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On the Operator $\Delta r^2 + \mu(\partial/\partial r)r + \lambda$

M. S. BAOUENDI*

Division of Mathematical Sciences, Purdue University, West Lafayette, Indiana 47907

C. GOULAOUIC

Department of Mathematics, University of Orsay, Orsay, France

AND

L. J. LIPKIN

Department of Mathematics, University of North Florida, Jacksonville, Florida 32216

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INTRODUCTION

We consider here an example of an elliptic operator in $\mathbb{R}^n$, singular at the origin.

Let $\lambda, \mu \in \mathbb{C}$, $x \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, and $r^2 = \sum_i x_i^2$. We consider the equation

$$Lu = L(\lambda, \mu)u = \Delta r^2 u + \mu(\partial/\partial r)\, ru + \lambda u = f, \tag{1}$$

where $\Delta = \sum_i \partial^2/\partial x_i^2$.

We prove first that if $f = 0$, any $\mathscr{C}^\infty$ function u satisfying (1) is necessarily analytic.

We also prove that if $(\lambda, \mu) \in \sum(L)$, where $\sum(L)$ is an exceptional set in $\mathbb{C}^2$, for any analytic function defined in a neighborhood of the origin in $\mathbb{R}^n$, there exists a unique function u, analytic in some neighborhood of 0 and satisfying (1). If $(\lambda, \mu) \in \sum(L)$ we give a complete description of the kernel of L and the compatibility conditions for the solvability of (1).

The operator L is not hypoelliptic, but we prove that if u is a $\mathscr{C}^\infty$ function such that Lu is analytic, then u is also analytic.

We hope that this particular example may explain and help in the study of more general equations, singular at one point.

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In Section 1 we give some results about the expansion of $\mathscr{C}^\infty$ and analytic functions with respect to harmonic polynomials.

In Section 2, we describe the set $\sum(L)$ with its properties, which are used in Section 3 where the main results are stated and proved.

1. Spherical Harmonics Expansion

Let S_{n-1} be the unit sphere in $\mathbb{R}^n$. If $x \in \mathbb{R}^n$, we denote

$$x = r\theta$$

with $r \geqslant 0$ and $\theta \in S_{n-1}$.

In $L^2(S_{n-1})$, we choose an orthonormal basis of spherical harmonics $P_{\ell,\alpha}$, with $\ell \in \mathbb{N}$ and $1 \leqslant \alpha \leqslant \alpha(\ell)$, where

$$\alpha(\ell) = \frac{(2\ell + n - 2)(n + \ell - 3)!}{(n-2)!\,\ell!} \qquad \text{for} \quad n \geqslant 2.$$

Therefore, $\tilde{P}_{\ell,\alpha}(x) = r^\ell P_{\ell,\alpha}(\theta)$ is a homogeneous harmonic polynomial of degree ℓ.

The laplacian can be written in spherical coordinates

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_\theta,$$

where Δ_θ is the Laplace–Beltrami operator on S_{n-1}. We have, in particular,

$$\Delta_\theta(P_{\ell,\alpha}(\theta)) = -\ell(\ell + n - 2)\, P_{\ell,\alpha}(\theta).$$

Let f be a $\mathscr{C}^\infty$ function defined on the closed ball: $r \leqslant r_0$ ($r_0 > 0$). We can write

$$\begin{aligned} f(x) &= \sum_{\ell=0}^{\infty} \sum_{\alpha=1}^{\alpha(\ell)} \tilde{f}_{\ell,\alpha}(r)\, P_{\ell,\alpha}(\theta), \\ \tilde{f}_{\ell,\alpha}(r) &= \int_{S_{n-1}} f(r\theta)\, \overline{P_{\ell,\alpha}(\theta)}\, d\theta. \end{aligned} \tag{2}$$

Each $\tilde{f}_{\ell,\alpha}$ is infinitely differentiable on $[0, r_0]$.

Using the ellipticity of Δ_θ on S_{n-1} and Sobolef's theorem, for $k > (n-1)/4$ and all (ℓ, α), we have

$$\sup_{\theta \in S_{n-1}} |P_{\ell,\alpha}(\theta)| \leqslant C_1(\|\Delta_\theta^k P_{\ell,\alpha}\|_{L^2(S_{n-1})} + \|P_{\ell,\alpha}\|_{L^2(S_{n-1})}),$$

where C_1 is a constant.

Therefore, there exist $C_0 > 0$ and $k_0 \in \mathbb{N}$ ($k_0 > (n-1)/2$) such that for all (ℓ, α),

$$\sup_{\theta \in S_{n-1}} | P_{\ell,\alpha}(\theta)| \leqslant C_0 \ell^{k_0}. \tag{3}$$

On the other hand, we have for any $k \in \mathbb{N}$,

$$\sum_{\ell,\alpha} | \tilde{f}_{\ell,\alpha}(r) \ell^k(\ell + n - 2)^k |^2 = \int_{S_{n-1}} | \Delta_\theta{}^k f(r\theta)|^2 \, d\theta.$$

Since Δ_θ is a partial differential operator with polynomial coefficients and f is $\mathscr{C}^\infty$ on $r \leqslant r_0$, we get, for any $k \in \mathbb{N}$,

$$\sup_{0 \leqslant r \leqslant r_0} \sum_{\ell,\alpha} | \tilde{f}_{\ell,\alpha}(r) \ell^k(\ell + n - 2)^k |^2 < \infty,$$

and hence, for each $k \in \mathbb{N}$, there exists $C_k > 0$ such that for all (ℓ, α),

$$\sup_{0 \leqslant r \leqslant r_0} | \tilde{f}_{\ell,\alpha}(r)| \leqslant C_k/\ell^k. \tag{4}$$

In particular, (4) together with (3) imply the normal convergence of series (2) on the ball $r \leqslant r_0$.

We define the collection of functions $\{f_{\ell,\alpha}\}$ by

$$f_{\ell,\alpha}(r^2) = r^{-\ell} \tilde{f}_{\ell,\alpha}(r) \qquad \text{for} \quad 0 \leqslant r \leqslant r_0. \tag{5}$$

We have the following result:

PROPOSITION 1. *Let f be a $\mathscr{C}^\infty$ function defined for $r \leqslant r_0$.*

(1) *For any $\ell \in \mathbb{N}$ and any α, $1 \leqslant \alpha \leqslant \alpha(\ell)$, the function $f_{\ell,\alpha}$ defined by (5) is infinitely differentiable on $[0, r_0{}^2]$.*

(2) *The function f is analytic in some neighborhood of 0 in $\mathbb{R}^n$ if and only if there exists $t_0 > 0$ and $M > 0$ such that, for any $\ell \in \mathbb{N}$, any $k \in \mathbb{N}$, and $1 \leqslant \alpha \leqslant \alpha(\ell)$,*

$$\sup_{0 \leqslant t \leqslant t_0} |(\partial/\partial t)^k g_{\ell,\alpha}(t)| \leqslant M^{\ell+k+1} k! \tag{6}$$

Proof. (1) Using Taylor expansion of f at the origin and the orthogonality properties of the spherical harmonics, we prove first that

$$r \mapsto r^{-\ell} \tilde{f}_{\ell,\alpha}(r)$$

is a $\mathscr{C}^\infty$ function on $[0, r_0]$.

We observe next that the latter function may be extended as an even $\mathscr{C}^\infty$ function on $[-r_0, r_0]$. The desired result follows immediately.

(2)(i) Let us assume that (6) holds. We shall prove that there exist $\rho > 0$ and $C > 0$ such that for any $k \in \mathbb{N}$,

$$\| \Delta^k f \|_{L^2(B(\rho))} \leqslant C^{k+1}(k!)^2 \tag{7}$$

[$B(\rho)$ is the open ball centered at 0 in $\mathbb{R}^n$, of radius ρ].

Using a very well-known result (see, for example, [1]), the analyticity of f in $B(\rho)$ follows from (7).

The condition (6) implies that $f_{\ell,\alpha}$ is analytic on a neighborhood of 0. We can write its Taylor series:

$$f_{\ell,\alpha}(t) = \sum_{j=0}^{\infty} f_{\ell,\alpha,j} t^j,$$

and we have

$$| f_{\ell,\alpha,j} | \leqslant M^{\ell+j+1} \qquad (\text{for any } (\ell, \alpha, j)).$$

Hence, we can write

$$f(x) = \sum_{\ell=0}^{\infty} \sum_{\alpha=1}^{\alpha(\ell)} \sum_{j=0}^{\infty} f_{\ell,\alpha,j} r^{2j} \tilde{P}_{\ell,\alpha}(x).$$

On the other hand, we have

$$\Delta(r^{2j}\tilde{P}_{\ell,\alpha}) = 2j(n + 2\ell + 2j - 2)\, r^{2j-2}\tilde{P}_{\ell,\alpha}\,.$$

Therefore, we get

$$\Delta^k(f(x)) = \sum_{\ell=0}^{\infty} \sum_{\alpha=1}^{\alpha(\ell)} \sum_{j=k}^{\infty} f_{\ell,\alpha,j} \frac{2^{2k} j! \left(j + \frac{n}{2} + \ell - 1\right)!}{(j-k)! \left(j + \frac{n}{2} + \ell - 1 - k\right)!} r^{2j-2k}\tilde{P}_{\ell,\alpha}(x).$$

Since

$$\| r^{2j-2k}\tilde{P}_{\ell,\alpha} \|^2_{L^2(B(\rho))} = \frac{\rho^{4j+2\ell-4k-n}}{4j + 2\ell - 4k + n},$$

we obtain

$$\| \Delta^k f \|_{L^2(B(\rho))} \leqslant \sum_{\ell=0}^{\infty} \sum_{\alpha=1}^{\alpha(\ell)} \sum_{i=0}^{\infty} M^{l-k+i+1} 2^{4k+2i+(n/2)+l-1} \frac{(k!)^2 \rho^{2i+l-n/2}}{(4i + 2\ell + n)^{1/2}},$$

and hence we proved (7) with suitable ρ and C.

(ii) Conversely, we prove now that the analyticity of f (in some neighborhood of 0 in $\mathbb{R}^n$) implies (6). In order to do so, we write the Taylor series of f at the origin:

$$f(x) = \sum_{\gamma\in\mathbb{N}^n} a_\gamma x^\gamma,$$

with

$$|a_\gamma| \leqslant A^{|\gamma|+1} \qquad (A \text{ is a positive constant}).$$

Hence, we get, for small positive t,

$$f_{\ell,\alpha}(t) = \sum_{\substack{\gamma\in\mathbb{N}^n \\ |\gamma|-\ell\in 2\mathbb{N}}} a_\gamma t^{(|\gamma|-\ell)/2} \int_{S_{n-1}} \theta^\gamma \langle P_{\ell,\alpha}(\theta)\rangle \, d\theta,$$

and then

$$\left|\left(\frac{\partial}{\partial t}\right)^k f_{\ell,\alpha}(t)\right| \leqslant \sum_{(|\gamma|-\ell)/2-k\in\mathbb{N}} \frac{A^{|\gamma|+1}\left(\dfrac{|\gamma|-\ell}{2}\right)!}{\left(\dfrac{|\gamma|-\ell}{2}-k\right)!} t^{(|\gamma|-\ell)/2-k} \left|\int_{S_{n-1}} \theta^\ell \, \overline{P_{\ell,\alpha}(\theta)}\, d\theta\right| .$$

By means of the Cauchy–Schwarz formula we can write

$$\left|\int_{S_{n-1}} \theta^l \overline{P_{\ell,\alpha}(\theta)}\, d\theta\right| \leqslant \sigma_n^{1/2},$$

where σ_n denotes the area of the unit sphere S_{n-1}.

We finally obtain

$$\left|\left(\frac{\partial}{\partial t}\right)^k f_{\ell,\alpha}(t)\right| \leqslant \sum_{j=0}^{\infty} \sigma_n^{1/2}(nA)^{\ell+2k+2j-1} k!\, 2^{k+j} t^j,$$

which proves (6) with suitable t_0 and M.

We have also shown the following:

PROPOSITION 2. *Let f be an analytic function defined in a neighborhood of the origin in $\mathbb{R}^n$. There exists a unique collection of complex numbers $(f_{\ell,\alpha,j})$, $\ell\in\mathbb{N}$, $j\in\mathbb{N}$, $1 \leqslant \alpha \leqslant \alpha(\ell)$, and a constant $M > 0$ such that*

$$|f_{\ell,\alpha,j}| \leqslant M^{\ell+j+1} \qquad \textit{for all } (\ell,\alpha,j) \tag{8}$$

and

$$f(x) = \sum_{\ell,\alpha,j} f_{\ell,\alpha,j} r^{2j} \tilde{P}_{\ell,\alpha}(x) \tag{9}$$

in a neighborhood of the origin.

Conversely, if $(f_{\ell,\alpha,j})$ *is a collection of complex numbers satisfying* (8), *then the series* (9) *defines an analytic function in some neighborhood of the origin.*

Let us point out that (8) and (3) imply the normal convergence of (9) on a neighborhood of the origin.

We also observe that $f_{\ell,\alpha,j}$ is a linear combination of partial derivatives of f at the origin of order $\ell + 2j$.

Remark 1. Let $\mathbb{C}[[X_1,..., X_n]]$ be the space of formal series in n variables with complex coefficients. It is easy to see that any $f \in \mathbb{C}[[X_1,..., X_n]]$ has a formal expansion (9), where $f_{\ell,\alpha,j}$ are complex numbers uniquely defined, each $f_{\ell,\alpha,j}$ is a linear combination of "partial derivatives of f at the origin" of order $\ell + 2j$. (Same formula as in the analytic case). Of course, (8) is not required here any more.

Conversely, any collection of complex numbers $(f_{\ell,\alpha,j})$ defines by means of (9) a unique $f \in \mathbb{C}[[X_1,..., X_n]]$.

Remark 2. Let f be as in Proposition 2 above. We define

$$H_j(x) = \sum_{\ell,\alpha} f_{\ell,\alpha,j} \tilde{P}_{\ell,\alpha}(x).$$

H_j is uniformly convergent near the origin and harmonic. We get from (9)

$$f(x) = \sum_{j=0}^{\infty} r^{2j} H_j(x),$$

the latter series being uniformly convergent for small $|x|$.

This is the Almansi expansion of analytic functions studied more extensively in [1].

2. The Spectrum of L

Let u and f be two $\mathscr{C}^\infty$ functions defined in a neighborhood of 0 in $\mathbb{R}^n$. We consider their spherical harmonics expansions defined in the previous section:

$$u(x) = \sum_{\ell,\alpha} u_{\ell,\alpha}(r^2)\, \tilde{P}_{\ell,\alpha}(x),$$

$$f(x) = \sum_{\ell,\alpha} f_{\ell,\alpha}(r^2)\, \tilde{P}_{\ell,\alpha}(x).$$

Let r_0 be a positive real number. We have

LEMMA 1. *The functions u and f satisfy* (1) *in* $B(r_0)$ *if and only if for each* $\ell \in \mathbb{N}$ *and each* α, $1 \leqslant \alpha \leqslant \alpha(\ell)$, $u_{\ell,\alpha}$ *and* $f_{\ell,\alpha}$ *satisfy the following differential equation in* $[0, r_0^2]$:

$$\begin{aligned} C&\left(\lambda, \mu, \ell, t\frac{d}{dt}\right) u_{\ell,\alpha} \\ &= \left\{4\left(t\frac{d}{dt}\right)^2 + 2(n+\mu+2\ell+2)\,t\frac{d}{dt} + \ell(4+\mu) + \lambda - \mu + 2n\right\} u_{\ell,\alpha} \\ &= f_{\ell,\alpha}. \end{aligned} \tag{10}$$

Proof. In spherical coordinates, L becomes

$$(\partial^2/\partial r^2)\,r^2 + ((n-1)/r)(\partial/\partial r)\,r^2 + \Delta_\theta + \mu(\partial/\partial r)\,r + \lambda.$$

An easy computation shows that

$$\begin{aligned} L&(u_{\ell,\alpha}(r^2)\,\tilde{P}_{\ell,\alpha}(x)) \\ &= \left(\left\{\left(r\frac{d}{dr}\right)^2 + (n+\mu+2\ell+2)\,r\frac{d}{dr} + (\ell(4+\mu) + \lambda + \mu + 2n)\right\} u_{\ell,\alpha}(r^2)\right) \\ &\quad \times P_{\ell,\alpha}(x). \end{aligned}$$

We point out that under the change of variable $t = r^2$, the operator $r(d/dr)$ becomes $2t(d/dt)$. The rest of the proof is straightforward by termwise differentiating the expansion of u.

Lemma 1 reduces the solvability and the uniqueness problems of (1) to the study of the ordinary differential equations (10), which are of Fuchs type.

The characteristic polynomial associated to Eq. (9) is

$$\begin{aligned} \sigma(\tau) &= \sigma(\lambda, \mu, \ell, \tau) \\ &= 4\tau^2 + 2(n+\mu+2\ell+2)\tau + \ell(4+\mu) + \lambda + \mu + 2n. \end{aligned}$$

The roots of $C(\tau) = 0$ (or characteristic roots) are

$$\begin{aligned} \tau_1 &= \tau_1(\lambda, \mu, \ell) = \frac{-(n+\mu+2\ell+2) + \delta^{1/2}}{4} \\ \tau_2 &= \tau_2(\lambda, \mu, \ell) = \frac{-(n+\mu+2\ell+2) - \delta^{1/2}}{4} \end{aligned} \tag{11}$$

with

$$\delta = (n + 2\ell + \mu + 2)^2 - 4(\ell(4 + \mu) + \lambda + \mu + 2n),$$

and the solution of the homogeneous equation

$$\sigma(\lambda, \mu, \ell, t(d/dt))\, u(t) = 0 \qquad \text{for} \quad t > 0,$$

is

$$C_1 t^{\tau_1} + C_2 t^{\tau_2} \qquad (C_1, C_2 \text{ are constant}),$$

when $\tau_1 \neq \tau_2$, and with the usual modification if $\tau_1 = \tau_2$. In particular, there exists a nontrivial C^∞ (near 0) solution of the homogeneous equation if and only if τ_1 or τ_2 is nonnegative integer j; that is to say,

$$\sigma(\lambda, \mu, \ell, j) = 0. \tag{12}$$

We define the following sets:

$$\textstyle\sum(L) = \{(\lambda, \mu) \in \mathbb{C}^2, \exists \ell \in \mathbb{N}, \exists j \in \mathbb{N}, \sigma(\lambda, \mu, \ell, j) = 0\}, \tag{13}$$

$$\textstyle\sum_0(L) = \{(\lambda, \mu) \in \mathbb{C}^2, \exists j \in \mathbb{N}, \mu = -4(j + 1), \lambda = 2(j + 1)(2j - n - 2)\}. \tag{14}$$

We have

$$\textstyle\sum_0(L) \subset \sum(L),$$

and we note

$$\textstyle\sum_1(L) = \sum(L) - \sum_0(L). \tag{15}$$

We say that the *exceptional set* $\sum(L)$ *is the spectrum of* L.

It is easy to prove the following lemmas which summarize the properties of $\sum(L)$.

LEMMA 2. (i) *If* $(\lambda, \mu) \in \mathbb{C}^2$, $(\lambda, \mu) \notin \sum(L)$, *there is no pair* $(\ell, j) \in \mathbb{N}^2$ *such that* $\sigma(\lambda, \mu, \ell, j) = 0$. *So, for* $i = 1, 2$ *and all* $\ell \in \mathbb{N}$,

$$\tau_i(\lambda, \mu, \ell) \notin \mathbb{N}.$$

(ii) *If* $(\lambda, \mu) \in \sum_1(L)$, *there exists only a finite number of pairs* $(\ell, j) \in \mathbb{N}^2$ *satisfying* $\sigma(\lambda, \mu, \ell, j) = 0$. *Then, for* $i = 1, 2$,

$$\tau_i(\lambda, \mu, \ell) \in \mathbb{N}$$

only for a finite number of ℓ's.

(iii) *If* $(\lambda, \mu) \in \sum_0(L)$, *then*

$$\sigma(\lambda, \mu, \ell, -((\mu + 4)/4)) = 0 \qquad \textit{for all } \ell \in \mathbb{N},$$

and there exists at most a finite number of pairs $(\ell, j) \in \mathbb{N}^2$, $j \neq -(\mu + 4)/4$, *such that* $\sigma(\lambda, \mu, \ell, j) = 0$.

In this case, $\tau_1(\lambda, \mu, \ell)$ *is the same integer* $-(\mu + 4)/4$ *for all* ℓ, *while* $\tau_2(\lambda, \mu, \ell)$ *belongs to* $\mathbb{N}$ *for at most a finite number of* ℓ.

LEMMA 3. *For any* $(\lambda, \mu) \in \mathbb{C}^2$, *there exists* $K_{\lambda,\mu} > 0$ *such that, for any* $(\ell, j) \in \mathbb{N}^2$, *one of the following relations holds*:

(i) $\sigma(\lambda, \mu, \ell, j) = 0$,

(ii) $|\sigma(\lambda, \mu, \ell, j)| \geqslant K_{\lambda,\mu}$.

3. UNIQUENESS, SOLVABILITY, AND REGULARITY

We are ready now to state and prove our main results about Eq. (1).

THEOREM 1. *Let* Ω *be a connected open set in* $\mathbb{R}^n$ *containing the origin. The kernel of* L *in* $\mathscr{C}^\infty(\Omega)$ *consists of analytic functions in* Ω. *More precisely, it is*:

(i) *zero if* $(\lambda, \mu) \notin \sum(L)$.

(ii) *a finite dimensional space of polynomials if* $(\lambda, \mu) \in \sum(L)$.

(iii) *the sum of a finite dimensional space of polynomials and* $r^{-(\mu-4)/2}H(\Omega)$, *where* $H(\Omega)$ *is the space of harmonic functions in* Ω, *if* $(\lambda, \mu) \in \sum_0(L)$.[1]

Proof. Because of the ellipticity of L in $\Omega - \{0\}$, if $u \in \mathscr{C}^\infty(\Omega)$ and satisfies $Lu = 0$, it is necessarily analytic in $\Omega - \{0\}$. Therefore, since Ω is connected, we can assume without loss of generality that Ω is an open ball $B(r_0)$ $(r_0 > 0)$.

Then, using Lemma 1, the problem is reduced to the investigation of the homogeneous solutions of Eq. (10); and Theorem 1 follows easily from the study of the spectrum $\sum(L)$ (Lemma 2).

Now, we are going to look at the solvability of (1) in the analytic case.

Let $\mathcal{O}_n$ be the space of convergent entire series in n variables, identified to the space of analytic functions (germs) at the origin.

If $f, u \cdots \in \mathcal{O}_n$, we denote by $(f_{\ell,\alpha,j}), (u_{\ell,\alpha,j}), \cdots$ the collection of complex numbers associated to $f, u, \ldots$ by Proposition 2. (See (9).)

[1] We recall here that if $(\lambda, \mu) \in \sum_0(L)$, we have $-(\mu + 4)/2 = 2j$ with $j \in \mathbb{N}$. (See (14).)

The differential operator L is a linear operator in $\mathcal{O}_n$. Its kernel is described in Theorem 1. The following theorem tells us about its range:

THEOREM 2. *An element $f \in \mathcal{O}_n$ is in $L\mathcal{O}_n$ if and only if*

$$f_{\ell,\alpha,j} = 0 \qquad \textit{for all } \ell, j$$

satisfying $\sigma(\lambda, \mu, \ell, j) = 0$ and $1 \leqslant \alpha \leqslant \alpha(\ell)$.
In particular,

(i) *If $(\lambda, \mu) \notin \sum(L)$, $L\mathcal{O}_n = \mathcal{O}_n$.*
(ii) *If $(\lambda, \mu) \in \sum_1(L)$, the codimension of $L\mathcal{O}_n$ in $\mathcal{O}_n$ is finite.*
(iii) *If $(\lambda, \mu) \in \sum_0(L)$, the codimension of $L\mathcal{O}_n$ is infinite.*

Proof. An easy computation shows that

$$L(r^{2j}\tilde{P}_{\ell,\alpha}(x)) = \sigma(\lambda, \mu, \ell, j)\, r^{2j}\tilde{P}_{\ell,\alpha}(x) \tag{16}$$

for all $(\ell, j) \in \mathbb{N}^2$ and $1 \leqslant \alpha \leqslant \alpha(\ell)$. A straightforward application of Proposition 2 and Lemma 3 together with (16) completes the proof of the first part of Theorem 2. Then, Lemma 2 takes care of the rest.

Remark 3. L is also a linear operator in $\mathbb{C}[[X_1, ..., X_n]]$. We consider the formal expansion (9) for any formal series (see Remark 1). Then we have:

Let f be a formal series. f belongs to the kernel of L in $\mathbb{C}[[X_1, ..., X_n]]$ if and only if

$$f_{\ell,\alpha,j} = 0 \qquad \text{for all } (\ell, j) \in \mathbb{N}^2$$

such that $\sigma(\lambda, \mu, \ell, j) \neq 0$ and $1 \leqslant \alpha \leqslant \alpha(\ell)$. Also, f belongs to $L\mathbb{C}[[X_1, ..., X_n]]$ if and only if

$$f_{\ell,\alpha,j} = 0 \qquad \text{for all } (\ell, j) \in \mathbb{N}^2$$

such that $\sigma(\lambda, \mu, \ell, j) = 0$ and $1 \leqslant \alpha \leqslant \alpha(\ell)$.

We state now the following regularity result:

THEOREM 3. *Let Ω be an open set in $\mathbb{R}^n$ and $u \in \mathscr{C}^\infty(\Omega)$ such that Lu is analytic in Ω; then u is also analytic in Ω.*

Proof. We must prove the analyticity of u only at the origin.

Remark 3 shows that necessarily $Lu \in L\mathcal{O}_n$. Then there exists $v \in \mathcal{O}_n$ satisfying $Lv = f$. Since $L(u - v) = 0$ in some neighborhood of 0, the use of Theorem 1 yields to the desired result.

Remark 4. Let us point out that for any $(\lambda, \mu) \in \mathbb{C}^2$, the operator L is not hypoelliptic.

More precisely, there exist $\ell \in \mathbb{N}$, $i \in \{1, 2\}$, such that the characteristic root, $\tau_i(\lambda, \mu, \ell)$, defined in (11) is not a nonnegative integer, therefore

$$u(x) = r^{2\tau_i} \tilde{P}_{\ell,\alpha}(x)$$

(defined in a suitable way as a distribution in $\mathbb{R}^n$) satisfies

$$Lu = 0 \text{ in } \mathbb{R}^n, \qquad u \notin \mathscr{C}^\infty(\mathbb{R}^n).$$

References

1. N. Aronszajn (in collaboration with T. M. Creese and L. J. Lipkin), Polyharmonic functions, to appear.