Higher order spectral shift

Ken Dykema 1, Anna Skripka ∗

Department of Mathematics, Texas A&M University, College Station, TX 77843-3368, USA

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Abstract

We construct higher order spectral shift functions, extending the perturbation theory results of M.G. Krein [M.G. Krein, On a trace formula in perturbation theory, Mat. Sb. 33 (1953) 597–626 (in Russian)] and L.S. Koplienko [L.S. Koplienko, Trace formula for perturbations of nonnuclear type, Sibirsk. Mat. Zh. 25 (1984) 62–71 (in Russian); translation in: Trace formula for nontrace-class perturbations, Siberian Math. J. 25 (1984) 735–743] on representations for the remainders of the first and second order Taylor-type approximations of operator functions. The higher order spectral shift functions represent the remainders of higher order Taylor-type approximations; they can be expressed recursively via the lower order (in particular, Krein’s and Koplienko’s) ones. We also obtain higher order spectral averaging formulas generalizing the Birman–Solomyak spectral averaging formula. The results are obtained in the semi-finite von Neumann algebra setting, with the perturbation taken in the Hilbert–Schmidt class of the algebra.

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1. Introduction

Let \( \mathcal{H} \) be a separable Hilbert space and \( B(\mathcal{H}) \) the algebra of bounded linear operators on \( \mathcal{H} \). Let \( \mathcal{M} \) be a semi-finite von Neumann algebra acting on \( \mathcal{H} \) and \( \tau \) a semi-finite normal faithful trace on \( \mathcal{M} \). We study how the value \( f(H_0) \) of a function \( f \) on a self-adjoint operator \( H_0 \) in \( \mathcal{M} \) changes under a perturbation \( V = V^* \in \mathcal{M} \) of the operator argument \( H_0 \). It is well known that
The value $f(H_0 + V)$ can be approximated by the Fréchet derivatives of the mapping $H^* = H \mapsto f(H)$ at point $H_0$.

**Theorem 1.1.** (Cf. [24, Theorem 1.43, Corollary 1.45].) Let $f : \mathbb{R} \to \mathbb{C}$ be a bounded function such that the mapping $H \mapsto f(H)$ defined on self-adjoint elements of $\mathcal{B}(\mathcal{H})$ is $p$ times continuously differentiable in the sense of Fréchet (and, hence, in the sense of Gâteaux). Let $H_0 = H_0^*$, $V = V^* \in \mathcal{B}(\mathcal{H})$ and denote

$$R_{p,H_0,V}(f) = f(H_0 + tV) - \sum_{j=0}^{p-1} \frac{1}{j!} \frac{d^j}{dt^j} f(H_0 + tV).$$

Then

$$R_{p,H_0,V}(f) = \frac{1}{(p-1)!} \int_0^1 (1-t)^{p-1} t^{p} f(H_0 + tV) dt$$

and

$$\| R_{p,H_0,V}(f) \| = O(\|V\|^{p}).$$

Theorem 1.1 generalizes the Taylor approximation theorem for scalar functions. It was proved in [8] that for $f \in C^{2p}(\mathbb{R})$, the operator function $f$ is Fréchet differentiable $p$ times on $\mathcal{B}(\mathcal{H})$, with the derivative written as an iterated operator integral. For $f \in \mathcal{W}_p$ (the set of functions $f \in C^p(\mathbb{R})$ such that for each $j = 0, \ldots, p$, the derivative $f^{(j)}$ equals the Fourier transform $\int_{\mathbb{R}} e^{it\lambda} d\mu_{f^{(j)}}(\lambda)$ of a finite Borel measure $\mu_{f^{(j)}}$) and a (possibly) unbounded $H_0$, the differentiability of $H \mapsto f(H)$ in the sense of Fréchet of order $p$ was established in [1]; in that case, the Gâteaux derivative $\frac{d^p}{dt^p} f(H_0 + tV)$ was represented as a Bochner-type multiple operator integral. For $f$ in the Besov class $B^1_{\infty,1}(\mathbb{R}) \cap B^p_{\infty,1}(\mathbb{R})$, it is known that the Gâteaux derivative of $f$ of order $p$ exists [23], but the bound (3) has not been proved.

In the scalar case ($\dim(\mathcal{H}) = 1$), we have that $\tau[R_{p,H_0,V}(f)]$ is a bounded functional on the space of functions $f^{(p)}$ and

$$\tau[R_{p,H_0,V}(f)] \leq \tau(|V|^p) \frac{f^{(p)}}{p!}.\|f^{(p)}\|_{\infty}.$$  

(4)

In the case of a nontrivial $\mathcal{H}$ ($\dim(\mathcal{H}) > 1$), it is generally hard to separate contribution of the perturbation $V$ to the estimate for the remainder (3) from contribution of the scalar function $f^{(p)}$. One of approaches to (4) is the estimate $\|R_{p,H_0,V}(f)\| \leq C(H_0, V) f^{(2p)} \|_{\infty}$, for $f \in C^{2p}(\mathbb{R})$ [8], with $C(H_0, V)$ a constant depending on bounded self-adjoint operators $H_0$ and $V$. Another approach is the estimate

$$\tau[R_{p,H_0,V}(f)] \leq \tau(|V|^p) \frac{\mu_{f^{(p)}}}{p!}.$$  

(5)

[10] (see [12] for an example when $\|\mu_{f^{(p)}}\|$ can be replaced with $\|\hat{f}(p)\|_1$), for $\tau$ the usual trace, $H_0 = H_0^*$ an operator in $\mathcal{H}$, $V = V^*$ an operator in the Schatten $p$-class, and $f \in \mathcal{W}_p$. If
$H_0 = H_0^*$ is affiliated with a semi-finite von Neumann algebra $\mathcal{M}$, $V = V^*$ is in the $\tau$-Schatten $p$-class of $\mathcal{M}$, and $f \in \mathcal{W}_p$, then the remainder $R_{p,H_0,V}(f)$ belongs to the Schatten $p$-class of $\mathcal{M}$ as well and

$$
\left[ \tau\left(\left|R_{p,H_0,V}(f)\right|^p\right)\right]^{1/p} + \left\| R_{p,H_0,V}(f) \right\| \leq \frac{(\tau(|V|^p))^{1/p} + \|V\|^p}{p!} \|\mu_{f(p)}\|
$$

see [1].

In the particular case of $p = 1$ or $p = 2$, the functional $\tau[R_{p,H_0,V}(f)]$ is bounded on the space of functions $f'$ or $f''$, respectively, and (4) holds. The measure representing the functional is absolutely continuous (with respect to Lebesgue's measure), with the density equal to Krein's spectral shift function $\xi_{H_0+V,H_0}$ or Koplienko's spectral shift function $\eta_{H_0,H_0+V}$, respectively. That is, we have

$$
\tau[R_{1,H_0,V}(f)] = \int_{\mathbb{R}} f'(t)\xi_{H_0+V,H_0}(t)\,dt, \quad \left| \tau[R_{1,H_0,V}(f)] \right| \leq \tau(|V|)\|f'\|_\infty \quad (6)
$$

and

$$
\tau[R_{2,H_0,V}(f)] = \int_{\mathbb{R}} f''(t)\eta_{H_0,H_0+V}(t)\,dt, \quad \left| \tau[R_{2,H_0,V}(f)] \right| \leq \frac{\tau(|V|^2)}{2}\|f''\|_\infty. \quad (7)
$$

Existence of $\xi_{H_0+V,H_0}$, with $\tau(|V|) < \infty$, satisfying (6) for $f \in \mathcal{W}_1$, was proved in the setting $\mathcal{M} = \mathcal{B}(\mathcal{H})$ in [16] (cf. also [17]) and extended to the setting of an arbitrary semi-finite von Neumann algebra $\mathcal{M}$ in [2,7]. Moreover, when $\mathcal{M} = \mathcal{B}(\mathcal{H})$, the trace formula in (6) is known to hold for $f \in B^1_{\infty,1}(\mathbb{R})$ [21]. In the setting $\mathcal{M} = \mathcal{B}(\mathcal{H})$, existence of $\eta_{H_0,H_0+V}$, with $V$ in the Hilbert–Schmidt class, satisfying (7) for bounded rational functions $f$ was proved in [15]. Later, it was proved in [22] that $\eta_{H_0,H_0+V}$ satisfies the trace formula in (7) for functions $f$ in $B^1_{\infty,1}(\mathbb{R}) \cap B^2_{\infty,1}(\mathbb{R})$. When $V$ is in the trace class, Koplienko’s spectral shift function can be written explicitly as

$$
\eta_{H_0,H_0+V}(t) = -\int_{-\infty}^t \xi_{H_0+V,H_0}(\lambda)\,d\lambda + \tau\left[E_{H_0}((-\infty,t))V\right], \quad (8)
$$

where $E_{H_0}$ is the spectral measure of $H_0$ [15]. In the context of a general $\mathcal{M}$, Koplienko’s spectral shift function $\eta_{H_0,H_0+V}$, with $\tau(|V|) < \infty$, and the representation (8) are discussed in [26].

For $p \geq 3$, $\mathcal{M} = \mathcal{B}(\mathcal{H})$, and $\tau(|V|^p) < \infty$, the distribution $\tau[R_{p,H_0,V}(f)]$ is given by an $L^2$-function $\gamma_{p,H_0,V}$ satisfying

$$
\tau[R_{p,H_0,V}(f)] = \frac{\tau(V^p)}{p!} f^{(p)}(0) + \int_{\mathbb{R}} f^{(p+1)}(t)\gamma_{p,H_0,V}(t)\,dt,
$$

for all $f \in \mathcal{W}_{p+1}$ [10]. It was conjectured in [15] that there exists a Borel measure $\nu_p$ with the total variation bounded by $\frac{\tau(|V|^p)}{p!}$ such that
\[ \tau[R_{p,H_0,V}(f)] = \int_{\mathbb{R}} f^{(p)}(t) \, dv_p(t), \tag{9} \]

for bounded rational functions \(f\). Unfortunately, the proof of (9) in [15] was based on the false claim that for \(V\) in the Schatten \(p\)-class, \(p > 2\), the set function defined on rectangles of \(\mathbb{R}^{p+1}\) by

\[ A_1 \times A_2 \times \cdots \times A_{p+1} \mapsto \tau[E(A_1)V E(A_2)V \cdots V E(A_{p+1})], \tag{10} \]

where \(E(\cdot)\) is a spectral measure on \(\mathbb{R}\) with values in \(B(\mathcal{H})\), extends to a (countably additive) measure of bounded variation (see a counterexample in Section 4). When \(V\) is in the Hilbert–Schmidt class of \(\mathcal{M} = B(\mathcal{H})\), the set function in (10) does extend to a (countably-additive) measure of bounded variation [5,20] and thus ideas of [15] can be applied to prove existence of a measure \(v_p\) satisfying (9) for bounded rational functions (see Section 7). In this case, the total variation of \(v_p\) is bounded by

\[ \|v_p\| \leq \frac{(\tau(|V|^2))^{p/2}}{p!}. \]

Adjusting techniques of [10] then extends (9) to the functions \(f \in \mathcal{W}_p\).

For \(\mathcal{M}\) a von Neumann algebra acting on an infinite-dimensional Hilbert space \(\mathcal{H}\), the set function in (10), with \(E(\cdot)\) the spectral measure attaining its values in \(\mathcal{M}\) and \(V \in \mathcal{M}\) satisfying \(\tau(|V|^2) < \infty\), may fail to extend to a finite measure on \(\mathbb{R}^{p+1}\) for \(p > 2\) even if \(\tau\) is finite (see a counterexample in Section 4). Therefore, the approach of [15] is not applicable in the proof of (9). When \(\mathcal{M}\) is a general semi-finite von Neumann algebra, we prove (9) for \(p = 3\) by relating \(R_3,H_0,V\) to \(R_2,H_0,V\), which allows to reduce the problem to the case of \(p = 2\) (see Sections 6 and 8). We also study the case when \(\mathcal{M}\) is finite and \(H_0,V \in \mathcal{M}\) are free with respect to the finite trace \(\tau\) (which is assumed normalized so that \(\tau(1) = 1\)). Freeness was introduced by Voiculescu (see, for example, [28]) and amounts to a specific prescription for the values of the mixed moments of \(H_0\) and \(V\) in terms of the individual moments of \(H_0\) and \(V\). Free perturbations have appeared in the study of quite general operators in finite von Neumann algebras, for example in the seminal work of Haagerup and Schultz [13]. Assuming freeness, we show that for all \(p\) the set function in (10) extends to a finite measure on \(\mathbb{R}^{p+1}\) (see Section 4), from which (9) can be derived.

Under the assumptions that we impose to prove existence of \(v_p\) satisfying (9) (see discussion in the two preceding paragraphs), we also construct a function \(\eta_p\), called the \(p\)th-order spectral shift function, such that \(d\eta_p(t) = \eta_p(t) \, dt\), provided \(H_0\) is bounded (see statements in Section 5 and proofs in Sections 7 and 8). The spectral shift function of order \(p\) admits the recursive representation

\[ \eta_p(t) = -\int_{-\infty}^{t} \eta_{p-1}(\lambda) \, d\lambda + \int_{\mathbb{R}^{p-1}} \text{spline}_{\lambda_1,\ldots,\lambda_{p-1}}(t) \, dm_{p-1,H_0,V}(\lambda_1,\ldots,\lambda_{p-1}), \tag{11} \]

where \(\text{spline}_{\lambda_1,\ldots,\lambda_{p-1}}\) is a piecewise polynomial of degree \(p - 2\) with breakpoints \(\lambda_1,\ldots,\lambda_{p-1}\) and \(dm_{p-1,H_0,V}(\lambda_1,\ldots,\lambda_{p-1})\) is a measure on \(\mathbb{R}^{p-1}\) determined by \(p - 1\) copies of the spectral
measure of $H_0$ intertwined with $p - 1$ copies of the perturbation $V$ (see Section 5 for the precise formula). As it is noticed in Section 5, the function $\eta_2$ given by (11) coincides with the function $\eta_{H_0, H_0 + V}$ given by (8), provided $\tau(|V|) < \infty$. The techniques of [15] that prove existence of $\nu_p$ when $M = B(H)$ do not give absolute continuity of $\nu_p$. We obtain $\eta_p$ by analyzing the Cauchy transform of the measure $\nu_p$ satisfying the trace formula (9) (see Section 6).

The approach of this paper, developed mainly for higher order spectral shift functions, contributes to the subject of Krein’s spectral shift function as well. In 1972, using Theorem 1.1, (2) and the double operator integral representation for the derivative

$$\frac{d}{dx} f(H_0 + xV) = \int_{\mathbb{R}^2} \Delta^{(1)}_{\lambda_1, \lambda_2}(f) E_{H_0 + xV}(d\lambda_1) V E_{H_0 + xV}(d\lambda_2),$$

M.Sh. Birman and M.Z. Solomyak [4] showed that

$$\tau\left[f(H_0 + V) - f(H_0)\right] = \int f'(t) \int_0^1 \tau\left[E_{H_0 + xV}(dt)\right] dx$$

(see [1, Theorem 6.3] for the analogous result in the context of von Neumann algebras), which along with Krein’s trace formula

$$\tau\left[f(H_0 + V) - f(H_0)\right] = \int f'(t) \xi_{H_0 + V, H_0}(t) dt$$

[2,7,16] implied the spectral averaging formula

$$\int_0^1 \tau\left[E_{H_0 + xV}(dt)\right] dx = \xi_{H_0 + V, H_0}(t) dt$$

(see [11,18,25,26] for generalizations and extensions). The operator $f(H_0 + V) - f(H_0)$ also admits a double operator integral representation

$$f(H_0 + V) - f(H_0) = \int_{\mathbb{R}^2} \Delta^{(1)}_{\lambda_1, \lambda_2}(f) E_{H_0 + V}(d\lambda_1) V E_{H_0}(d\lambda_2).$$

A natural question raised by M.Sh. Birman (see, e.g., [3]) asks if it is possible to deduce existence of $\xi_{H_0 + V, H_0}$, or equivalently, absolute continuity of the measure $\int_0^1 \tau\left[E_{H_0 + xV}(dt)\right] dx$, directly from the double operator integral representation (13). For $M$ a finite von Neumann algebra, we answer this question affirmatively and represent $\xi_{H_0 + V, H_0}$ as an integral of a basic spline straightforwardly from (13) (see Section 9). A general property of a basic spline is that it has the minimal support among all the splines with the same degree, smoothness, and domain properties (see, e.g., [9]). When dim($H$) $< \infty$, higher order spectral shift functions can be written as integrals of basic splines as well (see Section 9).

By combining different representations for the remainder $\tau[R_p, H_0, V(f)]$ in the setting of $M = B(H)$, we prove absolute continuity of the measure
\[ A \mapsto \int_0^1 (1-x)^{p-1} \tau \left[ \left( E_{H_0+x} V(A) V \right)^p \right] dx \]

and derive higher order analogs of the spectral averaging formula (12) (see Section 10).

Basic technical tools of the paper are discussed in Sections 2–4, main results are stated in Section 5 and then proved in Sections 6–8, additional representations for spectral shift functions are obtained in Section 9, and the Birman–Solomyak spectral averaging formula is generalized in Section 10. By saying “the standard setting” or “\( \tau \) is the standard trace,” we implicitly assume that \( \mathcal{M} = \mathcal{B}(\mathcal{H}) \) and \( \tau \) is the usual trace defined on the trace class operators of \( \mathcal{B}(\mathcal{H}) \). Let \( L_p(\mathcal{M}, \tau) \) denote the noncommutative \( L_p \)-space of \( (\mathcal{M}, \tau) \) with the norm \( \| V \|_p = \tau(|V|^p)^{1/p} \) and \( L_p(\mathcal{M}, \tau) = L_p(\mathcal{M}, \tau) \cap \mathcal{M} \) the Schatten \( p \)-class of \( (\mathcal{M}, \tau) \). The Schatten \( p \)-class is equipped with the norm \( \| \cdot \|_{p, \infty} = \| \cdot \|_p + \| \cdot \|_\infty \), where \( \| \cdot \|_\infty \) is the operator norm. Throughout the paper, \( H_0 \) and \( V \) denote self-adjoint operators in \( \mathcal{M} \) or affiliated with \( \mathcal{M} \); \( V \) is mainly taken to be an element of \( L_p(\mathcal{M}, \tau) \). Let \( \mathbb{N} \) denote the set of rational functions on \( \mathbb{R} \) with nonreal poles, \( \mathbb{N}_b \) the subset of \( \mathbb{N} \) of bounded functions. The symbol \( f_z \) is reserved for the function \( \mathbb{R} \ni \lambda \mapsto \frac{1}{z-\lambda} \), where \( z \in \mathbb{C} \setminus \mathbb{R} \).

2. Divided differences and splines

**Definition 2.1.** The divided difference of order \( p \) is an operation on functions \( f \) of one (real) variable, which we will usually call \( \lambda \), defined recursively as follows:

\[
\Delta_{\lambda_1}^{(0)}(f) := f(\lambda_1),
\]

\[
\Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}(f) := \begin{cases} 
\frac{\Delta_{\lambda_1, \ldots, \lambda_{p-1}}^{(p-1)}(f) - \lambda_p \Delta_{\lambda_1, \ldots, \lambda_{p-1}}^{(p-1)}(f)}{\lambda_p - \lambda_{p+1}} & \text{if } \lambda_p \neq \lambda_{p+1}, \\
\frac{\partial}{\partial \lambda} f_{\lambda=\lambda_p} & \text{if } \lambda_p = \lambda_{p+1}.
\end{cases}
\]

Below we state selected facts on the divided difference (see, e.g., [9]).

**Proposition 2.2.**

1. (See [9, Section 4.7, (a)].) \( \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}(f) \) is symmetric in \( \lambda_1, \lambda_2, \ldots, \lambda_{p+1} \).
2. (See [9, Section 4.7, (h)].) If all \( \lambda_1, \lambda_2, \ldots, \lambda_{p+1} \) are distinct, then

\[
\Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}(f) = \sum_{j=1}^{p+1} f(\lambda_j) \prod_{k \neq j} (\lambda_j - \lambda_k).
\]

3. (See [9, Section 4.7].) For \( f \) a sufficiently smooth function,

\[
\Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}(f) = \sum_{i \in \mathcal{I}} \sum_{j=0}^{m(\lambda_i)-1} c_{ij}(\lambda_1, \ldots, \lambda_{p+1}) f^{(j)}(\lambda_i).
\]

Here \( \mathcal{I} \) is the set of indices \( i \) for which \( \lambda_i \) are distinct, \( m(\lambda_i) \) is the multiplicity of \( \lambda_i \), and \( c_{ij}(\lambda_1, \ldots, \lambda_{p+1}) \in \mathbb{C} \).
(4) (See [9, Section 4.7].) \( \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}(a_p \lambda^p + a_{p-1} \lambda^{p-1} + \cdots + a_1 \lambda + a_0) = a_p \), where \( a_0, a_1, \ldots, a_p \in \mathbb{C} \).

(5) (See [9, Section 5.2, (2.3) and (2.6)].)

The basic spline with the break points \( \lambda_1, \ldots, \lambda_{p+1} \), where at least two of the values are distinct, is defined by

\[
 t \mapsto \begin{cases} 
 \frac{1}{|{\lambda_2 - \lambda_1}|} \chi_{[\min\{\lambda_1, \lambda_2\}, \max\{\lambda_1, \lambda_2\}]}(t) & \text{if } p = 1, \\
 \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}((\lambda - t)_{+}^{p-1}) & \text{if } p > 1.
\end{cases}
\]

Here the truncated power is defined by

\[
 x_k^\pm = \begin{cases} 
 x^k & \text{if } x \geq 0, \\
 0 & \text{if } x < 0,
\end{cases}
\]

for \( k \in \mathbb{N} \).

The basic spline is non-negative, supported in \( \min\{\lambda_1, \ldots, \lambda_{p+1}\}, \max\{\lambda_1, \ldots, \lambda_{p+1}\} \] and integrable with the integral equal to \( 1/p \). (Often the basic spline is normalized so that its integral equals 1).

(6) (See [9, Section 5.2, (2.2) and Section 4.7, (c)].)

For \( f \in C^p[\min\{\lambda_1, \ldots, \lambda_{p+1}\}, \max\{\lambda_1, \ldots, \lambda_{p+1}\}] \),

\[
 \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}(f) = \begin{cases} 
 \frac{1}{(p-1)!} \int_{-\infty}^{\infty} f^{(p)}(t) \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}((\lambda - t)_{+}^{p-1}) \, dt & \text{if } \exists i_1, i_2 \text{ such that } \lambda_{i_1} \neq \lambda_{i_2}, \\
 \frac{1}{p!} f^{(p)}(\lambda_1) & \text{if } \lambda_1 = \lambda_2 = \cdots = \lambda_{p+1}.
\end{cases}
\]

(7) (See [9, Section 4.7, (l)].) Let \( f \in C^p[a, b] \). Then, for \( \{\lambda_1, \ldots, \lambda_{p+1}\} \subset [a, b] \),

\[
 \left| \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}(f) \right| \leq \frac{1}{p!} \max_{\lambda \in [a, b]} |f^{(p)}(\lambda)|.
\]

Below we state useful properties of the divided difference to be used in the paper.

**Lemma 2.3.** For \( z \in \mathbb{C} \), with \( \operatorname{Im}(z) \neq 0 \),

\[
 \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)} \left( \frac{1}{z - \lambda} \right) = \prod_{j=1}^{p+1} \frac{1}{z - \lambda_j},
\]

where the divided difference is taken with respect to the real variable \( \lambda \).
**Proof.** We notice that by Definition 2.1,
\[
\Delta_{\lambda_1, \lambda_2}^{(1)} \left( \frac{1}{z - \lambda} \right) = \begin{cases} 
\left( \frac{1}{z - \lambda_1} - \frac{1}{z - \lambda_2} \right) \frac{1}{\lambda_1 - \lambda_2} = \frac{1}{(z - \lambda_1)(z - \lambda_2)} & \text{if } \lambda_1 \neq \lambda_2, \\
\frac{1}{(z - \lambda_1)^2} = \frac{1}{(z - \lambda_1)(z - \lambda_2)} & \text{if } \lambda_1 = \lambda_2.
\end{cases}
\]

By repeating the same argument, we obtain
\[
\Delta_{\lambda_1, \lambda_2, \lambda_3}^{(2)} \left( \frac{1}{z - \lambda} \right) = \frac{1}{(z - \lambda_1)(z - \lambda_2)(z - \lambda_3)}.
\]

The rest of the proof is accomplished by induction. \( \square \)

**Lemma 2.4.** Let \( D \) be a domain in \( \mathbb{C} \) and \( f \) a function continuously differentiable sufficiently many times on \( D \times \mathbb{R} \). Then for \( p \in \mathbb{N} \),

(i)
\[
\int \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}(f(z, \lambda)) \, dz = \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)} \left( \int f(z, \lambda) \, dz \right),
\]

with an appropriate choice of the constant of integration on the left-hand side;

(ii)
\[
\lim_{z \to z_0} \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}(f(z, \lambda)) = \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)} \left( \lim_{z \to z_0} f(z, \lambda) \right), \quad z_0 \in D;
\]

(iii)
\[
\frac{\partial}{\partial z} \left[ \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}(f(z, \lambda)) \right] = \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)} \left( \frac{\partial}{\partial z} f(z, \lambda) \right),
\]

where the divided difference is taken with respect to the variable \( \lambda \).

**Proof.** Follows immediately from Proposition 2.2(3). \( \square \)

**Corollary 2.5.** For \( p, k \in \mathbb{N} \),
\[
\frac{(-1)^k}{k!} \frac{\partial^k}{\partial z^k} \left( \prod_{j=1}^{p+1} \frac{1}{z - \lambda_j} \right) = \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)} \left( \frac{1}{(z - \lambda)^{k+1}} \right).
\]

**Proof.** Follows immediately from Lemma 2.3 and Lemma 2.4. \( \square \)
3. Remainders of Taylor-type approximations

In this section, we collect technical facts on derivatives of operator functions and remainders of the Taylor-type approximations.

The following lemma is routine.

Lemma 3.1. Let \( H_0 = H_0^* \) be an operator in \( \mathcal{H} \) and \( V = V^* \in \mathcal{B}(\mathcal{H}) \). Let \( H_x = H_0 + xV \), with \( x \in \mathbb{R} \). Then,

\[
\frac{d^p}{dx^p} \left( (zI - H_x)^{-k} \right) = p! \sum_{1 \leq k_0, k_1, \ldots, k_p \leq k} (zI - H_x)^{-k_0} V (zI - H_x)^{-k_1} V \cdots V (zI - H_x)^{-k_p}.
\]

If, in addition, \( H_0 \) is bounded, then

\[
\frac{d^p}{dx^p} \left( (H_x^k)^{-1} \right) = p! \sum_{0 \leq k_0, k_1, \ldots, k_p \leq k} H_x^{k_0} H_x^{k_1} V \cdots V H_x^{k_p}, \quad p \leq k.
\]

Lemma 3.2. Let \( H_0 = H_0^* \) be an operator affiliated with \( \mathcal{M} \) and \( V = V^* \in \mathcal{L}_2(\mathcal{M}, \tau) \). Then,

\[
\frac{(-1)^k}{k!} \frac{d^k}{dz^k} \tau \left[ (zI - H_0)^{-1} V (zI - H_0)^{-1} V (zI - H_0)^{-1} \right] = \frac{1}{2} \tau \left[ \frac{d^2}{dx^2} \right]_{x=0} \left( (zI - H_0 - xV)^{-k-1} \right). \quad (14)
\]

Proof. Firstly, we compute the left-hand side of (14). By cyclicity of the trace,

\[
\tau \left[ (zI - H_0)^{-1} V (zI - H_0)^{-1} V (zI - H_0)^{-1} \right] = \tau \left[ (zI - H_0)^{-2} V (zI - H_0)^{-1} V \right].
\]

By continuity of the trace in the norm \( \| \cdot \|_{1, \infty} \),

\[
\frac{d}{dz} \tau \left[ (zI - H_0)^{-2} V (zI - H_0)^{-1} V \right] = \tau \left[ \frac{d}{dz} \left( (zI - H_0)^{-2} V (zI - H_0)^{-1} V \right) \right].
\]

It is easy to see that

\[
\frac{d^k}{dz^k} \left( (zI - H_0)^{-2} V (zI - H_0)^{-1} V \right)
= \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} (-1)^j (j+1) (zI - H_0)^{-2-j} V (-1)^{k-j} (k-j) (zI - H_0)^{-1-(k-j)} V
= (-1)^k k! \sum_{j=0}^{k} (j+1) (zI - H_0)^{-2-j} V (zI - H_0)^{-1-(k-j)} V. \quad (15)
\]
Now we compute the right-hand side of (14). Let $H_x = H_0 + xV$. It is routine to see that

$$\frac{d}{dx} ((zI - H_x)^{-(k+1)}) = \sum_{i=1}^{k+1} (zI - H_x)^{-i} V (zI - H_x)^{-(k+2-i)},$$

and hence,

$$\frac{d^2}{dx^2} \left|_{x=0} \right. ((zI - H_x)^{-(k+1)})$$

$$= 2 \sum_{i=1}^{k+1} \frac{d}{dx} \left|_{x=0} \right. ((zI - H_x)^{-i}) V (zI - H_0)^{-(k+2-i)}$$

$$= 2 \sum_{i=1}^{k+1} \sum_{j=0}^{i-1} (zI - H_0)^{-(i-j)} V (zI - H_0)^{-(1-j)} V (zI - H_0)^{-(k+2-i)}.$$

Multiplying by $1/2$ and evaluating the trace in the latter expression provides

$$\frac{1}{2} \tau \left[ \frac{d^2}{dx^2} \left|_{x=0} \right. (zI - H_0 - xV)^{-(k-1)} \right]$$

$$= \sum_{i=1}^{k+1} \sum_{j=0}^{i-1} \tau [(zI - H_0)^{-(1-j)} V (zI - H_0)^{-(k+2-j)} V]$$

$$= \sum_{j=0}^{k} \sum_{i=j}^{k+1} \tau [(zI - H_0)^{-(1-j)} V (zI - H_0)^{-(2-(k-j))} V]$$

$$= \sum_{j=0}^{k} (k+1-j) \tau [(zI - H_0)^{-(1-j)} V (zI - H_0)^{-(2-(k-j))} V].$$

By changing the index of summation $i = k - j$ in the latter expression and by cyclicity of the trace, we obtain

$$\sum_{i=0}^{k} (i+1) \tau [(zI - H_0)^{-(2-i)} V (zI - H_0)^{-(1-(k-i))} V].$$

Comparing (15) and (16) completes the proof of the lemma. □

As a particular case of results of [23] we have the lemma below.

**Lemma 3.3.** Let $H_0 = H_0^*$ be an operator in $\mathcal{H}$ and $V = V^* \in \mathcal{B}(\mathcal{H})$. Denote $H_x = H_0 + xV$. For $f \in \mathcal{R}_b,$
\[
\frac{dp}{dx} f(H_0 + xV) = p! \int \int \ldots \int \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)} E_{H_0}(d\lambda_1)V E_{H_0}(d\lambda_2)V \ldots V E_{H_0}(d\lambda_{p+1}).
\] (17)

If, in addition, \( H_0 \) is bounded, then (17) holds for \( f \in \mathcal{R} \).

**Remark 3.4.** It was proved in [8, Theorem 2.2] that for \( H_0 \) a bounded operator, \( \frac{dp}{dt} f(H_0 + tV) \) is defined when \( f \in C^2_p(\mathbb{R}) \) and the derivative can be computed as an iterated operator integral (17). It was proved later in [23] that the Gâteaux derivative \( \frac{dp}{dt} f(H_0 + tV) \) is defined for \( f \) in the intersection of the Besov classes \( B^p_{\infty 1}(\mathbb{R}) \cap B^1_{\infty 1}(\mathbb{R}) \) and can be computed as a Bochner-type multiple operator integral.

The following lemma is a straightforward consequence of Lemma 3.1.

**Lemma 3.5.** Let \( H_0 = H_0^* \) be an operator in \( \mathcal{H} \) and \( V = V^* \in \mathcal{B}(\mathcal{H}) \). Then for \( f \) a polynomial of degree \( m \),

\[
R_{p, H_0, V}(f) = \sum_{k_0, k_1, \ldots, k_p \geq 0 \atop k_0 + k_1 + \ldots + k_p = m - p} a_{k_0, k_1, \ldots, k_p} H_0^{k_0} V H_0^{k_1} V \ldots V H_0^{k_p},
\]

with \( a_{k_0, k_1, \ldots, k_p} \) numbers.

**Lemma 3.6.** Let \( H_0 = H_0^* \) be an operator in \( \mathcal{H} \) and \( V = V^* \in \mathcal{B}(\mathcal{H}) \). Then,

\[
R_{p, H_0, V}(f_Z) = (zI - H_0 - V)^{-1} - \sum_{j=0}^{p-1} (zI - H_0)^{-1} (V(zI - H_0)^{-1})^j \quad (18)
\]

\[
= (zI - H_0 - V)^{-1} (V(zI - H_0)^{-1})^p. \quad (19)
\]

**Proof.** By Lemma 3.1,

\[
\frac{d^l}{dx^l} \bigg|_{x=x_0} (zI - H_0 - xV)^{-1} = j! (zI - H_0 - x_0 V)^{-1} (V(zI - H_0 - x_0 V)^{-1})^j,
\]

which gives (18). To derive (19) from (18), we use repeatedly the resolvent identity

\[
(zI - H_0 - V)^{-1} - (zI - H_0)^{-1} = (zI - H_0 - V)^{-1} V(zI - H_0)^{-1}.
\]

By combining \((zI - H_0 - V)^{-1}\) and the first summand of

\[
\sum_{j=0}^{p-1} (zI - H_0)^{-1} (V(zI - H_0)^{-1})^j,
\]
we obtain that (provided $p > 1$)

$$(zI - H_0 - V)^{-1} - \sum_{j=0}^{p-1} (zI - H_0)^{-1}(V(zI - H_0)^{-1})^j$$

$$= (zI - H_0 - V)^{-1}V(zI - H_0)^{-1} - \sum_{j=1}^{p-1} (zI - H_0)^{-1}(V(zI - H_0)^{-1})^j.$$

Repeating the reasoning above sufficiently many times completes the proof of (19). \qed

From (18) we have the following relation between the remainders of different order.

**Lemma 3.7.** Let $H_0 = H_0^*$ be an operator in $\mathcal{H}$ and $V = V^* \in \mathcal{B}(\mathcal{H})$. Then

$$R_{p+1,H_0,V}(f_z) = R_{p,H_0,V}(f_z) - \left((zI - H_0)^{-1}V\right)^p(zI - H_0)^{-1}.$$

The following lemma is a straightforward generalization of [10, Lemma 2.6].

**Lemma 3.8.** Let $H_0 = H_0^*$, $V = V^* \in \mathcal{B}(\mathcal{H})$, and $\Gamma = \{\lambda : |\lambda| = 1 + \|H_0\| + \|V\|\}$. Then, for every function $f$ analytic in a neighborhood of $D = \{\lambda : |\lambda| \leq 1 + \|H_0\| + \|V\|\}$,

$$R_{p,H_0,V}(f) = \frac{1}{2\pi i} \oint_{\Gamma} f(\lambda)(\lambda I - H_0)^{-1}(V(\lambda I - H_0)^{-1})^p(I - V(\lambda I - H_0)^{-1})^{-1} d\lambda.$$

Let $(S, \nu)$ be a measure space and let $L_{so}^\infty(S, \nu, L_1(M, \tau))$ denote the $\ast$-algebra of $\|\cdot\|$-bounded $so^*$-measurable functions $F : S \mapsto L_1(M, \tau)$ [19].

**Proposition 3.9.** (See [1, Lemma 3.10]). Let $F$ be a function in $L_{so}^\infty(S, \nu, L_1(M, \tau))$ uniformly $L_1(M, \tau)$-bounded. Then $\int_S F(s) d\nu(s) \in L_1(M, \tau)$, $\tau(F(\cdot))$ is measurable and

$$\tau\left(\int_S F(s) d\nu(s)\right) = \int_S \tau(F(s)) d\nu(s).$$

Similarly to [1, Lemma 4.5, Theorem 5.7], we have the following differentiation formula for an operator function $f(\cdot)$, with $f \in W_p$.

**Lemma 3.10.** Let $H_0 = H_0^*$ be an operator affiliated with $M$ and $V = V^* \in L_p(M, \tau)$. Let $H_x = H_0 + xV$, with $x \in \mathbb{R}$. Then, for $f \in W_p$ given by $f(\lambda) = \int_{\mathbb{R}} e^{i\lambda t} d\mu_f(t)$, the function $f(H_x)$ is $p$ times Fréchet differentiable in the norm $\|\cdot\|_{1,\infty}$ and the derivative equals the Bochner-type multiple operator integral

$$\frac{d^p}{dx^p} f(H_x) = p! \int_{\Pi(p)} e^{i(s_0-s_1)H_x} V \ldots V e^{i(s_{p-1}-s_p)H_x} V e^{i\lambda x} H_x d\sigma_f(s_0, \ldots, s_p).$$
Here

$$\Pi^{(p)} = \{(s_0, s_1, \ldots, s_p) \in \mathbb{R}^{p+1}: |s_p| \leq \cdots \leq |s_1| \leq |s_0|, \ \text{sign}(s_0) = \cdots = \text{sign}(s_p)\}$$

and

$$d\sigma^{(p)}(s_0, s_1, \ldots, s_p) = i^p \mu_f(ds_0)ds_1 \cdots ds_p.$$

In particular, for \( t \in \mathbb{R} \),

$$\frac{d^p}{dx^p} e^{itH_x} = i^p p! \int_{\Pi^{(p-1)}} e^{i(t-s_1)H_x} V \ldots V e^{i(s_{p-1}-s_p)H_x} V e^{is_pH_x} ds_p \cdots ds_1.$$

By applying Proposition 3.9 and Lemma 3.10, we obtain the following

Lemma 3.11. Let \( H_0 = H_0^* \) be an operator affiliated with \( \mathcal{M} \) and \( V = V^* \in \mathcal{L}_p(\mathcal{M}, \tau) \). Let \( H_x = H_0 + xV \), with \( x \in \mathbb{R} \). Then for \( f \in \mathcal{W}_p \), we have

$$\tau \left[ \frac{d^p}{dx^p} f(H_x) \right] = p! \int_{\Pi^{(p)}} \tau \left[ e^{i(s_0-s_1)H_x} V \ldots V e^{i(s_{p-1}-s_p)H_x} V e^{is_pH_x} \right] d\sigma^{(p)}(s_0, \ldots, s_p)$$

and

$$\left\| \frac{d^p}{dx^p} f(H_x) \right\|_1 \leq \|V\|_p^p \|\mu_f\|_p.$$

Corollary 3.12. Under the assumptions of Lemma 3.11,

$$\tau \left[ \frac{d^p}{dx^p} f(H_x) \right] = \int_{\mathbb{R}} \tau \left[ \frac{d^p}{dx^p} e^{itH_x} \right] d\mu_f(t).$$

Proof. The claim is proved by reducing the double integral to an iterated one and applying Lemmas 3.10 and 3.11. \( \square \)

Remark 3.13. By combining Lemma 3.11 and Theorem 1.1 (2), one obtains the estimate (5).

4. Multiple spectral measures

We will need the fact that certain finitely additive “multiple spectral measures” extend to countably additive measures.

Theorem 4.1. Let \( 2 \leq p \in \mathbb{N} \) and let \( E_1, E_2, \ldots, E_p \) be projection-valued Borel measures from \( \mathbb{R} \) into \( \mathcal{M} \). Suppose that \( V_1, \ldots, V_p \) belong to \( \mathcal{L}_2(\mathcal{M}, \tau) \). Assume that either \( \tau \) is the standard trace or \( p = 2 \). Then there is a unique (complex) Borel measure \( m \) on \( \mathbb{R}^p \) with total variation not exceeding the product \( \|V_1\|_2 \|V_2\|_2 \cdots \|V_p\|_2 \), whose value on rectangles is given by

$$m(A_1 \times A_2 \times \cdots \times A_p) = \tau \left[ E_1(A_1)V_1E_2(A_2)V_2 \cdots V_{p-1}E_p(A_p)V_p \right]$$

for all Borel subsets \( A_1, A_2, \ldots, A_p \) of \( \mathbb{R} \).
Proof. It is enough (see, e.g., [14, Theorem 2.12] for \( p = 2 \)) to prove that the variation of the set function \( m \) on the rectangles of \( \mathbb{R}^p \) is bounded by \( \|V_1\|_2 \|V_2\|_2 \cdots \|V_p\|_2 \), which can be accomplished completely analogously to the proof of [20, Theorem 1] (see also [5]). □

Remark 4.2. For \( \tau \) the standard trace, the bound for the total variation in Theorem 4.1 was proved in [20, Theorem 1]. Theorem 4.1 with \( \tau \) standard was also obtained in [5]. The proof in [5] is based on the facts that a Hilbert–Schmidt operator can be approximated by finite-rank operators in the norm \( \| \cdot \|_2 \) and that for rank-one perturbations \( V_1, \ldots, V_p \) and \( \tau \) the standard trace, the set function \( m \) decomposes into a product of scalar measures. It is classical that a direct product of countably additive measures always has a countably-additive extension to the \( \sigma \)-algebra generated by the direct product of the \( \sigma \)-algebras involved. The argument of [5] cannot be directly extended to the case of a general trace. For a general trace \( \tau \), the set function \( m \) is known to be of bounded variation only if \( p = 2 \). Technically, this constraint is explained by the fact that in general \( \| \cdot \|_p \) is not dominated by \( \| \cdot \|_2 \), as distinct from the particular case of the standard trace \( \tau \). A counterexample constructed further in this section demonstrates that \( p = 2 \) is not only a technical constraint.

Corollary 4.3. Let \( 2 \leq p \in \mathbb{N} \) and let \( E_1, E_2, \ldots, E_p \) be projection-valued Borel measures from \( \mathbb{R} \) to \( \mathcal{M} \). Suppose that \( V_1, \ldots, V_p \) belong to \( \mathcal{L}_2(\mathcal{M}, \tau) \). Assume that either \( \tau \) is the standard trace or \( p = 2 \). Then there is a unique (complex) Borel measure \( m_1 \) on \( \mathbb{R}^{p+1} \) with total variation not exceeding the product \( \|V_1\|_2 \|V_2\|_2 \cdots \|V_p\|_2 \), whose value on rectangles of \( \mathbb{R}^{p+1} \) is given by

\[
m_1(A_1 \times A_2 \times \cdots \times A_p \times A_{p+1}) = \tau [E_1(A_1)V_1E_2(A_2)V_2 \cdots V_{p-1}E_p(A_p)V_pE_1(A_{p+1})],
\]

for all Borel subsets \( A_1, A_2, \ldots, A_p, A_{p+1} \) of \( \mathbb{R} \).

Proof. It is straightforward to see that

\[
m_1(A_1 \times A_2 \times \cdots \times A_p \times A_{p+1}) = \tau [E_1(A_1 \cap A_{p+1})V_1E_2(A_2)V_2 \cdots V_{p-1}E_p(A_p)V_p].
\]

By repeating the argument of [5,20], one can see that the total variation of the set function \( m_1 \) is bounded on the rectangles of \( \mathbb{R}^{p+1} \) by \( \|V_1\|_2 \|V_2\|_2 \cdots \|V_p\|_2 \). Thus, \( m_1 \) extends to a unique complex Borel measure on \( \mathbb{R}^{p+1} \) with variation bounded by \( \|V_1\|_2 \|V_2\|_2 \cdots \|V_p\|_2 \). □

Corollary 4.4. Let \( 2 \leq p \in \mathbb{N} \), \( E_1, \ldots, E_p \) projection-valued Borel measures from \( \mathbb{R} \) into \( \mathcal{M} \), and \( \tau \) a finite trace. Suppose that \( V_1, \ldots, V_{p-1} \) belong to \( \mathcal{M} \). Assume that either \( \tau \) is the standard trace or \( p = 2 \). Then there is a unique complex Borel measure \( m_2 \) on \( \mathbb{R}^p \) with total variation not exceeding \( \|V_1\|_2 \|V_2\|_2 \cdots \|V_{p-1}\|_2 \tau(I)^{1/2} \), whose value on rectangles is given by

\[
m_2(A_1 \times A_2 \times \cdots \times A_p) = \tau [E_1(A_1)V_1E_2(A_2)V_2 \cdots V_{p-1}E_p(A_p)]
\]

for all Borel subsets \( A_1, A_2, \ldots, A_p \) of \( \mathbb{R} \).

Proof. It is an immediate consequence of Theorem 4.1 applied to \( V_p = I \). □
In the sequel, we will work with the set functions
\[
m_{p,H_0,V}(A_1 \times A_2 \times \cdots \times A_p) = \tau \left[ E_{H_0}(A_1) V E_{H_0}(A_2) V \cdots V E_{H_0}(A_p) V \right],
\]

and their countably-additive extensions (when they exist). Here \( A_j \) are measurable subsets of \( \mathbb{R} \), \( H_0 = H_0^* \) is affiliated with \( \mathcal{M} \), and \( V = V^* \in \mathcal{L}_2(\mathcal{M}, \tau) \).

In the next result, freeness of \((zI - H_0)^{-1}\) and \( V \) means freeness of the algebra generated by the spectral projections of \( H_0 \) and the unital algebra generated by \( V \).

**Theorem 4.5.** Let \( \tau \) be a finite trace normalized by \( \tau(I) = 1 \) and let \( H_0 = H_0^* \) be affiliated with \( \mathcal{M} \) and \( V = V^* \in \mathcal{M} \). Assume that \((zI - H_0)^{-1}\) and \( V \) are free. Then the set functions \( m_{p,H_0,V} \) and \( m^{(1)}_{p,H_0,V} \) extend to countably additive measures of bounded variation.

**Proof.** We prove the claim for the function \( m_{p,H_0,V} \); the case of \( m^{(1)}_{p,H_0,V} \) is completely analogous. Using the moment-cumulant formula (see [27, Theorem 2.17]), and that \( \prod_i E_{H_0}(A_i) = E_{H_0}(\bigcap_i A_i) \) we have
\[
m_{p,H_0,V}(A_1 \times \cdots \times A_p) = \tau \left[ E_{H_0}(A_1) V \cdots E_{H_0}(A_p) V \right]
\]

where \( \text{NC}(p) \) is the lattice of all noncrossing partitions of \( \{1, \ldots, p\} \) and where \( k_{K(\pi)}[V, \ldots, V] \) is the product of cumulants of \( V \), associated to the block structure of the Kreweras complement \( K(\pi) \) of \( \pi \); thus, \( k_{K(\pi)}[V, \ldots, V] \) is equal to a polynomial (that depends on \( \pi \) ) in \( p \) variables, evaluated at \( \tau(V), \tau(V^2), \ldots, \tau(V^p) \). Given \( \pi = \{B_1, \ldots, B_\ell\} \in \text{NC}(p) \), the measure
\[
\gamma_{p,\pi}: A_1 \times \cdots \times A_p \mapsto \prod_{i=1}^\ell \tau \left( E_{H_0} \left( \bigcap_{i \in B_j} A_i \right) \right)
\]

is the push-forward of the \( \ell \)-fold product \( x_1^\ell (\tau \circ E_{H_0}) \) of the spectral distribution measure of \( H_0 \) (with respect to \( \tau \) ) under the mapping of \( \mathbb{R}^\ell \) onto the product of diagonals in \( \mathbb{R}^p \) according to the block structure \( B_1, \ldots, B_\ell \). Each such push-forward is a probability measure. Thus, we see that \( m_{p,H_0,V} \) is a linear combination of probability measures, and has finite total variation. \( \square \)

In some cases used in the paper, the measure \( m_2 \) is known to be real-valued and the measure \( m \) non-negative.
Lemma 4.6. Let \( \tau \) be a finite trace. Let \( H_0 = H_0^* \) be affiliated with \( \mathcal{M} \) and \( V = V^* \in \mathcal{M} \). Then the measure \( m_{1,H_0,V}^{(2)} \) is real-valued.

Proof. For arbitrary measurable subsets \( A_1 \) and \( A_2 \) of \( \mathbb{R} \),

\[
\tau[E_{H_0+V}(A_1)V E_{H_0}(A_2)] = \tau[E_{H_0+V}(A_1)(H_0 + V) E_{H_0}(A_2)] - \tau[E_{H_0+V}(A_1)H_0 E_{H_0}(A_2)]
\]

\[
= \tau[E_{H_0}(A_2)(E_{H_0+V}(A_1)(H_0 + V)) E_{H_0}(A_2)] - \tau[E_{H_0+V}(A_1)(H_0 E_{H_0}(A_2)) E_{H_0+V}(A_1)].
\]

where the operators

\[
E_{H_0}(A_2)(E_{H_0+V}(A_1)(H_0 + V)) E_{H_0}(A_2) \quad \text{and} \quad E_{H_0+V}(A_1)(H_0 E_{H_0}(A_2)) E_{H_0+V}(A_1)
\]

are self-adjoint. Therefore, \( m_{1,H_0,V}^{(2)}(A_1 \times A_2) \in \mathbb{R} \), and the extension of \( m_{1,H_0,V}^{(2)} \) to the Borel subsets of \( \mathbb{R}^2 \) is real-valued. \qed

Lemma 4.7. Let \( H_0 = H_0^* \) be affiliated with \( \mathcal{M} \) and \( V = V^* \in \mathcal{L}_2(\mathcal{M}, \tau) \). Then the measure \( m_{2,H_0,V}^{(2)} \) is non-negative.

Proof. For arbitrary measurable subsets \( A_1 \) and \( A_2 \) of \( \mathbb{R} \),

\[
\tau[E_{H_0}(A_1)V E_{H_0}(A_2)V] = \tau[E_{H_0}(A_1)V E_{H_0}(A_2)V E_{H_0}(A_1)] \geq 0,
\]

since

\[
\{E_{H_0}(A_1)V E_{H_0}(A_2)V E_{H_0}(A_1) f, f\} = \{E_{H_0}(A_2)(V E_{H_0}(A_1) f), (V E_{H_0}(A_1) f)\} \geq 0,
\]

for any \( f \in \mathcal{H} \). \qed

Lemma 4.8. Let \( \tau \) be a finite trace. Let \( H_0 = H_0^* \) be an operator affiliated with \( \mathcal{M} \) and \( V = V^* \in \mathcal{M} \). Then \( m_{1,H_0,V}^{(2)} \) has no atoms on the diagonal \( D_{k+1} = \{(\lambda_1, \lambda_2, \ldots, \lambda_{k+1}) : \lambda_1 = \lambda_2 = \cdots = \lambda_{k+1} \in \mathbb{R}\} \) of \( \mathbb{R}^{k+1} \).

Proof. By definition of the measure \( m_{1,H_0,V}^{(2)} \),

\[
m_{1,H_0,V}^{(2)}(\{(\lambda, \lambda, \ldots, \lambda)\}) = \tau[E_{H_0+V}(\{\lambda\}) V E_{H_0}(\{\lambda\})(V E_{H_0}(\{\lambda\}))^{k-1}].
\]

We will show that \( E_{H_0+V}(\{\lambda\}) V E_{H_0}(\{\lambda\}) \) is the zero operator.

Let \( g \) be an arbitrary vector in \( \mathcal{H} \) and let \( h = E_{H_0}(\{\lambda\}) g \). Then \( H_0h = \lambda h \) and

\[
E_{H_0+V}(\{\lambda\}) V E_{H_0}(\{\lambda\}) g = E_{H_0+V}(\{\lambda\}) V h = E_{H_0+V}(\{\lambda\})(H_0 + V) h - E_{H_0+V}(\{\lambda\}) H_0 h
\]

\[
= E_{H_0+V}(\{\lambda\})(H_0 + V) h - \lambda E_{H_0+V}(\{\lambda\}) h = (H_0 + V - \lambda I) E_{H_0+V}(\{\lambda\}) h = 0. \qed
\]
Upon evaluating a trace, some iterated operator integrals can be written as Lebesgue integrals with respect to a “multiple spectral measure.”

**Lemma 4.9.** Assume the hypothesis of Theorem 4.1. Assume that the spectral measures $E_1, E_2, \ldots, E_p$ correspond to self-adjoint operators $H_0, H_1, \ldots, H_p$ affiliated with $\mathcal{M}$, respectively, and that $V_1, V_2, \ldots, V_p \in L_2(\mathcal{M}, \tau)$. Let $f_1, f_2, \ldots, f_p$ be functions in $C_0^\infty(\mathbb{R})$ (vanishing at infinity). Then

$$\tau\left[f_1(H_1)V_1 f_2(H_2)V_2 \ldots f_p(H_p)V_p\right] = \int_{\mathbb{R}^p} f_1(\lambda_1) f_2(\lambda_2) \ldots f_p(\lambda_p) \, dm(\lambda_1, \lambda_2, \ldots, \lambda_p),$$

with $m$ as in Theorem 4.1.

**Proof.** The result obviously holds for $f_1, f_2, \ldots, f_p$ simple functions. Uniform approximation of $f_1, f_2, \ldots, f_p \in C_0^\infty(\mathbb{R})$ by (totally bounded) simple functions completes the proof. 

**Remark 4.10.**

(i) The result analogous to the one of Theorem 4.1 holds for integrals with respect to the measures $m_1$ and $m_2$.

(ii) When the operators $H_0, H_1, \ldots, H_p$ are bounded, the functions $f_1, f_2, \ldots, f_p$ can be taken in $C_\infty(\mathbb{R})$.

**Corollary 4.11.** Let $H_0 = H_0^*$ be affiliated with $\mathcal{M}$ and $V = V^* \in L_2(\mathcal{M}, \tau)$. Denote $H_x := H_0 + xV$, $x \in [0, 1]$. Assume that either $\tau$ is standard or $p = 2$. Then for $f \in \mathfrak{R}_b$,

$$\tau\left[\frac{d^p}{dx^p} f(H_0 + xV)\right] = p! \int_{\mathbb{R}^{p+1}} \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}(f) \, dm_{p, H_x, V}(\lambda_1, \lambda_2, \ldots, \lambda_{p+1}).$$

**Proof.** It is enough to prove the result for $f(\lambda) = (z - \lambda)^{-k}$, $k \in \mathbb{N}$. By Lemma 3.3,

$$\frac{d^p}{dx^p} ((zI - H_x)^{-k}) = p! \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}((z - \lambda)^{-k}) \, E_{H_x}(d\lambda_1) \, E_{H_x}(d\lambda_2) \ldots \, V \, E_{H_x}(d\lambda_{p+1}). \quad (22)$$

By Corollary 2.5, the function $\Delta_{\lambda_1, \ldots, \lambda_{p+1}}^{(p)}((z - \lambda)^{-k})$ is a linear combination of products $\prod_{j=1}^{p+1} f_j(\lambda_j)$, with $f_j$ in $C_0^\infty(\mathbb{R})$ for $1 \leq j \leq p + 1$, and hence, the trace of the expression in (22) can be written as a linear combination of integrals like in Remark 4.10 (i), where $m_1 = m^{(1)}_{p, H_x, V}(\lambda_1, \lambda_2, \ldots, \lambda_{p+1})$. 

**Remark 4.12.** One also has

$$\tau\left[((zI - H_0)^{-1}V)^p\right] = \int_{\mathbb{R}^p} \Delta_{\lambda_1, \ldots, \lambda_p}^{(p-1)}(f_z) \, dm_{p, H_0, V}(\lambda_1, \lambda_2, \ldots, \lambda_p).$$
\[
\tau[(zI - H_0 - V)^{-1} (V(zI - H_0)^{-1})^p] = \int_{\mathbb{R}^{p+1}} \Delta(z_1, \ldots, z_{p+1}, f) \, dm(z_1, \ldots, z_{p+1}, f).
\]

**Remark 4.13.** If \( H_0 \) is bounded, then Corollary 4.11 also holds for \( f \) a polynomial and for \( f \in C^\infty(\mathbb{R}) \) such that \( f \mid_{[a,b]} \) is a polynomial, where \([a, b] \supset \sigma(H_0) \cup \sigma(H_0 + V)\).

**A counterexample.** Let \( p \geq 2 \) be an integer. Let \( V \) be a self-adjoint operator on a Hilbert space \( \mathcal{H} \) and assume \( V \) belongs to the Schatten \( p \)-class, with respect to the usual trace \( \text{Tr} \). Let \( E \) be a spectral measure. A crucial estimate in [15] is of the total variation of the function that is defined on product sets by

\[
A_1 \times \cdots \times A_p \mapsto \text{Tr}(a_1 E(A_1) V E(A_2) V \cdots E(A_p) V).
\]

Unfortunately, the estimate result in [15] is false when \( p \geq 3 \). In this section, we provide an example, based on Hadamard matrices, having unbounded total variation. We also give a version for finite traces.

Consider the self-adjoint unitary \( 2 \times 2 \) matrix

\[
V_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.
\]

When \( n = 2^k \), consider the self-adjoint unitary \( n \times n \) matrix

\[
V_n = V_2 \otimes \cdots \otimes V_2, \quad k \text{ times}
\]

Then each such \( V_n \) is a multiple of a Hadamard matrix.

Let \((e_{jk})_{1 \leq j, k \leq n}\) be the usual system of matrix units for \( M_n(\mathbb{C}) \). Let \( E_n \) be the spectral measure on the set \([1, \ldots, n]\) taking values in \( M_n(\mathbb{C}) \), defined by \( E_n([j]) = e_{jj} \).

The following lemma can be proved directly for all \( n \), or first in the case \( n = 2 \) and then by observing how the total variation behaves under taking tensor products of matrices and spectral measures.

**Lemma 4.14.** For every integer \( p \geq 2 \), and every \( n \) that is a power of 2, the set function

\[
A_1 \times \cdots \times A_p \mapsto \text{Tr}(E_n(A_1) V_n E_n(A_2) V \cdots E_n(A_p) V_n)
\]

has total variation \( n^{p/2} \).
Consider the von Neumann algebra \( \mathcal{M} \) with normal trace \( \tau \) given by

\[
(\mathcal{M}, \tau) = \bigoplus_{k=1}^{\infty} \frac{M_{n(k)}(\mathbb{C})}{\alpha(k)}
\]

where \( \alpha(k) > 0 \) and this notation indicates that \( \mathcal{M} \) is the \( \ell^\infty \)-direct sum of the matrix algebras \( M_{n(k)}(\mathbb{C}) \), where every \( n(k) \) is a power of 2 and where \( \tau \) is the trace determined by

\[
\tau \left( 0 \oplus \cdots \oplus 0 \oplus I_{n(k)} \oplus 0 \oplus 0 \oplus \cdots \right) = \alpha(k).
\]

We will be interested in the following two cases:

(I) \( \mathcal{M} \) embeds in \( \mathcal{B}(\mathcal{H}) \), for \( \mathcal{H} \) a separable, infinite-dimensional Hilbert space, in such a way that \( \tau \) is the restriction of the usual trace \( \text{Tr} \) on \( \mathcal{B}(\mathcal{H}) \); this is equivalent to \( \alpha(k) \) being an integer multiple of \( n(k) \) for all \( k \).

(II) \( \mathcal{M} \) is a finite von Neumann algebra and \( \tau \) is normalized to take value 1 on the identity; this is equivalent to \( \sum_{k=1}^{\infty} \alpha(k) = 1 \).

**Example 4.15.** Consider

\[
T = t_1 V_{n(1)} \oplus t_2 V_{n(2)} \oplus \cdots \in \mathcal{M},
\]

for a bounded sequence of \( t_k \geq 0 \). Then

\[
|T| = t_1 I_{n(1)} \oplus t_2 I_{n(2)} \oplus \cdots,
\]

\[
\|T\|_p = \sum_{k=1}^{\infty} t_k^p \alpha(k).
\]  

(23)

Taking the obvious diagonal spectral measure \( E \) defined on the set \( \{(k, j) \in \mathbb{N}^2 \mid j \leq n(k)\} \) by

\[
E\left(\{(k, j)\}\right) = 0 \oplus \cdots \oplus 0 \oplus e_{jj} \oplus 0 \oplus 0 \oplus \cdots
\]

k-1 times

and using the result of Lemma 4.14, we find that the total variation of the set function

\[
A_1 \times \cdots \times A_p \mapsto \tau \left( E(A_1)T E(A_2)T \cdots E(A_p)T \right)
\]

is

\[
\sum_{k=1}^{\infty} t_k^p \alpha(k) \frac{n(k)^{p/2}}{n(k)} = \sum_{k=1}^{\infty} t_k^p \alpha(k) n(k)^{(p/2) - 1}.
\]  

(24)
Now assuming \( n(k) \) is unbounded as \( k \to \infty \) and fixing an integer \( p \geq 3 \), it is easy to choose values of \( \alpha(k) \) and \( t_k \) such that the \( p \)-norm in (23) is finite while the total variation (24) is infinite, in both cases (I) and (II) above.

**Remark 4.16.** The above examples also work to show that the set function

\[
A_1 \times \cdots \times A_{p+1} \mapsto \tau[E(A_1)T \cdots T E(A_{p+1})]
\]

has infinite total variation.

### 5. Main results

In this section we state the main results which will be proved in the next three sections.

**Theorem 5.1.** Let \( 2 < p \in \mathbb{N} \). Let \( H_0 \) be a self-adjoint operator affiliated with \( \mathcal{M} = \mathcal{B}(\mathcal{H}) \) and \( V \) a self-adjoint operator in \( \mathcal{L}_2(\mathcal{M}, \tau) \). Then, the following assertions hold.

(i) There is a unique finite real-valued measure \( \nu_p \) on \( \mathbb{R} \) such that the trace formula

\[
\tau[R_{p,H_0,V}(f)] = \int_{\mathbb{R}} f^{(p)}(t) \, d\nu_p(t) \tag{25}
\]

holds for \( f \in \mathcal{W}_p \). The total variation of \( \nu_p \) is bounded by

\[
\text{var} (\nu_p) \leq c_p := \frac{1}{p!} \| V \|^p_2.
\]

(ii) If, in addition, \( H_0 \) is bounded, then \( \nu_p \) is finite and supported in \( [a,b] \supset \sigma(H_0) \cup \sigma(H_0 + V) \). The density \( \eta_p \) of \( \nu_p \) can be computed recursively by

\[
\eta_p(t) = \frac{\tau(V^{p-1})}{(p-1)!} - \nu_{p-1}((-\infty, t]) - \frac{1}{(p-1)!} \int_{\mathbb{R}^{p-1}} \Delta_{\lambda_1,\ldots,\lambda_{p-1}}^{(p-2)}((\lambda - t)^{p-2}) \, dm_{p-1,H_0,V}(\lambda_1, \ldots, \lambda_{p-1}), \tag{26}
\]

for a.e. \( t \in \mathbb{R} \). In this case, (25) also holds for \( f \in \mathcal{R} \).

**Theorem 5.2.** Let \( p \in \{2,3\} \). Let \( H_0 = H_0^* \) be an operator affiliated with a von Neumann algebra \( \mathcal{M} \) with normal faithful semi-finite trace \( \tau \) and \( V = V^* \) an operator in \( \mathcal{L}_2(\mathcal{M}, \tau) \). Then, the following assertions hold.

(i) There is a unique real-valued measure \( \nu_p \) on \( \mathbb{R} \) such that the trace formula (25) holds for \( f \in C_c^\infty(\mathbb{R}) \). If \( H_0 \) is bounded, then \( \nu_p \) is finite and (25) also holds for \( f \in \mathcal{W}_p \cup \mathcal{R} \).

(ii) The measure \( \nu_2 \) is absolutely continuous. If, in addition, \( H_0 \) is bounded, then \( \nu_p \) is absolutely continuous for \( p = 3 \).
(iii) Assume, in addition, that \( V \in \mathcal{L}_1(\mathcal{M}, \tau) \) if \( p = 2 \) or that \( H_0 \) is bounded if \( p = 3 \). Then the density \( \eta_p \) of \( \nu_p \) can be computed recursively by (26).

**Remark 5.3.** When \( V \in \mathcal{L}_1(\mathcal{M}, \tau) \), Koplienko’s spectral shift function \( \eta_2 = \eta_{H_0, H_0 + V} \) can be represented by (26), which reduces to the known formula (see [15,26])

\[
\eta_2(t) = \tau(V) - \int_{-\infty}^{t} \xi_{H_0 + V, H_0}(\lambda) d\lambda - \int_{t}^{\infty} d\tau\left(E_{H_0}(\lambda)V\right)
= -\int_{-\infty}^{t} \xi_{H_0 + V, H_0}(\lambda) d\lambda + \tau\left[E_{H_0}((-\infty, t))V\right].
\]

For \( V \) in the standard Hilbert–Schmidt class, no explicit formula for \( \eta_{H_0, H_0 + V} \) is known; existence of Koplienko’s spectral shift function is proved implicitly by approximation of \( V \) by finite-rank operators.

**Remark 5.4.** Representation (8) of Koplienko’s spectral shift function via Krein’s spectral shift function was obtained by integrating by parts in the trace formula in (6) [15,26]. When \( V \in \mathcal{L}_1(\mathcal{M}, \tau) \) and \( f \in \mathcal{R}_b \) (or \( f \in \mathcal{R} \) if \( H_0 \) is bounded), one can see from Lemma 3.1 that \( \tau\left(\frac{d}{dt}|_{t=0} f(H_0 + tV)\right) = \tau[Vf'(H_0)] \), and thus,

\[
\tau\left[R_{2, H_0, V}(f)\right] = \tau[f(H_0 + V) - f(H_0) - Vf'(H_0)].
\]

When \( \mathcal{M} \) is finite and \( H_0 \) is bounded (so that \( \eta_2 \) is integrable and supported in a segment containing the spectra of \( H_0 \) and \( H_0 + V \)), integrating by parts in Koplienko’s trace formula in (7) gives

\[
\tau\left[f(H_0 + V) - f(H_0) - Vf'(H_0) - \frac{1}{2}V^2 f''(H_0)\right]
= \int_{\mathbb{R}} f'''(t) \left(-\int_{-\infty}^{t} \eta_2(\lambda) d\lambda + \frac{1}{2} \tau\left[V^2 E_{H_0}((-\infty, t))\right]\right) dt.
\]

The bound for the remainder in the approximation formula (27) is \( \mathcal{O}(\|V\|_2^2) \) since \( \|\eta_2\|_1 = \frac{\|V\|_2^2}{2} \) and \( \eta_2 \geq 0 \) (see [15,26] for properties of \( \eta_2 \)).

**Corollary 5.5.** Let \( H_0 \) be a self-adjoint operator in \( \mathcal{M} \) and \( V \) a self-adjoint operator in \( \mathcal{L}_p(\mathcal{M}, \tau) \), where \( 2 < p \in \mathbb{N} \) if \( \mathcal{M} = \mathcal{B}(\mathcal{H}) \) or \( p = 3 \) if \( \mathcal{M} \) is a general semi-finite von Neumann algebra. Then, there exists a sequence \( \{\eta_{p,n}\}_n \) of \( L_\infty \)-functions such that

\[
\tau\left[R_{p, H_0, V}(f)\right] = \lim_{n \to \infty} \int_{\mathbb{R}} f^{(p)}(t) \eta_{p,n}(t) dt,
\]

for \( f \in \mathcal{W}_p \).
Proof. Given $V \in \mathcal{L}_p(\mathcal{M}, \tau)$, there exists a sequence of operators $V_n \in \mathcal{M}$, which are linear combinations of $\tau$-finite projections in $\mathcal{M}$ (or just finite rank operators when $\mathcal{M} = \mathcal{B}(\mathcal{H})$) such that $\lim_{n \to \infty} \|V - V_n\|_p = 0$. Then by Lemma 3.11 and Proposition 3.9 for $f \in \mathcal{W}_p$,

$$\lim_{n \to \infty} \tau[R_{p,H_0,V_n}(f)] = \tau[R_{p,H_0,V}(f)].$$

(28)

By Theorem 5.1 in the case of $\mathcal{M} = \mathcal{B}(\mathcal{H})$ or Theorem 5.2 in the case of a general $\mathcal{M}$, respectively, applied to the $\tau$-Hilbert–Schmidt perturbations $V_n$, there exists a sequence $\{\eta_{p,n}\}_n$ of $L_\infty$-functions such that

$$\tau[R_{p,H_0,V_n}(f)] = \int_{\mathbb{R}} f^{(p)}(t)\eta_{p,n}(t)\,dt.$$  

(29)

Combining (28) and (29) completes the proof.  

Theorem 5.6. Let $\tau$ be finite and let $H_0 = H_0^\ast$ be affiliated with the algebra $\mathcal{M}$ and $V = V^* \in \mathcal{M}$. Assume that $(zI - H_0)^{-1}$ and $V$ are free in the noncommutative space $(\mathcal{M}, \tau)$. Then for $p \geq 3$ the following assertions hold.

(i) There is a unique finite real-valued measure $\nu_p$ on $\mathbb{R}$ such that the trace formula (25) holds for $f \in \mathcal{W}_p$.

(ii) If, in addition, $H_0$ is bounded, then $\nu_p$ is absolutely continuous and supported in $[a,b] \supset \sigma(H_0) \cup \sigma(H_0 + V)$. The density $\eta_p$ of $\nu_p$ can be computed recursively by (26). In this case, (25) also holds for $f \in \mathfrak{R}$.

6. Recursive formulas for the Cauchy transform

Let $H_0$ and $V$ be self-adjoint operators in $\mathcal{M}$. Assume, in addition, that $V \in \mathcal{L}_2(\mathcal{M}, \tau)$. In this section we investigate a measure $\nu_p = \nu_p,H_0,V$ as defined in (25) for $f = f_z$ and $f(t) = t^p$. We derive properties of the Cauchy transform of the measure $\nu_p$ which will be used in Section 7 to show that the measure $\nu_{p+1} = \nu_{p+1},H_0,V$ satisfying (25) for $f \in \mathfrak{R}$ is absolutely continuous and its density can be determined explicitly via the density of $\nu_p$ and an integral of a spline function against a certain multiple spectral measure. In addition, for a general trace $\tau$, the results of this section will be used in Section 8 to prove existence of an absolutely continuous measure $\nu_3$ satisfying (25) for $p = 3$ and find an explicit formula for the density of $\nu_3$.

Let $G_\nu$ denote the Cauchy transform of a finite measure $\nu$:

$$G_\nu(z) = \int_{\mathbb{R}} \frac{1}{z - t} \,d\nu(t), \quad \text{Im}(z) \neq 0.$$  

(30)

The goal of this section is to prove the theorem below.

Theorem 6.1. Let $H_0 = H_0^\ast \in \mathcal{M}$ and $V = V^* \in \mathcal{L}_2(\mathcal{M}, \tau)$. Suppose that $\nu_p$ is a real-valued absolutely continuous measure satisfying (25) for $f = f_z$ and $f(t) = t^p$. Let $G : \mathbb{C}_+ \to \mathbb{C}$ be the analytic function satisfying
\begin{align}
G^{(p+1)}(z) &= -G_v^{(p)}(z) - (-1)^{(p+1)} \tau \left[ ((zI - H_0)^{-1} V)^p (zI - H_0)^{-1} \right], \quad (31) \\
\lim_{|z| \to \infty} G(z) &= 0. \quad (32)
\end{align}

Then \( G(z) \) is the Cauchy transform of the measure \( \nu_{p+1} \) satisfying (25) for \( f = f_z \), which is absolutely continuous with the density given by

\[ \eta_{p+1}(t) = \frac{1}{p!} \left( \tau(V^p) - p! \nu_p((-\infty, t]) - \int_{\mathbb{R}^p} \Delta_{\lambda_1, \ldots, \lambda_p}^{(p-1)}((\lambda - t)^{p-1}) \, dm_{p, H_0, V}(\lambda_1, \ldots, \lambda_p) \right), \]

for a.e. \( t \in \mathbb{R} \).

**Lemma 6.2.** Let \( \nu_p \) be a measure satisfying (25) for \( f(t) = t^p \). Then

\[ \int_{\mathbb{R}} dv_p(t) = \frac{1}{p!} \tau(V^p). \]

**Proof.** Applying the trace formula (25) to the polynomial \( f(t) = t^p \) and applying Lemma 3.1 give

\[ p! \int_{\mathbb{R}} dv_p(t) = \tau \left( (H_0 + V)^p - \sum_{j=0}^{p-1} \sum_{k_0, k_1, \ldots, k_j \geq 0} H_0^{p_0} V H_0^{p_1} V \ldots V H_0^{p_j} \right) = \tau(V^p). \]

**Lemma 6.3.** Let \( H_0 = H_0^* \in \mathcal{M} \) and \( V = V^* \in \mathcal{L}_p(\mathcal{M}, \tau) \). Let \( \nu_p \) and \( \nu_{p+1} \) be compactly supported measures. Then \( \nu_p \) and \( \nu_{p+1} \) satisfy (25) for \( f = f_z \) if and only if

\[ G_{\nu_{p+1}}^{(p+1)}(z) = -G_{\nu_p}^{(p)}(z) - (-1)^{(p+1)} \tau \left[ ((zI - H_0)^{-1} V)^p (zI - H_0)^{-1} \right]. \]

**Proof.** The result follows immediately from Lemma 3.7 upon employing the straightforward equality

\[ (-1)^p G_{\nu_p}^{(p)}(z) = \tau \left[ R_{p, H_0, V}(f_z) \right]. \]

Lemma 6.3 will be used to construct an absolutely continuous measure \( \nu_{p+1} \) satisfying (25) for \( f = f_z \) based on the existence of an absolutely continuous measure \( \nu_p \) satisfying (25) for \( f = f_z \).

**Lemma 6.4.** Let \( H_0 = H_0^* \in \mathcal{M} \) and \( V = V^* \in \mathcal{L}_p(\mathcal{M}, \tau) \). Let \( \nu_p \) be a measure satisfying (25) for \( f = f_z \) and \( f(t) = t^p \). Assume, in addition, that \( \nu_p \) is absolutely continuous with the density \( \eta_p \) compactly supported in \([a, b]\). Assume that \( G : \mathbb{C}_+ \to \mathbb{C} \) is an analytic function satisfying (31). Then \( G \) is determined by
\[ G(z) = -\log(z - b) \frac{1}{p!} \tau(V^p) - \int_{\mathbb{R}} \frac{1}{z - \lambda} \chi_{[a,b]}(\lambda) \int_{a}^{\lambda} \eta_p(t) \, dt \, d\lambda \]

\[ + (-1)^{(p+1)} \frac{1}{p} \int \cdots \int \tau\left[ ((z I - H_0)^{-1} V)^p \right] dz^p, \]

up to a polynomial of degree \( p \).

**Proof.** We note that

\[ -\tau\left[ ((z I - H_0)^{-1} V)^p (z I - H_0)^{-1} \right] = \frac{d}{dz} \left( \frac{1}{p!} \tau\left[ ((z I - H_0)^{-1} V)^p \right] \right). \]

Then by Lemma 6.3,

\[ G(z) = -\int G_{\nu_p}(z) \, dz + (-1)^{(p+1)} \frac{1}{p} \int \cdots \int \tau\left[ ((z I - H_0)^{-1} V)^p \right] dz^p. \quad (33) \]

By the assumption of the lemma, \( d\nu_p(\lambda) = \eta_p(\lambda) \, d\lambda \), and hence,

\[ G_{\nu_p}(z) = \int_{a}^{b} \frac{1}{z - \lambda} \eta_p(\lambda) \, d\lambda. \]

Integrating the latter expression by parts gives

\[ G_{\nu_p}(z) = \left( \frac{1}{z - \lambda} \int_{a}^{\lambda} \eta_p(t) \, dt \right)\bigg|_{a}^{b} - \int_{a}^{b} \frac{1}{(z - \lambda)^2} \int_{a}^{\lambda} \eta_p(t) \, dt \, d\lambda. \quad (34) \]

By Lemma 6.2, the first summand in (34) equals

\[ \frac{1}{z - b} \int_{a}^{b} \eta_p(t) \, dt = \frac{1}{z - b} \frac{1}{p!} \tau(V^p). \quad (35) \]

The second summand in (34) equals

\[ \frac{d}{dz} \left( \int_{a}^{b} \frac{1}{z - \lambda} \int_{a}^{\lambda} \eta_p(t) \, dt \, d\lambda \right). \quad (36) \]

Combining (33)–(36) completes the proof. \( \square \)
Theorem 6.5. Let $H_0 = H_0^* \in \mathcal{M}$ and $V = V^* \in \mathcal{L}_2(\mathcal{M}, \tau)$. Let $[a, b]$ be a segment containing $\sigma(H_0) \cup \sigma(H_0 + V)$. Assume that either $\tau$ is standard or $p = 2$. Let $\nu_p$ be a measure compactly supported in $[a, b]$ and satisfying (25) for $f = f_z$ and $f(t) = t^p$. Assume, in addition, that $\nu_p$ is absolutely continuous with the density $\eta_p$. Then the function

$$G(z) = \frac{1}{p!} \int_{\mathbb{R}} \frac{1}{z - t} \left( \tau(V^p) - p! \nu_p\big((\infty, t] \big) \right)$$

$$- \int_{\mathbb{R}^p} \Delta_{\lambda_1, \ldots, \lambda_p}^{(p-1)} \left( \frac{1}{z - \lambda} \right) dm_{p,H_0,V}(\lambda_1, \ldots, \lambda_p) \, dt$$

satisfies (31) and (32).

Proof. Since $d\nu_p(t) = \eta_p(t) \, dt$, we have

$$\chi_{[a,b]}(\lambda) \int_a^\lambda \eta_p(t) \, dt = \nu_p\big((\infty, \lambda]\big) \chi_{[a,b]}(\lambda). \quad (37)$$

By Remark 4.12, we obtain the representation

$$\tau\left[ ((zI - H_0)^{-1})^p \right] = \int_{\mathbb{R}^p} \Delta_{\lambda_1, \ldots, \lambda_p}^{(p-1)} \left( \frac{1}{z - \lambda} \right) dm_{p,H_0,V}(\lambda_1, \ldots, \lambda_p). \quad (38)$$

Since $\sigma(H_0) \cup \sigma(H_0 + V) \subset [a, b]$, the measure $m_{p,H_0,V}$ is supported in $[a, b]$. By Lemma 2.4(i), we can interchange the order of integration in

$$\int \cdots \int \tau\left[ ((zI - H_0)^{-1})^p \right] \, dz^p$$

$$= \int \cdots \int \left( \int_{[a,b]^p} \Delta_{\lambda_1, \ldots, \lambda_p}^{(p-1)} \left( \frac{1}{z - \lambda} \right) dm_{p,H_0,V}(\lambda_1, \ldots, \lambda_p) \right) \, dz^p$$

and obtain

$$\int \cdots \int \tau\left[ ((zI - H_0)^{-1})^p \right] \, dz^p$$

$$= \int_{[a,b]^p} \Delta_{\lambda_1, \ldots, \lambda_p}^{(p-1)} \left( \int \cdots \int \frac{1}{z - \lambda} \, dz^p \right) dm_{p,H_0,V}(\lambda_1, \ldots, \lambda_p), \quad (39)$$

with a suitable choice of constants of integration on the left-hand side of (39). For a reason to become clear later, we choose the antiderivatives in (39) with real constants of integration. Since

$$\int \cdots \int \frac{1}{z - \lambda} \, dz^p = (z - \lambda)^{p-1} \log(z - \lambda) + \alpha_{p-1} z^{p-1} + \text{pol}_{p-2}(z),$$
with \( \text{pol}_{p-2}(z) \) a polynomial of degree \( p-2 \) and a constant \( \alpha_{p-1} \in \mathbb{R} \) to be fixed later, we obtain by Proposition 2.2(4) that the expression in (39) equals

\[
\frac{1}{(p-1)!} \int_{[a,b]^p} \left( \Delta_{\lambda_1,\ldots,\lambda_p}^{(p-1)}((z-\lambda)^{p-1} \log(z-\lambda)) + \alpha_{p-1} \right) \, dm_{p,H_0,V}(\lambda_1,\ldots,\lambda_p).
\]

By Lemma 6.4 and (37)–(40),

\[
G(z) = -\int_a^b \frac{1}{z-t} v_p((-\infty, t]) \, dt + \frac{(-1)^{p+1}}{p!} \int_{[a,b]^p} \left( \Delta_{\lambda_1,\ldots,\lambda_p}^{(p-1)}((z-\lambda)^{p-1} \log(z-\lambda)) \right. \\
\left. + (-1)^p \log(z-b) + \alpha_{p-1} \right) \, dm_{p,H_0,V}(\lambda_1,\ldots,\lambda_p).
\]

Now we will represent the second integral in (41) as the Cauchy transform of an absolutely continuous measure. If not all \( \lambda_1, \lambda_2, \ldots, \lambda_p \) coincide, then by Proposition 2.2(6) and (5),

\[
\Delta_{\lambda_1,\ldots,\lambda_p}^{(p-1)}((z-\lambda)^{p-1} \log(z-\lambda)) = \frac{1}{(p-2)!} \int_{\mathbb{R}} \frac{\partial^{p-1}}{\partial t^{p-1}} ((z-t)^{p-1} \log(z-t)) \Delta_{\lambda_1,\ldots,\lambda_p}^{(p-1)}((\lambda-t)^{p-2}) \, dt \\
= \frac{1}{(p-2)!} \int_{\mathbb{R}} ((-1)^{p-1}(p-1)! \log(z-t) + \gamma_{p-1}) \Delta_{\lambda_1,\ldots,\lambda_p}^{(p-1)}((\lambda-t)^{p-2}) \, dt \\
= (-1)^{p-1}(p-1) \int_{\mathbb{R}} \log(z-t) \Delta_{\lambda_1,\ldots,\lambda_p}^{(p-1)}((\lambda-t)^{p-2}) \, dt + \frac{1}{(p-1)!} \gamma_{p-1},
\]

with \( \gamma_{p-1} \in \mathbb{R} \). By (42) and Proposition 2.2(5), we obtain

\[
J_{\lambda_1,\ldots,\lambda_p}(z) = \Delta_{\lambda_1,\ldots,\lambda_p}^{(p-1)}((z-\lambda)^{p-1} \log(z-\lambda)) + (-1)^p \log(z-b) + \alpha_{p-1} \\
= (-1)^{p-1}(p-1) \int_{\mathbb{R}} \log(z-t) - \log(z-b) \Delta_{\lambda_1,\ldots,\lambda_p}^{(p-1)}((\lambda-t)^{p-2}) \, dt \\
+ \frac{1}{(p-1)!} \gamma_{p-1} + \alpha_{p-1}.
\]

Since in (41) we need only to consider \( \lambda_1, \ldots, \lambda_p \in (a, b) \) and \( \Delta_{\lambda_1,\ldots,\lambda_p}^{(p-1)}((\lambda-t)^{p-2}) \) is supported in \([\min\{\lambda_1, \ldots, \lambda_p\}, \max\{\lambda_1, \ldots, \lambda_p\}]\), we obtain that in (43) it is enough to take \( t \in [a, b] \). By standard computations, for \( t < b \),

the function \( z \mapsto \log(z-t) - \log(z-b) \) maps \( \mathbb{C}_+ \) to \( \mathbb{C}_- \).
and

\[ \lim_{y \to \infty} iy \left( \log(iy - t) - \log(iy - b) \right) = b - t. \quad (45) \]

Let \( \alpha_{p-1} = -\frac{1}{(p-1)!} \gamma_{p-1} \). Then (44) and (45) along with Proposition 2.2(5) imply that \( J_{\lambda_1, \ldots, \lambda_p} \) in (43) maps \( \mathbb{C}_+ \) to \( \mathbb{C}_\pm \) (depending on the sign of \((-1)^{p-1}\)) and \( \lim_{y \to \infty} iy J_{\lambda_1, \ldots, \lambda_p}(iy) \in \mathbb{R} \). By the classical theory of analytic functions, \( J_{\lambda_1, \ldots, \lambda_p} \) is the Cauchy transform of a finite real-valued measure. If \( \lambda_1 = \lambda_2 = \cdots = \lambda_p \), then

\[
J_{\lambda_1, \ldots, \lambda_1}(z) = \Delta_{\lambda_1, \ldots, \lambda_1}^{(p-1)} ((z - \lambda)^{p-1} \log(z - \lambda)) + (-1)^p \log(z - b) + \alpha_{p-1} \\
= (-1)^{p-1} \left( \log(z - \lambda_1) - \log(z - b) \right) + \alpha_{p-1}. \quad (46)
\]

By (44) and (45), the function \( J_{\lambda_1, \ldots, \lambda_1} \) is also the Cauchy transform of a finite real-valued measure. Below we show that the measure generating \( J_{\lambda_1, \ldots, \lambda_p} \) is absolutely continuous.

If all \( \lambda_1, \lambda_2, \ldots, \lambda_p \) are distinct, then by Proposition 2.2(2),

\[
\Delta_{\lambda_1, \ldots, \lambda_p}^{(p-1)} ((z - \lambda)^{p-1} \log(z - \lambda)) = \sum_{k=1}^{p} (z - \lambda_k)^{p-1} \log(z - \lambda_k) \prod_{j \neq k} (\lambda_k - \lambda_j).
\]

Since \( \Delta_{\lambda_1, \ldots, \lambda_p}^{(p-1)} ((z - \lambda)^{p-1} \log(z - \lambda)) \) is symmetric in \( \lambda_1, \lambda_2, \ldots, \lambda_p \), we may assume without loss of generality that \( \lambda_1 < \lambda_2 < \cdots < \lambda_p \). Then

\[
\phi(t) := \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \Im \left( (-1)^p \left( \log(t + i\varepsilon - b) + \alpha_{p-1} \right) \right) \\
= \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \Im \left( \Delta_{\lambda_1, \ldots, \lambda_p}^{(p-1)} ((t + i\varepsilon - \lambda)^{p-1} \log(t + i\varepsilon - \lambda)) \right) \\
= \left\{ \begin{array}{ll}
(-1)^{p+1} + (-1)^p \sum_{k=1}^{p} \frac{(\lambda_k - t)^{p-1}}{\prod_{j \neq k} (\lambda_k - \lambda_j)} & \text{if } t < \lambda_1, \\
(-1)^p + (-1)^p \sum_{k=m}^{p} \frac{(\lambda_k - t)^{p-1}}{\prod_{j \neq k} (\lambda_k - \lambda_j)} & \text{if } \lambda_{m-1} \leq t < \lambda_m \text{, for } 2 \leq m \leq p, \\
(-1)^{p+1} & \text{if } \lambda_p \leq t < b, \\
0 & \text{if } t \geq b.
\end{array} \right. \quad (47)
\]

By Proposition 2.2(2) and (4),

\[
(-1)^{p+1} + (-1)^p \sum_{k=1}^{p} \frac{(\lambda_k - t)^{p-1}}{\prod_{j \neq k} (\lambda_k - \lambda_j)} = (-1)^{p+1} + (-1)^p \Delta_{\lambda_1, \ldots, \lambda_p}^{(p-1)} ((\lambda - t)^{p-1}) = 0, \quad (48)
\]

and hence, \( \phi \) is supported in \([a, b]\). Combining (47) and (48) gives

\[
\phi(t) = (-1)^{p+1} \chi_{(-\infty, b]}(t) + (-1)^p \Delta_{\lambda_1, \ldots, \lambda_p}^{(p-1)} ((\lambda - t)^{p-1})_+. \quad (49)
\]
Similarly, with the use of Definition 2.1 and Lemma 2.4 (ii), one can see that (49) holds when some of the values $\lambda_1, \lambda_2, \ldots, \lambda_p$ repeat. Combining (41) and (49) gives

\[
G(z) = -\int_a^b \frac{1}{z-t} v_p((\infty, t]) \, dt \\
+ \frac{1}{p!} \int_a^b \int \frac{1}{z-t} \left( \chi_{(-\infty, b]}(t) - \Delta^{(p-1)}_{\lambda_1, \ldots, \lambda_p} ((\lambda - t)^{p-1}_+) \right) dt \, dm_{p, H_0, V}(\lambda_1, \ldots, \lambda_p).
\]

(50)

Changing the order of integration in the second integral in (50) and applying Lemma 6.2 along with the fact that $v_p$ is supported in $[a, b]$ imply the representation

\[
G(z) = \int \frac{1}{z-t} \left( \chi_{(-\infty, b]}(t) \left( \tau(V^p) - v_p((\infty, t]) \right) \\
- \frac{1}{p!} \int \Delta^{(p-1)}_{\lambda_1, \ldots, \lambda_p} ((\lambda - t)^{p-1}_+) \, dm_{p, H_0, V}(\lambda_1, \ldots, \lambda_p) \right) dt \\
= \int \frac{1}{z-t} \left( \tau(V^p) - v_p((\infty, t]) \\
- \frac{1}{p!} \int \Delta^{(p-1)}_{\lambda_1, \ldots, \lambda_p} ((\lambda - t)^{p-1}_+) \, dm_{p, H_0, V}(\lambda_1, \ldots, \lambda_p) \right) dt.
\]

Proof of Theorem 6.1. In view of Theorem 6.5, it is enough to prove that the function

\[
t \mapsto \frac{1}{p!} \left( \tau(V^p) - p! v_p((\infty, t]) - \int \Delta^{(p-1)}_{\lambda_1, \ldots, \lambda_p} ((\lambda - t)^{p-1}_+) \, dm_{p, H_0, V}(\lambda_1, \ldots, \lambda_p) \right)
\]

(51)
is real-valued. The integral

\[
\int \Delta^{(p-1)}_{\lambda_1, \ldots, \lambda_p} ((\lambda - t)^{p-1}_+) \, dm_{p, H_0, V}(\lambda_1, \ldots, \lambda_p)
\]

(52)
can be written as

\[
\int \Delta^{(p-1)}_{\lambda_1, \ldots, \lambda_p} ((\lambda - t)^{p-1}_+) \, d \text{Re}(m_{p, H_0, V}(\lambda_1, \ldots, \lambda_p)) \\
+ i \int \Delta^{(p-1)}_{\lambda_1, \ldots, \lambda_p} ((\lambda - t)^{p-1}_+) \, d \text{Im}(m_{p, H_0, V}(\lambda_1, \ldots, \lambda_p)).
\]

(53)
It is easy to see that
\[
m_{p,H_0,V}(d\lambda_1, d\lambda_2, \ldots, d\lambda_{p-1}, d\lambda_p) = m_{p,H_0,V}(d\lambda_p, d\lambda_{p-1}, \ldots, d\lambda_2, d\lambda_1),
\]
and hence,
\[
\text{Im}(m_{p,H_0,V}(d\lambda_1, d\lambda_2, \ldots, d\lambda_{p-1}, d\lambda_p)) = -\text{Im}(m_{p,H_0,V}(d\lambda_p, d\lambda_{p-1}, \ldots, d\lambda_2, d\lambda_1)). \quad (54)
\]
Along with symmetry of the divided difference \(\Delta_{(\lambda, t)}^{(p-1)}((\lambda - t)^{p-1})\) in \(\lambda_1, \ldots, \lambda_p\), the equality (54) implies that the second integral in (53) equals 0, and thus (52) is real-valued. We have that \(v_1\) and \(\eta_1\) are real-valued. By induction, we obtain that \(v_p\) and \(\eta_p\) are real-valued for every \(p \in \mathbb{N}\). Therefore, (51) is real-valued. \(\Box\)

7. Spectral shift functions for \(\mathcal{M} = \mathcal{B}(\mathcal{H})\)

Proof of Theorem 5.1(i). Let \(H_x = H_0 + xV\). The proof of the theorem will proceed in several steps.

Step 1. Assume first that \(H_0\) is bounded and \(f \in \mathfrak{R}\). Let \([a, b]\) be a segment containing \(\sigma(H_0) \cup \sigma(H_0 + V)\). By Corollary 4.4, the finitely additive measure defined on rectangles by
\[
m^{(1)}_{p,H_x,V}(A_1 \times A_2 \times \cdots \times A_p \times A_{p+1}) = \tau[E_{H_0}(A_1)V E_{H_0}(A_2)V \cdots E_{H_0}(A_p)V E_{H_0}(A_{p+1})],
\]
with \(A_1, \ldots, A_{p+1}\) Borel subsets of \(\mathbb{R}\), extends to a countably additive measure with total variation not exceeding \(\|V\|_2^p\). It follows from Corollary 4.11 and Remark 4.13 that
\[
\tau\left[\frac{d^p}{dx^p} f(H_0 + xV)\right] = p! \int_{\mathbb{R}^{p+1}} \Delta^{(p)}_{(\lambda_1, \ldots, \lambda_{p+1})}(f) \, dm^{(1)}_{p,H_x,V}(\lambda_1, \lambda_2, \ldots, \lambda_{p+1}). \quad (55)
\]
By Proposition 2.2(7),
\[
|\Delta^{(p)}_{(\lambda_1, \ldots, \lambda_{p+1})}(f)| \leq \frac{1}{p!} \max_{\lambda \in [a, b]} |f^{(p)}(\lambda)|,
\]
which along with (55) ensures that
\[
\left|\tau\left[\frac{d^p}{dx^p} f(H_0 + xV)\right]\right| \leq \|V\|_2^p \max_{\lambda \in [a, b]} |f^{(p)}(\lambda)|.
\]
Applying the latter estimate to the integrand in (2) guarantees that \(R_{p,H_0,V}(f)\) is a bounded functional on the space of \(f^{(p)}\) with the norm not exceeding \(\frac{1}{p!}\|V\|_2^p\). Therefore, there exists a measure \(v_{p,H_0,V}\) supported in \([a, b]\) and of variation not exceeding \(\frac{1}{p!}\|V\|_2^p\) such that
\[
\tau[R_{p,H_0,V}(f)] = \int_{a}^{b} f^{(p)}(t) \, dv_{p,H_0,V}(t), \quad (56)
\]
for all \(f \in \mathfrak{R}\).
Step 2. We prove the claim of the theorem for $H_0$ bounded and $f \in \mathcal{W}_p$. Repeating the reasoning of [10, Theorem 2.8], one extends (56) from $\mathbb{R}$ to the set of functions $\mathbb{R} \ni \lambda \mapsto e^{it\lambda}$, $t \in \mathbb{R}$, as follows. By Runge’s Theorem, there exists a sequence of rational functions $r_n$ with poles off $D = \{\lambda: |\lambda| \leq 1 + \|H_0\| + \|V\|\}$ such that

$$r_n^{(k)}(\lambda) \to (it)^ke^{it\lambda}, \quad \lambda \in D, \ k = 0, 1, 2, \ldots,$$

where the convergence is understood in the uniform sense. Making use of Lemma 3.8 and passing to the limit on both sides of (56) written for $f \in \mathbb{R}$ proves (56) for $f(\lambda) = e^{it\lambda}$, with the same measure $\nu_{p,H_0,V}$ as at the previous step. Finally, applying Corollary 3.12 extends (56) to the class of $f \in \mathcal{W}_p$, with the same measure $\nu_{p,H_0,V}$.

Step 3. Now we extend (25) to the case of an unbounded operator $H_0$ and $f \in \mathcal{W}_p$. This is done similarly to [10, Lemma 2.7], with replacement of iterated operator integrals by multiple operator integrals. Let $H_{0,n} = E_{H_0}((-n,n))H_0$ and $H_{x,n} = H_{0,n} + xV$. It follows from (2) of Theorem 1.1 that

$$R_{p,H_0,V}(f) - R_{p,H_{0,n},V}(f) = \frac{1}{(p-1)!} \int_0^1 \left( \frac{d^p}{dx^p} f(H_x) - \frac{d^p}{dx^p} f(H_{x,n}) \right) (1-x)^{p-1} \, dx.$$

There exists a finite Borel measure $\mu_f$ such that $f(\lambda) = \int_{\mathbb{R}} e^{it\lambda} \, d\mu_f(t)$. On the strength of Lemma 3.11,

$$\tau \left[ \frac{d^p}{dx^p} f(H_x) - \frac{d^p}{dx^p} f(H_{x,n}) \right]$$

$$= p! \int_{\Pi^{(p)}} \tau \left[ e^{i(s_0-s_1)H_0} \ldots V e^{i(s_{p-1})H_{x,n}} - e^{i(s_0-s_1)H_0} \ldots V e^{i(s_{p-1})H_{x,n}} \right] d\sigma_f^{(p)}(s_0, \ldots, s_p). \quad (57)$$

Proposition 3.9 implies that the integrand in (57) converges to 0, and hence, the whole expression in (57) converges to 0 as $n \to \infty$. Then applying Proposition 3.9 yields

$$\lim_{n \to \infty} \tau \left[ R_{p,H_0,V}(f) - R_{p,H_{0,n},V}(f) \right]$$

$$= \lim_{n \to \infty} \frac{1}{(p-1)!} \int_0^1 \tau \left[ \frac{d^p}{dx^p} f(H_x) - \frac{d^p}{dx^p} f(H_{x,n}) \right] (1-x)^{p-1} \, dx = 0. \quad (58)$$

By the result of the previous step applied to the bounded operators $H_{0,n}$, there is a sequence of measures $\nu_{p,H_{0,n},V}$ of variation bounded by $c_p$, representing the functionals $R_{p,H_{0,n},V}(f)$ for $f \in \mathcal{W}_p$. Denote by $F_n$ the distribution function of $\nu_{p,H_{0,n},V}$. By Helly’s selection theorem, there is a subsequence $\{F_{n_k}\}_k$ and a function $F$ of variation not exceeding $c_p$ such that $F_{n_k}$ converges to $F$ pointwise and in $L^1_{\text{loc}}(\mathbb{R})$. The trace formula (25) for bounded operators and the convergence in (58) ensure that the measure with the distribution $F$ satisfies (25) for $f \in \mathcal{W}_p$. □

Proof of Theorem 5.1(ii). It is an immediate consequence of Theorem 5.1(i) and Theorem 6.1. □
8. Spectral shift functions for an arbitrary semi-finite $\mathcal{M}$

Proof of Theorem 5.2 for $p = 2$. Due to Theorem 4.1 and Corollary 4.11, the proof of existence of Koplienko’s spectral shift function $\eta_2$ for a Hilbert–Schmidt perturbation $V$ [15, Lemma 3.3] (cf. also [6]) can be extended to the case of a $\tau$-Hilbert–Schmidt perturbation.

The proof of Theorem 5.2 for $p = 3$ will be based on the fact (see the lemma below) that if a measure (possibly complex-valued) satisfies (25) for $f = f_z$, then it satisfies (25) for any $f \in \mathcal{R}_b$.

Lemma 8.1. Let $H_0 = H_0^*$ be an operator affiliated with $\mathcal{M}$ and $V = V^* \in L_2(\mathcal{M}, \tau)$. Let $\nu_p$, with $p = 3$, be a Borel measure satisfying

$$R_{p,H_0,V}(f_z) = p! \int_{\mathbb{R}} \frac{1}{(z-t)^{p+1}} \, d\nu_p(t).$$

Then, for all $f \in \mathcal{R}_b$,

$$R_{p,H_0,V}(f) = \int_{\mathbb{R}} f^{(p)}(t) \, d\nu_p(t). \quad (59)$$

If, in addition, $H_0$ is bounded and $\nu_p$ is compactly supported, then (59) holds for $f \in \mathcal{R}$.

To prove Lemma 8.1, we need a simple lemma below.

Lemma 8.2. Assume that the trace formula (25) holds for $f = f_z$ with a finite measure $\nu_p$. Then,

$$G^{(p)}_{\nu_p}(z) = (-1)^p \tau \left[ (zI - H_0 - V)^{-1} - \sum_{j=0}^{p-1} (zI - H_0)^{-1} (V(zI - H_0)^{-1})^j \right] \quad (60)$$

$$= (-1)^p \tau \left[ (zI - H_0 - V)^{-1} (V(zI - H_0)^{-1})^p \right]. \quad (61)$$

Proof. Differentiating the integral in (30) gives

$$G^{(p)}_{\nu_p}(z) = (-1)^p p! \int_{\mathbb{R}} \frac{1}{(z-t)^{p+1}} \, d\nu_p(t), \quad \text{Im}(z) \neq 0. \quad (62)$$

Applying the trace formula (25) to $f = f_z$ ensures

$$\tau \left[ (zI - H_0 - V)^{-1} - \sum_{j=0}^{p-1} (zI - H_0)^{-1} (V(zI - H_0)^{-1})^j \right] = p! \int_{\mathbb{R}} \frac{1}{(z-t)^{p+1}} \, d\nu_p(t). \quad (63)$$

Comparing (63) with (62) completes the proof of (60); comparing (63) with (19) of Lemma 3.6 completes the proof of (61).
Proof of Lemma 8.1. Step 1. Assume that $H_0$ is bounded. We prove the claim for $f$ a polynomial. For $z \in \mathbb{C} \setminus \mathbb{R}$, with $|z|$ large enough,

$$G_{vp}(z) = \sum_{k=0}^{\infty} z^{-(k+1)} \int_{\mathbb{R}} t^k \, dv_p(t),$$

and hence,

$$(-1)^p G_{vp}^{(p)}(z) = \sum_{k=0}^{\infty} z^{-(k+p+1)}(k+1)(k+2)\ldots(k+p) \int_{\mathbb{R}} t^k \, dv_p(t). \quad (64)$$

On the other hand,

$$(-1)^p G_{vp}^{(p)}(z) = \tau \left[ (zI - H_0 - V)^{-1} - \sum_{j=0}^{p-1} (zI - H_0)^{-1} \left( V(zI - H_0)^{-1} \right)^j \right]$$

$$= \tau \left[ \frac{1}{z} \left( I - \frac{H_0 + V}{z} \right)^{-1} - \sum_{j=0}^{p-1} \frac{1}{z^{j+1}} \left( I - \frac{H_0}{z} \right)^{-1} \left( V \left( I - \frac{H_0}{z} \right)^{-1} \right)^j \right]. \quad (65)$$

Employing the power series expansion in (65) gives

$$(-1)^p G_{vp}^{(p)}(z) = \tau \left[ \frac{1}{z} \sum_{m=0}^{\infty} \left( \frac{H_0 + V}{z} \right)^m \right.$$

$$- \sum_{j=0}^{p-1} \sum_{i=0}^{p-1} \sum_{k_0,k_1,...,k_j \geq 0} \left( H_0 \right)^{i \cdot k_0} V \left( H_0 \right)^{k_1} V \ldots V \left( H_0 \right)^{k_j} \left] \right. \left. \right.$$

$$= \tau \left[ \sum_{m=0}^{\infty} z^{-(m+1)}(H_0 + V)^m \right.$$

$$- \sum_{j=0}^{p-1} \sum_{i=0}^{p-1} \sum_{k_0,k_1,...,k_j \geq 0} z^{-i} H_0^{k_0} V H_0^{k_1} V \ldots V H_0^{k_j} \right]. \quad (66)$$

By expanding $(H_0 + V)^m$ one can see that

$$\tau \left[ \sum_{m=0}^{p-1} z^{-(m+1)}(H_0 + V)^m - \sum_{j=0}^{p-1} \sum_{i=0}^{p-1-j} \sum_{k_0,k_1,...,k_j \geq 0} z^{-i} H_0^{k_0} V H_0^{k_1} V \ldots V H_0^{k_j} \right]$$

$$= 0. \quad (67)$$
Subtracting (67) from (66) yields

\[
(-1)^p G^{(p)}_{\nu_p}(z) = \tau \left[ \sum_{m=p}^{\infty} \sum_{i=0}^{p-1} \sum_{k_0, k_1, \ldots, k_j \geq 0} H_0^{k_0} V H_0^{k_1} V \ldots V H_0^{k_j} \right].
\]

By the continuity of the trace \( \tau \), (68) can be rewritten as

\[
(-1)^p G^{(p)}_{\nu_p}(z) = \tau \left[ \sum_{m=p}^{\infty} \sum_{i=0}^{p-1} \sum_{k_0, k_1, \ldots, k_j \geq 0} H_0^{k_0} V H_0^{k_1} V \ldots V H_0^{k_j} \right].
\]

By comparing the representations for \((-1)^p G^{(p)}_{\nu_p}(z)\) of (64) and (69), we obtain that for any \( k \in \{0 \} \cup \mathbb{N} \),

\[
\tau \left[ (H_0 + V)^k - \sum_{j=0}^{p-1} \sum_{k_0, k_1, \ldots, k_j \geq 0} H_0^{k_0} V H_0^{k_1} V \ldots V H_0^{k_j} \right]
= (k+1)(k+2) \ldots (k+p) \int_{\mathbb{R}} t^k \, d\nu_p(t),
\]

along with Lemma 3.1 proving the trace formula (25) for all polynomials. We note that under the assumptions of Step 1, \( p \) can be any natural number.

Step 2. Assume that \( f \in \mathcal{R}_b \), with \( H_0 \) not necessarily bounded. It is enough to prove the statement for \( f(t) = \frac{1}{(z-t)^{p+1}} \), \( k \in \{0 \} \cup \mathbb{N} \). Applying Lemma 3.7 gives

\[
p! \int_{\mathbb{R}} \frac{1}{(z-t)^{p+1}} \, d\nu_p(t) = (p-1)! \int_{\mathbb{R}} \frac{1}{(z-t)^{p}} \, d\nu_{p-1}(t) - \tau \left[ ((zI - H_0)^{-1} V)^{p-1} (zI - H_0)^{-1} \right].
\]

Differentiating (70) \( k \) times with respect to \( z \) gives
\(-1\)^k (p + k) \int_\mathbb{R} \frac{1}{(z - t)^{p+k}} d\nu_p(t)
\]
\[= (-1)^k (p - 1 + k) \int_\mathbb{R} \frac{1}{(z - t)^{p+k}} d\nu_{p-1}(t) - \frac{d^k}{dz^k} \tau \left[ ((zI - H_0)^{-1}V)^{p-1}(zI - H_0)^{-1} \right].
\]

(71)

Dividing by \(-1\)^k k! on both sides of (71) implies
\[
\frac{(p + k)!}{k!} \int_\mathbb{R} \frac{1}{(z - t)^{p+k}} d\nu_p(t)
\]
\[= \frac{(p - 1 + k)!}{k!} \int_\mathbb{R} \frac{1}{(z - t)^{p+k}} d\nu_{p-1}(t) - \frac{(-1)^k}{k!} \frac{d^k}{dz^k} \tau \left[ ((zI - H_0)^{-1}V)^{p-1}(zI - H_0)^{-1} \right].
\]

(72)

Making use of the representation
\[R_{p-1,H_0,V}\left(\frac{1}{(z - t)^{k+1}}\right) = \frac{(p - 1 + k)!}{k!} \int_\mathbb{R} \frac{1}{(z - t)^{p+k}} d\nu_{p-1}(t)
\]
(see Theorem 5.2 for Koplienko’s spectral shift function) and Lemma 3.2 converts (72) to
\[
\frac{(p + k)!}{k!} \int_\mathbb{R} \frac{1}{(z - t)^{p+k}} d\nu_p(t)
\]
\[= R_{p-1,H_0,V}\left(\frac{1}{(z - t)^{k+1}}\right) - \frac{1}{2} \left[ \frac{d^2}{dx^2} \left|_{x=0} \right. (zI - H_0 - xV)^{-(k-1)} \right].
\]

(73)

By (1),
\[R_{p,H_0,V}\left(\frac{1}{(z - t)^{k+1}}\right)
\]
\[= R_{p-1,H_0,V}\left(\frac{1}{(z - t)^{k+1}}\right) - \frac{1}{2} \left[ \frac{d^2}{dx^2} \left|_{x=0} \right. (zI - H_0 - xV)^{-(k-1)} \right].
\]

(74)

Comparing (73) and (74) completes the proof of (25) for \(f(t) = \frac{1}{(z - t)^{k+1}}\).

**Proof of Theorem 5.2 for \(p = 3\).** When \(H_0\) is bounded, Lemma 8.1 and Theorem 6.1 prove the theorem for \(f \in \mathfrak{F}\). Repeating the argument of Step 2 from the proof of Theorem 5.1(i) for \(\tau\) the standard trace extends (ii) and (iii) of Theorem 5.2 to \(f \in \mathcal{W}_{p}\) for \(H_0\) bounded. Repeating the argument of Step 3 from the proof of Theorem 5.1(i) on each segment of \(\mathbb{R}\) extends (i) to \(f \in C_c^\infty(\mathbb{R})\) for \(H_0\) unbounded. □
Proof of Theorem 5.6. (i) Due to Theorem 4.5, there exists a bounded measure \( v_p \) satisfying the trace formula (25) for \( f \in \mathcal{W}_p \). The proof repeats the proof of Theorem 5.1 for the standard trace.

(ii) Using the moment-cumulant formula (see [27, Theorem 2.17]), we have

\[
\tau \left[ \left( (zI - H_0)^{-1}V \right)^{p-1} \right] = \sum_{\pi = \{B_1, \ldots, B_\ell\} \in \text{NC}(p-1)} k_{K(\pi)}[V, \ldots, V] \prod_{j=1}^{\ell} \tau \left[ (zI - H_0)^{-1}B_j \right],
\]

where (see the proof of Theorem 4.5 for a bit of explanation, or [27, Theorem 2.17] for a thorough description) \( k_{K(\pi)}[V, \ldots, V] \) is a polynomial of \( \tau(V), \tau(V^2), \ldots, \tau(V^{p-1}) \). Since for \( b \geq 1 \),

\[
\tau \left[ (zI - H_0)^{-b} \right] = \int_{\mathbb{R}^b} \frac{1}{(z - \lambda_1) \cdots (z - \lambda_b)} \tau \left( E_{H_0}(d\lambda_1) \cdots E_{H_0}(d\lambda_1) \right),
\]

we have

\[
\prod_{j=1}^{\ell} \tau \left[ (zI - H_0)^{-b} \right] = \int_{\mathbb{R}^{b-1}} \frac{1}{(z - \lambda_1) \cdots (z - \lambda_b)} d\gamma_{p-1, \pi}(\lambda_1, \ldots, \lambda_p),
\]

where \( \gamma_{p-1, \pi} \) is the measure described at (21). Combining (75) and (20) gives

\[
\tau \left[ \left( (zI - H_0)^{-1}V \right)^{p-1} \right] = \int_{\mathbb{R}^{p-1}} \Delta^{(p-2)}_{\lambda_1, \ldots, \lambda_{p-1}} \left( \frac{1}{z - \lambda} \right) dm_{p-1, H_0, V}(\lambda_1, \ldots, \lambda_{p-1}).
\]

Following the lines in the proof of Theorem 6.1 completes the proof of the absolute continuity of \( v_p \) and repeating the proof of Lemma 8.1, Step 1, proves (25) for \( f \) a polynomial. \( \square \)

9. Spectral shift functions via basic splines

We represent the density of the measure \( v_p \) provided by Theorem 5.1 as an integral of a basic spline against a certain multiple spectral measure when \( H_0 \) and \( V \) are matrices. In addition, we show that existence of Krein’s spectral shift function can be derived from the representation of the Cauchy transform via basic splines when \( \mathcal{M} \) is finite. The representation of the Cauchy transform via basic splines, in its turn, follows from the double integral representation of \( f(H_0 + V) - f(H_0) \).

Lemma 9.1. Let \( \dim(\mathcal{H}) < \infty \) and \( H_0 = H_0^*, V = V^* \in \mathcal{M} = \mathcal{B}(\mathcal{H}) \). Then the Cauchy transform of the measure \( v_p \) satisfying (25) equals

\[
G_{v_p}^{(p)}(z) = \frac{d^p}{dz^p} \left[ (-1)^p \int_{\mathbb{R}^{p+1}} \Delta^{(p)}_{\lambda_1, \ldots, \lambda_{p+1}} \left( \frac{1}{(p-1)!} (z - \lambda)^{p-1} \log(z - \lambda) \right) dm_{p, H_0, V}(\lambda_1, \lambda_2, \ldots, \lambda_{p+1}) \right], \quad \text{Im}(z) \neq 0.
\]
Proof. Upon applying Remark 4.12 and Lemma 8.2, we obtain
\[
G_{\nu_p}(z) = (-1)^p \int_{\mathbb{R}^{p+1}} \Delta_{\lambda_1,\ldots,\lambda_{p+1}}^{(p)} \left( \frac{1}{z - \lambda} \right) \, dm_{p,H_0,V}(\lambda_1, \lambda_2, \ldots, \lambda_{p+1}).
\]

By Lemma 2.4, one of the antiderivatives of order \(p\) of the function
\[
z \mapsto \frac{1}{(p-1)!} (z - \lambda)^{p-1} \log(z - \lambda) - c_{p-1}(z - \lambda)^{p-1}
\]
equals
\[
\Delta_{\lambda_1,\ldots,\lambda_{p+1}}^{(p)} \left( \frac{1}{(p-1)!} (z - \lambda)^{p-1} \log(z - \lambda) - c_{p-1}(z - \lambda)^{p-1} \right),
\]
where \(c_{p-1}\) is a constant. Applying Proposition 2.2(4) gives
\[
\Delta_{\lambda_1,\ldots,\lambda_{p+1}}^{(p)} (c_{p-1}(z - \lambda)^{p-1}) = 0,
\]
completing the proof of the lemma. \(\square\)

Lemma 9.2. Let \(D_{p+1} = \{ (\lambda_1, \lambda_2, \ldots, \lambda_{p+1}) : \lambda_1 = \lambda_2 = \cdots = \lambda_{p+1} \in \mathbb{R} \}\). Then, for any \((\lambda_1, \lambda_2, \ldots, \lambda_{p+1}) \in \mathbb{R}^{p+1} \setminus D_{p+1}\) and \(z \in \mathbb{C} \setminus \mathbb{R}\),
\[
\Delta_{\lambda_1,\ldots,\lambda_{p+1}}^{(p)} \left( \frac{1}{(p-1)!} (z - \lambda)^{p-1} \log(z - \lambda) \right)
= \frac{1}{(p-1)!} \int_{\mathbb{R}} \frac{(-1)^p}{z - t} \Delta_{\lambda_1,\ldots,\lambda_{p+1}}^{(p)} ((\lambda - t)^{p-1}) \, dt.
\]

Proof. By Proposition 2.2(6),
\[
\Delta_{\lambda_1,\ldots,\lambda_{p+1}}^{(p)} \left( \frac{1}{(p-1)!} (z - \lambda)^{p-1} \log(z - \lambda) \right)
= \frac{1}{(p-1)!} \int_{\mathbb{R}} \frac{\partial^p}{\partial t^p} \left( \frac{1}{(p-1)!} (z - t)^{p-1} \log(z - t) \right) \Delta_{\lambda_1,\ldots,\lambda_{p+1}}^{(p)} ((\lambda - t)^{p-1}) \, dt
= \frac{1}{(p-1)!} \int_{\mathbb{R}} \frac{(-1)^p}{z - t} \Delta_{\lambda_1,\ldots,\lambda_{p+1}}^{(p)} ((\lambda - t)^{p-1}) \, dt. \quad \square
\]

Theorem 9.3. Let \(\dim(\mathcal{H}) < \infty\) and \(H_0 = H_0^*\), \(V = V^* \in \mathcal{M} = \mathcal{B}(\mathcal{H})\). Then the Cauchy transform of the measure \(\nu_p\) satisfying (25) equals
\[ G_{\nu_p}(z) = \int_{\mathbb{R}} \frac{1}{z-t} \left( \frac{1}{(p-1)!} \int_{\mathbb{R}^{p+1}\setminus D_{p+1}} \Delta^{(p)}_{\lambda_1,\ldots,\lambda_{p+1}} ((\lambda - t)_+^{p-1}) dm^{(2)}_{p,H_0,V} (\lambda_1, \lambda_2, \ldots, \lambda_{p+1}) \right) dt. \]

**Proof.** By Lemmas 9.1 and 9.2,

\[ G_{\nu_p}(z) = \operatorname{pol}_p(z) + \frac{1}{(p-1)!} \int_{\mathbb{R}^{p+1}\setminus D_{p+1}} \left( \int_{\mathbb{R}} \frac{1}{z-t} \Delta^{(p)}_{\lambda_1,\ldots,\lambda_{p+1}} ((\lambda - t)_+^{p-1}) \right) dt \, dm^{(2)}_{p,H_0,V} (\lambda_1, \lambda_2, \ldots, \lambda_{p+1}) + \frac{1}{(p-1)!} \int_{D_{p+1}} \frac{1}{z-\lambda} \, dm^{(2)}_{p,H_0,V} (\lambda, \lambda, \ldots, \lambda), \tag{76} \]

where \( \operatorname{pol}_p(z) \) is a polynomial of degree \( \leq p \). As stated in Proposition 2.2(5), the basic spline \( \Delta^{(p)}_{\lambda_1,\ldots,\lambda_{p+1}} ((\lambda - t)_+^{p-1}) \) is non-negative and integrable, with the \( L_1 \)-norm equal to \( 1/p \). By Corollary 4.4, the measure \( m^{(2)}_{p,H_0,V} \) has bounded variation. On one hand, it guarantees that the first integral in (76) is \( O(1/\operatorname{Im}(z)) \) as \( \operatorname{Im}(z) \to +\infty \). On the other hand, it allows to change the order of integration in the first integral in (76). By Lemma 4.8, the second integral in (76) equals 0. Comparing the asymptotics of \( G_{\nu_p}(z) \) and the integrals in (76) as \( \operatorname{Im}(z) \to +\infty \) implies that \( \operatorname{pol}_p(z) = 0 \), completing the proof of the theorem. \( \square \)

**Corollary 9.4.** Let \( \dim(\mathcal{H}) < \infty \) and \( H_0 = H_0^* \), \( V = V^* \in \mathcal{M} = \mathcal{B}(\mathcal{H}) \). Then the density of the measure \( \nu_p \) satisfying (25) equals

\[ \eta_p(t) = \frac{1}{(p-1)!} \int_{\mathbb{R}^{p+1}\setminus D_{p+1}} \Delta^{(p)}_{\lambda_1,\ldots,\lambda_{p+1}} ((\lambda - t)_+^{p-1}) dm^{(2)}_{p,H_0,V} (\lambda_1, \lambda_2, \ldots, \lambda_{p+1}), \]

for a.e. \( t \in \mathbb{R} \).

**Proof.** By Theorem 9.3, the Cauchy transforms of \( \nu_p \) and \( \eta_p(t)dt \) coincide. This implies (see Step 1 in the proof of Lemma 8.1) that the functionals given by \( \nu_p \) and \( \eta_p(t)dt \) coincide on the polynomials defined on \( [a,b] \), where \( [a,b] \) contains the spectra of \( H_0 \) and \( H_0 + V \). Hence, \( d\nu_p = \eta_p(t)dt \). \( \square \)

Below, we prove absolute continuity of \( \nu_1 \) by techniques different from those of [16].

**Theorem 9.5.** Let \( \tau \) be finite. Let \( H_0 = H_0^* \) be an operator affiliated with \( \mathcal{M} \) and \( V = V^* \in \mathcal{M} \). The trace formula (25) with \( p = 1 \) holds for every \( f \in \mathcal{W}_1 \), with \( \nu_1 \) absolutely continuous. The density \( \eta_1 \) of \( \nu_1 \) is given by the formula

\[ \eta_1(t) = \frac{1}{(p-1)!} \int_{\mathbb{R}^{p+1}\setminus D_{p+1}} \Delta^{(p)}_{\lambda_1,\ldots,\lambda_{p+1}} ((\lambda - t)_+^{p-1}) dm^{(2)}_{p,H_0,V} (\lambda_1, \lambda_2, \ldots, \lambda_{p+1}), \]

for a.e. \( t \in \mathbb{R} \).\]
\begin{equation}
\eta_1(t) = \int_{\mathbb{R}^2 \setminus D_2} \frac{1}{|\mu - \lambda|} \chi(\min(\lambda, \mu), \max(\lambda, \mu))(t) \, dm_{1, H_0, V}(\lambda, \mu),
\end{equation}

for a.e. \( t \in \mathbb{R} \). If, in addition, \( H_0 \) is bounded, then (25) holds for \( f \in \mathcal{R} \).

**Proof.** Repeating the argument in the proof of Theorem 9.3 leads to the formula (76). By Lemma 4.6, the measure \( m_{1, H_0, V} \) is real-valued. Then by Lemma 4.8 and the Poisson inversion, for any \( x \in \mathbb{R} \),

\[
\lim_{\varepsilon \to 0^+} \Im \left( \int_{D_2} \frac{1}{x + i\varepsilon - \lambda} \, dm_{1, H_0, V}(\lambda, \lambda) \right) = 0,
\]

proving Krein’s trace formula for \( f = f_z \) with \( \eta_1 \) given by (77). Adjusting the argument in the proof of Lemma 8.1, Step 2, extends (25) to \( f \in \mathcal{R}_b \). Repeating the argument in the proof of [29, Lemma 8.3.2] extends the result of the theorem from \( f \in \mathcal{R} \) to \( f \in \mathcal{W}_1 \) with the same absolutely continuous measure \( d\nu_1(t) = \eta_1(t) \, dt \).

10. Higher order spectral averaging formulas

**Theorem 10.1.** Assume that \( H_0 = H_0^* \in \mathcal{M} \) and either \( \tau \) is standard or \( p = 2 \). Let \( V \in \mathcal{L}_2(\mathcal{M}, \tau) \). Then the measure

\[
\int_0^1 (1 - x)^{p-1} \tau \left[ (E_{H_0 + xV}(dt)V)^p \right] \, dx
\]

is absolutely continuous with the density equal to

\[
\eta_p(t)(p-1)! - p \int_0^1 (1 - x)^{p-1} \int_{\mathbb{R}^{p+1} \setminus D_{p+1}} \Delta_{\lambda_1, \ldots, \lambda_{p+1}}(\lambda - t)^{p-1} \, dm_{1, H_0, V}(\lambda_1, \ldots, \lambda_{p+1}) \, dx.
\]

**Proof.** Let \([a, b] \supset \sigma(H_0) \cup \sigma(H_0 + V)\). Then by Theorem 1.1 (2) and Remark 4.13,

\[
\tau \left[ R_{p, H_0, V}(f) \right] = \frac{1}{(p-1)!} \int_0^1 (1 - x)^{p-1} \tau \left[ \frac{d^p}{dx^p} f(H_0 + xV) \right] \, dx
\]

\[
= \frac{1}{(p-1)!} \int_0^1 (1 - x)^{p-1} \int_{\mathbb{R}^{p+1}} \Delta_{\lambda_1, \ldots, \lambda_{p+1}}(f) \, dm_{1, H_0, V}(\lambda_1, \ldots, \lambda_{p+1}) \, dx.
\]
for \( f \in C^\infty_c(\mathbb{R}) \) such that \( f\big|_{[a,b]} \) coincides with a polynomial. Applying Proposition 2.2(6) and then changing the order of integration yield

\[
\tau \left[ R_{p,H_0,V}(f) \right] = \frac{p}{(p-1)!} \int_0^1 (1-x)^{p-1} \times \int_{\mathbb{R}^{p+1}\setminus D_{p+1}} \left( \int_{\mathbb{R}} f^{(p)}(t) \Delta_{\lambda_1,\ldots,\lambda_{p+1}}^{(p)}(\lambda - t)^{p-1}_+ dt \right) \, dm^{(1)}_{p,H_0+xV,V}(\lambda_1,\ldots,\lambda_{p+1}) \, dx \]

\[
+ \frac{1}{(p-1)!} \int_0^1 (1-x)^{p-1} \int_{D_{p+1}} f^{(p)}(\lambda) \, dm^{(1)}_{p,H_0+xV,V}(\lambda,\ldots,\lambda) \, dx. \]

\[
\tau \left[ R_{p,H_0,V}(f) \right] = \int_{\mathbb{R}} f^{(p)}(t) \left( \frac{p}{(p-1)!} \int_0^1 (1-x)^{p-1} \times \int_{\mathbb{R}^{p+1}\setminus D_{p+1}} \Delta_{\lambda_1,\ldots,\lambda_{p+1}}^{(p)}(\lambda - t)^{p-1}_+ \, dm^{(1)}_{p,H_0+xV,V}(\lambda_1,\ldots,\lambda_{p+1}) \, dx \right) \right) \, dt \]
Remark 10.3. The argument in the proof of Theorem 10.1 can be repeated for \( p = 1 \), provided \( V \in L_1(\mathcal{M}, \tau) \). Since \( m_{1,H_0,x,V}(A_1 \times A_2) = \tau[EH_0+xV(A_1 \cap A_2)V] \), for \( A_1, A_2 \in \mathbb{R} \), one has that \( m_{1,H_0,x,V}(\mathbb{R}^{p+1} \setminus D_{p+1}) = 0 \). Therefore, (78) converts to

\[
\tau \left[ f(H_0 + V) - f(H_0) \right] = \int_0^1 \int_\mathbb{R} f'(t) \tau \left[ EH_0 + xV(dt)V \right] dx
\]

Along with Krein’s trace formula the latter implies that \( \int_0^1 \tau \left[ EH_0 + xV(dt)V \right] dx = \eta_1(t) dt \).

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References


