Fixed-Point Theorems and Morse's Lemma for Lipschitzian Functions

JEAN-MARC BONNISSEAU AND BERNARD CORNET

CORE, 34, Voie du Roman Pays, 1348 Louvain-la-Neuve, Belgium

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We prove a fixed-point theorem for set-valued mappings defined on a nonempty compact subset $X$ of $\mathbb{R}^n$ which can be represented by inequality constraints, i.e., $X = \{ x \in \mathbb{R}^n \mid f(x) \leq 0 \}$, $f$ locally Lipschitzian and satisfying a nondegeneracy assumption outside of $X$. This class of sets extends significantly the class of convex, compact sets with a nonempty interior.

Topological properties of such sets $X$ are proved (continuous deformation retract of a ball, acyclicity) as a consequence of a generalization of Morse's lemma for Lipschitzian real-valued function defined on $\mathbb{R}^n$ a result also of interest for itself.

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1. INTRODUCTION

In this paper, we prove a fixed-point theorem for set-valued mappings $S$ defined on a compact subset of a finite dimensional Euclidean space $E$, i.e., $S$ is a mapping from $X$ to the set of all nonempty subsets of $E$. Here we shall consider the class $\mathcal{C}$ of compact subsets $X$ of $E$ which can be defined by inequality constraints, i.e., $X = \{ x \in E \mid f(x) \leq 0 \}$ where $f$ is a real-valued locally Lipschitzian function which satisfies a nondegeneracy assumption outside the interior of $X$. This class of sets, which is of particular importance for applications extends significantly the class of convex, compact subsets of $E$ with a nonempty interior; in this case, one can take $f = \gamma_x - 1$, where $\gamma_x$ is the gauge function of $X$ (with respect to some element in its interior), which is a real-valued locally Lipschitzian function defined on $E$ but is not continuously differentiable, in general (see [Rockafellar, 1970]). Other examples of nonconvex sets in the class $\mathcal{C}$ can be found in [Bonnisseau–Cornet, forthcoming]. The proof of our fixed-point theorem will be a consequence of (i) Kakutani's Theorem and (ii) a topological property of the sets $X$ in the class $\mathcal{C}$, which will be shown to be continuous deformation retracts of a ball $B$ in $E$, hence acyclic sets.
This last topological property will be proved as a consequence of our
Theorem 2.4, also of interest for itself, which provides a nonsmooth
generalization of the "noncritical neck principle" of Morse's theory
[Milnor, 1963, Theorem 3.1, or Schwartz, 1969, Theorem 4.7.1], namely,
to the case of locally Lipschitzian real valued functions defined on E, a
finite dimensional Euclidean space. As in the smooth case, Theorem 2.4
states that if \( a, b \), are real numbers, \( b > a \), if \( f^{-1}(a, b) \) is compact
and contains no (generalized) critical points of \( f \), then the set
\( M_a = \{ x \in E \mid f(x) \leq a \} \) is a continuous deformation retract of
\( M_b = \{ x \in E \mid f(x) \leq b \} \). Here, we call (generalized) critical point \( x \) of \( f \), every
element \( x \) in \( E \) such that 0 belongs to \( \partial f(x) \), the generalized gradient in the
sense of [Clarke, 1975]. Furthermore, we shall also prove a property of the
retraction mapping on the neighborhood of the set \( \{ x \in E \mid f(x) = a \} \),
which will be of fundamental use in our fixed-point theorem.

Unlike fixed-point theorems for set-valued mappings defined on an
acyclic domain, such as [Eilenberg-Montgomery, 1946], we shall use in
this paper only techniques from analysis and mainly from the theory of
generalized gradients of Lipschitzian functions, also called nonsmooth
analysis (see [Clarke, 1983; Rockafellar, 1982]). In the end of the introduc-
tion, we recall some general notations and definitions used throughout the
paper. Our main results are stated precisely in the next section. The proof
of the nonsmooth Morse's Lemma is given in Section 3 and the proof of
our fixed-point theorem is given in Section 4.

In all the following, \( E \) will be a finite dimensional Euclidean space and,
if \( x, y \) are elements in \( E \), we denote by \( x \cdot y \) the scalar product of \( E \), and
by \( \|x\| = (x \cdot x)^{1/2} \) its Euclidean norm. If \( A \) and \( B \) are subsets of \( E \), we let
\( \text{cl} A, \text{int} A, \partial A, \text{co} A \) be, respectively, the closure, the interior, the bound-
ary, the convex hull of \( A \), we let \( A^0 = \{ v \in E \mid \text{for all} \ a \in A, v \cdot a \leq 0 \} \) be the
negative polar cone of \( A \) and we let \( \partial A = \{ a \mid \text{for all} \ a \in A, b \in B \} \); if \( A \) is
nonempty, for \( x \) in \( E \), we let \( \inf x \cdot A = \inf \{ x \cdot a \mid a \in A \} \), \( \sup x \cdot A = \sup \{ x \cdot a \mid a \in A \} \), \( d_A(x) = \inf \{ \| x - a \| \mid a \in A \} \) and, for \( r \geq 0 \), \( B(A, r) = \{ x \in E \mid d_A(x) \leq r \} \). A set-valued mapping \( S \), from \( X \), a nonempty subset of
\( E \), to \( E \) is said to be upper semicontinuous (usc) if, for every open subset
\( V \) of \( E \), the set \( \{ x \in X \mid S(x) \subset V \} \) is open in \( X \) (for its relative topology).
\( S \) is said to be upper hemicontinuous (uhc) if, for every \( y \) in \( E \), and every
real number \( y \), the set \( \{ x \in X \mid \sup y \cdot S(x) < y \} \) is open in \( X \) (for its relative
topology) or, equivalently, if for every \( y \) in \( E \), the function \( x \rightarrow \sup y \cdot S(x) \),
from \( X \) to \( \mathbb{R} \cup \{ + \infty \} \) is upper semicontinuous. Clearly, if \( S \) is usc, it is
uhc. Furthermore, if \( S \) is upper demicontinuous (udc) in the sense of [Ky
Fan, 1972], then \( S \) is also uhc and the three notions, usc, udc, uhc, are
equivalent under the further assumption that there exists a compact subset
\( K \) of \( E \) such that, for all \( x \) in \( X \), \( S(x) \subset K \).

We now let \( V \) be an open subset of \( E \) and \( f: V \rightarrow \mathbb{R} \) be a locally
Lipschitzian function, i.e., for all $x$ in $V$, there exists $k > 0$ and an open neighborhood $N$ of $x$ such that, for all $x_1, x_2$ in $N$, $|f(x_1) - f(x_2)| \leq k \|x_1 - x_2\|$. Following [Clarke, 1975], we define the generalized gradient $\partial f(x)$ of $f$ at an element $x$ in $V$, to be the set

$$\partial f(x) = \operatorname{cl} \operatorname{co} \{\lim_{x_q \to x} \nabla f(x_q) | \forall \{x_q\} \subset \Omega_f, \{x_q\} \to x\},$$

where $\Omega_f$ denotes the set on which $f$ is differentiable and $\nabla f(x_q)$ denotes the gradient of $f$ at $x_q$. From Rademacher's Theorem, a locally Lipschitzian real-valued function $f$ is almost everywhere (for the Lebesgue measure) differentiable, so that $\Omega_f$ is dense in $V$ and $\partial f(x)$ is nonempty, for every $x$. We further recall that the set-valued mapping $x \to \partial f(x)$ is USC with nonempty, convex, compact values and that, if $f$ is continuously differentiable on an open neighborhood $N$ of $x_0$, then $\partial f(x) = \{\nabla f(x)\}$ on $N$. For these definitions and properties, we refer to [Clarke, 1983].

2. STATEMENT OF THE RESULTS

We first state our fixed-point theorem in which we shall assume that the subset $X$ of $E$ satisfies:

**Assumption 2.1.** $X$ is a nonempty compact subset of $E$ such that

$$X = \{x \in E | f(x) \leq 0\},$$

where

$$f: E \to \mathbb{R}$$

is locally Lipschitzian.

If $b = \max \{f(x) | x \in \text{co } X\}$, then $f^{-1}([0, b])$ is compact and

$$0 \notin \partial f(x), \quad \text{for all } x \in f^{-1}([0, b]).$$

**Theorem 2.1.** Let $X$ satisfy Assumption 2.1 and let $T$ be an uhc set-valued mapping, from $X$ to $E$, such that, for all $x$ in $X$, $T(x)$ is a nonempty, closed, convex subset of $E$ and one of the two following conditions is satisfied:

(I) for all $x \in \partial X$, $T(x) \cap \{x\} + \partial f(x)^0 \neq \emptyset$, or

(O) for all $x \in \partial X$, $T(x) \cap \{x\} - \partial f(x)^0 \neq \emptyset$.

Then, there exists $x^*$ in $X$ such that $x^* \in T(x^*)$.

The conditions (I) and (O), which are respectively called "inward" and "outward," are discussed hereafter and a geometric characterization of them is given when $X$ is further convex. We first give an equivalent
reformulation of Theorem 2.1 in terms of the existence of critical points of a set-valued mapping.

**Theorem 2.2.** Let $X$ satisfy Assumption 2.1 and let $S$ be an uhc set-valued mapping from $X$ to $E$ such that, for all $x$ in $E$, $S(x)$ is a nonempty, closed, convex subset of $E$ and

$$(T.C.) \quad \text{for all } x \in \partial X, \ S(x) \cap \partial f(x)^0 \neq \emptyset.$$ 

Then, there exists $x^*$ in $X$ such that $0 \in S(x^*)$.

To prove that Theorems 2.1 and 2.2 are equivalent we note first that the set-valued mapping $Id$, from $X$ to $E$, defined by $Id(x) = x$ is clearly uhc and second that, if $F, G$ are two uhc set-valued mappings from $X$ to $E$, and $\lambda, \mu$ are real numbers, then the set-valued mapping $\lambda F + \mu G$, from $X$ to $E$, defined by $[\lambda F + \mu G](x) = \lambda F(x) + \mu G(x)$, is also uhc. Hence, Theorem 2.1 is deduced from Theorem 2.2 by letting $S = T - Id$ or $S = Id - T$ and Theorem 2.2 is deduced from Theorem 2.1 by letting $T = S + Id$.

From the above theorems, in the convex case, we can deduce the:

**Corollary 2.3.** Let $X$ be a convex, compact, subset of $E$ with a nonempty interior and let $T$ be an uhc set-valued mapping from $X$ to $E$ such that, for all $x$ in $X$, $T(x)$ is a nonempty, closed, convex subset of $E$ and one of the following conditions is satisfied:

1. for all $x \in \partial X$, $T(x) \cap \text{cl}\{x + \lambda(y - x) | \lambda > 0, y \in X\} \neq \emptyset$, or
2. for all $x \in \partial X$, $T(x) \cap \text{cl}\{x - \lambda(y - x) | \lambda > 0, y \in X\} \neq \emptyset$, or
3. for all $x \in \partial X$, $T(x) \subset X$.

Then, there exists $x^*$ in $X$ such that $x^* \in T(x^*)$.

The above Corollary 2.3, has been proven in the more general setting of real Hausdorff locally convex topological vector spaces, under different variants [Bergman–Halpern, 1968; Browder, 1968; Fan, 1969; Halpern, 1970; Reich, 1978, 1979]; the above statement for uhc set-valued mapping is taken from [Cornet, 1975]. Hence our fixed-point Theorem 2.1 provides a generalization of fixed-point theorems for inward and outward set-valued mappings, defined on nonconvex domains. We further point out that fixed-point theorems for set-valued mappings defined on an acyclic subset $X$ of $E$, such as in [Eilenberg–Montgomery, 1946] do assume tangential conditions of the type (K) and even the stronger condition of Kakutani that, for all $x \in X$, $T(x) \subset X$. Finally we mention two possible extensions of our results which are worthy of interest. The first one concerns the generalization to an infinite dimensional setting and the second one concerns
the weakening of the assumption that \( S(x) \) is a convex subset of \( E \), for example, by the one of acyclicity.

We now furnish a proof of Corollary 2.3.

**Proof of Corollary 2.3.** Without any loss of generality, we assume that \( 0 \in \text{int} X \) and we let \( \gamma_X \) be the gauge function of \( X \), i.e., \( \gamma_X(x) = \inf \{ \lambda > 0 | x \in \lambda X \} \), for \( x \in E \). From [Rockafellar, 1970], \( X = \{ x \in E | \gamma_X(x) \leq 1 \} \), \( \partial X = \{ x \in E | \gamma_X(x) = 1 \} \), \( \gamma_X \) is convex, locally Lipschitzian, and, for \( x \in E \), the three following conditions are equivalent: (i) \( 0 \in \partial \gamma_X(x) \), (ii) \( \gamma_X(x) = \min \{ \gamma_X(y) | y \in E \} \), and (iii) \( x = 0 \). Consequently, Assumption (2.1) is satisfied by \( \gamma = \gamma_X - 1 \), \( a = 0 \), and \( b = 0 \). But, from [Clarke, 1983, Theorem 2.4.7], for all \( x \in \partial X = \{ x \in E | \gamma_X(x) = 1 \} \), one has \( \partial \gamma_X(x)^0 = T_X(x) \), Clarke's tangent cone to \( X \) at \( x \), which equates the set \( \text{cl} \{ \lambda (v - x) | \lambda > 0, v \in X \} \) since \( X \) is convex [Clarke, 1983]. Hence, from above, Corollary 2.3 is a direct consequence of Theorem 2.1 (noting that assumption (K) is stronger than (I)).

**Remark 2.1.** If \( X \) is no longer assumed to be convex, under Assumption (2.1), one clearly has \( \partial X \subset \{ x \in E | f(x) = 0 \} \) (and in fact the equality holds, see Section 4) so that, by [Clarke, 1983, Theorem 2.4.7], one has the inclusion

\[
\partial f(x)^0 \subset T_X(x), \quad \text{for all} \quad x \in \partial X,
\]

where \( T_X(x) \) denotes Clarke's tangent cone to \( X \) at \( x \), but the equality does not hold in general. However, the equality holds if we further assume that \( f \) is convex (as in Corollary 2.3) or that \( f \) is continuously differentiable and, more generally, if \( f \) is tangentially regular in the sense of [Clarke, 1983]. Under this latter assumption, the "inward" and "outward" conditions (I) and (O) of Theorem 2.1 then have a natural geometric interpretation.

We now come to our second result on Morse's Lemma for a locally Lipschitzian function \( f: E \to \mathbb{R} \). If \( a, b \) are real numbers, \( a < b \), we let

\[
M_a = \{ x \in E | f(x) \leq a \}, \quad M_b = \{ x \in E | f(x) \leq b \},
\]

\[
M_{ab} = f^{-1}([a, b]).
\]

**Theorem 2.4.** Let \( f: E \to \mathbb{R} \) be a locally Lipschitzian function, let \( a, b \) be real numbers, \( a < b \) such that (i) the set \( M_{ab} \) is nonempty and compact, (ii) there exists an uhc set-valued mapping \( \delta \) from \( E \) to \( E \) such that, for all \( x \) in \( E \), \( \delta(x) \) is nonempty, closed and convex, and (iii) for all \( x \) in \( M_{ab} \), \( \partial f(x) \subset \delta(x) \) and 0 does not belong to \( \delta(x) \).

(a) There exists a neighborhood \( M \) of \( M_b \), and a locally Lipschitzian mapping \( r \) from \( M \) to \( M_a \) such that
(i) \( r(x) = x \), for all \( x \in M_a \).

(ii) \( f(r(x)) = a \) (or equivalently \( r(x) \in \partial M_a \), for all \( x \in M \setminus M_a \).

(b) There exists \( \varepsilon \in (0, b - a) \) such that for all \( x \) in \( f^{-1}((a, a + \varepsilon]) \), for all \( y \) in \( f^{-1}([a, a + \varepsilon]) \), with \( r(x) = r(y) \), then

\[
0 < (x - r(x)) \cdot \delta, \quad \text{for all} \quad \delta \in \delta(y).
\]

The above theorem, together with Theorem 2.5, generalizes parts of the "noncritical neck principle" as discussed by Milnor [1963, Theorem 3.1] and Schwartz [1969, Theorems 4.7.1 and 4.7.2], who assume that \( f \) is a \( C^p \) function (\( p \geq 2 \)) that, for all \( x \), \( \delta(x) = df(x) \) (which, in this case, reduces to \( \nabla f(x) \) the gradient of \( f \) at \( x \)) but allow \( E \) to be an (infinite dimensional) complete Riemannian manifold.

**Remark 2.2.** An important (and first) example of a set-valued mapping \( \delta \) satisfying the above assumption (ii) of the theorem is clearly given by the generalized gradient \( \partial f(\cdot) \). The introduction of the set-valued mapping \( \delta \) in Theorem 2.4 is not only done for a matter of generality, it will be of fundamental use later in the proof of our fixed-point theorem. Other examples of set-valued mappings \( \delta \) are naturally introduced, as in [Bonnissieu-Cornet, forthcoming], as follows. We let \( f_i : E \to \mathbb{R} \) (\( i = 1, \ldots, n \)) be locally Lipschitzian functions and we let \( f : E \to \mathbb{R} \) be defined by \( f(x) = \sum_{i=1}^{n} f_i(x) \), then \( f \) is locally Lipschitzian and, by [Clarke, 1983, Theorem 2.3.3], one has

\[
\partial f(x) \subset \partial f_1(x) + \cdots + \partial f_n(x) =: \delta(x), \quad \text{for all} \quad x \in E,
\]

but the equality does not hold, in general. It holds, however, when the \( f_i \)'s are assumed to be tangentially regular in the sense of [Clarke, 1983]. Then, clearly, the set-valued mapping \( \delta \), from \( E \) to \( E \), defined above, is usc and, for all \( x \) in \( E \), \( \delta(x) \) is nonempty, convex, compact, and contains \( \partial f(x) \). Many other examples can be constructed as the one above, among which we only point out the following one. Let \( f_i \) (\( i = 1, \ldots, n \)) be as above and let \( F : E \to \mathbb{R} \) be defined by \( F(x) = \sup \{ f_i(x), \ i = 1, \ldots, n \} \), then. \( F \) is locally Lipschitzian and, by [Clarke, 1983, Theorem 2.3.12],

\[
\partial F(x) \subset \text{co} \bigcup_{i \in I(x)} \partial f_i(x) =: A(x), \quad \text{for all} \quad x \in E,
\]

where

\[
I(x) = \{ i \in \{ 1, \ldots, n \} \mid f_i(x) = \sup \{ f_j(x) \mid j = \{ 1, \ldots, n \} \} \}.
\]

Then, again, the set-valued \( A \), from \( E \) to \( E \), defined above, is usc and, for all \( x \) in \( E \), \( A(x) \) is nonempty, convex, compact, and contains \( \partial F(x) \).
Remark 2.3. Condition (b) of Theorem 2.4 gives a behavior of the mapping \( r \), outside \( M_a \), and on a neighborhood of the set \( f^{-1}(\{a\}) \) which will be of fundamental use in the proof of our fixed-point theorems. We note that condition (b) implies, in particular, that

\[
  r(x) - x \in \text{int}([\delta(r(x))]^0), \quad \text{for all } x \in f^{-1}((a, a + \varepsilon]),
\]

which also implies the more geometric condition,

\[
  r(x) - x \in \text{int} T_{M_a}(r(x)), \quad \text{for all } x \in f^{-1}((a, a + \varepsilon]),
\]

where \( T_{M_a}(r(x)) \) denotes Clarke's tangent cone to \( M_a \) at \( r(x) \) (see [Clarke, 1983]). This last property is a direct consequence of the first one and the fact that, under the assumption of Theorem 2.4, by [Clarke, 1983, Theorem 2.4.7], \( \delta(r(x))^0 \subset \partial f(r(x))^0 \subset T_{M_a}(r(x)) \).

We end this section by the following theorem which will not be used here but is worth pointing out. The proof of it is left to the reader and can be obtained along the same lines as [Schwartz, 1969, Corollary 4.721].

**Theorem 2.5.** Under the assumptions of Theorem 2.4, there exists a (homotopy) continuous mapping \( H: E \times [0, 1] \to E \) such that

(i) for every \( t \in [0, 1] \), \( H_t: x \to H(t, x) \) is a homeomorphism from \( E \) onto \( E \);

(ii) for all \( x \in E \), \( H_0(x) = x \);

(iii) \( H_1(M_a) = H_1(M_b) \).

We note that Theorem 2.5 says, in particular, that the sets \( M_a \) and \( M_b \) are homeomorphic.

### 3. Proof of Theorem 2.4.

In all this section, we posit the assumptions of Theorem 2.4. We first prepare the proof of the theorem by three lemmas.

**Lemma 3.1.** There exists a bounded open neighborhood \( \Omega \) of \( M_{ab} \), two positive real numbers \( \alpha, \eta \) and an infinitely differentiable mapping \( F: \Omega \to E \) such that

\[
  0 < \alpha < \inf F(y) \cdot \delta(x), \quad \text{for all } x, y \in \Omega, \|x - y\| \leq \eta.
\]

**Proof.** We first claim that there exist \( \varepsilon > 0 \) and \( \alpha > 0 \) such that

for all \( x \in B(M_{ab}, \varepsilon) \), \( A(x, \alpha) := \{ u \in E | \alpha < \inf u \cdot \delta(x) \} \neq \emptyset \).
Suppose it is not true. Then there exists a sequence \( \{x_q\} \subset E \) such that, for all \( q, x_q \in B(M_{ab}, 1/q) \) and the set \( A(x_q, 1/q) \) is empty. Since \( \{x_q\} \subset B(M_{ab}, 1) \), a compact set, without any loss of generality, we can suppose that \( \{x_q\} \) converges to some element \( \bar{x} \), which clearly belongs to \( M_{ab} \). Since 0 does not belong to \( \delta(\bar{x}) \), a nonempty, closed, convex subset of \( E \), by a separation theorem, there exist \( \bar{u} \in E \) and a real number \( \gamma \) such that \( 0 < \gamma < \inf \bar{u} \cdot \delta(\bar{x}) \). Since \( \delta \) is uhc, the set \( \{x \in E \mid \gamma < \inf \bar{u} \cdot \delta(x)\} \) is open and contains \( \bar{x} \). Hence, for \( q \) large enough, \( 1/q < \gamma < \inf \bar{u} \cdot \delta(x_q) \) which contradicts that \( A(x_q, 1/q) \) is empty and ends the proof of the claim.

We now let \( \varepsilon > 0 \) and \( \alpha > 0 \) be defined as in the above claim, we let \( \Omega = \text{int} B(M_{ab}, \varepsilon) \), and we define, for \( u \in E \),

\[
A^{-1}(u) := \{ x \in E \mid u \in A(x, \alpha) \} = \{ x \in E \mid \alpha < \inf u \cdot \delta(x) \},
\]

which is clearly an open subset of \( E \) since \( \delta \) is uhc. From the above claim, \( B(M_{ab}, \varepsilon) \subset \bigcup_{u \in E} A^{-1}(u) \) and, from the compactness of \( B(M_{ab}, \varepsilon) \), there exists a finite number of elements \( \{u_1, \ldots, u_n\} \subset E \) such that \( B(M_{ab}, \varepsilon) \subset \bigcup_{i=1}^n A^{-1}(u_i) =: V \). Hence, there exists a \( C^\infty \) partition of the unity \( \{\lambda_i \mid i = 1, \ldots, n\} \) subordinate to the open covering \( \{A^{-1}(u_i) \mid i = 1, \ldots, n\} \) of \( V \) (an open set), hence a \( C^\infty \) submanifold of \( E \), see, for example, [Hirsch, 1976, Theorem 2.1], i.e., for all \( i \), (a) \( \lambda_i : V \to [0, 1] \) is \( C^\infty \), (b) \( V \cap \text{cl}\{x \in V \mid \lambda_i(x) > 0\} \subset A^{-1}(u_i) \) (where, we recall, that "cl" denotes the closure in \( E \)) and (c) for all \( x \in V \), \( \sum_{i=1}^n \lambda_i(x) = 1 \). We now define, for all \( i \), the set

\[
K_i = B(M_{ab}, \varepsilon) \cap \text{cl}\{x \in V \mid \lambda_i(x) > 0\},
\]

which is clearly compact and is also a subset of \( A^{-1}(u_i) \) since \( B(M_{ab}, \varepsilon) \subset V \). Clearly, there exists \( \eta > 0 \), such that

\[
B(K_i, \eta) \subset A^{-1}(u_i), \quad \text{for} \quad i = 1, \ldots, n.
\]

We now let \( F : V \to E \) be defined by

\[
F(y) = \sum_{i=1}^n \lambda_i(y) u_i, \quad \text{for} \quad y \in V,
\]

which is clearly a \( C^\infty \) mapping. We end the proof of the lemma by showing that, for all \( x, y \) in \( \Omega \) such that \( \|x - y\| \leq \eta \) one has \( F(y) \in A(x, \alpha) \). Indeed, let \( I(y) = \{i = 1, \ldots, n \mid \lambda_i(y) > 0\} \). Then, for \( i \in I(y) \), by the above condition (b), \( y \) belongs to \( K_i \) and, since \( \|x - y\| \leq \eta \), \( x \in B(K_i, \eta) \subset A^{-1}(u_i) \), hence \( u_i \in A(x, \alpha) \). Noting that \( A(x, \alpha) \) is a convex subset, since for \( i \in I(y) \), \( \lambda_i(y) > 0 \) and \( \sum_{i \in I(y)} \lambda_i(y) = 1 \), one deduces that \( F(x) = \sum_{i \in I(y)} \lambda_i(y) u_i \in A(x, \alpha) \). This ends the proof of the lemma.
Let \( F: \Omega \to E \) be defined as in Lemma 3.1, for \( x \in \Omega \), we denote by \( \varphi(\cdot, x) \) the maximal solution of

\[
(D.E.) \quad \dot{x}(t) = F(x(t)), \quad x(0) = x,
\]

and we let \( I(x) \subset \mathbb{R} \) be the maximal interval of definition of \( \varphi(\cdot, x) \). Hence, for all \( x \in \Omega \), \( \varphi(\cdot, x) \) is a \( C^\infty \) mapping from \( I(x) \) to \( \Omega \) (see, for example, [Hirsch–Smale, 1974]). In fact, we shall only use later the fact that \( F \) is locally Lipschitzian (since it is \( C^\infty \)) which implies that, for all \( x \in \Omega \), there exists a unique solution of (D.E.) (Cauchy Lipschitz's Theorem).

**Lemma 3.2.** There exists a compact neighborhood \( K \) of \( M_{ab} \) and a Lipschitzian function \( \tau: K \to \mathbb{R} \) such that,

\[\tau(x) \in I(x), \quad |\tau(x)| \leq |a - f(x)|/\alpha,\]

and

\[f(\varphi(\tau(x), x)) = a, \quad \text{for all } x \in K.\]

**Proof.** Let \( \Omega \) and \( \alpha > 0 \) be as in Lemma 3.1, we first claim that, for all \( x \in \Omega \),

\[f(\varphi(t_2, x)) - f(\varphi(t_1, x)) \geq \alpha(t_2 - t_1),\]

for all \( t_1, t_2 \in I(x), \ t_2 \geq t_1 \).

Indeed, for all \( x \in \Omega \), the function \( t \mapsto g(t) = f(\varphi(t, x)) \) is locally Lipschitzian on \( I(x) \). Hence, for all \( t_1, t_2 \in I(x), \ t_2 > t_1 \), by the mean value theorem for locally Lipschitzian functions [Lebourg, 1975], see also [Clarke, 1983, Theorem 2.3.7] one has

\[g(t_2) - g(t_1) \in \{u \cdot (t_2 - t_1) \mid u \in \partial g(t), \ t \in (t_1, t_2)\}.\]

Clearly, to end the proof of the claim, it suffices to show that, for all \( t \in I(x) \), \( \partial g(t) \subset [\alpha, +\infty) \). Indeed, let \( t \in I(x) \), by the chain rule [Clarke, 1983, Theorem 2.3.10], using the fact that \( \varphi(\cdot, x) \) is the solution of (D.E.), one has

\[\partial g(t) \subset \left\{u \cdot \frac{\partial \varphi}{\partial t}(t, x) \mid u \in \partial f(\varphi(t, x))\right\}
\subset \left\{u \cdot F(\varphi(t, x)) \mid u \in \delta(\varphi(t, x))\right\},\]

and, by Lemma 3.1, \( \partial g(t) \subset [\alpha, +\infty) \), which ends the proof of the claim.

We now choose real numbers \( a', b' \) such that \( a' < a \leq b < b' \) and the set
$K = M_{a,b'} \cap \Omega$ is closed in $E$, hence compact. We prove that such a choice of $a', b'$ is possible by contraposition. Suppose that, for all integer $q$, the set $K_q = \{ x \in E \mid a - 1/q \leq f(x) \leq b + 1/q \} \cap \Omega$ is not closed, then there exists a sequence $\{ x_q^\alpha \mid n \in \mathbb{N} \} \subset K_q$, converging to some element $x_q^\alpha \notin K_q$. From the continuity of $f$, for all $q$, $a - 1/q \leq f(x_q^\alpha) \leq b + 1/q$, hence $x_q^\alpha \in \partial \Omega$. Since $\Omega$ is bounded, without any loss of generality, we can assume that the sequence $\{ x_q^\alpha \}$ converges to some element $\tilde{x}$ in $\partial \Omega$, but, from above, when $q \to \infty$, one gets $a \leq f(\tilde{x}) \leq b$, hence $\tilde{x} \in \Omega$, a contradiction.

Let $K$ be defined as above, we now show that, for all $x \in K$, there exists a unique real number $\tau(x)$ in $I(x)$ such that $|\tau(x)| \leq |a - f(x)|/\alpha$ and $f(\varphi(\tau(x), x)) = a$. Clearly, the uniqueness part and the fact that $|\tau(x)| \leq |a - f(x)|/\alpha$ (if it exists) are both obtained as a direct consequence of the above claim. We now prove the existence part by contraposition. Suppose that, for some $x$ in $K$, and all $t$ in $Z(x)$, one has $f(\varphi(t, x)) \neq a$. We further suppose that $f(x) > a$, (and the case $f(x) < a$, left to the reader, is proved similarly). From the above claim, the function $t \mapsto f(\varphi(t, x))$ is increasing in $I(x)$, hence for all $t \in I(x) \cap (-\infty, 0]$, $\varphi(t, x) \in K = M_{a,b'} \cap \Omega$ since $a < f(\varphi(t, x)) \leq f(\varphi(0, x)) = f(x) \leq b'$. Since $K$ is compact, the solution $\varphi(t, x)$ can be extended on $(-\infty, 0]$, or, in other words, $(-\infty, 0] \subset I(x)$, see, for example, [Hirsh-Smale, 1974]. But, for $t < [a' - f(x)]/\alpha$, by the above claim, $f(\varphi(t, x)) \leq f(\varphi(0, x)) + \alpha t = f(x) + \alpha t < a'$, which contradicts that $\varphi(t, x) \in M_{a,b'} \cap \Omega$.

We now claim that the restriction of $\tau: K \to \mathbb{R}$ to the compact set $K_+ = \{ x \in K \mid f(x) \geq a \} = \{ x \in E \mid a \leq f(x) \leq b' \} \cap \Omega$ is Lipschitzian. We first note that, since $f$ and $F$ are locally Lipschitzian, respectively on $E$ and $\Omega$, and, since $K$ is compact and included in $\Omega$, then, in fact, $f$ and $F$ are Lipschitzian on $K$ with constants $k_2 > 0$ and $k_1 > 0$, respectively. We let $x_1, x_2$ be in $K_+$ and we let $t_1 = \tau(x_1)$ and $t_2 = \tau(x_2)$, then $t_1 \leq 0$ and $t_2 \leq 0$. Without any loss of generality, we can suppose that $t_1 \leq t_2 \leq 0$ so that $|\tau(x_1) - \tau(x_2)| = t_2 - t_1$. From the above claim, since $f(\varphi(t_1, x_1)) = f(\varphi(t_2, x_2)) = a$, one gets

\[
\alpha |\tau(x_2) - \tau(x_1)| \leq f(\varphi(t_2, x_1)) - f(\varphi(t_1, x_1)) = f(\varphi(t_2, x_1)) - f(\varphi(t_2, x_2)).
\]

Clearly, $\varphi(t_2, x_2) \in M_{a,b'} \cap \Omega \subset K$ and also $\varphi(t_2, x_1) \in M_{a,b'} \cap \Omega \subset K$ since $t_1 \leq t_2$ and, by the above claim, $a = f(\varphi(t_1, x_1)) \leq f(\varphi(t_2, x_1)) \leq f(\varphi(0, x_1)) \leq b'$. Consequently,

\[
\alpha |\tau(x_2) - \tau(x_1)| \leq k_2 \| \varphi(t_2, x_1) - \varphi(t_2, x_2) \|
\]

and we end the proof of the claim by showing that

\[
\| \varphi(t_2, x_1) - \varphi(t_2, x_2) \| \leq k \| x_1 - x_2 \|
\]
with
\[ k = \exp[k \cdot (b' - a)/x]. \]

Indeed, from the first claim of the lemma, since \( t_1 \leq t_2 \leq 0 \), for all \( t \in [t_2, 0] \), both \( \varphi(t, x_2) \) and \( \varphi(t, x_1) \) belong to \( M_{ab'} \cap \Omega \), and we let \( u(t) = \|\varphi(t, x_2) - \varphi(t, x_1)\|^2 \exp(2k_1 t) \). Then,
\[
\frac{d}{dt} u(t) = \exp(2k_1 t) \cdot 2[\varphi(t, x_2) - \varphi(t, x_1)] \cdot [F(\varphi(t, x_2)) - F(\varphi(t, x_1))]
\]
\[
+ \exp(2k_1 t) \cdot 2k_1 \|\varphi(t, x_2) - \varphi(t, x_1)\|^2
\]
\[
\geq \exp(2k_1 t) \|\varphi(t, x_2) - \varphi(t, x_1)\|^2 (-2k_1 + 2k_1) = 0.
\]

Hence the function \( u(t) \) is increasing and
\[
\|\varphi(t_2, x_2) - \varphi(t_2, x_1)\|^2
\]
\[
\leq \exp(-2k_1 t_2) \|\varphi(0, x_2) - \varphi(0, x_1)\|^2 \leq k^2 \|x_2 - x_1\|^2,
\]
which ends the proof of the claim.

Similarly, one shows that the restriction of \( \tau: K \to \mathbb{R} \) to the compact set \( K^- = \{x \in K \mid f(x) \leq a\} = \{x \in E \mid a' \leq f(x) \leq a\} \cap \Omega \) is also Lipschitzian. Consequently, the function \( \tau: K \to \mathbb{R} \) is continuous. Furthermore, \( \tau \) is Lipschitzian by the following lemma.

**Lemma 3.3.** Let \( E, F \) be two finite dimensional Euclidean spaces, let \( A_1, A_2 \) be two closed subsets of \( E \), let \( g_1: A_1 \to F, g_2: A_2 \to F \) be two locally Lipschitzian mappings such that, for all \( x \in A_1 \cap A_2 \), \( g_1(x) = g_2(x) \), and let \( g: A_1 \cup A_2 \to F \) be defined by \( g(x) = g_1(x) \) if \( x \in A_1 \) and \( g(x) = g_2(x) \) if \( x \in A_2 \).

Then, if \( A_1 \cap A_2 \subseteq \text{int}(A_1 \cup A_2) \), the mapping \( g \) is locally Lipschitzian.

The proof of the above lemma is straightforward. We now come to the proof of Theorem 2.4.

**Proof of Theorem 2.4.** Part (a). Let \( \Omega, a', b' \) be defined as above, we recall that \( a' < a \leq b \leq b' \) and that \( K_+ = \{x \in E \mid a \leq f(x) \leq b'\} \cap \Omega \). Then, we let \( M = \{x \in E \mid f(x) \leq a\} \cup \{x \in \Omega \mid a \leq f(x) \leq b'\} = M_a \cup K_+ \), hence \( M \) is a neighborhood of \( M_b \) and we define \( r: M \to M_a \) by
\[
r(x) = \varphi(\tau(x), x), \quad \text{if} \quad x \in K_+ = \{x \in \Omega \mid a \leq f(x) \leq b'\},
\]
\[
r(x) = x, \quad \text{if} \quad f(x) \leq a.
\]

From the definition of \( \tau(x) \), for all \( x \in K_+ \), \( f(r(x)) = f(\varphi(\tau(x), x)) = a \), so the proof of Part (a) will be complete if we show that \( r \) is locally Lipschitz-
zian. Since \( r|_{M_a} \) the restriction of \( r \) to \( M_a \), is clearly locally Lipschitzian and \( M_a \cap K_+ = \{ x \in E \mid f(x) = a \} \subset \text{int } M \), from Lemma 3.3, it suffices to show that \( r|_{K_+} \), the restriction of \( r \) to \( K_+ \) is locally Lipschitzian. But, if we let \( \text{Dom } \phi = \{(t, x) \in \mathbb{R} \times \Omega \mid t \in I(x) \} \), then \( \text{Dom } \phi \) is an open subset of \( \mathbb{R} \times \mathbb{E} \) and the mapping \( \phi: \text{Dom } \phi \rightarrow \Omega \) is \( C^\infty \) (see, for example, [Hirsh-Smale, 1974]) hence is locally Lipschitzian. Consequently, by Lemma 3.2, \( r|_{K_+} \) is also locally Lipschitzian. This ends the proof of Part (a).

Part (b). We first claim that there exists \( \varepsilon \in (0, b - a) \), such that, for all \( x, y \) in \( f^{-1}([a, a + \varepsilon]) \) with \( r(x) = r(y) \), then \( \| x - y \| < \eta \), where \( \eta \) is defined as in Lemma 3.1. Indeed, if the claim is not satisfied, there exist sequences \( \{ x_q \}, \{ y_q \} \) in \( E \) such that, for \( q \geq 1/(b - a) \), \( x_q \) and \( y_q \) belong to \( f^{-1}([a, a + 1/q]) \), \( r(x_q) = r(y_q) \) and \( \| x_q - y_q \| \geq \eta \). Since \( M_{ab} \) is compact, without any loss of generality, we can suppose that the sequence \( \{(x_q, y_q)\} \) converges to \( (\bar{x}, \bar{y}) \). Clearly, \( f(\bar{x}) = f(\bar{y}) = a \), \( r(\bar{x}) = r(\bar{y}) \), and \( \| \bar{x} - \bar{y} \| \geq \eta \). But, from Part (a), \( x = r(x) = r(j) = \bar{x} \), a contradiction, with \( \| x - j \| \leq q \). Now, let \( x \in f^{-1}([a, a + \varepsilon]) \), \( y \in f^{-1}([a, a + \varepsilon]) \) with \( r(x) = r(y) \) and let \( \delta \in \delta(y) \). We consider the mapping \( \psi \) from \([\tau(x), 0]\) to \( \mathbb{R} \) defined by \( \psi(t) = (\phi(0, x) - \phi(t, x)) \cdot \delta \). Clearly, \( \psi(0) = 0 \) and \( \psi(\tau(x)) = (x - r(x)) \cdot \delta \). Furthermore, \( \psi'(t) = -F(\phi(t, x)) \cdot \delta \) and, for all \( t \in [\tau(x), 0] \), \( \phi(t, x) \in f^{-1}([a, a + \varepsilon]) \) and \( r(\phi(t, x)) = r(x) \). From our choice of \( \varepsilon \), \( \| \phi(t, x) - y \| < \eta \), hence, from Lemma 3.1, \( \varepsilon > F(\phi(t, x)) \cdot \delta = \psi'(t) \). Consequently,

\[
(x - r(x)) \cdot \delta = \psi(\tau(x)) > -\varepsilon \tau(x) > 0.
\]

This ends the proof of Part (b) and the proof of Theorem 2.4.

4. PROOF OF THEOREM 2.2.

We prepare the proof of the theorem by a lemma.

**Lemma 4.1.** Under the assumptions of Theorem 2.2, \( \partial X = \{ x \in E \mid f(x) = 0 \} \).

**Proof.** By the continuity of \( f \), clearly \( \partial X \subset \{ x \in E \mid f(x) = 0 \} \). Conversely, let \( x \in E \), such that \( f(x) = 0 \). If \( x \not\in \partial X \), then \( x \in \text{int } X \) and, for all \( x' \in \text{int } X \), \( f(x') \leq f(x) = 0 \), hence \( f(x) = \max \{ f(x') \mid x' \in \text{int } X \} \). Hence, by [Clarke, 1983, Proposition 2.3.2], \( 0 \in \partial f(x) \), which contradicts Assumption 2.1.

**Proof of Theorem 2.2.** By contraposition. Let us suppose that, for all \( x \)
in $X$, 0 does not belong to $S(x)$. Then, by a separation theorem, for all $x \in X$, there exists $p$ in $E$ such that $\sup p \cdot S(x) < 0$. Hence,

$$X = \bigcup_{p \in E} V(p), \quad \text{where} \quad V(p) = \{x \in X \mid \sup p \cdot S(x) < 0\}.$$  

Since $S$ is uhc, for all $p$, $V(p)$ is an open subset of $X$ (for its relative topology) and, since $X$ is compact, there exists a finite subset \{\(p_1, \ldots, p_n\)\} in $E$ such that $X = \bigcup_{i=1}^n V(p_i)$. Let $\lambda_1, \ldots, \lambda_n$ be a continuous partition of unity subordinate to the open covering $V(p_i)$, i.e., for all $i$, (a) $\lambda_i: X \to [0, 1]$ is continuous, (b) $\text{cl}\{x \in X \mid \lambda_i(x) > 0\} \subset V(p_i)$, and (c) for all $x \in X$, $\sum \lambda_i(x) = 1$. We now define the mapping $p: X \to E$ by $p(x) = \sum \lambda_i(x) p_i$. Then, clearly, $p$ is continuous and we now claim that

$$\text{for all } x \in X, \quad \sup p(x) \cdot S(x) < 0,$$

$$\text{for all } x \in \partial X, \quad -p(x) \notin \bigcup_{\lambda \geq 0} \lambda \partial f(x).$$

To prove the first assertion, let $x \in X$, and let $I = \{i = 1, \ldots, n \mid \lambda_i(x) > 0\}$. For all $i \in I$, then $x \in V(p_i)$, hence,

$$\text{for all } i \in I, \quad \sup p_i \cdot S(x) < 0.$$

Multiplying each inequality by $\lambda_i(x)$ and summing up one gets

$$\sup p(x) \cdot S(x) = \sup \left[ \sum_{i \in I} \lambda_i(x) p_i \right] \cdot S(x) \leq \sum_{i \in I} \lambda_i(x) \sup p_i \cdot S(x) < 0,$$

which ends the proof of the first part of the claim. We prove the second part by contraposition. Suppose that there exist $x \in \partial X$ and $\lambda \geq 0$ such that $-p(x) \in \lambda \partial f(x)$. From assumption (T.C.) of Theorem 2.2, there exists $s \in S(x) \cap \partial f(x)^0$, hence $s \cdot (-p(x)) \leq 0$, which contradicts the first part of the claim.

We now define the set-valued mapping $\delta$, from $E$ to $E$, by

$$\delta(x) = p(x), \quad \text{if } f(x) < 0,$$

$$\delta(x) = \text{co} \left[ \{ p(x) \} \cup \partial f(x) \right], \quad \text{if } f(x) = 0,$$

$$\delta(x) = \partial f(x), \quad \text{if } f(x) > 0.$$

We now check that $f, a = 0, b$ and $\delta$, as defined above and in Assumption 2.1, satisfy the assumptions of Theorem 2.4. Clearly, $\delta$ is uhc, for all $x \in E$, $\delta(x)$ is a nonempty, convex, compact subset of $E$ and, for all $x \in M_{ab} = f^{-1}([a, b])$, $\partial f(x) \subset \delta(x)$. We now show that, for all $x \in M_{ab}$, 0 does not belong to $\delta(x)$. Suppose, on the contrary, that, for some
x \in M_{ab}, 0 \in \delta(x). Then, either \( f(x) > a \), and \( 0 \in \delta(x) = \partial f(x) \) contradicts Assumption 2.1, or \( f(x) = a \), and there exists \( t \in [0, 1] \) such that \( 0 \in tp(x) + (1 - t) \partial f(x) \). One then must have \( t > 0 \) since, by Assumption 2.1, \( 0 \) does not belong to \( \partial f(x) \). Hence \( -p(x) \in \bigcup_{\lambda \geq 0} \lambda \partial f(x) \) and \( x \in \partial X \), by Lemma 4.1. These two last assertions contradict the second part of the above claim.

Consequently, by Theorem 2.4, there exists an open neighborhood \( M \) of \( M_b \) and a continuous mapping \( r: M \to M_a \) such that, for all \( x \in M_a \), \( r(x) = x \) and which satisfies condition (b) of Theorem 2.4. We now define, for \( \alpha > 0 \), the set-valued mapping \( \phi_\alpha \), from \( M \) to \( E \), by

\[
\phi_\alpha(x) = r(x) - \alpha \delta(r(x)).
\]

Clearly, one can choose \( \alpha > 0 \) small enough so that

\[
C := B(\co X, \alpha) \subset M;
\]

\[
\phi_\alpha(x) \subset C, \quad \text{for all } x \in C;
\]

\[
\phi_\alpha(x) \subset \{ y \in E \mid f(y) < a + \epsilon \}, \quad \text{for all } x \in M;
\]

where \( \epsilon > 0 \) is defined as in Theorem 2.4b.

Consequently, from Kakutani’s Theorem, (4.1) and the fact that \( \phi_\alpha \) is usc, and, for all \( x \) in \( C \), \( \phi_\alpha(x) \) is nonempty, convex, compact subset of \( C \), one deduces that there exists \( x^* \) in \( C \), such that \( x^* \in \phi_\alpha(x^*) \) or, equivalently, there exists \( \delta^* \) in \( \delta(r(x^*)) \) such that \( x^* = r(x^*) - \alpha \delta^* \). We now distinguish the two cases: \( x^* \in X \) and \( x^* \notin X \). Suppose first that \( x^* \in X \), then \( r(x^*) = x^* \), hence \( 0 = \delta^* \in \delta(x^*) \); but we have shown above that, if \( f(x^*) \geq a \), then \( 0 \notin \delta(x^*) \); hence we must have \( f(x^*) < a \) but \( 0 \in \delta(x^*) = \{ p(x^*) \} \) contradicts our first claim that \( \sup p(x^*) \cdot S(x) < 0 \). We now suppose that \( x^* \notin X \); since \( x^* \in \phi_\alpha(x^*) \), by (4.1), one has \( x^* \in f^{-1}((a, a + \epsilon)] \) and, by condition (b) of Theorem 2.4, since \( \delta^* \in \delta(r(x^*)) \), one must have \( 0 < (x^* - r(x^*)) \cdot \delta^* = -\alpha \| \delta^* \|^2 \), a contradiction. This ends the proof of Theorem 2.2.

References


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**FIXED-POINT THEOREM**

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