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The Lie Dimension Subgroup Conjecture*

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For every $n \geqslant 9$ a group G_n is constructed so that its nth Lie dimension subgroup contains properly its nth lower central subgroup, settling negatively the Lie dimension subgroup conjecture. The question remains open for $n=7$ and 8.

1. INTRODUCTION

Let $\Delta(G)$ be the augmentation ideal of an integral group ring $\mathbb{Z}G$. Define the Lie powers $\Delta^{(i)}(G)$ of $\Delta(G)$ inductively by $\Delta^{(1)}(G)=\Delta(G)$, $\Delta^{(m+1)}(G)=(\Delta^{(m)}(G), \Delta(G))\mathbb{Z}G$, the ideal generated by the Lie products $(x, y)=xy-yx$, $x \in \Delta^{(m)}(G)$, $y \in \Delta(G)$. Let $\gamma_n(G)$ be the nth term of the lower central series of G. Denote by $D_n(G)$ and $D_{(n)}(G)$ the nth dimension subgroup and the nth Lie dimension subgroup respectively of G, namely,

$$D_n(G)=G \cap (1+\Delta^n(G)), \qquad D_{(n)}(G)=G \cap (1+\Delta^{(n)}(G)).$$

Denoting by $[x, y]$ the group commutator $x^{-1}y^{-1}xy$, we have

$$\begin{aligned}[x, y]-1 &= x^{-1}y^{-1}(xy-yx)\\ &= x^{-1}y^{-1}[(x-1)(y-1)-(y-1)(x-1)]\\ &= x^{-1}y^{-1}(x-1, y-1).\end{aligned}$$

It follows by induction that

$$\gamma_n(G) \subseteq D_{(n)}(G) \subseteq D_n(G).$$

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The dimension subgroup conjecture, $D_n(G)=\gamma_n(G)$, was disproved for $n=4$ by means of an example by Rips [7]. N. Gupta [2] has recently produced a metabelian 2-group $\mathcal{G}$ (depending on n) such that $D_n(\mathcal{G})\neq\gamma_n(\mathcal{G})$ for all $n\geqslant 4$. The dimension subgroup conjecture is however known to be true for $n\leqslant 3$ and for $n=4$ if G is of odd order (see [1, 5]). The Lie dimension subgroup conjecture refers to the equality $D_{(n)}(G)=\gamma_n(G)$. This is known to be true for $n\leqslant 6$ and for metabelian groups [8]. We disprove the conjecture for all $n\geqslant 9$ by constructing an example which is a modification of the Gupta group. The question remains open for $n=7$ and 8.

In view of our example one might expect that $\gamma_n(G)$ is the group associated with the restricted Lie powers of $\Delta(G)$. These Lie powers are defined inductively,

$$\Delta^{[1]}(G)=\Delta(G), \qquad \Delta^{[m+1]}(G)=(\Delta^{[m]}(G), \Delta(G)),$$

the additive group generated by the Lie products. Then it is a nontrivial result of Gupta and Levin [3] that

$$D_{[n]}(G)=G\cap(1+\Delta^{[n]}(G)\mathbb{Z}G)$$

contains $\gamma_n(G)$. Thus $\gamma_n(G)\subseteq D_{[n]}(G)\subseteq D_{(n)}(G)\subseteq D_n(G)$ for all n. We point out in Section 5 by means of an example G that for $n\geqslant 14$, $D_{[n]}(G)\neq\gamma_n(G)$.

We construct our group G in the next section. For the definition and properties of basic commutators we refer to [4]. Recall that a simple basic commutator is one of the form $[x_{i_1}, x_{i_2}, ..., x_{i_n}]$ with $i_1>i_2\leqslant i_3\leqslant \cdots \leqslant i_n$.

2. Construction of G

Let F be a free group generated by a set of generators Y. Let n be an integer $\geqslant 9$. Define $t=(n-3)/2$ if n is odd and $t=(n-4)/2$ if n is even. Let a, b, c be distinct simple basic commutators of weight t and assume $a>b>c$. Let r be a simple basic commutator of weight 2 when n is odd and of weight 3 when n is even. Assume that there is at least one symbol in each of a, b, c, r which is not in the others. When $n=10$ and $t=3$ assume $r<c$ in the ordering of the basic commutators of weight 3. A more precise specification of these basic commutators will be given in Section 3. Let $p=n-t-1$. Define

$$x_1=[a, r], \qquad y_1=[b, r], \qquad z_1=[c, r]$$
$$x_2=[a, r, a], \qquad y_2=[b, r, b], \qquad z_2=[c, r, c].$$

Then x_1, y_1, z_1 are basic commutators of weight p and x_2, y_2, z_2 are basic commutators of weight $(n-1)$. We construct $G = F/R$ in a number of stages. Write $\gamma_s = \gamma_s(F)$. Set

$$W = \{[a, b], [b, c], [c, a], x_1, y_1, z_1, x_2, y_2, z_2\}.$$

Define

$$R_1 = \gamma_n. \tag{1}$$

Then the elements in W generate a free abelian group modulo γ_n. Let $H = \langle [a, b, x], [b, c, x], [a, c, x] : x \in F \rangle^F$. Set

$$R_2 = \gamma_n \cdot H. \tag{2}$$

Each element in H is of weight $\geqslant (2t+1)$ and contains the symbols from two of $\{a, b, c\}$. Hence each element of H may be written as a product of basic commutators each of weight $\geqslant (2t+1)$ and containing the symbols of two of $\{a, b, c\}$. Thus in F/R_2 the elements of W generate a free abelian group.

Let $w_1 = [a^{\alpha(6)}, r]$, $w_2 = [b^{\alpha(4)}, r]$, $w_3 = [c^{\alpha(2)}, r]$, where $\alpha(k) = 2^k$. Define

$$R_3 = \langle w_1, w_2, w_3 \rangle^F \cdot R_2. \tag{3}$$

Recalling the notation

$$[x, ny] = [x, \underbrace{y, y, ..., y}_{n}]$$

we have the

LEMMA 1. *If* $[x, y], [x, 2y], ..., [x, sy]$ *commute then*

$$[x, y^s] = [x, y]^s \, [x, 2y]^{\binom{s}{2}} \cdots [x, sy].$$

Proof. Induction on s. ∎

A direct consequence is the

LEMMA 2. *In* F/R_3, $[a, r, a^s] = [a, r, a]^s = [a^s, r, a]$.

In F/R_3 we have:

LEMMA 3. (i) *The order of* x_2 *is* $\alpha(6)$, *of* y_2 *is* $\alpha(4)$, *and of* z_2 *is* $\alpha(2)$.

(ii) $x_1^{\alpha(6)} = x_2^{\alpha(5)}$; $y_1^{\alpha(4)} = y_2^{\alpha(3)}$; $z_1^{\alpha(2)} = z_2^{\alpha(1)}$.

(iii) *The order of* x_1 *is* $\alpha(7)$, *of* y_1 *is* $\alpha(5)$, *and of* z_1 *is* $\alpha(3)$.

Proof. We give the proof for x_2 and x_1. The other cases are similar. By Lemma 2, $x_2^{\alpha(6)} \equiv [a, r, a^{\alpha(6)}] \equiv [a^{\alpha(6)}, r, a] \equiv 1 \bmod R_3$. We now show that $x_2^{\alpha(5)} \not\equiv 1 \bmod R_3$. If $x_2^{\alpha(5)} \in R_3$ then $x_2^{\alpha(5)} \equiv w \bmod \gamma_n$ with $w \in \langle w_1 \rangle^F$. Also,

$$w \equiv w_1^s \Pi [w_1, y_i]^{\pm 1} \qquad \bmod \gamma_n \quad \text{with} \quad y_i \in F. \tag{$*$}$$

By Lemma 1,

$$w_1 \equiv [a, r]^{\alpha(6)} [a, r, a]^{\binom{\alpha(6)}{2}} \qquad \bmod \gamma_n.$$

Since $x_2 \in \gamma_{n-1}$ this implies that $s = 0$ in $(*)$. Further, by Lemma 1,

$$[w_1, y_i] \equiv [[a, r]^{\alpha(6)}, y_i] \equiv [a, r, y_i]^{\alpha(6)} \qquad \bmod \gamma_n.$$

It follows from $(*)$ that $x_2^{\alpha(5)}$ is congruent $\bmod \gamma_n$ to a product of basic commutators each to the power $\alpha(6)$ which is impossible as x_2 is a basic commutator.

(ii) By Lemma 2,

$$\begin{aligned} x_1^{\alpha(6)} = [a, r]^{\alpha(6)} &\equiv [a^{\alpha(6)}, r][a, r, a]^{-\binom{\alpha(6)}{2}} \\ &\equiv [a, r, a]^{-\alpha(5)(\alpha(6)-1)} \qquad \bmod R_3 \\ &\equiv [a, r, a]^{\alpha(5)} \qquad (\text{as } x_2^{\alpha(6)} \equiv 1), \\ &= x_2^{\alpha(5)}. \end{aligned}$$

(iii) That the order of x_1 is $\alpha(7)$ follows from (i) and (ii). ∎

Let $w_4 = z_2^{-1} y_2^{\alpha(2)}$; $w_5 = y_2^{-1} x_2^{\alpha(2)}$; $w_6 = [a, b]^{-\alpha(4)} x_2^{\alpha(2)}$; $w_7 = [b, c]^{-\alpha(2)} x_2^{\alpha(2)}$; $w_8 = [c, a]^{\alpha(2)} x_2^{\alpha(1)}$. Define

$$R_4 = \langle w_4, w_5, w_6, w_7, w_8 \rangle^F \cdot R_3. \tag{4}$$

Note that $[R_4, F] \subseteq R_3$. This is because $[w_i, x] \in R_3$ for $x \in F$, $i = 4$ to 8.

LEMMA 4. *The order of* $[a, b]$ *in* F/R_4 *is* $\alpha(8)$.

Proof. This follows because W generates a free abelian group in F/R_2 and the order of x_2 in F/R_3 is $\alpha(6)$. Observe that $[a, b]$, $[b, c]$, $[c, a]$ generate a free abelian group in F/R_3. ∎

Let $w_9 = a^{-\alpha(6)} y_1^{\alpha(2)} z_1^{\alpha(1)}$; $w_{10} = b^{-\alpha(4)} x_1^{-\alpha(2)} z_1$; and $w_{11} = c^{-\alpha(2)} x_1^{-\alpha(1)} y_1^{-1}$. Define

$$R = \langle w_9, w_{10}, w_{11} \rangle^F \cdot R_4. \tag{5}$$

Let $G = F/R$. This completes the construction of G.

3. Order of $[a, b]$

The purpose of this section is to show that the order of $[a, b]$ in G is precisely $\alpha(8)$. Set $W = \{x_1, x_2, y_1, y_2, z_1, z_2, [a, b], [b, c], [c, a]\}$ and $T = \{a, b, c, r\}$. Modulo γ_n, the relators (2), (3), and (4) are all relations between elements of W (e.g.,

$$[a^{\alpha(6)}, r] \equiv [a, r]^{\alpha(6)} [a, r, a]^{\binom{\alpha(6)}{2}}).$$

We are interested in relations modulo γ_n involving elements of W which are deducible from relators (5). We claim that these are already deducible from relators (2), (3), and (4). Then it will follow from Lemma 4 that the order of $[a, b]$ in F/R is $\alpha(8)$. We may factor out any commutator y for which y and $[y, x]$, $x \in F$ when written as a product of basic commutators modulo γ_n does not involve an element of W.

We now specify more precisely the elements a, b, c, and r. Let F be free on a_2, a_1, b_2, c_2, r_2 with the ordering $a_1 < r_2 < c_2 < b_2 < a_2$. Define

$$a = [a_2, (t-1)a_1], \qquad b = [b_2, (t-1)a_1], \qquad c = [c_2, (t-1)a_1]$$

and

$$r = [r_2, a_1] \text{ if } n \text{ is odd and } = [r_2, a_1, a_1] \text{ if } n \text{ is even.}$$

Assume also that r is the smallest basic commutator of its weight. The consequences of any commutator involving more than one occurrence of r_2 will also involve r_2 more than once. The elements of W involve r_2 once only and hence we can assume that any commutator involving r_2 more than once is one.

LEMMA 5. *$[y, y_1 y_2 \cdots y_n] = \Pi[y, y_{i_1}, y_{i_2}, ..., y_{i_s}]$ with $i_1 < i_2 < \cdots < i_s$.*

Proof. By a standard commutator identity

$$[y, y_1 y_2 \cdots y_n] = [y, y_2 \cdots y_n][y, y_1][y, y_1, y_2 \cdots y_n].$$

Applying induction to $[y, y_2 \cdots y_n]$ and $[[y, y_1], y_2 \cdots y_n]$ we obtain the result. ∎

Going mod γ_{t+1} it is easily seen that we have only to concentrate on consequences of the form $[w, x]$, $w = w_9, w_{10}, w_{11}$, $x \in F$. Consider consequences of the word $w = w_9 = a^{-\alpha(6)} y_1^{\alpha(2)} z_1^{\alpha(1)}$ involving elements from W modulo γ_n. Then for $y \in F$,

$$\begin{aligned}[w, y] &= [a^{-\alpha(6)} y_1^{\alpha(2)} z_1^{\alpha(1)}, y] \\ &\equiv [a^{-\alpha(6)}, y][y_1^{\alpha(2)}, y][z_1^{\alpha(1)}, y] \qquad \text{mod } \gamma_n.\end{aligned}$$

For any $y \in F$, $y \equiv \beta_1^{\alpha_1}\beta_2^{\alpha_2}\cdots\beta_l^{\alpha_l}$ modulo γ_{n-t}, where β_i are basic commutators with $\beta_1 < \beta_2 < \cdots < \beta_l$. Hence by Lemma 5, all commutators $[w, y]$ are a product modulo γ_n of elements of the form

$$[a^{-\alpha(6)}, \beta_1, \beta_2, ..., \beta_s][y_1^{\alpha(2)}, \beta_1, \beta_2, ..., \beta_s][z_1^{\alpha(1)}, \beta_1, \beta_2, ..., \beta_s] \quad (**)$$

with β_i basic commutators satisfying $\beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_s$ and their inverses. (Note that the negative α_i can be eliminated using $[y, x^{-1}]^{-1} = [y, x][y, x, x^{-1}]$).

Define *constituents* of a basic commutator as follows. The constituents of a generator of F is the null set. Suppose c is a basic commutator with $c = [l, q]$ and the constituents of l and q have been defined inductively. Then the constituents of c are l, q and the constituents of l and q.

Also if c is a basic commutator and $c = [l, q]$ we term l the *first component* of c and q the *second component* of c.

LEMMA 6. *Suppose $[l, q]$ is a basic commutator. Suppose the second components of l and q are $\leqslant y$. Then $[l, q, y]$ is a product modulo γ_m of basic commutators in which l and q are constituents where $m = 2$ weight $l +$ weight $q +$ weight y.*

Proof. If $q \leqslant y$ then $[l, q, y]$ is a basic commutator. Otherwise, suppose $q > y$. Then

$$\begin{aligned}[l, q, y] &= [l; q, y]^{[l,y][l,q]}[l, q; l, y][l, q; q, y]^{[l,y]}[l, y; q, y][l, y, q]^{[q,y]} \\ &\equiv [l; q, y][l, q; q, y][l, y; q, y][l, y, q][l, y, q; q, y] \qquad \bmod \gamma_m.\end{aligned}$$

All commutators, on the right are basic and each involves l and q as constituents. This proves the lemma.

Now, we return to (**). We note that $[a, \beta_1, ..., \beta_s]$ is always a basic commutator or the inverse of one when $s = 1$ and $\beta_1 > a$. Consider then $[y_1, \beta_1] = [b, r, \beta_1]$ and $[z_1, \beta_1] = [c, r, \beta_1]$. When $r \leqslant \beta_1$ these are basic. Otherwise suppose $r > \beta_1$ and consider

$$\begin{aligned}[b, r, \beta_1] &= [b; r, \beta_1]^{[b,\beta_1][b,r]}[b, r; b, \beta_1][b, r; r, \beta_1]^{[b,\beta_1]} \\ &\quad \times [b, \beta_1; r, \beta_1][b, \beta_1, r]^{[r,\beta_1]} \\ &\equiv [b; r, \beta_1][b, \beta_1; r, \beta_1][b, \beta_1, r].\end{aligned}$$

Now by Lemma 6, $[b, r, \beta_1, ..., \beta_s]$ is congruent to a product with either $[r, \beta_1]$ or $[b, \beta_1]$ as a constituent. Note that weight of $\beta_1 \leqslant$ weight of r.

Similarly $r \leqslant \beta_1$ or else $[z_1, \beta_1, ..., \beta_s]$ is congruent to a product with either $[z, \beta_1]$ or $[c, \beta_1]$ as a constituent. Also note that when $\beta_1 < r$ then $[a^{-\alpha(6)}, \beta_1, ..., \beta_s]$ is congruent to a product of basic commutators none contained in W. Thus the consequences of w involving elements of W are deducible from

$$[a^{-\alpha(6)}, \beta_1, ..., \beta_s][y_1^{\alpha(2)}, \beta_1, ..., \beta_s][z_1^{\alpha(1)}, \beta_1, ..., \beta_s],$$

where $r \leqslant \beta_1$. If $s \geqslant 2$, the basic commutators produced are not in W. So we need only consider the case $\beta_1 \in T$. We work modulo basic commutators not in W.

(i) Suppose $\beta_1 = a$ then $[a^{-\alpha(6)}, a][y_1^{\alpha(2)}, a][z_1^{\alpha(1)}, a] \equiv 1$.

(ii) Suppose $\beta_1 = b$ then

$$\begin{aligned}&[a^{-\alpha(6)}, b][y_1^{\alpha(2)}, b][z_1^{\alpha(1)}, b]\\ &\quad\equiv [a, b]^{-\alpha(6)}\,[b, r, b]^{\alpha(2)} \equiv [a, b]^{-\alpha(6)}\,y_2^{\alpha(2)}\\ &\quad\equiv ([a, b]^{-\alpha(4)}\,y_2)^{\alpha(2)} = ([a, b]^{-\alpha(4)}x_2^{\alpha(2)}w_5^{-1})^{\alpha(2)}\\ &\quad= (w_6 w_5^{-1})^{\alpha(2)}.\end{aligned}$$

(iii) Suppose $\beta_1 = c$ then

$$\begin{aligned}&[a^{-\alpha(6)}, c][y_1^{\alpha(2)}, c][z_1^{\alpha(1)}, c] \equiv [a, c]^{-\alpha(6)} z_2^{\alpha(1)}\\ &\quad\equiv ([a, c]^{-\alpha(5)} z_2)^{\alpha(1)} = ([a, c]^{-\alpha(5)}\,y_2^{\alpha(2)} w_4^{-1})^{\alpha(1)}\\ &\quad= ([a, c]^{-\alpha(5)} x_2^{\alpha(4)} w_5 w_4^{-1})^{\alpha(1)} = (w_8^{-\alpha(3)} w_5 w_4^{-1})^{\alpha(1)}.\end{aligned}$$

(iv) Finally, let $\beta_1 = r$ then $[a^{-\alpha(6)}, r][y_1^{\alpha(2)}, r][z_1^{\alpha(1)}, r] \equiv (w_1)^{-a^{-\alpha(6)}}$.

It follows similarly that the consequences of w_{10} and w_{11} modulo γ_n involving elements of W are all deducible from relators (2), (3), and (4). This concludes the proof that the order of $[a, b]$ in F/R is $\alpha(8)$.

We need one more lemma. Recall that $p = (n - t - 1)$.

LEMMA 7. (i) $b^{\alpha(7)}c^{\alpha(6)} \in \gamma_p^{\alpha(6)} \cdot \gamma_{n-1} \cdot R$.

(ii) $a^{\alpha(7)}c^{-\alpha(5)} \in \gamma_p^{\alpha(4)} \cdot \gamma_{n-1} \cdot R$.

(iii) $a^{\alpha(6)}b^{\alpha(5)} \in \gamma_p^{\alpha(2)} \cdot \gamma_{n-1} \cdot R$.

Proof. (i) Let

$$\begin{aligned}b^{\alpha(7)}c^{\alpha(6)} &= b^{\alpha(4)\alpha(3)}c^{\alpha(2)\alpha(4)}\\ &\equiv (x_1^{-\alpha(2)}z_1)^{\alpha(3)}\,(x_1^{-\alpha(1)}\,y_1^{-1})^{\alpha(4)} && (\text{mod } R) \quad (\text{by relators } (5))\\ &\equiv x_1^{-\alpha(5)} z_1^{\alpha(3)} x_1^{-\alpha(5)}\,y_1^{-\alpha(4)} && (\text{mod } R)\\ &\equiv x_1^{-\alpha(6)}\,y_1^{-\alpha(4)} z_1^{\alpha(3)} && (\text{mod } R)\\ &\equiv 1 && \text{mod}(\gamma_{n-1} \cdot R) \quad (\text{by Lemma } 3).\end{aligned}$$

(ii) Let

$$a^{\alpha(7)}c^{-\alpha(5)} = a^{\alpha(6)\alpha(1)}c^{-\alpha(2)\alpha(3)}$$

$$\equiv (y_1^{\alpha(2)}z_1^{\alpha(1)})^{\alpha(1)}(x_1^{\alpha(1)}y_1)^{\alpha(3)} \quad (\text{mod } R) \quad (\text{by relators (5)})$$

$$\equiv x_1^{\alpha(4)}y_1^{\alpha(4)}z_1^{\alpha(2)} \quad (\text{mod } R)$$

$$\equiv x_1^{\alpha(4)} \quad (\text{mod } \gamma_{n-1}\cdot R) \quad (\text{by Lemma 3})$$

$$\equiv 1 \quad (\text{mod } \gamma_p^{\alpha(4)}\cdot\gamma_{n-1}\cdot R).$$

(ii) Let

$$a^{\alpha(6)}b^{\alpha(5)} = a^{\alpha(6)}b^{\alpha(4)\alpha(1)}$$

$$\equiv y_1^{\alpha(2)}z_1^{\alpha(1)}(x_1^{-\alpha(2)}z_1)^{\alpha(1)} \quad (\text{mod } R) \quad (\text{by relators (5)})$$

$$\equiv x_1^{-\alpha(3)}y_1^{\alpha(2)}z_1^{\alpha(2)} \quad (\text{mod } R)$$

$$\equiv (x_1^{-\alpha(1)}y_1)^{\alpha(2)} \quad (\text{mod } \gamma_{n-1}\cdot R) \quad (\text{by Lemma 3})$$

$$\equiv 1 \quad (\text{mod}(\gamma_p)^{\alpha(2)}\cdot\gamma_{n-1}\cdot R). \quad \blacksquare$$

4. Calculations in the Group Ring

Recall that $\Delta(F)$ is the augmentation ideal of $\mathbb{Z}F$ and $\Delta(F, R)$ is the ideal $\mathbb{Z}F(R-1)$. We show that there exists a word $g \in F$ such that $g \equiv [a, b]^{\alpha(7)}$ mod R and $g-1 \equiv 0$ mod$(\Delta F)^{(n)} + \Delta(F, R)$. Since we already know that $[a, b]^{\alpha(7)} \neq 1$ mod R it follows that for $\bar{g}$, the image of g in $F/R = G$, we have $1 \neq \bar{g} \in \gamma_n(G)$. Therefore, $D_{(n)}(G) \neq \gamma_n(G)$. We shall use throughout the well known identity (see [6])

$$\Delta^{(m)}(G)\cdot\Delta^{(k)}(G) \subseteq \Delta^{(m+k-1)}(G).$$

Lemma 8. (a) $\alpha(3)(c-1)$ *and* $\alpha(5)(b-1)$ *belong to* $(\Delta F)^{(p)} + \Delta(F, R)$.

(b) *Modulo* $(\Delta F)^{(n)} + \Delta(F, R)$ *we have*

(i) $\alpha(6)(c-1)(a-1) \equiv (c-1)(a^{\alpha(6)}-1)$

(ii) $\alpha(6)(a-1)(c-1) \equiv (a-1)(c^{\alpha(6)}-1)$

(iii) $\alpha(7)(b-1)(a-1) \equiv (b-1)(a^{\alpha(7)}-1)$

(iv) $\alpha(7)(a-1)(b-1) \equiv (a-1)(b^{\alpha(7)}-1)$

(v) $\alpha(5)(b-1)(c-1) \equiv (b-1)(c^{\alpha(5)}-1)$

(vi) $\alpha(5)(c-1)(b-1) \equiv (c-1)(b^{\alpha(5)}-1)$.

Proof. (a) Let

$$c^{\alpha(3)}-1\equiv\alpha(3)(c-1)+\binom{\alpha(3)}{2}(c-1)^2 \qquad \mathrm{mod}(\Delta F)^{(p)}+\Delta(F,R)$$
$$\equiv\alpha(3)(c-1)+7(c^{\alpha(2)}-1)(c-1).$$

Since $c^{\alpha(2)}\in\gamma_p(F)$ it follows that the second term on the right side and the left hand side term both belong to $(\Delta F)^{(p)}$. It follows that $\alpha(3)(c-1)\in(\Delta F)^{(p)}+\Delta(F,R)$. The second part is similar.

(b) we prove (i) the remaining congruences follow similarly.

$$\begin{aligned}\alpha(6)(c-1)(a-1)&\equiv(c-1)[(a^{\alpha(6)}-1)+k_1\alpha(5)(a-1)^2\\&\quad+k_2\alpha(4)(a-1)^3],\qquad k_i\in\mathbb{Z}\\&\equiv(c-1)(a^{\alpha(6)}-1)+k_1\alpha(5)(c-1)(a-1)^2\\&\quad+k_2\alpha(4)(c-1)(a-1)^3\\&\equiv(c-1)(a^{\alpha(6)}-1)\qquad\text{by (a).}\ \blacksquare\end{aligned}$$

Consider the element $g=[a,b]^{\alpha(7)}[a,c]^{\alpha(6)}[b,c]^{\alpha(5)}$ in F.

LEMMA 9. $g\equiv[a,b]^{\alpha(7)}$ mod R.

Proof. By relators (4), $[a,c]^{\alpha(2)}\equiv x_2^{\alpha(1)}$ and thus $[a,c]^{\alpha(6)}\equiv x_2^{\alpha(5)}$. Also, $[b,c]^{\alpha(5)}\equiv x_2^{\alpha(5)}$. Therefore, $[a,c]^{\alpha(6)}[b,c]^{\alpha(5)}\equiv x_2^{\alpha(6)}\equiv 1$ by Lemma 3(i). Hence $g\equiv[a,b]^{\alpha(7)}$ as claimed.

LEMMA 10. $g-1\in(\Delta F)^{(n)}+\Delta(F,R)$.

Proof. We work modulo $(\Delta F)^{(n)}+\Delta(F,R)$. Using the identity

$$xy-1=(x-1)+(y-1)+(x-1)(y-1)$$

and the fact that $[a,b]^{\alpha(7)}$, $[a,c]^{\alpha(6)}$, $[b,c]^{\alpha(5)}$ belong to $\gamma_{n-4}(F)$ we see that

$$(g-1)\equiv([a,b]^{\alpha(7)}-1)+([a,c]^{\alpha(6)}-1)+([b,c]^{\alpha(5)}-1).$$

We now show that

$$\begin{aligned}&[a,b]^{\alpha(7)}-1\equiv\alpha(7)((a-1)(b-1)-(b-1)(a-1)\\&[a,c]^{\alpha(6)}-1\equiv\alpha(6)((a-1)(c-1)-(c-1)(a-1))\\&[b,c]^{\alpha(5)}-1\equiv\alpha(5)((b-1)(c-1)-(c-1)(b-1)).\end{aligned}$$

We have $[a, b]^{\alpha(7)}-1 \equiv \alpha(7)([a, b]-1)$ and

$$[a, b]-1=(a^{-1}b^{-1}-1)\{(a-1)(b-1)-(b-1)(a-1)\} + \{(a-1)(b-1)-(b-1)(a-1)\}.$$

We need to show that $\alpha(7)(a^{-1}b^{-1}-1)\{(a-1)(b-1)-(b-1)(a-1)\} \equiv 0$. This follows from part (a) of Lemma 8. We have proved the first of the three congruences. The other two follow similarly. Hence

$$\begin{aligned}(g-1) &\equiv \alpha(7)\{(a-1)(b-1)-(b-1)(a-1)\} \\ &\quad + \alpha(6)\{(a-1)(c-1)-(c-1)(a-1)\} \\ &\quad + \alpha(5)\{(b-1)(c-1)-(c-1)(b-1)\} \\ &\equiv (a-1)(b^{\alpha(7)}-1)-(b-1)(a^{\alpha(7)}-1) \\ &\quad + (a-1)(c^{\alpha(6)}-1)-(c-1)(a^{\alpha(6)}-1) \\ &\quad + (b-1)(c^{\alpha(5)}-1)-(c-1)(b^{\alpha(5)}-1) \qquad \text{(by Lemma 8)} \\ &\equiv (a-1)(b^{\alpha(7)}c^{\alpha(6)}-1)-(b-1)(a^{\alpha(7)}c^{-\alpha(5)}-1) \\ &\quad - (c-1)(a^{\alpha(6)}b^{\alpha(5)}-1) \qquad \text{(by relators (5))} \\ &\equiv (a-1)(d_a^{\alpha(6)}-1)-(b-1)(d_b^{\alpha(4)}-1)-(c-1)(d_c^{\alpha(2)}-1)\end{aligned}$$

from Lemma 7 with $d_a, d_b, d_c \in \gamma_p(F)$,

$$\begin{aligned} &\equiv \alpha(6)(a-1)(d_a-1)-\alpha(4)(b-1)(d_b-1) \\ &\quad - \alpha(2)(c-1)(d_c-1) \\ &\equiv (a^{\alpha(6)}-1)(d_a-1)-(b^{\alpha(4)}-1)(d_b-1) \\ &\quad - (c^{\alpha(2)}-1)(d_c-1) \equiv 0 \qquad \text{(by relators (5)).}\end{aligned}$$

5. Restricted Lie Dimension Subgroups

To show that for $n \geqslant 14$ there exists a group G such that $\gamma_n(G) \neq (D_{[n]}(G))$, we proceed as follows.

Let $n \geqslant 14$ and $t=(n-5)/2$ if n is odd and $t=(n-4)/2$ if n is even. Let a, b, c be distinct simple basic commutators of weight t with $a>b>c$. Let r be a simple basic commutator of weight 3 when n is even and of weight 4 when n is odd. Assume there is one symbol in each of a, b, c, r which is not in the others. We define $x_1, y_1, z_1, x_2, y_2, z_2$ as before and construct G in a similar manner. The element g is given before and an analysis of

Lemma 10 shows that $g-1 \in (\Delta F)^{[n]} + \Delta(FR)$. This uses the result (see [3])

$$(\Delta F)^{[k]} \cdot (\Delta F)^{[l]} \subseteq (\Delta F)^{[k+l-2]}.$$

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