Note
An improvement of the Dulmage–Mendelsohn theorem
Jian Shen
Department of Mathematics, University of Science and Technology of China, Hefei, Anhui 230026, China
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Abstract

An $n \times n$ nonnegative matrix $A$ is called primitive if for some positive integer $k$, every entry in the matrix $A^k$ is positive or, in notation, $A^k \gg 0$. The exponent of primitivity of $A$ is defined to be $\gamma(A) = \min\{k \in \mathbb{Z}_+ : A^k \gg 0\}$, where $\mathbb{Z}_+$ denotes the set of positive integers. The well known Dulmage–Mendelsohn theorem is that $\gamma(A) \leq n + s(n - 2)$, where $s$ is the shortest circuit in $D(A)$, the directed graph of $A$. In this paper we prove that $\gamma(A) \leq D + 1 + s(D - 1)$, where $D$ is the diameter of $D(A)$.

1. Some notations

Associated with an $n \times n$ nonnegative matrix $A = (a_{ij})$ we shall consider its directed graph $D(A)$ which consists of a set $V$ of $n$ vertices, labeled conveniently, $1, 2, \ldots, n$ and a set of directed edges $E$ with a directed edge from vertex $i$ to vertex $j$ if and only if $a_{ij} \neq 0$. We shall use the notations $i \xrightarrow{d} j$ and $i \xrightarrow{d^+} j$ to denote, respectively, that there is a path of length $d$ from vertex $i$ to vertex $j$ and that there is no path of length $d$ linking vertex $i$ to vertex $j$. $D$ and $D_\kappa$ denote, respectively, the diameter of $D(A)$ and the diameter of $D(A^k)$. $d(i, j)$ and $d_\kappa(i, j)$ are, respectively, the distance from vertex $i$ to vertex $j$ in $D(A)$ and in $D(A^k)$. Note the following fact:

$A^k \gg 0 \iff i \xrightarrow{k} j \quad \text{for every } i, j \in V(D(A)).$

2. Main result

Lemma 1 (Shen [6]). If $A$ is primitive, then $D_\kappa \leq D$ for any integer $k \geq 1$.
Theorem 1. Suppose $A$ is primitive and $(a_0, a_1, a_2, \ldots, a_{s-1}, a_0)$ is an $s$-circuit in $D(A)$, where $s \geq 2$. Let $k = D - [(D + 1 - \varepsilon)/s]$, where $\varepsilon = 1$ if $s = 2$ and $\varepsilon = 0$ if $s > 2$, $b \in V(D(A))$ and $1 \leq m \leq s - 1$. Then either of the following two cases is true:

1. $a_0 \xrightarrow{k_1} b$;
2. $a_m \xrightarrow{k_2} b$.

Proof. We suppose otherwise: $d_\varepsilon(a_0, b) = k_1 \geq k + 1$ and $d_\varepsilon(a_m, b) = k_2 \geq k + 1$ since $a_0$ and $a_m$ each have a loop in $D(A)$ and without loss of generality we can suppose $m \leq s - m$.

First we will prove $k_1 = k_2$. Since $d_\varepsilon(a_0, b) = k_1 \geq k + 1$, we have $a_0 \xrightarrow{k_1} b$ in $D(A)$, then there exist $A_0, A_1, \ldots, A_{s-1} \in V(D(A))$ such that $a_0 = A_0 \xrightarrow{D+1} A_1 \xrightarrow{D+1} \cdots \xrightarrow{D+1} A_{s-1} \xrightarrow{k_1-(D+1)(s-1)} b$. Let $x_i = d(A_{i-1}, A_i)$ for any $1 \leq i \leq s - 1$, so $x_i \leq D$ and $A_0 \xrightarrow{x_1} A_1 \xrightarrow{x_2} A_2 \xrightarrow{\cdots} \xrightarrow{x_{s-1}} A_{s-1}$. We consider the set $T(x_1, x_2, \ldots, x_{s-1}) = \{(D + 1 - x_1); l = 1, 2, \ldots, s - 1\}$. Suppose either of the following two cases is true:

Case 1: $\sum_{i=1}^{s-1} (D + 1 - x_i) \equiv 0 \pmod{s}$ for some $l_1$.
Case 2: $\sum_{i=1}^{s-1} (D + 1 - x_i) \equiv \sum_{i=1}^{s-1} (D + 1 - x_i) \pmod{s}$ for some $l_1 \neq l_2$.

Then there exist some $l_1 \leq l_2$ such that $\sum_{i=1}^{l_1} (D + 1 - x_i) \equiv 0 \pmod{s}$. We consider the path: $a_0 = A_0 \xrightarrow{D+1} A_1 \xrightarrow{D+1} \cdots \xrightarrow{D+1} A_{l_1-1} \xrightarrow{x_{l_1}} A_{l_1} \xrightarrow{x_{l_1+1}} \cdots \xrightarrow{x_{s-1}} A_{s-1} \xrightarrow{k_1-(D+1)(s-1)} b$, the length of which, less than $k_1 s$, but is $k_1 s - \sum_{i=1}^{l_1} (D + 1 - x_i) \equiv 0 \pmod{s}$, is a contradiction to $d_\varepsilon(a_0, b) = k_1$. So neither case 1 nor case 2 is true, i.e., $T(x_1, x_2, \ldots, x_{s-1}) \equiv \{1, 2, \ldots, s - 1\} \pmod{s}$.

Similarly, we have $T(x_1, x_2, \ldots, x_{s-1}) \equiv \{1, 2, \ldots, s - 1\} \pmod{s}$ by comparing the elements of these two sets. Since $T(x_1, x_2, \ldots, x_{s-1}) \equiv \{l(D + 1 - x); l = 1, 2, \ldots, s - 1\} \pmod{s}$, it is easy to see that $\gcd(D + 1 - x, s) = 1$. Therefore for any $1 \leq i \leq s - 1$, there exists some $l_1 \leq l_2 \leq s - 1$, such that $\{l(D + 1 - x) \equiv i \pmod{s}\}$. We consider the path: $a_0 = A_0 \xrightarrow{D+1} A_1 \xrightarrow{D+1} \cdots \xrightarrow{D+1} A_{l_1-1} \xrightarrow{x_{l_1}} A_{l_1} \xrightarrow{x_{l_1+1}} \cdots \xrightarrow{x_{s-1}} A_{s-1} \xrightarrow{k_1-(D+1)(s-1)} b$, the length of which, less than $k_1 s, k_1 s - \sum_{i=1}^{l_1} (D + 1 - x_i) \equiv -l(D + 1 - x) \equiv i \pmod{s}$, so we have $a_0 \xrightarrow{(k_1-1)s+i} b$ for any $1 \leq i \leq s$, but $a_0 \xrightarrow{(k_1-1)s+i} b$ as $d_\varepsilon(a_0, b) = k_1$. Since $a_m \xrightarrow{s-m} a_0$, then $a_m \xrightarrow{s-m+(k_1-1)s+i} b$ for any $1 \leq i \leq s$ and we have $a_m \xrightarrow{k_2} b$ by letting $i = m$. Therefore $d_\varepsilon(a_m, b) = k_2 \geq k_1$. Similarly we also have $k_1 \leq k_2$ and hence $k_1 = k_2$.

Since $a_m \xrightarrow{s-m} a_0 \xrightarrow{(k_1-1)s+i} b$, there exist $B_0, B_1, \ldots, B_{s-1} \in V(D(A))$ such that $a_m \xrightarrow{s-m} a_0 \xrightarrow{(k_1-1)s+i} b$,

$$a_m \xrightarrow{s-m} a_0 \xrightarrow{(k_1-1)s+i} b$$,

where $B_0 = a_m$ if $e = 0$ and $B_0 = a_0$ if $e = 1$. Let $y_i = d(B_{i-1}, B_i)$ for any $1 \leq i \leq s - 1$. Note that $d_\varepsilon(a_m, b) = k_2 = k_1$. As above we can prove $y_1 \equiv y_2 \equiv \cdots \equiv y_{s-1} \equiv y(s-1) \equiv 0 \pmod{s}$, $\gcd(D + 1 - y, s) = 1$ and $a_m \xrightarrow{(k_1-1)s+i} b$ for any $1 \leq i \leq s$.}

Case 1: $e = 1$, then $s = 2, m = 1$ and $B_0 = a_0$, i.e., $a_0 \xrightarrow{(k_1-1)s+i} b$ for any $1 \leq i \leq s - 1$, so $a_0 \xrightarrow{(k_1-1)s+i} b$ by letting $i = 1$, contradicting the fact $d_\varepsilon(a_0, b) = k_1$. 


Case 2: $e = 0$, then $s > 2$ and $B_0 = a_m$. Since $m \leq s - m$, then $m \leq s/2 < s - 1$. Since there exists some $l$, $1 \leq l \leq s - 1$, such that $l(D + 1 - y) \equiv m \pmod{s}$, similarly as above we have $a_m^{k_s - 1} \equiv b$. Since $a_0 \xrightarrow{m} a_m$, then $a_0 \xrightarrow{m + k_s - l(D + 1 - y)} b$. Since $d_A(a_0, b) = k_1$ and $m + k_1 s - l(D + 1 - y) \equiv 0 \pmod{s}$, we have $m + k_1 s - l(D + 1 - y) \equiv 0 \pmod{s}$, i.e., $l \leq l(D + 1 - y) \equiv m < s - 1$. We consider the path: $a_0 \xrightarrow{D+1-s-m} B_1 \xrightarrow{y_1} B_2 \xrightarrow{y_1} \cdots \xrightarrow{y_i} B_i \xrightarrow{y_i} B_{i+1} \xrightarrow{D+1} B_{i+2} \xrightarrow{D+1} \cdots B_{i+1-k_1} \xrightarrow{k_1-s-(D+1)(s-1)} b$, the length of which, less than $k_1 s$, is $D + 1 - (s - m) + \sum_{i=2}^{i+1} y_i + (D + 1)$ $(s - l - 2) + k_1 s - (D + 1)(s - 1) = (k_1 - 1) s + m - \sum_{i=2}^{i+1} (D + 1 - y_i) \equiv m - l (D + 1 - y) \equiv 0 \pmod{s}$, a contradiction to $d_A(a_0, b) = k_1$.

Combining case 1 and case 2, we have either $d_A(a_0, b) = k_1 \leq k$ or $d_A(a_m, b) = k_2 \leq k$, and we have either $a_0 \xrightarrow{ks} b$ or $a_m \xrightarrow{ks} b$ in $D(A)$ since $a_0, a_m$ each have a loop in $D(A^2)$.

**Theorem 2.** Suppose $A$ is primitive and $s$ is the shortest circuit in $D(A)$, then

$$\gamma(A) \leq D + 1 + s(D - 1).$$

**Proof.** $D = 1$ is trivial, so we suppose $D \geq 2$ and $(a_0, a_1, \ldots, a_{s-1}, a_0)$ is an $s$-circuit in $D(A)$. It is sufficient to prove $b \xrightarrow{D+1+s(D-1)} c$ for any $b, c \in V(D(A))$. Let $x_i = d(b, a_i)$ for any $0 \leq i \leq s - 1$. We consider the following two cases:

Case 1: There exists some $i$ such that $x_i \leq D + 1 - s$. By Lemma 1 we have $D_0 \leq D$, it is easy to prove $a_i \xrightarrow{sD} c$ since $a_i$ lies in an $s$-circuit in $D(A)$. Then $b \xrightarrow{x_i} a_i \xrightarrow{sD} c$, i.e., $b \xrightarrow{sD+x} c$. Since $c$ is arbitrary, $b \xrightarrow{D+1+s(D-1)} c$.

Case 2: $D + 2 - s \leq x_i \leq D$ for any $0 \leq i \leq s - 1$. Then $s \geq 2$ and there exist $i \neq j$, $0 \leq i, j \leq s - 1$, such that $x_i = x_j$ since $X_i$ only have $s$ values. By Theorem 1 we have either $b \xrightarrow{x_i} a_i \xrightarrow{k_1} c$ or $b \xrightarrow{x_i} a_j \xrightarrow{ks} c$, where $k = D - \lceil(D + 1 - \varepsilon)/s\rceil$, so we always have $b \xrightarrow{x_i+ks} c$ since $x_i = x_j$. Since $x_i + ks \leq D + s(D - \lceil(D + 1 - \varepsilon)/s\rceil)] \leq D + s(D - 1)$, we again have $b \xrightarrow{D+1+s(D-1)} c$.

**Postscript.** In a recent paper [7] we have proved $\gamma(A) \leq D^2 + 1$ for any primitive matrix $A$; so combining Theorem 2 we have $\gamma(A) \leq D + 1 + (D - 1) \min(s, D)$.

**References**


