,\dots$;
- for each ordered pair (a,b) of objects of A, a category $C(a,b)$;
- for each ordered triple (a,b,c) of objects A, a functor $K_{a,b,c}:C(b,c)\times C(a,b)\to C(a,c)$, called composition functor. This composition functor have to satisfy the associativity law;
- for each object a, a functor $U_a:\mathbf{1}\to C(a,a)$ where $\mathbf{1}$ is the terminal category (i.e. the category with one object, one arrow), called unit functor. This unit functor has to provide a left and right identity for the composition functor.

This set of axioms for a 2-category is exactly like the set of axioms for a category in which the arrows-sets $\mathrm{Hom}(a,b)$ have been replaced by the categories $C(a,b)$. We call the categories $C(a,b)$ (with $a,b\in A$) the *categories of morphisms* of the 2-category $\mathcal{A}$: the objects of $C(a,b)$ are the 1-*arrows* of $\mathcal{A}$ and the arrows of $C(a,b)$ are the 2-*arrows* of $\mathcal{A}$.

Let $\mathcal{A}=(A,C(a,b),K_{a,b,c},U_a)_{a,b,c\in A}$ and $\mathcal{A}'=(A',C(a',b'),K_{a',b',c'},U_{a'})_{a',b',c'\in A'}$ be two 2-categories. A 2-*functor* (called also a *morphism of* 2-*categories*)

$$(F,F_{a,b})_{a,b\in A}:\mathcal{A}\longrightarrow\mathcal{A}'$$

consists of

- an application $F:A\to A'$ between the objects of $\mathcal{A}$ and the objects of $\mathcal{A}'$,
- a family of functors $F_{a,b}:C(a,b)\to C(F(a),F(b))$ (with $a,b\in A$) which are compatible with the composition functors and with the unit functors of $\mathcal{A}$ and $\mathcal{A}'$.

1. Recall on strictly commutative Picard stacks

Let $\mathbf{S}$ be a site. For the notions of $\mathbf{S}$-pre-stack, $\mathbf{S}$-stack and morphisms of $\mathbf{S}$-stacks we refer to [G71, Chapter II 1.2].

A *strictly commutative Picard* $\mathbf{S}$-*stack* is an $\mathbf{S}$-stack of groupoids $\mathcal{P}$ endowed with a functor $+:\mathcal{P}\times_{\mathbf{S}}\mathcal{P}\to\mathcal{P}$, $(a,b)\mapsto a+b$, and two natural isomorphisms of associativity σ and of commutativity τ, which are described by the functorial isomorphisms

$$\sigma_{a,b,c}:(a+b)+c\xrightarrow{\cong}a+(b+c)\quad\forall a,b,c\in\mathcal{P},\tag{1.1}$$

$$\tau_{a,b}:a+b\xrightarrow{\cong}b+a\quad\forall a,b\in\mathcal{P};\tag{1.2}$$

such that for any object U of $\mathbf{S}$, $(\mathcal{P}(U),+,\sigma,\tau)$ is a strictly commutative Picard category (i.e. it is possible to make the sum of two objects of $\mathcal{P}(U)$ and this sum is associative and commutative, see [D73, 1.4.2] for more details). Here "strictly" means that $\tau_{a,a}$ is the identity for all $a\in\mathcal{P}$. Any strictly commutative Picard $\mathbf{S}$-stack admits a global neutral object e and the sheaf of automorphisms of the neutral object $\underline{\mathrm{Aut}}(e)$ is abelian.

Let $\mathcal{P}$ and $\mathcal{Q}$ be two strictly commutative Picard **S**-stacks. An *additive functor* $(F,\Sigma):\mathcal{P}\to\mathcal{Q}$ between strictly commutative Picard **S**-stacks is a morphism of **S**-stacks (i.e. a cartesian **S**-functor, see [G71, Chapter I 1.1]) endowed with a natural isomorphism Σ which is described by the functorial isomorphisms

$$\Sigma_{a,b}:F(a+b)\xrightarrow{\cong}F(a)+F(b)\quad\forall a,b\in\mathcal{P}$$

and which is compatible with the natural isomorphisms σ and τ of $\mathcal{P}$ and $\mathcal{Q}$. A *morphism of additive functors* $u:(F,\Sigma)\to(F',\Sigma')$ is an **S**-morphism of cartesian **S**-functors (see [G71, Chapter I 1.1]) which is compatible with the natural isomorphisms Σ and Σ' of F and F' respectively. We denote by $\mathbf{Add}_{\mathbf{S}}(\mathcal{P},\mathcal{Q})$ the category whose objects are additive functors from $\mathcal{P}$ to $\mathcal{Q}$ and whose arrows are morphisms of additive functors. The category $\mathbf{Add}_{\mathbf{S}}(\mathcal{P},\mathcal{Q})$ is a groupoid, i.e. any morphism of additive functors is an isomorphism of additive functors.

An *equivalence of strictly commutative Picard* **S***-stacks* between $\mathcal{P}$ and $\mathcal{Q}$ is an additive functor $(F,\Sigma):\mathcal{P}\to\mathcal{Q}$ with F an equivalence of **S**-stacks. Two strictly commutative Picard **S**-stacks are *equivalent as strictly commutative Picard* **S***-stacks* if there exists an equivalence of strictly commutative Picard **S**-stacks between them.

To any strictly commutative Picard **S**-stack $\mathcal{P}$, we associate the sheafification $\pi_0(\mathcal{P})$ of the presheaf which associates to each object U of **S** the group of isomorphism classes of objects of $\mathcal{P}(U)$, the sheaf $\pi_1(\mathcal{P})$ of automorphisms $\underline{\mathrm{Aut}}(e)$ of the neutral object of $\mathcal{P}$, and an element $\varepsilon(\mathcal{P})$ of $\mathrm{Ext}^2(\pi_0(\mathcal{P}),\pi_1(\mathcal{P}))$. Two strictly commutative Picard **S**-stacks $\mathcal{P}$ and $\mathcal{P}'$ are equivalent as strictly commutative Picard **S**-stacks if and only if $\pi_i(\mathcal{P})$ is isomorphic to $\pi_i(\mathcal{P}')$ for $i=0,1$ and $\varepsilon(\mathcal{P})=\varepsilon(\mathcal{P}')$ (see Remark 1.3).

A *strictly commutative Picard* **S***-pre-stack* is an **S**-pre-stack of groupoids $\mathcal{P}$ endowed with a functor $+:\mathcal{P}\times_{\mathbf{S}}\mathcal{P}\to\mathcal{P}$ and two natural isomorphisms of associativity σ (1.1) and of commutativity τ (1.2), such that for any object U of **S**, $(\mathcal{P}(U),+,\sigma,\tau)$ is a strictly commutative Picard category. If $\mathcal{P}$ is a strictly commutative Picard **S**-pre-stack, there exists modulo a unique equivalence one and only one pair $(a\mathcal{P},j)$ where $a\mathcal{P}$ is a strictly commutative Picard **S**-stack and $j:\mathcal{P}\to a\mathcal{P}$ is an additive functor. $(a\mathcal{P},j)$ is *the strictly commutative Picard* **S***-stack generated by* $\mathcal{P}$.

To each complex $K=[K^{-1}\xrightarrow{d}K^0]$ of $\mathcal{K}^{[-1,0]}(\mathbf{S})$, we associate a strictly commutative Picard **S**-stack $st(K)$ which is the **S**-stack generated by the following strictly commutative Picard **S**-pre-stack $pst(K)$: for any object U of **S**, the objects of $pst(K)(U)$ are the elements of $K^0(U)$, and if x and y are two objects of $pst(K)(U)$ (i.e. $x,y\in K^0(U)$), an arrow of $pst(K)(U)$ from x to y is an element f of $K^{-1}(U)$ such that $df=y-x$. A morphism of complexes $g:K\to L$ induces an additive functor $st(g):st(K)\to st(L)$ between the strictly commutative Picard **S**-stacks associated to the complexes K and L.

In [D73, §1.4] Deligne proves the following links between strictly commutative Picard **S**-stacks and complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$, between additive functors and morphisms of complexes and between morphisms of additive functors and homotopies of complexes:

- for any strictly commutative Picard **S**-stack $\mathcal{P}$ there exists a complex K of $\mathcal{K}^{[-1,0]}(\mathbf{S})$ such that $\mathcal{P}=st(K)$;
- if K, L are two complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$, then for any additive functor $F:st(K)\to st(L)$ there exists a quasi-isomorphism $k:K'\to K$ and a morphism of complexes $l:K'\to L$ such that F is isomorphic as additive functor to $st(l)\circ st(k)^{-1}$;
- if $f,g:K\to L$ are two morphisms of complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$, then

$$\mathrm{Hom}_{\mathbf{Add}_{\mathbf{S}}(st(K),st(L))}\big(st(f),st(g)\big)\cong\{\text{homotopies } H:K\longrightarrow L\mid g-f=dH+Hd\}.\tag{1.3}$$

Denote by Picard(**S**) the category whose objects are small strictly commutative Picard **S**-stacks and whose arrows are isomorphism classes of additive functors. We can summarize the above links between strictly commutative Picard **S**-stacks and complexes of abelian sheaves on **S** with the following theorem:

Theorem 1.1. *The functor*

$$\begin{aligned} st : \mathcal{D}^{[-1,0]}(\mathbf{S}) &\longrightarrow \mathrm{Picard}(\mathbf{S}), \\ K &\longmapsto st(K), \\ K \xrightarrow{f} L &\longmapsto st(K) \xrightarrow{st(f)} st(L) \end{aligned} \tag{1.4}$$

is an equivalence of categories.

We denote by [ ] the inverse equivalence of st.

Let $\mathcal{P}icard(\mathbf{S})$ be the 2-category of strictly commutative Picard $\mathbf{S}$-stacks whose objects are strictly commutative Picard $\mathbf{S}$-stacks and whose categories of morphisms are the categories $\mathbf{Add}_\mathbf{S}(\mathcal{P}, \mathcal{Q})$ (i.e. the 1-arrows are additive functors between strictly commutative Picard $\mathbf{S}$-stacks and the 2-arrows are morphisms of additive functors).

Theorem 1.2. *Via the functor st, there exists a 2-functor between*

(a) *the 2-category whose objects and 1-arrows are the objects and the arrows of the category $\mathcal{K}^{[-1,0]}(\mathbf{S})$ and whose 2-arrows are the homotopies between 1-arrows (i.e. H such that $g - f = dH + Hd$ with $f, g : K \to L$ 1-arrows),*
(b) *the 2-category $\mathcal{P}icard(\mathbf{S})$.*

Remark 1.3. Let $K = [K^{-1} \xrightarrow{d} K^0]$ be a complex of $\mathcal{D}^{[-1,0]}(\mathbf{S})$. The long exact sequence

$$0 \longrightarrow \mathrm{H}^{-1}(K) \longrightarrow K^{-1} \xrightarrow{d} K^0 \longrightarrow \mathrm{H}^0(K) \longrightarrow 0$$

is an element of $\mathrm{Ext}^2(\mathrm{H}^0(K), \mathrm{H}^{-1}(K))$ that we denote by $\varepsilon(K)$. The sheaves H^0, H^{-1} and the element ε of $\mathrm{Ext}^2(\mathrm{H}^0, \mathrm{H}^{-1})$ classify objects of $\mathcal{D}^{[-1,0]}(\mathbf{S})$ modulo isomorphisms. Through the equivalence of categories (1.4), the above classification of objects of $\mathcal{D}^{[-1,0]}(\mathbf{S})$ is equivalent to the classification of strictly commutative Picard $\mathbf{S}$-stacks via the sheaves π_0, π_1 and the invariant $\varepsilon \in \mathrm{Ext}^2(\pi_0, \pi_1)$. In particular $\pi_0(\mathcal{P}) = \mathrm{H}^0([\mathcal{P}])$, $\pi_1(\mathcal{P}) = \mathrm{H}^{-1}([\mathcal{P}])$, $\varepsilon(\mathcal{P}) = \varepsilon([\mathcal{P}])$.

Example. Let $\mathcal{P}$ and $\mathcal{Q}$ be two strictly commutative Picard $\mathbf{S}$-stacks. Let

$$\mathrm{HOM}(\mathcal{P}, \mathcal{Q})$$

be the following strictly commutative Picard $\mathbf{S}$-stack:

- for any object U of $\mathbf{S}$, the objects of the category $\mathrm{HOM}(\mathcal{P}, \mathcal{Q})(U)$ are additive functors from $\mathcal{P}_{|U}$ to $\mathcal{Q}_{|U}$ and its arrows are morphisms of additive functors;
- the functor $+ : \mathrm{HOM}(\mathcal{P}, \mathcal{Q}) \times \mathrm{HOM}(\mathcal{P}, \mathcal{Q}) \to \mathrm{HOM}(\mathcal{P}, \mathcal{Q})$ is defined by the formula

$$(F_1 + F_2)(a) = F_1(a) + F_2(a) \quad \forall a \in \mathcal{P}$$

and the natural isomorphism

$$\Sigma : (F_1 + F_2)(a + b) \xrightarrow{\cong} (F_1 + F_2)(a) + (F_1 + F_2)(b)$$

is given by the commutative diagram

$$\begin{tikzcd}
(F_1+F_2)(a+b) \arrow[r, "\Sigma"] \arrow[d, equal] & (F_1+F_2)(a)+(F_1+F_2)(b) \arrow[r, equal] & F_1(a)+F_2(a)+F_1(b)+F_2(b) \\
F_1(a+b)+F_2(a+b) \arrow[rr, "\Sigma_{F_1}+\Sigma_{F_2}"'] & & F_1(a)+F_1(b)+F_2(a)+F_2(b); \arrow[u, "Id+\tau_{F_1(b),F_2(a)}+Id"']
\end{tikzcd}$$

- the natural isomorphisms of associativity σ and of commutativity τ of $\mathrm{HOM}(\mathcal{P},\mathcal{Q})$ are defined via the analogous natural isomorphisms of $\mathcal{Q}$.

Because of isomorphism (1.3) and of equivalence of categories (1.4) we have the following equality in the derived category $\mathcal{D}^{[-1,0]}(\mathbf{S})$.

Lemma 1.4.

$$\big[\mathrm{HOM}(\mathcal{P},\mathcal{Q})\big]=\tau_{\leqslant 0}\mathrm{RHom}\big([\mathcal{P}],[\mathcal{Q}]\big).$$

We can define the following bifunctor on $\mathrm{Picard}(\mathbf{S})\times\mathrm{Picard}(\mathbf{S})$

$$\begin{aligned}\mathrm{HOM}:\mathrm{Picard}(\mathbf{S})\times\mathrm{Picard}(\mathbf{S}) &\longrightarrow \mathrm{Picard}(\mathbf{S}),\\ (\mathcal{P},\mathcal{Q}) &\longmapsto \mathrm{HOM}(\mathcal{P},\mathcal{Q}).\end{aligned}$$

2. Operations on strictly commutative Picard stacks

We start defining the product of two strictly commutative Picard **S**-stacks. Let $\mathcal{P}$ and $\mathcal{Q}$ be two strictly commutative Picard **S**-stacks.

Definition 2.1. The *product of* $\mathcal{P}$ *and* $\mathcal{Q}$ is the strictly commutative Picard **S**-stack $\mathcal{P}\times\mathcal{Q}$ defined as follows:

- for any object U of **S**, an object of the category $\mathcal{P}\times\mathcal{Q}(U)$ is a pair (p,q) of objects with p an object of $\mathcal{P}(U)$ and q an object of $\mathcal{Q}(U)$;
- for any object U of **S**, if (p,q) and (p',q') are two objects of $\mathcal{P}\times\mathcal{Q}(U)$, an arrow of $\mathcal{P}\times\mathcal{Q}(U)$ from (p,q) to (p',q') is a pair (f,g) of arrows with $f:p\to p'$ an arrow of $\mathcal{P}(U)$ and $g:q\to q'$ an arrow of $\mathcal{Q}(U)$;
- the functor $+:(\mathcal{P}\times\mathcal{Q})\times(\mathcal{P}\times\mathcal{Q})\to\mathcal{P}\times\mathcal{Q}$ is defined by the formula

$$(p,q)+\big(p',q'\big)=\big(p+p',q+q'\big)$$

  for any $p,p'\in\mathcal{P}$ and $q,q'\in\mathcal{Q}$;
- the natural isomorphisms of associativity σ and of commutativity τ of $\mathcal{P}\times\mathcal{Q}$ are defined via the analogous natural isomorphisms of $\mathcal{P}$ and $\mathcal{Q}$.

In the derived category $\mathcal{D}^{[-1,0]}(\mathbf{S})$ we have the following equality

$$[\mathcal{P}\times\mathcal{Q}]=[\mathcal{P}]+[\mathcal{Q}]$$

which implies the following equality of abelian sheaves

$$\pi_i(\mathcal{P}\times\mathcal{Q})=\pi_i(\mathcal{P})+\pi_i(\mathcal{Q})\quad\text{for } i=0,1.$$

Now we define the fibered sum (called also the push-down) and the fibered product (called also the pull-back) of strictly commutative Picard **S**-stacks (in the sense of bilimits). Let $G:\mathcal{Q}\to\mathcal{P}$ and $F:\mathcal{Q}\to\mathcal{P}'$ be additive functors between strictly commutative Picard **S**-stacks.

Definition 2.2. The *fibered sum of $\mathcal{P}$ and $\mathcal{P}'$ under $\mathcal{Q}$ via F and G* is the strictly commutative Picard **S**-stack $\mathcal{P}+^{\mathcal{Q}}\mathcal{P}'$ generated by the following strictly commutative Picard **S**-pre-stack $\mathcal{D}$:

- for any object U of **S**, the objects of the category $\mathcal{D}(U)$ are the objects of the category $(\mathcal{P}\times\mathcal{P}')(U)$, i.e. pairs (p,p') with p an object of $\mathcal{P}(U)$ and p' an object of $\mathcal{P}'(U)$;
- for any object U of **S**, if (p_1,p'_1) and (p_2,p'_2) are two objects of $\mathcal{D}(U)$, an arrow of $\mathcal{D}(U)$ from (p_1,p'_1) to (p_2,p'_2) is an equivalence class of triplets (q,α,β) with q an object of $\mathcal{Q}(U)$, $\alpha:p_1+G(q)\to p_2$ an arrow of $\mathcal{P}(U)$ and $\beta:p'_1\to F(q)+p'_2$ an arrow of $\mathcal{P}'(U)$; two triplets (q_1,α_1,β_1) and (q_2,α_2,β_2) are equivalent it there is an arrow $\gamma:q_1\to q_2$ in $\mathcal{Q}(U)$ such that $\alpha_2\circ(id+G(\gamma))=\alpha_1$ and $(F(\gamma)+id)\circ\beta_1=\beta_2$;
- the functor $+:\mathcal{D}\times\mathcal{D}\to\mathcal{D}$ is induced by the functors $+:\mathcal{P}\times\mathcal{P}\to\mathcal{P}$ and $+:\mathcal{P}'\times\mathcal{P}'\to\mathcal{P}'$;
- the natural isomorphisms of associativity σ and of commutativity τ are induced by the analogous natural isomorphisms of $\mathcal{P}$ and $\mathcal{P}'$.

The fibered sum $\mathcal{P}+^{\mathcal{Q}}\mathcal{P}'$ is also called the *push-down $F_*\mathcal{P}$ of $\mathcal{P}$ via $F:\mathcal{Q}\to\mathcal{P}'$* or the *push-down $G_*\mathcal{P}'$ of $\mathcal{P}'$ via $G:\mathcal{Q}\to\mathcal{P}$*. It is endowed with two additive functors $In_1:\mathcal{P}\to\mathcal{P}+^{\mathcal{Q}}\mathcal{P}'$ and $In_2:\mathcal{P}'\to\mathcal{P}+^{\mathcal{Q}}\mathcal{P}'$ and with an isomorphism of additive functors $\iota:In_1\circ G\Rightarrow In_2\circ F$. Moreover it satisfies the following universal property: given two additive functors $H:\mathcal{P}\to\mathcal{O}$ and $K:\mathcal{P}'\to\mathcal{O}$, and given an isomorphism of additive functors $\phi:H\circ G\Rightarrow K\circ F$, then there exists an additive functor $U:\mathcal{P}+^{\mathcal{Q}}\mathcal{P}'\to\mathcal{O}$ and two isomorphisms of additive functors $\phi^H:U\circ In_1\Rightarrow H$ and $\phi^K:U\circ In_2\Rightarrow K$ such that the following diagram commutes

$$\begin{tikzcd} U\circ In_1\circ G \arrow[r, Rightarrow, "\phi^H\circ G"] \arrow[d, Rightarrow, "U\circ\iota"'] & H\circ G \arrow[d, Rightarrow, "\phi"] \\ U\circ In_2\circ F \arrow[r, Rightarrow, "\phi^K\circ F"] & K\circ F; \end{tikzcd}$$

moreover, if $\overline{U}:\mathcal{P}+^{\mathcal{Q}}\mathcal{P}'\to\mathcal{O}$ and $\bar{\phi}^H:\overline{U}\circ In_1\Rightarrow H$ and $\bar{\phi}^K:\overline{U}\circ In_2\Rightarrow K$ satisfy the same condition, then there is a unique isomorphism of additive functors $\psi:\overline{U}\Rightarrow U$ such that the following diagrams commute

$$\begin{tikzcd} \overline{U}\circ In_1 \arrow[rr, Rightarrow, "\psi\circ In_1"] \arrow[dr, Rightarrow, "\bar{\phi}^H"'] & & U\circ In_1 \arrow[dl, Rightarrow, "\phi^H"] \\ & H & \end{tikzcd} \qquad \begin{tikzcd} \overline{U}\circ In_2 \arrow[rr, Rightarrow, "\psi\circ In_2"] \arrow[dr, Rightarrow, "\bar{\phi}^K"'] & & U\circ In_2 \arrow[dl, Rightarrow, "\phi^K"] \\ & K. & \end{tikzcd}$$

The following square formed by the fibered sum $\mathcal{P}+^{\mathcal{Q}}\mathcal{P}'$

$$\begin{tikzcd} \mathcal{Q} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{P}' \arrow[d, "In_2"] \\ \mathcal{P} \arrow[r, "In_1"'] \arrow[ur, Rightarrow, "\iota"'] & \mathcal{P}+^{\mathcal{Q}}\mathcal{P}' \end{tikzcd}$$

is called a *push-down square* or a *cocartesian square*.

If $[\mathcal{P}]=[K^{-1}\xrightarrow{d_K}K^0]$, $[\mathcal{P}']=[L^{-1}\xrightarrow{d_L}L^0]$ and $[\mathcal{Q}]=[M^{-1}\xrightarrow{d_M}M^0]$, in the derived category $\mathcal{D}^{[-1,0]}(\mathbf{S})$ we have the following equality

$$[\mathcal{P}+^{\mathcal{Q}}\mathcal{P}']=[K^{-1}+^{M^{-1}}L^{-1}\xrightarrow{d_K+^{d_M}d_L}K^0+^{M^0}K^0]$$

where for $i=-1,0$ the abelian sheaf $K^i+^{M^i}L^i$ is the fibered sum of K^i and of L^i under M^i and the morphism of abelian sheaves $d_K+^{d_M}d_L$ is given by the universal property of the fibered product $K^{-1}+^{M^{-1}}K^{-1}$.

Remark that we have the exact sequences of abelian sheaves

$$\pi_0(\mathcal{P})+\pi_0(\mathcal{P}')\longrightarrow\pi_0(\mathcal{P}+^{\mathcal{Q}}\mathcal{P}')\longrightarrow 0.$$

Now we introduce the dual notion of fibered sum: the fibered product. Let $G:\mathcal{P}\to\mathcal{Q}$ and $F:\mathcal{P}'\to\mathcal{Q}$ be additive functors between strictly commutative Picard **S**-stacks.

Definition 2.3. The *fibered product of $\mathcal{P}$ and $\mathcal{P}'$ over $\mathcal{Q}$ via F and G* is the strictly commutative Picard **S**-stack $\mathcal{P}\times_{\mathcal{Q}}\mathcal{P}'$ defined as follows:

- for any object U of **S**, the objects of the category $(\mathcal{P}\times_{\mathcal{Q}}\mathcal{P}')(U)$ are triplets (p,p',f) where p is an object of $\mathcal{P}(U)$, p' is an object of $\mathcal{P}'(U)$ and $f:G(p)\xrightarrow{\cong}F(p')$ is an isomorphism of $\mathcal{Q}(U)$ between $G(p)$ and $F(p')$;
- for any object U of **S**, if (p_1,p'_1,f) and (p_2,p'_2,g) are two objects of $(\mathcal{P}\times_{\mathcal{Q}}\mathcal{P}')(U)$, an arrow of $(\mathcal{P}\times_{\mathcal{Q}}\mathcal{P}')(U)$ from (p_1,p'_1,f) to (p_2,p'_2,g) is a pair (f,g) of arrows with $\alpha:p_1\to p_2$ of arrow of $\mathcal{P}(U)$ and $\beta:p'_1\to p'_2$ an arrow of $\mathcal{P}'(U)$ such that $g\circ G(\alpha)=F(\beta)\circ f$;
- the functor $+:(\mathcal{P}\times_{\mathcal{Q}}\mathcal{P}')\times(\mathcal{P}\times_{\mathcal{Q}}\mathcal{P}')\to\mathcal{P}\times_{\mathcal{Q}}\mathcal{P}'$ is induced by the functors $+:\mathcal{P}\times\mathcal{P}\to\mathcal{P}$ and $+:\mathcal{P}'\times\mathcal{P}'\to\mathcal{P}'$;
- the natural isomorphisms of associativity σ and of commutativity τ are induced by the analogous natural isomorphisms of $\mathcal{P}$ and $\mathcal{P}'$.

The fibered product $\mathcal{P}\times_{\mathcal{Q}}\mathcal{P}'$ is also called the *pull-back $F^*\mathcal{P}$ of $\mathcal{P}$ via $F:\mathcal{P}'\to\mathcal{Q}$* or the *pull-back $G^*\mathcal{P}'$ of $\mathcal{P}'$ via $G:\mathcal{P}\to\mathcal{Q}$*. It is endowed with two additive functors $Pr_1:\mathcal{P}\times_{\mathcal{Q}}\mathcal{P}'\to\mathcal{P}$ and $Pr_2:\mathcal{P}\times_{\mathcal{Q}}\mathcal{P}'\to\mathcal{P}'$ and with an isomorphism of additive functors $\pi:G\circ Pr_1\Rightarrow F\circ Pr_2$. Moreover it satisfies the dual universal property of the fibered sum (we leave to the reader to write down explicitly this universal property). The following square formed by the fibered product $\mathcal{P}\times_{\mathcal{Q}}\mathcal{P}'$

$$\begin{tikzcd}
\mathcal{P}\times_{\mathcal{Q}}\mathcal{P}' \arrow[r, "Pr_2"] \arrow[d, "Pr_1"'] & \mathcal{P}' \arrow[d, "F"] \\
\mathcal{P} \arrow[r, "G"'] \arrow[ur, Rightarrow, "\pi"] & \mathcal{Q}
\end{tikzcd}$$

is called a *pull-back square* or a *cartesian square*.

If $[\mathcal{P}]=[K^{-1}\xrightarrow{d_K}K^0]$, $[\mathcal{P}']=[L^{-1}\xrightarrow{d_L}L^0]$ and $[\mathcal{Q}]=[M^{-1}\xrightarrow{d_M}M^0]$, in the derived category $\mathcal{D}^{[-1,0]}(\mathbf{S})$ we have the following equality

$$[\mathcal{P}\times_{\mathcal{Q}}\mathcal{P}']=[K^{-1}\times_{M^{-1}}L^{-1}\xrightarrow{d_K\times_{d_M}d_L}K^0\times_{M^0}K^0]$$

where for $i=-1,0$ the abelian sheaf $K^i\times_{M^i}L^i$ is the fibered product of K^i and of L^i over M^i and the morphism of abelian sheaves $d_K\times_{d_M}d_L$ is given by the universal property of the fibered product $K^0\times_{M^0}K^0$.

Remark that we have the exact sequences of abelian sheaves

$$0 \longrightarrow \pi_1\big(\mathcal{P} \times_{\mathcal{Q}} \mathcal{P}'\big) \longrightarrow \pi_1(\mathcal{P}) + \pi_1\big(\mathcal{P}'\big).$$

3. Extensions of strictly commutative Picard stacks

Let $\mathcal{P}$ and $\mathcal{Q}$ be two strictly commutative Picard **S**-stacks. Consider an additive functor $F:\mathcal{P} \to \mathcal{Q}$. Denote by $\mathbf{1}$ the strictly commutative Picard **S**-stack such that for any object U of **S**, $\mathbf{1}(U)$ is the category with one object and one arrow.

Definition 3.1. The *kernel of* F, $\ker(F)$, is the fibered product $\mathcal{P} \times_{\mathcal{Q}} \mathbf{1}$ of $\mathcal{P}$ and $\mathbf{1}$ over $\mathcal{Q}$ via $F:\mathcal{P} \to \mathcal{Q}$ and the additive functor $\mathbf{1}:\mathbf{1} \to \mathcal{Q}$.

The *cokernel of* F, $\mathrm{coker}(F)$, is the fibered sum $\mathbf{1} +^{\mathcal{P}} \mathcal{Q}$ of $\mathbf{1}$ and $\mathcal{Q}$ under $\mathcal{P}$ via $F:\mathcal{P} \to \mathcal{Q}$ and the additive functor $\mathbf{1}:\mathcal{P} \to \mathbf{1}$.

We have the cartesian and cocartesian squares

$$\begin{tikzcd}
\ker(F) \arrow[r] \arrow[d] & \mathbf{1} \arrow[d, "\mathbf{1}"] \\
\mathcal{P} \arrow[r, "F"'] & \mathcal{Q}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P} \arrow[r, "F"] \arrow[d, "\mathbf{1}"'] & \mathcal{Q} \arrow[d] \\
\mathbf{1} \arrow[r] & \mathrm{coker}(F)
\end{tikzcd}$$

and the exact sequences of abelian sheaves

$$0 \longrightarrow \pi_1\big(\ker(F)\big) \longrightarrow \pi_1(\mathcal{P}), \qquad \pi_0(\mathcal{Q}) \longrightarrow \pi_0\big(\mathrm{coker}(F)\big) \longrightarrow 0. \tag{3.1}$$

Explicitly, according to Definition 2.3 the kernel of F is the strictly commutative Picard **S**-stack $\ker(F)$ where

- for any object U of **S**, the objects of the category $\ker(F)(U)$ are pairs (p, f) where p is an object of $\mathcal{P}(U)$ and $f:F(p) \xrightarrow{\cong} e$ is an isomorphism between $F(p)$ and the neutral object e of $\mathcal{Q}$;
- for any object U of **S**, if (p, f) and (p', f') are two objects of $\ker(F)(U)$, an arrow $\alpha:(p, f) \to (p', f')$ of $\ker(F)(U)$ is an arrow $\alpha:p \to p'$ of $\mathcal{P}(U)$ such that $f' \circ F(\alpha) = f$.

By Definition 2.2 the cokernel of F is the strictly commutative Picard **S**-stack $\mathrm{coker}(F)$ generated by the following strictly commutative Picard **S**-pre-stack $\mathrm{coker}'(F)$ where

- for any object U of **S**, the objects of $\mathrm{coker}'(F)(U)$ are the objects of $\mathcal{Q}(U)$;
- for any object U of **S**, if q and q' are two objects of $\mathrm{coker}'(F)(U)$ (i.e. objects of $\mathcal{Q}(U)$), an arrow of $\mathrm{coker}'(F)(U)$ from q to q' is an equivalence class of pairs (p, α) with p an object of $\mathcal{P}(U)$ and $\alpha:q + F(p) \to q'$ an arrow of $\mathcal{Q}(U)$; two pairs (p_1, α_1) and (p_2, α_2) are equivalent if there is an arrow $\beta:p_1 \to p_2$ of $\mathcal{P}(U)$ such that $\alpha_2 \circ (id + F(\beta)) = \alpha_1$.

Definition 3.2. An *extension* $\mathcal{E} = (\mathcal{E}, I, J)$ *of* $\mathcal{P}$ *by* $\mathcal{Q}$

$$\mathcal{Q} \xrightarrow{I} \mathcal{E} \xrightarrow{J} \mathcal{P} \tag{3.2}$$

consists of

(1) a strictly commutative Picard **S**-stack $\mathcal{E}$,
(2) two additive functors $I : \mathcal{Q} \to \mathcal{E}$ and $J : \mathcal{E} \to \mathcal{P}$,
(3) an isomorphism of additive functors between the composite $J \circ I$ and the trivial additive functor: $J \circ I \cong 0$,

such that the following equivalent conditions are satisfied:

(a) $\pi_0(J) : \pi_0(\mathcal{E}) \to \pi_0(\mathcal{P})$ is surjective and I induces an equivalence of strictly commutative Picard **S**-stacks between $\mathcal{Q}$ and $\ker(J)$;
(b) $\pi_1(I) : \pi_1(\mathcal{Q}) \to \pi_1(\mathcal{E})$ is injective and J induces an equivalence of strictly commutative Picard **S**-stacks between $\mathrm{coker}(I)$ and $\mathcal{P}$.

The additive functors $I : \mathcal{Q} \to \mathcal{E}$ and $J : \mathcal{E} \to \mathcal{P}$ induce the sequences of abelian sheaves

$$0 \longrightarrow \pi_1(\mathcal{Q}) \xrightarrow{\pi_1(I)} \pi_1(\mathcal{E}) \xrightarrow{\pi_1(J)} \pi_1(\mathcal{P}),$$
$$\pi_0(\mathcal{Q}) \xrightarrow{\pi_0(I)} \pi_0(\mathcal{E}) \xrightarrow{\pi_0(J)} \pi_0(\mathcal{P}) \longrightarrow 0$$

which are exact in $\pi_1(\mathcal{Q})$ and $\pi_0(\mathcal{P})$ because of the equivalences of strictly commutative Picard **S**-stacks $\mathcal{Q} \cong \ker(J)$ and $\mathrm{coker}(I) \cong \mathcal{P}$. According to [AN09, Proposition 6.2.6], we can say more about these two sequences: in fact there exists a connecting morphism of abelian sheaves

$$\delta : \pi_1(\mathcal{P}) \longrightarrow \pi_0(\mathcal{Q})$$

leading to the long exact sequence

$$0 \longrightarrow \pi_1(\mathcal{Q}) \xrightarrow{\pi_1(I)} \pi_1(\mathcal{E}) \xrightarrow{\pi_1(J)} \pi_1(\mathcal{P}) \xrightarrow{\delta} \pi_0(\mathcal{Q}) \xrightarrow{\pi_0(I)} \pi_0(\mathcal{E}) \xrightarrow{\pi_0(J)} \pi_0(\mathcal{P}) \longrightarrow 0. \tag{3.3}$$

Explicitly the connecting morphism $\delta : \pi_1(\mathcal{P}) \to \pi_0(\mathcal{Q})$ is defined as follows: if $f : e_{\mathcal{P}} \to e_{\mathcal{P}}$ is an element of $\pi_1(\mathcal{P})(U)$ with U an object of **S**, then $\delta(f)$ represents the isomorphism class of the element

$$(e_{\mathcal{E}}, f \circ 1_J)$$

of $\ker(J)(U) \cong \mathcal{Q}(U)$, where $1_J : J(e_{\mathcal{E}}) \xrightarrow{\cong} e_{\mathcal{P}}$ is the isomorphism resulting from the additivity of the functor $J : \mathcal{E} \to \mathcal{P}$ (here $e_{\mathcal{E}}$ and $e_{\mathcal{P}}$ are the neutral objects of $\mathcal{E}$ and $\mathcal{P}$ respectively).

Let $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{P}'$ and $\mathcal{Q}'$ be strictly commutative Picard **S**-stacks. Let $\mathcal{E} = (\mathcal{E}, I, J)$ be an extension of $\mathcal{P}$ by $\mathcal{Q}$ and let $\mathcal{E}' = (\mathcal{E}', I', J')$ be an extension of $\mathcal{P}'$ by $\mathcal{Q}'$.

Definition 3.3. A *morphism of extensions*

$$(F, G, H) : \mathcal{E} \longrightarrow \mathcal{E}'$$

consists of

(1) three additive functors $F : \mathcal{E} \to \mathcal{E}'$, $G : \mathcal{P} \to \mathcal{P}'$, $H : \mathcal{Q} \to \mathcal{Q}'$,
(2) two isomorphisms of additive functors $J' \circ F \cong G \circ J$ and $F \circ I \cong I' \circ H$,

which are compatible with the isomorphisms of additive functors $J \circ I \cong 0$ and $J' \circ I' \cong 0$ underlying the extensions $\mathcal{E}$ and $\mathcal{E}'$, i.e. the composite

$$0 \xleftrightarrow{\cong} G \circ 0 \xleftrightarrow{\cong} G \circ J \circ I \xleftrightarrow{\cong} J' \circ F \circ I \xleftrightarrow{\cong} J' \circ I' \circ H \xleftrightarrow{\cong} 0 \circ H \xleftrightarrow{\cong} 0$$

should be the identity.

The three additive functors F, G and H furnish the following commutative diagram modulo isomorphisms of additive functors

$$\begin{tikzcd} \mathcal{Q} \arrow[r, "I"] \arrow[d, "H"'] & \mathcal{E} \arrow[r, "J"] \arrow[d, "F"'] & \mathcal{P} \arrow[d, "G"] \\ \mathcal{Q}' \arrow[r, "I'"'] & \mathcal{E}' \arrow[r, "J'"'] & \mathcal{P}'. \end{tikzcd}$$

Fix two strictly commutative Picard **S**-stacks $\mathcal{P}$ and $\mathcal{Q}$. The extensions of $\mathcal{P}$ by $\mathcal{Q}$ form a 2-category

$$\mathcal{E}xt(\mathcal{P}, \mathcal{Q})$$

where

- the objects are extensions of $\mathcal{P}$ by $\mathcal{Q}$,
- the 1-arrows are morphisms of extensions, i.e. additive functors between extensions,
- the 2-arrows are morphisms of additive functors.

Now we show which objects of the derived category $\mathcal{D}^{[-1,0]}(\mathbf{S})$ correspond via the equivalence of categories (1.4) to the strictly commutative Picard **S**-stacks $\ker(F)$, $\mathrm{coker}(F)$ and $\mathcal{P} = (\mathcal{P}, I, J)$.

Lemma 3.4. *Let $K = [K^{-1} \xrightarrow{d^K} K^0]$ and $L = [L^{-1} \xrightarrow{d^L} L^0]$ be complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$. Let $F : st(K) \to st(L)$ be an additive functor induced by a morphism of complexes $f = (f^{-1}, f^0) : K \to L$. The strictly commutative Picard **S**-stacks* $\ker(F)$ *and* $\mathrm{coker}(F)$ *correspond via the equivalence of categories* (1.4) *to the following complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$:*

$$\big[\ker(F)\big] = \tau_{\leqslant 0}\big(MC(f)[-1]\big) = \big[K^{-1} \xrightarrow{(f^{-1}, -d^K)} \ker\big(d^L, f^0\big)\big], \tag{3.4}$$

$$\big[\mathrm{coker}(F)\big] = \tau_{\geqslant -1} MC(f) = \big[\mathrm{coker}\big(f^{-1}, -d^K\big) \xrightarrow{(d^L, f^0)} L^0\big] \tag{3.5}$$

where τ denotes the good truncation and $MC(f)$ is the mapping cone of the morphism f.

Proof. It is enough to show that the strictly commutative **S**-pre-stacks associated to $\mathrm{coker}(F)$ is equivalent to the one associated to $\tau_{\geqslant -1} MC(f)$, since for each strictly commutative **S**-pre-stack $\mathcal{P}$, the strictly commutative **S**-stack generated by $\mathcal{P}$ is unique modulo a unique equivalence (idem for $\ker(F)$). Explicitly $MC(f)$ is the complex

$$0 \longrightarrow K^{-1} \xrightarrow{(f^{-1}, -d^K)} L^{-1} + K^0 \xrightarrow{(d^L, f^0)} L^0 \longrightarrow 0$$

concentrated in degree -2, -1 and 0.

Let U be an object of **S**. The objects of $pst(\tau_{\geqslant -1} MC(f))(U)$ are the elements of $L^0(U)$ and so they are the same objects of $pst(L)(U)$. Moreover, if l and l' are two objects of $pst(\tau_{\geqslant -1} MC(f))(U)$, an arrow of $pst(\tau_{\geqslant -1} MC(f))(U)$ from l to l' is an equivalence class of pairs (α, k) with k an object of $K^0(U)$ and α an object of $L^{-1}(U)$ such that

$$\big(d^L, f^0\big)(\alpha, k) = l' - l.$$

This equality can be rewritten as $d^L(\alpha) = l' - (l + f^0(k))$. Therefore an arrow from l to l' is an equivalence class of pairs (α, k) with k an object of $pst(K)(U)$ and $\alpha : l + F(k) \to l'$ an arrow of $pst(L)(U)$. According to Definition 3.1, we can conclude that $pst(\tau_{\geqslant -1} MC(f)) \cong \mathrm{coker}'(F)$.

The objects of $pst(\tau_{\leqslant 0}(MC(f)[-1]))(U)$ are pairs (f,k) with k an object of $pst(K)(U)$ and $f : F(k) \to e_{pst(L)}$ an isomorphism from $F(k)$ to the neutral object $e_{pst(L)}$ of $pst(L)$. If (f,k) and (f',k') are two objects of $pst(\tau_{\leqslant 0}(MC(f)[-1]))(U)$, an arrow of $pst(\tau_{\leqslant 0}(MC(f)[-1]))(U)$ from (f,k) to (f',k') is an element g of $K^{-1}(U)$ such that

$$\left(f^{-1}, -d^K\right)(g) = \left(f', k'\right) - (f,k).$$

This equality implies the equalities $f^{-1}(g) = f' - f$ and $-d^K(g) = k' - k$. Therefore $g : k' \to k$ is an arrow of $pst(K)(U)$ such that the following diagram is commutative:

$$\begin{tikzcd} F(k') \arrow[rr, "F(g)"] \arrow[dr, "f"'] & & F(k) \arrow[dl, "f'"] \\ & e_{pst(L)}. & \end{tikzcd}$$

According to Definition 3.1, we can conclude that $pst(\tau_{\leqslant 0}(MC(f)[-1])) \cong \ker(F)$. $\square$

By the above lemma, the following notion of extension in $\mathcal{K}^{[-1,0]}(\mathbf{S})$ is equivalent to Definition 3.2 through the equivalence of categories (1.4). Let K and M be two complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$.

Definition 3.5. An *extension $L = (L, i, j)$ of M by K*

$$K \xrightarrow{i} L \xrightarrow{j} M$$

consists of

(1) a complex L of $\mathcal{K}^{[-1,0]}(\mathbf{S})$,
(2) two morphisms of complexes $i : K \to L$ and $j : L \to M$ of $\mathcal{K}^{[-1,0]}(\mathbf{S})$,
(3) a homotopy between $j \circ i$ and 0,

such that the following equivalent conditions are satisfied:

(a) $\mathrm{H}^0(j) : \mathrm{H}^0(L) \to \mathrm{H}^0(M)$ is surjective and i induces a quasi-isomorphism between K and $\tau_{\leqslant 0}(MC(j)[-1])$;
(b) $\mathrm{H}^{-1}(i) : \mathrm{H}^{-1}(K) \to \mathrm{H}^{-1}(L)$ is injective and j induces a quasi-isomorphism between $\tau_{\geqslant -1} MC(i)$ and M.

Remark 3.6. Consider a short exact sequence of complexes in $\mathcal{K}^{[-1,0]}(\mathbf{S})$

$$0 \longrightarrow K \xrightarrow{i} L \xrightarrow{j} M \longrightarrow 0.$$

It exists a distinguished triangle $K \xrightarrow{i} L \xrightarrow{j} M \to +$ in $\mathcal{D}(\mathbf{S})$, and M is isomorphic to $MC(i)$ in $\mathcal{D}(\mathbf{S})$. Therefore a short exact sequence of complexes in $\mathcal{K}^{[-1,0]}(\mathbf{S})$ is an extension of complexes of $\mathcal{K}^{[-1,0]}(\mathbf{S})$ according to the above definition.

Remark 3.7. If $K = [K^{-1} \xrightarrow{0} K^0]$ and $M = [M^{-1} \xrightarrow{0} M^0]$, then an extension of M by K consists of an extension of M^0 by K^0 and an extension of M^{-1} by K^{-1}.

Remark 3.8. Assume that $st(L) = (st(L), I, J)$ is an extension of $st(M)$ by $st(K)$. Since $\mathrm{H}^{-1}(i)$ is injective and $\mathrm{H}^1(K) = 0$, the distinguished triangle $K \xrightarrow{i} L \to MC(i) \to +$ furnishes the long exact sequence

$$0 \longrightarrow \mathrm{H}^{-1}(K) \xrightarrow{\mathrm{H}^{-1}(i)} \mathrm{H}^{-1}(L) \longrightarrow \mathrm{H}^{-1}\big(\tau_{\geqslant -1}\, MC(i)\big)$$
$$\longrightarrow \mathrm{H}^{0}(K) \xrightarrow{\mathrm{H}^{0}(i)} \mathrm{H}^{0}(L) \longrightarrow \mathrm{H}^{0}\big(\tau_{\geqslant -1}\, MC(i)\big) \longrightarrow 0.$$

Because of the equality $\tau_{\geqslant -1}\, MC(i) = M$ in $\mathcal{D}(\mathbf{S})$, we see that the above long exact sequence is just the long exact sequence (3.3).

4. Operations on extensions of strictly commutative Picard stacks

Using the results of Section 2 we define the pull-back and the push-down of extensions of strictly commutative Picard **S**-stacks via additive functors. Let $\mathcal{E} = (\mathcal{E}, I : \mathcal{Q} \to \mathcal{E}, J : \mathcal{E} \to \mathcal{P})$ be an extension of $\mathcal{P}$ by $\mathcal{Q}$.

Definition 4.1. The *pull-back $F^*\mathcal{E}$ of the extension $\mathcal{E}$ via an additive functor $F : \mathcal{P}' \to \mathcal{P}$* is the fibered product $\mathcal{E} \times_{\mathcal{P}} \mathcal{P}'$ of $\mathcal{E}$ and $\mathcal{P}'$ over $\mathcal{P}$ via J and F.

Lemma 4.2. *The pull-back $F^*\mathcal{E}$ of $\mathcal{E}$ via F is an extension of $\mathcal{P}'$ by $\mathcal{Q}$.*

Proof. Denote by $Pr : F^*\mathcal{E} \to \mathcal{P}'$ the additive functor underlying the pull-back of $\mathcal{E}$ via F. Composing the equivalence of strictly commutative Picard **S**-stacks $\mathcal{Q} \cong \ker(J) = \mathcal{E} \times_{\mathcal{P}} \mathbf{1}$ with the natural equivalence of strictly commutative Picard **S**-stacks $\mathcal{E} \times_{\mathcal{P}} \mathbf{1} \cong \mathcal{E} \times_{\mathcal{P}} \mathcal{P}' \times_{\mathcal{P}'} \mathbf{1} = \ker(Pr)$, we get that $\mathcal{Q}$ is equivalent to the strictly commutative Picard **S**-stack $\ker(Pr)$. Moreover the surjectivity of $\pi_0(J) : \pi_0(\mathcal{E}) \to \pi_0(\mathcal{P})$ implies the surjectivity of $\pi_0(Pr) : \pi_0(F^*\mathcal{E}) \to \pi_0(\mathcal{P}')$. Hence $(F^*\mathcal{E}, I, Pr)$ is an extension of $\mathcal{P}'$ by $\mathcal{Q}$. □

Definition 4.3. The *push-down $G_*\mathcal{E}$ of the extension $\mathcal{E}$ via an additive functor $G : \mathcal{Q} \to \mathcal{Q}'$* is the fibered sum $\mathcal{E} +^{\mathcal{Q}} \mathcal{Q}'$ of $\mathcal{E}$ and $\mathcal{Q}'$ under $\mathcal{Q}$ via G and I.

Lemma 4.4. *The push-down $G_*\mathcal{E}$ of $\mathcal{E}$ via G is an extension of $\mathcal{P}$ by $\mathcal{Q}'$.*

Proof. Denote by $In : \mathcal{Q}' \to G_*\mathcal{E}$ the additive functor underlying the push-down of $\mathcal{E}$ via G. Composing the equivalence of strictly commutative Picard **S**-stacks $\mathrm{coker}(I) \cong \mathcal{P} = \mathbf{1} +^{\mathcal{Q}} \mathcal{E}$ with the natural equivalence of strictly commutative Picard **S**-stacks $\mathbf{1} +^{\mathcal{Q}} \mathcal{E} \cong \mathbf{1} +^{\mathcal{Q}'} \mathcal{Q}' +^{\mathcal{Q}} \mathcal{E} = \mathrm{coker}(In)$, we get that $\mathcal{P}$ is equivalent to the strictly commutative Picard **S**-stack $\mathrm{coker}(In)$. Moreover the injectivity of $\pi_1(I) : \pi_0(\mathcal{Q}) \to \pi_1(\mathcal{E})$ implies the injectivity of $\pi_1(In) : \pi_1(\mathcal{Q}') \to \pi_1(G_*\mathcal{E})$. Hence $(G_*\mathcal{E}, In, P)$ is an extension of $\mathcal{Q}'$ by $\mathcal{P}$. □

Before to define a group law for extensions of $\mathcal{P}$ by $\mathcal{Q}$, we need the following

Lemma 4.5. *Let $\mathcal{E}$ be an extension of $\mathcal{P}$ by $\mathcal{Q}$ and let $\mathcal{E}'$ be an extension of $\mathcal{P}'$ by $\mathcal{Q}'$. Then $\mathcal{E} \times \mathcal{E}'$ is an extension of $\mathcal{P} \times \mathcal{P}'$ by $\mathcal{Q} \times \mathcal{Q}'$.*

Proof. Via the equivalence of categories (1.4), we have that the complex $[\mathcal{E}] = ([\mathcal{E}], i, j)$ (resp. $[\mathcal{E}'] = ([\mathcal{E}'], i', j')$) is an extension of $[\mathcal{P}]$ by $[\mathcal{Q}]$ (resp. an extension of $[\mathcal{P}']$ by $[\mathcal{Q}']$) in the derived category $\mathcal{D}(\mathbf{S})$. Therefore $\mathrm{H}^0(j + j') = \mathrm{H}^0(j) + \mathrm{H}^0(j') : \mathrm{H}^0([\mathcal{E}] + [\mathcal{E}']) \to \mathrm{H}^0([\mathcal{P}] + [\mathcal{P}'])$ is surjective. Moreover $i + i'$ induces an isomorphism in $\mathcal{D}(\mathbf{S})$ between $[\mathcal{Q}] + [\mathcal{Q}']$ and

$$\tau_{\leqslant 0}\big(MC(j)[-1]\big) + \tau_{\leqslant 0}\big(MC(j')[-1]\big) = \tau_{\leqslant 0}\big(MC\big(j + j'\big)[-1]\big).$$

Hence we can conclude that $[\mathcal{E} \times \mathcal{E}'] = ([\mathcal{E} \times \mathcal{E}'], i + i', j + j')$ is an extension of $[\mathcal{P} \times \mathcal{P}']$ by $[\mathcal{Q} \times \mathcal{Q}']$. □

Let $\mathcal{E}$ and $\mathcal{E}'$ be two extensions of $\mathcal{P}$ by $\mathcal{Q}$. According to the above lemma, the product $\mathcal{E} \times \mathcal{E}'$ is an extension of the product $\mathcal{P} \times \mathcal{P}$ by the product $\mathcal{Q} \times \mathcal{Q}$.

Definition 4.6. The *sum $\mathcal{E} + \mathcal{E}'$ of the extensions $\mathcal{E}$ and $\mathcal{E}'$* is the following extension of $\mathcal{P}$ by $\mathcal{Q}$

$$D^* +_* \left(\mathcal{E} \times \mathcal{E}'\right) \tag{4.1}$$

where $D : \mathcal{P} \to \mathcal{P} \times \mathcal{P}$ is the diagonal additive functor and $+ : \mathcal{Q} \times_{\mathbf{S}} \mathcal{Q} \to \mathcal{Q}$ is the functor underlying the strictly commutative Picard **S**-stack $\mathcal{Q} = (\mathcal{Q}, +, \sigma, \tau)$.

Lemma 4.7. *The above notion of sum of extensions defines on the set of equivalence classes of extensions of $\mathcal{P}$ by $\mathcal{Q}$ an associative, commutative group law with neutral object, that we denote $\mathcal{P} \times \mathcal{Q}$.*

Proof. Neutral object: it is the product $\mathcal{P} \times \mathcal{Q}$ of the extension $\mathcal{P} = (\mathcal{P}, \mathbf{1} : \mathbf{1} \to \mathcal{P}, Id : \mathcal{P} \to \mathcal{P})$ of $\mathcal{P}$ by $\mathbf{1}$ with the extension $\mathcal{Q} = (\mathcal{Q}, Id : \mathcal{Q} \to \mathcal{Q}, \mathbf{1} : \mathcal{Q} \to \mathbf{1})$ of $\mathbf{1}$ by $\mathcal{Q}$. Lemma 4.5 provides that such a product is an extension of $\mathcal{P} \times \mathbf{1} \cong \mathcal{P}$ by $\mathcal{Q} \times \mathbf{1} \cong \mathcal{Q}$. Commutativity: it is clear from the formula (4.1). Associativity: Consider three extensions $\mathcal{E}$, $\mathcal{E}'$, $\mathcal{E}''$ of $\mathcal{P}$ by $\mathcal{Q}$. Using the functor $+ : \mathcal{Q} \times_{\mathbf{S}} \mathcal{Q} \times_{\mathbf{S}} \mathcal{Q} \to \mathcal{Q}$ and the diagonal functor $D : \mathcal{P} \to \mathcal{P} \times \mathcal{P} \times \mathcal{P}$, it is enough to show that the extensions $(\mathcal{E} + \mathcal{E}') + \mathcal{E}''$ and $\mathcal{E} + (\mathcal{E}' + \mathcal{E}'')$ are equivalent. We left this computation to the reader. □

This last lemma implies that if $\mathcal{O}$, $\mathcal{P}$ and $\mathcal{Q}$ are three strictly commutative Picard **S**-stacks, we have the following equivalence of 2-categories

$$\mathcal{E}xt(\mathcal{O} \times \mathcal{P}, \mathcal{Q}) \cong \mathcal{E}xt\,(\mathcal{O}, \mathcal{Q}) \times \mathcal{E}xt(\mathcal{P}, \mathcal{Q}),$$

$$\mathcal{E}xt(\mathcal{O}, \mathcal{P} \times \mathcal{Q}) \cong \mathcal{E}xt\,(\mathcal{O}, \mathcal{P}) \times \mathcal{E}xt(\mathcal{O}, \mathcal{Q}).$$

A 2-groupoid is a 2-category whose 1-arrows are invertible up to a 2-arrow and whose 2-arrows are strictly invertible. An **S**-2-stack in 2-groupoids $\mathbb{P}$ is a fibered 2-category in 2-groupoids over **S** such that

- for every pair of objects X, Y of the 2-category $\mathbb{P}(U)$, the fibered category of morphisms $\mathrm{Arr}_{\mathbb{P}(U)}(X, Y)$ of $\mathbb{P}(U)$ is a U-stack (called the U-stack of morphisms);
- 2-descent is effective for objects in $\mathbb{P}$.

See [B09, §6] for more details. A strictly commutative Picard **S**-2-stack is the 2-analog of a strictly commutative Picard **S**-stack, i.e. it is an **S**-2-stack in 2-groupoids $\mathbb{P}$ endowed with a morphism of **S**-2-stacks $+ : \mathbb{P} \times_{\mathbf{S}} \mathbb{P} \to \mathbb{P}$ and with associative and commutative constraints (see [T09, Definition 2.3] for more details). With these notation Lemma 4.7 implies that extensions of $\mathcal{P}$ by $\mathcal{Q}$ form a strictly commutative Picard **S**-2-stack $\underline{\mathcal{E}xt}(\mathcal{P}, \mathcal{Q})$ where

- for any object U of **S**, the objects of the 2-category $\underline{\mathcal{E}xt}(\mathcal{P}, \mathcal{Q})(U)$ are extensions of $\mathcal{P}_{|U}$ by $\mathcal{Q}_{|U}$, its 1-arrows are additive functors between such extensions and its 2-arrows are morphisms of additive functors. In particular if $\mathcal{E}$ and $\mathcal{E}'$ are two objects of $\underline{\mathcal{E}xt}(\mathcal{P}, \mathcal{Q})(U)$, the U-stack of morphisms from $\mathcal{E}$ to $\mathcal{E}'$ is the U-stack $\mathrm{HOM}(\mathcal{E}, \mathcal{E}')$;
- the functor $+ : \underline{\mathcal{E}xt}(\mathcal{P}, \mathcal{Q}) \times \underline{\mathcal{E}xt}(\mathcal{P}, \mathcal{Q}) \to \underline{\mathcal{E}xt}(\mathcal{P}, \mathcal{Q})$ is defined by the formula (4.1).

As for strictly commutative Picard **S**-stacks and complexes of abelian sheaves concentrated in degrees -1 and 0, in [T09] Tatar proves that there is a dictionary between strictly commutative Picard **S**-2-stacks and complexes of abelian sheaves concentrated in degrees -2, -1 and 0. The complex of abelian sheaves associated to the strictly commutative Picard **S**-2-stack $\underline{\mathcal{E}xt}(\mathcal{P}, \mathcal{Q})$ is $\tau_{\leqslant 0}\mathrm{RHom}([\mathcal{P}], [\mathcal{Q}][1])$.

5. Proof of the main theorem

In this section we use the same notation as in the introduction.

Definition 5.1. Two extensions $\mathcal{E}$ and $\mathcal{E}'$ of $\mathcal{P}$ by $\mathcal{Q}$ are *equivalent as extensions of* $\mathcal{P}$ *by* $\mathcal{Q}$ if there is

(1) an additive functor $F : \mathcal{E} \to \mathcal{E}'$ and
(2) two isomorphisms of additive functors $J' \circ F \cong Id_{\mathcal{P}} \circ J$ and $F \circ I \cong I' \circ Id_{\mathcal{Q}}$,

which are compatible with the isomorphisms of additive functors $J \circ I \cong 0$ and $J' \circ I' \cong 0$ underlying the extensions $\mathcal{E}$ and $\mathcal{E}'$ (see Definition 3.3).

The additive functor F furnishes the following commutative diagram modulo isomorphisms of additive functors

$$\begin{tikzcd}
\mathcal{Q} \arrow[r, "I"] \arrow[d, "Id_{\mathcal{Q}}"'] & \mathcal{E} \arrow[r, "J"] \arrow[d, "F"'] & \mathcal{P} \arrow[d, "Id_{\mathcal{P}}"] \\
\mathcal{Q} \arrow[r, "I'"'] & \mathcal{E}' \arrow[r, "J'"'] & \mathcal{P}.
\end{tikzcd}$$

Definition 5.2. An extension of $\mathcal{P}$ by $\mathcal{Q}$ is *split* if it is equivalent as extension of $\mathcal{P}$ by $\mathcal{Q}$ to the neutral object $\mathcal{P} \times \mathcal{Q}$ of the group law defined in Definition 4.6.

Proof of Theorem 0.1 (b) and (c). Let $\mathcal{E} = (I : \mathcal{Q} \to \mathcal{E}, J : \mathcal{E} \to \mathcal{P})$ be an extension of $\mathcal{P}$ by $\mathcal{Q}$. The strictly commutative Picard **S**-stacks $\mathrm{HOM}(\mathcal{P}, \mathcal{Q})$ and $\mathrm{HOM}(\mathcal{E}, \mathcal{E})$ are equivalent as strictly commutative Picard **S**-stacks via the following additive functor

$$\mathrm{HOM}(\mathcal{P}, \mathcal{Q}) \longrightarrow \mathrm{HOM}(\mathcal{E}, \mathcal{E}),$$
$$F \longmapsto \big(a \longmapsto a + IFJ(a)\big).$$

By Lemma 1.4 we can conclude that $[\mathrm{HOM}(\mathcal{E}, \mathcal{E})] = \tau_{\leqslant 0}\mathrm{RHom}([\mathcal{P}], [\mathcal{Q}])$, i.e. the group of isomorphism classes of additive functors from $\mathcal{E}$ to itself is isomorphic to the group $\mathrm{Hom}_{\mathcal{D}(\mathbf{S})}([\mathcal{P}], [\mathcal{Q}])$, and the group of automorphisms of an additive functor from $\mathcal{E}$ to itself is isomorphic to the group $\mathrm{Hom}_{\mathcal{D}(\mathbf{S})}([\mathcal{P}], [\mathcal{Q}][-1])$ (for this last isomorphism see in particular (1.3)). □

Proof of Theorem 0.1 (a). First we construct a morphism from the group $\mathcal{E}xt^1(\mathcal{P}, \mathcal{Q})$ of equivalence classes of extensions of $\mathcal{P}$ by $\mathcal{Q}$ to the group $\mathrm{Ext}^1([\mathcal{P}], [\mathcal{Q}])$

$$\Theta : \mathcal{E}xt^1(\mathcal{P}, \mathcal{Q}) \longrightarrow \mathrm{Ext}^1\big([\mathcal{P}], [\mathcal{Q}]\big)$$

and a morphism from the group $\mathrm{Ext}^1([\mathcal{P}], [\mathcal{Q}])$ to the group $\mathcal{E}xt^1(\mathcal{P}, \mathcal{Q})$

$$\Psi : \mathrm{Ext}^1\big([\mathcal{P}], [\mathcal{Q}]\big) \longrightarrow \mathcal{E}xt^1(\mathcal{P}, \mathcal{Q}).$$

Then we check that $\Theta \circ \Psi = Id = \Psi \circ \Theta$ and that Θ is a homomorphism of groups.

(1) Construction of Θ: Consider an extension $\mathcal{E} = (I : \mathcal{Q} \to \mathcal{E}, J : \mathcal{E} \to \mathcal{P})$ of $\mathcal{P}$ by $\mathcal{Q}$ and denote by $L = (i : K \to L, j : L \to M)$ the corresponding extension of complexes in $\mathcal{D}^{[-1,0]}(\mathbf{S})$. By definition we

have the equality $K = \tau_{\leqslant 0}(MC(j)[-1])$ in the category $\mathcal{D}^{[-1,0]}(\mathbf{S})$. Hence the distinguished triangle $MC(j)[-1] \to L \xrightarrow{j} M \to +$ furnishes the long exact sequence

$$\cdots \to \mathrm{Hom}(M,K) \to \mathrm{Hom}(M,L) \xrightarrow{j\circ} \mathrm{Hom}(M,M) \xrightarrow{\partial} \mathrm{Ext}^1(M,K) \to \cdots. \tag{5.1}$$

We set

$$\Theta(\mathcal{E}) = \partial(id_M).$$

The naturality of the connecting map ∂ implies that $\Theta(\mathcal{E})$ depends only on the equivalence class of the extension $\mathcal{E}$.

Lemma 5.3. *If* $\mathrm{Ext}^1([\mathcal{P}],[\mathcal{Q}]) = 0$, *then every extension of* $\mathcal{P}$ *by* $\mathcal{Q}$ *is split.*

Proof. By the long exact sequence (5.1), if the cohomology group $\mathrm{Ext}^1(M,K)$ is zero, the identity morphism $id_M : M \to M$ lifts to a morphism $f : M \to L$ of $\mathcal{D}^{[-1,0]}(\mathbf{S})$ which corresponds via the equivalence of categories (1.4) to an isomorphism classes of additive functors $F : \mathcal{P} \to \mathcal{E}$ such that $J \circ F \cong Id_{\mathcal{P}}$. Hence $\mathcal{E}$ is a split extension of $\mathcal{P}$ by $\mathcal{Q}$. □

The above lemma means that $\Theta(\mathcal{E})$ is an obstruction for the extension $\mathcal{E}$ to be split: $\mathcal{E}$ is split if and only if $id_M : M \to M$ lifts to $\mathrm{Hom}(M,L)$ if and only if $\Theta(\mathcal{E})$ vanishes in $\mathrm{Ext}^1([\mathcal{P}],[\mathcal{Q}])$.

(2) Construction of Ψ: Choose two complexes $P = [P^{-1} \xrightarrow{d_P} P^0]$ and $N = [N^{-1} \xrightarrow{d_N} N^0]$ of $\mathcal{D}^{[-1,0]}(\mathbf{S})$ such that P^{-1}, P^0 are projective and the three complexes N, P, M build a short exact sequence in $\mathcal{D}^{[-1,0]}(\mathbf{S})$

$$0 \longrightarrow N \xrightarrow{s} P \xrightarrow{t} M \longrightarrow 0 \tag{5.2}$$

(because of the projectivity of the P^i for $i = -1, 0$, there exists a surjective morphism of complexes $P \to M$ and then, in order to get a short exact sequence, choose $N^i = \ker(P^i \to M^i)$ for $i = -1, 0$). By Remark 3.6 the above exact sequence furnishes an extension of strictly commutative Picard $\mathbf{S}$-stacks

$$st(N) \xrightarrow{S} st(P) \xrightarrow{T} \mathcal{P}$$

where S and T are the isomorphism classes of additive functors corresponding to the morphisms s and t. Applying $\mathrm{Hom}(-, K[1])$ to the distinguished triangle $N \to P \to M \to +$ associated to the short exacts sequence (5.2) we get the long exact sequence

$$\cdots \longrightarrow \mathrm{Hom}(M,K) \longrightarrow \mathrm{Hom}(P,K) \xrightarrow{\circ s} \mathrm{Hom}(N,K) \xrightarrow{\partial} \mathrm{Ext}^1(M,K) \longrightarrow 0. \tag{5.3}$$

Given an element x of $\mathrm{Ext}^1(M,K)$, choose an element u of $\mathrm{Hom}(N,K)$ such that $\partial(u) = x$. We set

$$\Psi(x) = U_* st(P),$$

i.e. $\Psi(x)$ is the push-down $U_* st(P)$ of the extension $st(P)$ via one representative of the isomorphism class $U : st(N) \to \mathcal{Q}$ of additive functors corresponding to the morphism $u : N \to K$ of $\mathcal{D}^{[-1,0]}(\mathbf{S})$. By Lemma 4.4 the strictly commutative Picard $\mathbf{S}$-stack $\Psi(x)$ is an extension of $\mathcal{P}$ by $\mathcal{Q}$. Now we check that the morphism Ψ is well defined, i.e. $\Psi(x)$ doesn't depend on the lift of x. If $u' \in \mathrm{Hom}(N,K)$ is another lift of x, then there exists an element f of $\mathrm{Hom}(P,K)$ such that $u' - u = f \circ s$. Consider the push-down $(U' - U)_* st(P)$ of the extension $st(P)$ via one representative of the isomorphism class $U' - U : st(N) \to \mathcal{Q}$ of additive functors (as for u, we denote here by $U' : st(N) \to \mathcal{Q}$ the isomorphism class corresponding to the morphism $u' : N \to K$ of $\mathcal{D}^{[-1,0]}(\mathbf{S})$). Since $u' - u = f \circ s$, by the universal property of the push-down there exists a unique additive functor $H : (U' - U)_* st(P) \to \mathcal{Q}$ such that

$H \circ In \cong Id_{\mathcal{Q}}$, where $In : \mathcal{Q} \to (U' - U)_* st(P)$ is the additive functor underlying the extension $(U' - U)_* st(P)$ of $\mathcal{P}$ by $\mathcal{Q}$. Hence the extension $(U' - U)_* st(P)$ of $\mathcal{P}$ by $\mathcal{Q}$ is split and so the extensions $U'_* st(P)$ and $U_* st(P)$ are equivalent.

(3) $\Theta \circ \Psi = Id$: With the notation of (2), given an element x of $\mathrm{Ext}^1(M, K)$, choose an element u of $\mathrm{Hom}(N, K)$ such that $\partial(u) = x$. By definition $\Psi(x) = U_* st(P)$. Because of the naturality of the connecting map ∂, the following diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}(M,K) \arrow[r] & \mathrm{Hom}(P,K) \arrow[r] & \mathrm{Hom}(N,K) \arrow[r, "\partial"] & \mathrm{Ext}^1(M,K) \\
\mathrm{Hom}(M,K) \arrow[r] \arrow[u, equal] & \mathrm{Hom}([U_* st(P)],K) \arrow[r] \arrow[u] & \mathrm{Hom}(K,K) \arrow[r, "\partial"] \arrow[u, "\circ u"] & \mathrm{Ext}^1(M,K). \arrow[u, equal]
\end{tikzcd}$$

Therefore $\Theta(U_* st(P)) = x$, i.e. Θ surjective.

(4) $\Psi \circ \Theta = Id$: Consider an extension $\mathcal{E} = (I : \mathcal{Q} \to \mathcal{E}, J : \mathcal{E} \to \mathcal{P})$ of $\mathcal{P}$ by $\mathcal{Q}$ and denote by $L = (i : K \to L, j : L \to M)$ the corresponding extension of complexes in $\mathcal{D}^{[-1,0]}(\mathbf{S})$. Choose two complexes $P = [P^{-1} \to P^0]$ and $N = [N^{-1} \to N^0]$ as in (2). The lifting property of the complex P furnishes a lift $u : P \to L$ of the morphism of complexes $t : P \to M$ and hence a commutative diagram

$$\begin{tikzcd}
st(N) \arrow[r, "S"] \arrow[d, "U_{|}"'] & st(P) \arrow[r, "T"] \arrow[d, "U"'] & \mathcal{P} \arrow[d, "Id_{\mathcal{P}}"] \\
\mathcal{Q} \arrow[r, "I"'] & \mathcal{E} \arrow[r, "J"'] & \mathcal{P},
\end{tikzcd}$$

where $U : st(P) \to \mathcal{E}$ is the isomorphism class of additive functors corresponding to the lift $u : P \to L$ and $U_{|} : st(N) \to \mathcal{Q}$ is the restriction of U to $st(N)$. Consider now the push-down $(U_{|})_* st(P)$ of the extension $st(P)$ via a representative of $U_{|}$. Because of the universal property of the push-down, there exists a unique additive functor $H : (U_{|})_* st(P) \to \mathcal{E}$ such that the following diagram commutes

$$\begin{tikzcd}
st(N) \arrow[rr, "S"] \arrow[dr, "U_{|}"] \arrow[dd, "U_{|}"'] & & st(P) \arrow[rr, "T"] \arrow[dr] \arrow[dd, bend right, "U"' near start] & & \mathcal{P} \arrow[dr, "Id_{\mathcal{P}}"] \arrow[dd, bend right, "Id_{\mathcal{P}}"' near start] & \\
& \mathcal{Q} \arrow[rr] \arrow[dl, "Id_{\mathcal{Q}}"] & & \mathcal{Q} +^{st(N)} st(P) \arrow[rr] \arrow[dl, dotted, "H"] & & \mathcal{P} \arrow[dl, "Id_{\mathcal{P}}"] \\
\mathcal{Q} \arrow[rr, "I"'] & & \mathcal{E} \arrow[rr, "J"'] & & \mathcal{P}. &
\end{tikzcd}$$

Hence we have that the extensions $(U_{|})_* st(P)$ and $\mathcal{E}$ are equivalent, which implies that $\Psi(\Theta(\mathcal{E})) = \Psi(\Theta((U_{|})_* st(P))) = (U_{|})_* st(P) \cong \mathcal{E}$, i.e. Θ injective.

(5) Θ is a homomorphism of groups: Consider two extensions $\mathcal{E}, \mathcal{E}'$ of $\mathcal{P}$ by $\mathcal{Q}$. With the notations of (2) we can suppose that $\mathcal{E} = U_* st(P)$ and $\mathcal{E}' = U'_* st(P)$ with $U, U' : st(N) \to \mathcal{Q}$ two isomorphism classes of additive functors corresponding to two morphisms $u, u' : N \to K$ of $\mathcal{D}^{[-1,0]}(\mathbf{S})$. Now by Definition 4.6

$$\begin{aligned}
\mathcal{E} + \mathcal{E}' &= D^*(+_{\mathcal{Q}})_*\big(U_* st(P) \times U'_* st(P)\big) \\
&= D^*(+_{\mathcal{Q}})_*\big(U \times U'\big)_*\big(st(P) \times st(P)\big)
\end{aligned}$$

$$\begin{aligned} &= \big(U+U'\big)_* D^*(+_{st(N)})_*\big(st(P)\times st(P)\big) \\ &= \big(U+U'\big)_*\big(st(P)+st(P)\big) \end{aligned}$$

where $D:\mathcal{P}\to\mathcal{P}\times\mathcal{P}$ is the diagonal additive functor and $+_{\mathcal{Q}}:\mathcal{Q}\times_{\mathbf{S}}\mathcal{Q}\to\mathcal{Q}$ (resp. $+_{st(N)}:st(N)\times_{\mathbf{S}} st(N)\to st(N)$) is the functor underlying the strictly commutative Picard **S**-stack $\mathcal{Q}$ (resp. $st(N)$). If $\partial:\mathrm{Hom}(N,K)\to\mathrm{Ext}^1(M,K)$ is the connecting map of the long exact sequence (5.3), we get

$$\Theta\big(\mathcal{E}+\mathcal{E}'\big)=\partial\big(u+u'\big)=\partial(u)+\partial\big(u'\big)=\Theta(\mathcal{E})+\Theta\big(\mathcal{E}'\big). \qquad \square$$

Remark 5.4. In the construction of $\Psi:\mathrm{Ext}^1([\mathcal{P}],[\mathcal{Q}])\to\mathcal{E}xt^1(\mathcal{P},\mathcal{Q})$, instead of two complexes $P=[P^{-1}\to P^0]$ and N of $\mathcal{D}^{[-1,0]}(\mathbf{S})$ such that P^{-1}, P^0 are projective and the three complexes N, P, M build a short exact sequence $0\to N\to P\to M\to 0$, we can consider two complexes $I=[I^{-1}\to I^0]$ and N' of $\mathcal{D}^{[-1,0]}(\mathbf{S})$ such that I^{-1}, I^0 are injective and the three complexes K, I, N' build a short exact sequence $0\to K\to I\to N'\to 0$. In this case instead of applying $\mathrm{Hom}(-,K[1])$ we apply $\mathrm{Hom}(M,-)$ and instead of considering push-downs of extensions we consider pull-backs. This two way to construct Ψ with projectives or with injectives are dual.

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