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A Chebyshev spectral method based on operational matrix for initial and boundary value problems of fractional order

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ABSTRACT

We are concerned with linear and nonlinear multi-term fractional differential equations (FDEs). The shifted Chebyshev operational matrix (COM) of fractional derivatives is derived and used together with spectral methods for solving FDEs. Our approach was based on the shifted Chebyshev tau and collocation methods. The proposed algorithms are applied to solve two types of FDEs, linear and nonlinear, subject to initial or boundary conditions, and the exact solutions are obtained for some tested problems. Numerical results with comparisons are given to confirm the reliability of the proposed method for some FDEs.

1. Introduction

It is known that many phenomena in several branches of science can be described very successfully by models using mathematical tools from fractional calculus. Methods of solutions of problems for fractional differential equations have been studied extensively by many researchers (see, e.g., [1–3] and the references given therein). The analytic results on the existence and uniqueness of solutions to the FDEs have been investigated by many authors; among them, [4,5]. In general, most of FDEs do not have exact analytic solutions, so approximation and numerical techniques must be used.

Finding accurate and efficient methods for solving FDEs has become an active research undertaking. There are several analytic methods such as the Adomian decomposition method [6,7], the homotopy-perturbation method [8], the variational iteration method [9] and the homotopy analysis method [10]. From the numerical point of view, Diethelm et al. [11] presented the predictor–corrector method for numerical solutions of FDEs. In [12], the authors have proposed an approximate method for the numerical solution of a class of FDEs which are expressed in terms of Caputo type fractional derivatives. In fact, the method presented in [12] takes advantage of FDEs converting into Volterra-integral equations. In [7], analytical and numerical methods are used to solve a multi-term nonlinear fractional differential equation. Furthermore, the generalization of the Legendre operational matrix to the fractional calculus has been studied in [13].

The main advantage of spectral methods lies in their accuracy for a given number of unknowns. For smooth problems in simple geometries, they offer exponential rates of convergence/spectral accuracy (see, e.g., [14–17]). In the present paper, we extend the application of spectral methods with generalization of Chebyshev operational matrix (COM) to the fractional calculus for developing direct solution techniques for solution of linear multi-term FDEs.

Doha et al. [18] proposed an efficient spectral tau and collocation methods based on the Chebyshev polynomials for solving multi-term linear and nonlinear fractional differential equations subject to nonhomogeneous initial conditions.

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Furthermore, Bhrawy et al. [19] proved a new formula expressing explicitly any fractional-order derivatives of shifted Legendre polynomials of any degree in terms of shifted Legendre polynomials themselves. Extension of the tau method for solving some multi-order fractional differential equation variable coefficients is treated using the shifted Legendre Gauss–Lobatto quadrature (see, [19]). In [20,21], the authors have presented the spectral tau method for numerical solutions of some FDEs. Recently, Esmaeili and Shamsi [22] introduced a direct solution technique for obtaining the spectral solution of a special family of fractional initial value problems using a pseudo-spectral method, and in [23], Pedas and Tamme developed the spline collocation methods for solving FDEs. The algorithms in the present work are somewhat related to the ideas used by Saadatmandi and Dehghan [13], Doha et al. [18] and Bhrawy et al. [19] in developing accurate algorithms for various purposes.

The main aim of this paper is to propose a suitable way to approximate linear multi-term FDEs with constant coefficients using a shifted Chebyshev tau method based on COM such that it can be implemented efficiently and at the same time has a good convergence property.

Dealing with nonlinear multi-order fractional initial or boundary value problems on the interval $(0, L)$, we propose a spectral shifted Chebyshev collocation method based on COM to find the solution $u_N(x)$. The nonlinear FDE is collocated only at the $(N-m+1)$ points. For suitable collocation points, we use the $(N-m+1)$ nodes of the shifted Chebyshev–Gauss interpolation on $(0, L)$. These equations, together with m initial conditions or m boundary conditions, generate $(N+1)$ nonlinear algebraic equations which can then be solved using Newton's iterative method. Finally, the accuracy of the proposed algorithms is demonstrated by test problems.

The rest of the paper is organized as follows. In Section 2, we introduce some mathematical preliminaries of the fractional calculus theory and some relevant properties of the Chebyshev polynomials. In Section 3, the COM of fractional derivative is obtained and proved. Section 4 is devoted to applying the spectral methods for solving multi-order linear and nonlinear FDEs using the COM of fractional derivative. Some numerical experiments are presented in Section 5. Finally, we conclude the paper with some remarks.

2. Preliminaries

For m to be the smallest integer that is greater than or equal to ν, the Caputo's fractional derivative operator of order $\nu > 0$ is defined as:

$$D^{\nu}f(x) = \begin{cases} J^{m-\nu}D^{m}f(x), & \text{if } m-1<\nu<m, \\ D^{m}f(x), & \text{if } \nu = m,\ m \in N, \end{cases} \tag{2.1}$$

where

$$J^{\nu}f(x) = \frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}f(t)\mathrm{d}t, \quad \nu>0,\ x>0.$$

For the Caputo derivative, we have

$$D^{\nu}x^{\beta} = \begin{cases} 0, & \text{for } \beta \in N_0 \text{ and } \beta < \lceil \nu \rceil, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\nu)}\, x^{\beta-\nu}, & \text{for } \beta \in N_0 \text{ and } \beta \geq \lceil \nu \rceil \text{ or } \beta \notin N \text{ and } \beta > \lfloor \nu \rfloor. \end{cases} \tag{2.2}$$

We use the ceiling function $\lceil \nu \rceil$ to denote the smallest integer greater than or equal to ν, and the floor function $\lfloor \nu \rfloor$ to denote the largest integer less than or equal to ν. Also $N = \{1, 2, \dots\}$ and $N_0 = \{0, 1, 2, \dots\}$. Recall that for $\nu \in N$, the Caputo differential operator coincides with the usual differential operator of an integer order.

The Chebyshev polynomials $\{T_i(t); i = 0, 1, \dots\}$ are defined on the interval $(-1, 1)$. In order to use these polynomials on the interval $x \in (0, L)$, we defined the so-called shifted Chebyshev polynomials by introducing the change of variable $t = \frac{2x}{L} - 1$. Let the shifted Chebyshev polynomials $T_i(\frac{2x}{L} - 1)$ be denoted by $T_{L,i}(x)$, satisfying the orthogonality relation

$$\int_0^L T_{L,j}(x)T_{L,k}(x)w_L(x)\mathrm{d}x = \delta_{kj}h_k, \tag{2.3}$$

where $w_L(x) = \frac{1}{\sqrt{Lx-x^2}}$ and $h_k = \frac{\epsilon_k}{2}\pi$, $\epsilon_0 = 2$, $\epsilon_k = 1, k \geq 1$.

The analytic form of the shifted Chebyshev polynomials $T_{L,i}(x)$ of degree i is given by

$$T_{L,i}(x) = i\sum_{k=0}^{i}(-1)^{i-k}\frac{(i+k-1)!\,2^{2k}}{(i-k)!\,(2k)!\,L^k}x^k, \tag{2.4}$$

where $T_{L,i}(0) = (-1)^i$ and $T_{L,i}(L) = 1$.

In this form, $T_{L,i}(x)$ may be generated with the aid of the following recurrence formula:

$$T_{L,i+1}(x) = 2\left(\frac{2x}{L}-1\right)T_{L,i}(x) - T_{L,i-1}(x), \quad i = 1, 2, \dots, \tag{2.5}$$

where $T_{L,0}(x) = 1$ and $T_{L,1}(x) = \frac{2x}{L} - 1$.

A function $u(x)$, square integrable in $(0, L)$, may be expressed in terms of the shifted Chebyshev polynomials as

$$u(x) = \sum_{j=0}^{\infty} c_j T_{L,j}(x),$$

where the coefficients c_j are given by

$$c_j = \frac{1}{h_j} \int_0^L u(x) T_{L,j}(x) w_L(x) \mathrm{d}x, \quad j = 0, 1, 2, \ldots. \tag{2.6}$$

In practice, only the first $(N+1)$-terms shifted Chebyshev polynomials are considered. Hence, if we write

$$u_N(x) \simeq \sum_{j=0}^{N} c_j T_{L,j}(x) = C^T \phi(x), \tag{2.7}$$

where the shifted Chebyshev coefficient vector C and the shifted Chebyshev vector $\phi(x)$ are given by

$$\begin{aligned} C^T &= [c_0, c_1, \ldots, c_N], \\ \phi(x) &= [T_{L,0}(x), T_{L,1}(x), \ldots, T_{L,N}(x)]^T, \end{aligned} \tag{2.8}$$

then the derivative of the vector $\phi(x)$ can be expressed by

$$\frac{\mathrm{d}\phi(x)}{\mathrm{d}x} = \mathbf{D}^{(1)} \phi(x), \tag{2.9}$$

where $\mathbf{D}^{(1)}$ is the $(N+1) \times (N+1)$ operational matrix of derivative given by

$$\mathbf{D}^{(1)} = (d_{ij}) = \begin{cases} \dfrac{4i}{\epsilon_j L}, & j = 0, 1, \ldots, \; i = j + k, \; \begin{cases} k = 1, 3, 5, \ldots, N, & \text{if } N \text{ is odd}, \\ k = 1, 3, 5, \ldots, N-1, & \text{if } N \text{ is even}, \end{cases} \\ 0, & otherwise, \end{cases}$$

for example for even N, we have

$$\mathbf{D}^{(1)} = \frac{2}{L} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 3 & 0 & 6 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 8 & 0 & 8 & 0 & \cdots & 0 & 0 \\ 5 & 0 & 10 & 0 & 10 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ N-1 & 0 & 2N-2 & 0 & 2N-2 & \cdots & 0 & 0 \\ 0 & 2N & 0 & 2N & 0 & \cdots & 2N & 0 \end{pmatrix}.$$

3. COM for fractional derivatives

The main objective of this section is to generalize the COM of derivatives for the fractional calculus. By using Eq. (2.9), it is clear that

$$\frac{\mathrm{d}^n \phi(x)}{\mathrm{d}x^n} = (\mathbf{D}^{(1)})^n \phi(x), \tag{3.1}$$

where $n \in N$ and the superscript, in $\mathbf{D}^{(1)}$, denotes matrix powers. Thus

$$\mathbf{D}^{(n)} = (\mathbf{D}^{(1)})^n, \quad n = 1, 2, \ldots. \tag{3.2}$$

Lemma 3.1. *Let $T_{L,i}(x)$ be a shifted Chebyshev polynomial; then*

$$D^{\nu} T_{L,i}(x) = 0, \quad i = 0, 1, \ldots, \lceil \nu \rceil - 1, \; \nu > 0. \tag{3.3}$$

Proof. This lemma can be easily proved by making use of relation (2.2) and (2.4). □

Theorem 3.2. *Let $\phi(x)$ be the shifted Chebyshev vector defined in Eq. (2.8) and suppose $\nu > 0$; then*

$$D^{\nu}\phi(x) \simeq \mathbf{D}^{(\nu)}\phi(x), \tag{3.4}$$

where $\mathbf{D}^{(\nu)}$ is the $(N+1)\times(N+1)$ COM of derivatives of order ν in the Caputo sense and is defined as follows:

$$\mathbf{D}^{(\nu)} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ S_{\nu}(\lceil\nu\rceil,0) & S_{\nu}(\lceil\nu\rceil,1) & S_{\nu}(\lceil\nu\rceil,2) & \cdots & S_{\nu}(\lceil\nu\rceil,N) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ S_{\nu}(i,0) & S_{\nu}(i,1) & S_{\nu}(i,2) & \cdots & S_{\nu}(i,N) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ S_{\nu}(N,0) & S_{\nu}(N,1) & S_{\nu}(N,2) & \cdots & S_{\nu}(N,N) \end{pmatrix} \tag{3.5}$$

where

$$S_{\nu}(i,j) = \sum_{k=\lceil\nu\rceil}^{i} \frac{(-1)^{i-k}\, 2i(i+k-1)!\, \Gamma\left(k-\nu+\frac{1}{2}\right)}{\epsilon_j\, L^{\nu}\, \Gamma\left(k+\frac{1}{2}\right)(i-k)!\, \Gamma(k-\nu-j+1)\, \Gamma(k+j-\nu+1)}.$$

Note that in $\mathbf{D}^{(\nu)}$, the first $\lceil\nu\rceil$ rows are all zero.

Proof. The analytic form of the shifted Chebyshev polynomials $T_{L,i}(x)$ of degree i is given by (2.4), Using Eqs. (2.2) and (2.4), we have

$$\begin{aligned} D^{\nu}T_{L,i}(x) &= i\sum_{k=0}^{i}(-1)^{i-k}\frac{(i+k-1)!\,2^{2k}}{(i-k)!(2k)!\,L^{k}}\,D^{\nu}x^{k} \\ &= i\sum_{k=\lceil\nu\rceil}^{i}(-1)^{i-k}\frac{(i+k-1)!\,2^{2k}\,k!}{(i-k)!(2k)!\,L^{k}\,\Gamma(k-\nu+1)}x^{k-\nu}, \quad i=\lceil\nu\rceil,\lceil\nu\rceil+1,\ldots,N. \end{aligned} \tag{3.6}$$

Now, approximating $x^{k-\nu}$ by $(N+1)$ terms of shifted Chebyshev series, we have

$$x^{k-\nu} = \sum_{j=0}^{N} b_{kj}T_{L,j}(x), \tag{3.7}$$

where b_{kj} is given from (2.6) with $u(x) = x^{k-\nu}$, and

$$b_{kj} = \begin{cases} \dfrac{1}{\sqrt{\pi}}\dfrac{L^{k-\nu}\,\Gamma\left(k-\nu+\frac{1}{2}\right)}{\Gamma(k-\nu+1)}, & j=0, \\ \dfrac{j\,L^{k-\nu}}{\sqrt{\pi}}\displaystyle\sum_{r=0}^{j}(-1)^{j-r}\frac{(j+r-1)!\,2^{2r+1}\,\Gamma\left(k+r-\nu+\frac{1}{2}\right)}{(j-r)!(2r)!\,\Gamma(k+r-\nu+1)}, & j=1,2,\ldots,N. \end{cases} \tag{3.8}$$

Employing Eqs. (3.6)–(3.8), we get

$$D^{\nu}T_{L,i}(x) = \sum_{j=0}^{N} S_{\nu}(i,j)T_{L,j}(x), \quad i=\lceil\nu\rceil,\lceil\nu\rceil+1,\ldots,N, \tag{3.9}$$

where $S_{\nu}(i,j) = \sum_{k=\lceil\nu\rceil}^{i}\theta_{ijk}$, and

$$\theta_{ijk} = \begin{cases} \dfrac{i(-1)^{i-k}(i+k-1)!\,2^{2k}\,k!\,\Gamma\left(k-\nu+\frac{1}{2}\right)}{L^{\nu}(i-k)!(2k)!\,\sqrt{\pi}\,(\Gamma(k-\nu+1))^{2}}, & j=0, \\ \dfrac{(-1)^{i-k}\,i\,j(i+k-1)!\,2^{2k+1}\,k!}{L^{\nu}(i-k)!\,(2k)!\,\Gamma(k-\nu+1)\,\sqrt{\pi}} & \\ \times\displaystyle\sum_{r=0}^{j}\frac{(-1)^{j-r}\,(j+r-1)!\,2^{2r}\,\Gamma\left(k+r-\nu+\frac{1}{2}\right)}{(j-r)!(2r)!\,\Gamma(k+r-\nu+1)}, & j=1,2,\ldots. \end{cases}$$

After some lengthy manipulation, $\theta_{i,j,k}$ may be put in the form

$$\theta_{ijk} = \frac{(-1)^{i-k}\, 2i(i+k-1)!\, \Gamma\left(k-\nu+\frac{1}{2}\right)}{\epsilon_j\, L^{\nu}\, \Gamma\left(k+\frac{1}{2}\right)(i-k)!\, \Gamma(k-\nu-j+1)\, \Gamma(k+j-\nu+1)}, \quad j = 0, 1, \ldots, N. \tag{3.10}$$

Rewriting Eq. (3.9) as a vector form, we have

$$D^{\nu} T_{L,i}(x) \simeq \Big[S_{\nu}(i,0), S_{\nu}(i,1), S_{\nu}(i,2), \ldots, S_{\nu}(i,N)\Big]\phi(x), \quad i = \lceil\nu\rceil, \lceil\nu\rceil+1, \ldots, N. \tag{3.11}$$

Also, according to Lemma 3.1, we can write

$$D^{\nu} T_{L,i}(x) \simeq \Big[0, 0, 0, \ldots, 0\Big]\phi(x), \quad i = 0, 1, \ldots, \lceil\nu\rceil - 1. \tag{3.12}$$

A combination of Eqs. (3.11) and (3.12) leads to the desired result. □

Remark. If $\nu = n \in N$, then Theorem 3.2 gives the same result as Eq. (3.1).

4. Applications of spectral methods based on COM for FDEs

In this section, in order to show the fundamental importance of COM of fractional derivatives, we apply it to solve multi-order FDEs. For the existence and uniqueness and continuous dependence of the solution to the problem, see [24].

4.1. Linear multi-order initial FDEs

Consider the linear FDE

$$D^{\nu} u(x) = \sum_{j=1}^{k} \gamma_j D^{\beta_j} u(x) + \gamma_{k+1} u(x) + g(x), \quad \text{in } I = (0, L), \tag{4.1}$$

with initial conditions

$$u^{(i)}(0) = d_i, \quad i = 0, \ldots, m-1, \tag{4.2}$$

where γ_j $(j = 1, \ldots, k+1)$ are real constant coefficients and also $m-1 < \nu \le m,\ 0 < \beta_1 < \beta_2 < \cdots < \beta_k < \nu$. Moreover $D^{\nu}u(x) \equiv u^{(\nu)}(x)$ denotes the Caputo fractional derivative of order ν for $u(x)$, the values of d_i $(i = 0, \ldots, m-1)$ describe the initial state of $u(x)$, and $g(x)$ is a given source function.

To solve the initial value problem (4.1)–(4.2), we approximate $u(x)$ and $g(x)$ by the shifted Chebyshev polynomials as

$$u(x) \simeq \sum_{i=0}^{N} c_i T_{L,i}(x) = C^T \phi(x), \tag{4.3}$$

$$g(x) \simeq \sum_{i=0}^{N} g_i T_{L,i}(x) = G^T \phi(x), \tag{4.4}$$

where vector $G = [g_0, \ldots, g_N]^T$ is known, but $C = [c_0, \ldots, c_N]^T$ is an unknown vector.

By using Theorem 3.2 (relation (3.4)) and (4.3), we get

$$D^{\nu} u(x) \simeq C^T D^{\nu} \phi(x) \simeq C^T \mathbf{D}^{(\nu)} \phi(x), \tag{4.5}$$

$$D^{\beta_j} u(x) \simeq C^T D^{\beta_j} \phi(x) \simeq C^T \mathbf{D}^{(\beta_j)} \phi(x), \quad j = 1, \ldots, k. \tag{4.6}$$

Employing Eqs. (4.3)–(4.6), the residual $R_N(x)$ for Eq. (4.1) can be written as

$$R_N(x) = \left(C^T \mathbf{D}^{(\nu)} - C^T \sum_{j=1}^{k} \gamma_j \mathbf{D}^{(\beta_j)} - \gamma_{k+1} C^T - G^T\right)\phi(x). \tag{4.7}$$

As in a typical tau method (see [14,13]), we generate $(N-m+1)$ linear equations by applying

$$\langle R_N(x), T_{L,j}(x)\rangle = \int_0^L R_N(x) T_{L,j}(x) \mathrm{d}x = 0, \quad j = 0, 1, \ldots, N-m. \tag{4.8}$$

Also, by substituting Eqs. (3.1) and (4.3) in Eq. (4.2), we get

$$u^{(i)}(0) = C^T \mathbf{D}^{(i)} \phi(0) = d_i, \quad i = 0, 1, \ldots, m-1. \tag{4.9}$$

Eqs. (4.8) and (4.9) generate $(N-m+1)$ and (m) set of linear equations, respectively. These linear equations can be solved for unknown coefficients of the vector C. Consequently, $u(x)$ given in Eq. (4.3) can be calculated, which gives a solution of Eq. (4.1) with the initial conditions (4.2).

4.2. Treatment of nonhomogeneous boundary conditions

To solve Eq. (4.1) with respect to the following boundary conditions (for m is even),

$$u^{(i)}(0) = a_i, \qquad u^{(i)}(L) = b_i, \quad i = 0, 1, \ldots, \frac{m}{2} - 1. \tag{4.10}$$

We apply the same technique described in Section 4.1, but the (m) set of linear equations resulting from (4.9) is changed to be obtained from

$$u^{(i)}(0) = C^T \mathbf{D}^{(i)} \phi(0) = a_i, \qquad u^{(i)}(L) = C^T \mathbf{D}^{(i)} \phi(L) = b_i, \quad i = 0, 1, \ldots, \frac{m}{2} - 1. \tag{4.11}$$

Eqs. (4.8) and (4.11) generate $(N + 1)$ system of linear equations. This system can be solved to determine the unknown coefficients of the vector C.

4.3. Nonlinear multi-order FDEs

Extension of COM for nonlinear multi-order FDEs are treated using the shifted Chebyshev collocation method.

4.3.1. Initial value problem

Let us consider the following nonlinear FDE

$$D^{\nu} u(x) = F(x, u(x), D^{\beta_1} u(x), \ldots, D^{\beta_k} u(x)), \tag{4.12}$$

with initial conditions (4.2), where F can be nonlinear in general.

In order to use COM for this problem, we first approximate $u(x)$, $D^{\nu} u(x)$ and $D^{\beta_j} u(x)$ $(j = 1, \ldots, k)$ as Eqs. (4.3), (4.5) and (4.6), respectively. By substituting these equations in Eq. (4.12), we get

$$C^T \mathbf{D}^{(\nu)} \phi(x) \simeq F(x, C^T \phi(x), C^T \mathbf{D}^{(\beta_1)} \phi(x), \ldots, C^T \mathbf{D}^{(\beta_k)} \phi(x)). \tag{4.13}$$

Also, by substituting Eqs. (3.1) and (4.3) in Eq. (4.2), we obtain

$$u^{(i)}(0) = C^T \mathbf{D}^{(i)} \phi(0) = d_i, \quad i = 0, 1, \ldots, m - 1. \tag{4.14}$$

Eq. (4.13) is satisfied exactly at the collocation points $x_{L,N-m+1,k}$, $k = 0, 1, \ldots, N - m$. In other words, we have to collocate Eq. (4.13) at the $(N - m + 1)$ shifted Chebyshev roots $x_{L,N-m+1,k}$. These equations together with Eq. (4.14) generate $(N + 1)$ nonlinear equations, which can be solved using Newton's iterative method. Consequently, the approximate solution $u(x)$ can be obtained (for more details, see [18,13]).

4.3.2. Boundary value problem

Consider the nonlinear FDE (4.12) with boundary conditions (4.10). We apply the same technique described in Section 4.3.1, but Eq. (4.14) shall be changed to be (4.11). After using the collocation method with the aid of COM for fractional derivatives at the $(N - m + 1)$ nodes, we have a system of $(N + 1)$ nonlinear algebraic equations, which can be solved using Newton's iterative method.

5. Numerical results

To illustrate the effectiveness of the proposed methods in the present paper, several test examples are carried out in this section. Comparisons of the results obtained by the present methods with that obtained by other methods reveal that the present methods are very effective and convenient.

Example 1. As the first example, we consider the equation (see [25]):

$$D^2 u(x) + D^{\frac{1}{2}} u(x) + u(x) = x^2 + 2 + \frac{2.6666666667}{\Gamma(0.5)} x^{1.5}, \qquad u(0) = 0,$$
$$u'(0) = 0, \quad x \in [0, L], \tag{5.1}$$

whose exact solution is given by $u(x) = x^2$.

Ford and Connolly [25] applied three alternative decomposition methods for the approximate solution of Eq. (5.1) using the Caputo form of the fractional derivative. Methods 1 and 2 produce different systems of equations. In the case of method 1, the system may be of quite high dimension. Method 2 keeps the dimension of the system reasonably small and independent of the precise orders in the equation. However, method 3 decomposes the multi-term equation into a system of fractional equations of varying orders. Regarding problem (5.1), in [25], the best result is achieved with 512 steps and the maximum absolute errors are 7.0×10^{-5}, 1.74×10^{-5} and 1.7×10^{-5} by using method 1, method 2 and method 3, respectively. Our method is more accurate than the three decomposition methods [25]; see Table 2 in [25].

By applying the technique described in Section 4.1 with $N = 2$, we may write the approximate solution and the right hand side in the forms

$$u(x) = \sum_{i=0}^{2} c_i T_{L,i}(x) = C^T \phi(x)$$

$$g(x) \simeq \sum_{i=0}^{2} g_i T_{L,i}(x) = G^T \phi(x).$$

Here, we have

$$\mathbf{D}^{(2)} = \frac{1}{L^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 16 & 0 & 0 \end{pmatrix}, \qquad \mathbf{D}^{\left(\frac{1}{2}\right)} = \frac{8}{\pi\sqrt{\pi L}}\begin{pmatrix} 0 & 0 & 0 \\ 1 & \frac{2}{3} & \frac{-2}{15} \\ \frac{-4}{9} & \frac{8}{5} & \frac{8}{7} \end{pmatrix}, \qquad G = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}.$$

Therefore, using Eq. (4.8) we obtain

$$c_0 + \frac{8}{\pi\sqrt{\pi L}} c_1 + \left(\frac{16}{L^2} - \frac{32}{9\pi\sqrt{\pi L}}\right) c_2 - g_0 = 0. \tag{5.2}$$

Now, by applying Eq. (4.9), we have

$$\begin{aligned} C^T \phi(0) &= c_0 - c_1 + c_2 = 0, \\ C^T \mathbf{D}^{(1)} \phi(0) &= \frac{2}{L} c_1 - \frac{8}{L} c_2 = 0. \end{aligned} \tag{5.3}$$

Finally, by solving linear system of three equations, (5.2)–(5.3), we obtain

$$c_0 = \frac{3L^2}{8}, \qquad c_1 = \frac{L^2}{2}, \qquad c_2 = \frac{L^2}{8}.$$

Therefore, we can write

$$u(x) = \left(\frac{3L^2}{8}, \quad \frac{L^2}{2}, \quad \frac{L^2}{8}\right)\begin{pmatrix} 1 \\ \frac{2x}{L} - 1 \\ \frac{8x^2}{L^2} - \frac{8x}{L} + 1 \end{pmatrix} = x^2,$$

which is the exact solution.

Example 2. Consider the following boundary Bagely–Torvik equation; see [26].

$$D^2 u(x) + D^{\frac{3}{2}} u(x) + u(x) = g(x), \qquad u(0) = 0, \qquad u(L) = L^2, \quad x \in [0, L], \tag{5.4}$$

where $g(x) = x^2 + 2 + 4\sqrt{\frac{x}{\pi}}$ and the exact solution is $u(x) = x^2$.

Now, we can apply our technique described in Section 4.2 in Eq. (5.4) with $N = 2$.
Then, the 3 unknown coefficients will be in the form

$$c_0 = \frac{3L^2}{8}, \qquad c_1 = \frac{L^2}{2}, \qquad c_2 = \frac{L^2}{8}.$$

Thus, we can write

$$u(x) = \sum_{i=0}^{3} c_i T_{L,i}(x) = x^2.$$

Numerical results will not be presented since the exact solution is obtained.

Example 3. Consider the following boundary Bagely–Torvik equation

$$D^{\frac{5}{2}} u(x) + D^2 u(x) - 2D^{\frac{1}{2}} u(x) + 4u(x) = g(x), \qquad u(0) = 0, \qquad u'(0) = 0, \quad x \in [0, 10],$$

where $g(x) = 4x^9 + \frac{131\,027}{12\,155} x^{\frac{17}{2}} + 72x^7 + \frac{49\,152}{143\sqrt{\pi}} x^{\frac{13}{2}}$ and the exact solution is $u(x) = x^9$.

Table 5.1
Maximum absolute error with various choices of N.

N	6	8	9	10	12	14
Our method	2.54	0.043	0.00	0.00	0.00	0.00

Table 5.2
Maximum absolute error for $N = 4, 8, 12, 16$.

N	α	Our method	α	Our method
4		4.4×10^{-5}		2.6×10^{-2}
8		4.8×10^{-11}		2.9×10^{-6}
12	1	9.7×10^{-18}	π	5.7×10^{-11}
16		6.2×10^{-25}		3.5×10^{-16}

In Table 5.1, we display maximum absolute error, using tau spectral method based on COM for various choices of N. It is noticed that our method reaches the exact solution at $N = 9$.

Example 4. Consider the initial value problem

$$u^{\frac{3}{2}}(x) + 7u^{\frac{1}{4}}(x) = f(x), \quad \text{in } I = (0, 1), \qquad u(0) = 1, \qquad u'(0) = 0, \tag{5.5}$$

with an exact solution $u(x) = \cos(\alpha x)$.

Table 5.2 lists the maximum absolute error using tau method based on COM with two choices of α and various choices of N. It is noticed that only a small number of shifted Chebyshev polynomials is needed to obtain a satisfactory result.

Example 5. We next consider the following nonlinear initial value problem; see [27].

$$D^3u(x) + D^{\frac{5}{2}}u(x) + u^2(x) = x^4, \qquad u(0) = u'(0) = 0, \qquad u''(0) = 2, \quad x \in [0, L].$$

The exact solution of this problem is $u(x) = x^2$.

By applying the technique described in Section 4.3.1 with $N = 3$, we approximate the solution as

$$u(x) = \sum_{i=0}^{3} c_i T_{L,i}(x) = C^T \phi(x).$$

Here, we have

$$\mathbf{D}^{(1)} = \frac{1}{L}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 6 & 0 & 12 & 0 \end{pmatrix}, \qquad \mathbf{D}^{(2)} = \frac{1}{L^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 16 & 0 & 0 & 0 \\ 0 & 96 & 0 & 0 \end{pmatrix},$$

$$\mathbf{D}^{(3)} = \frac{1}{L^3}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 192 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{D}^{(\frac{5}{2})} = \frac{1}{L(\pi L)^{\frac{3}{2}}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 768 & 512 & \frac{-512}{5} & \frac{1536}{35} \end{pmatrix}, \qquad C = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Using Eq. (4.12) yields

$$C^T\mathbf{D}^{(3)}\phi(x) + C^T\mathbf{D}^{(\frac{5}{2})}\phi(x) + [\phi(x)]^2 - x^4 = 0. \tag{5.6}$$

Now collocate Eq. (5.6) at the first root of $T_{L,4}(x)$, i.e.

$$x_0 = \frac{L}{2} + \frac{L}{2}\cos\left(\frac{\pi}{8}\right).$$

Also, by using Eq. (4.9), we get

$$\begin{aligned} C^T\phi(0) &= c_0 - c_1 + c_2 - c_3 = 0, \\ C^T\mathbf{D}^{(1)}\phi(0) &= \frac{2}{L}c_1 - \frac{8}{L}c_2 + \frac{18}{L}c_3 = 0, \\ C^T\mathbf{D}^{(2)}\phi(0) &= \frac{16}{L^2}c_2 - \frac{96}{L^2}c_3 = 2. \end{aligned} \tag{5.7}$$

By solving Eqs. (5.6) and (5.7) yields

$$c_0 = \frac{3L^2}{8}, \qquad c_1 = \frac{L^2}{2}, \qquad c_2 = \frac{L^2}{8}, \qquad c_3 = 0.$$

Therefore

$$u(x) = \begin{pmatrix} \frac{3L^2}{8}, & \frac{L^2}{2}, & \frac{L^2}{8}, & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{2x}{L} - 1 \\ \frac{8x^2}{L^2} - \frac{8x}{L} + 1 \\ \frac{32x^3}{L^3} - \frac{48x^2}{L^2} + \frac{8x}{L} - 1 \end{pmatrix} = x^2.$$

Numerical results will not be presented since the exact solution is obtained.

6. Conclusion

We derived a general formulation for the Chebyshev operational matrix of fractional derivatives, which is used to approximate the numerical solution of a class of fractional differential equations. Our approach was based on the shifted Chebyshev tau and collocation methods. The fractional derivatives are described in the Caputo sense because the Caputo fractional derivative allows traditional initial and boundary conditions to be included in the formulation of the problem. The results given in the previous section demonstrate the good accuracy of these algorithms. Moreover, only a small number of shifted Chebyshev polynomials is needed to obtain a satisfactory result.

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