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Letter to the Editor

A note on SOR(ω) splitting of an M-matrix[☆]

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Abstract

Recently, Wang and Huang (J. Comput. Appl. Math. 135 (2001) 325, Corollary 4.7) assumed that SOR(ω) splitting with $\omega \geqslant 1$ is a weak regular splitting of an M-matrix. In this note, we point out that SOR(ω) splitting with $\omega > 1$ can never be a weak regular splitting of an M-matrix. This shows that Chang's counterexample (J. Comput. Appl. Math. 167 (2004) 251, Example 1) is not an appropriate one since the SOR(ω) splitting with $\omega > 1$ given there is not a weak regular splitting.

MSC: 65F10

Keywords: SOR(ω) method; Spectral radius; M-matrix

1. SOR(ω) splitting

Consider a linear system of the form

$$Ax = b, \quad x, b \in \mathbb{R}^n, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is a large sparse M-matrix. A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called an *M-matrix* if $a_{ij} \leqslant 0$ for $i \neq j$ and $A^{-1} \geqslant 0$.

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Let

$$A = D - L - U,$$

where $D = \mathrm{diag}(A)$ which denotes a diagonal matrix whose diagonal part coincides with the diagonal part of A, L and U are strictly lower and strictly upper triangular matrices of A, respectively. For a relaxation parameter $\omega > 0$, SOR(ω) splitting of a matrix A is given by

$$A = \frac{1}{\omega}(D - \omega L) - \frac{1}{\omega}((1-\omega)D + \omega U). \tag{2}$$

The SOR(ω) method associated with SOR(ω) splitting (2) for solving the linear system (1) is described by

$$x_{k+1} = T_\omega x_k + \omega(D - \omega L)^{-1} b, \tag{3}$$

where $T_\omega = (D - \omega L)^{-1}((1-\omega)D + \omega U)$ is SOR(ω) iteration matrix.

For a matrix A, $\rho(A)$ denotes the *spectral radius* of A, and $A \geqslant 0$ ($A > 0$) denotes that all components of A are nonnegative (positive). A splitting $A = M - N$ is called a *weak regular splitting* if $M^{-1} \geqslant 0$ and $M^{-1}N \geqslant 0$. Recently, Wang and Huang [2] showed the following result.

Theorem 1.1 (*Wang and Huang [2], Corollary 4.7*). *Suppose that A is an M-matrix. If* SOR(ω) *splitting with $\omega \geqslant 1$ is a weak regular splitting, then*

$$\rho(T_\omega) \leqslant 1 - \omega + \omega\rho(B), \tag{4}$$

where $B = D^{-1}(L + U)$.

The following theorem shows that SOR(ω) splitting with $\omega > 1$ can never be a weak regular splitting of an M-matrix.

Theorem 1.2. *Suppose that A is an M-matrix. If $\omega > 1$, then the* SOR(ω) *splitting* (2) *is not a weak regular splitting of A.*

Proof. Let $M_\omega = (1/\omega)(D - \omega L)$ and $N_\omega = (1/\omega)((1-\omega)D + \omega U)$. Since A is an M-matrix, $L \geqslant 0$, $D \geqslant 0$ is nonsingular, and thus $M_\omega^{-1} \geqslant 0$. However, simple calculation shows that the $(1, 1)$-entry of $T_\omega = M_\omega^{-1} N_\omega$ is $1 - \omega$. It follows that SOR(ω) splitting $A = M_\omega - N_\omega$ is not a weak regular splitting of A for $\omega > 1$.

Theorem 1.2 shows that the spectral bound (4) for SOR(ω) iteration matrix is not in general valid unless $\omega = 1$, and that Chang's counterexample given in [1] is not an appropriate one since the SOR(ω) splitting is not a weak regular splitting of A for $\omega > 1$.

References

[1] D.-W. Chang, A note on the upper bound of the spectral radius for SOR iteration matrix, J. Comput. Appl. Math. 167 (2004) 251–253.
[2] C.-L. Wang, T.-Z. Huang, New convergence results for alternating methods, J. Comput. Appl. Math. 135 (2001) 325–333.