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Journal of Algebra 315 (2007) 874–884

**JOURNAL OF
Algebra**

www.elsevier.com/locate/jalgebra

Extrinsic properties of automorphism groups of formal groups

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Received 18 October 2006

Available online 6 March 2007

Communicated by Susan Montgomery

Abstract

We prove two conjectures on the automorphism group of a one-dimensional formal group law defined over a field of positive characteristic. The first is that if a series commutes with a nontorsion automorphism of the formal group law, then that series is already an automorphism. The second is that the group of automorphisms is its own normalizer in the group of all invertible series over the ground field. A consequence of these results is that a formal group law in positive characteristic is determined by any one of its nontorsion automorphisms.

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Keywords: Formal groups; p -divisible groups; p -adic dynamical systems

Introduction

In this paper we deal with formal group laws over a field of characteristic $p > 0$, in other words with coordinatized one-dimensional formal groups. The theory of one-dimensional formal groups in positive characteristic is rather different in flavor from that in characteristic zero. In particular the fact that a formal group in characteristic zero is determined by one of its endomorphisms is so immediate that it hardly deserves notice. From $f \in \text{End}(G)$, if G is a formal group with coefficients lying in any \mathbb{Q} -algebra, one can derive by a simple degree-by-degree

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computation the logarithm $\log_G : G \rightarrow \mathcal{A}$, where \mathcal{A} is the additive group law $x + y$, and from this the coefficients of $G(x, y)$ drop out, again degree by degree.

In characteristic p , however, the picture is entirely different: for instance, for x^p to be an endomorphism, it is only necessary that $G(x, y) \in \mathbb{F}_p[[x, y]]$. So if there is any hope of characterizing a formal group law by one or more of its endomorphisms, these should be automorphisms. Further, a series such as $\alpha x \in k[[x]]$ will be an automorphism of $G(x, y)$ if and only if the only nonzero coefficients of G occur in degrees congruent to 1 modulo $m - 1$, for m the multiplicative period of α . So we mostly worry about nontorsion automorphisms of G .

The first result along these lines was Theorem 7 of [9], and it states that if u is a nontorsion element of the center \mathbb{Z}_p^* of the absolute automorphism group of a formal group law G and $\psi(x)$ is a series commuting with u , then ψ is an endomorphism of G . A more recent one is due to Li, who generalized the methods of [9] in [6] to the case where $u'(0)$ generates \mathbb{F}_{p^h} .

The current paper’s first main result is the strongest possible theorem of this type: in the group $k[[x]]^\circ$ of all invertible series over k , the centralizer of a nontorsion automorphism of G is contained in $\text{Aut}(G)$. This is certainly of interest in itself, but is also an important ingredient in the proof of the other main result. This theorem says that $\text{Aut}(G)$ is its own normalizer in $k[[x]]^\circ$.

And in turn, this theorem shows that, as a subgroup of $k[[x]]^\circ$, a given automorphism group comes from only one formal group law; this can then be used to see that a single nontorsion automorphism determines completely the formal group law that it belongs to. One must hasten to add that there seems to be no calculation of any effective nature that would allow computation of any of the coefficients of this mysterious group law.

1. Notations, conventions, background

Our formal group laws G will all be of dimension one, defined over a field k of characteristic $p > 0$, and of finite height greater than 1. Since our interest throughout this paper is in coordinatized objects, we will feel free to omit the word “law” and speak somewhat inaccurately of “formal groups.” Indeed, the significance of the word “extrinsic” in our title is that we are dealing with the group of automorphisms of G as series, and how these groups may sit inside the larger group of all invertible power series over k .

The one-dimensionality of our formal groups implies that they are commutative, so that the set of endomorphisms of G , written $\text{End}_k(G)$, is a ring, the addition being G -addition of series, and multiplication being composition. The natural map $\mathbb{Z} \rightarrow \text{End}_k(G)$ is denoted $n \mapsto [n]_G$; and this map extends canonically to \mathbb{Z}_p , so that the endomorphism ring has a natural structure of \mathbb{Z}_p -algebra. Over any algebraically closed field Ω containing k , it has been known since Dieudonné [1] that the endomorphism ring is isomorphic to the maximal order in a central division algebra over \mathbb{Q}_p of rank h^2 and invariant $1/h$, where h is the height of G . When dealing with such a formal group G , we will further assume of k that this field is so large that $\text{End}_k(G) = \text{End}_\Omega(G)$. Such a field k must necessarily contain \mathbb{F}_{p^h} , but this condition may not be sufficient to catch all the endomorphisms of G .

One may use $G(x, y)$ to add not only endomorphisms, but arbitrary series in $xk[[x]]$: if φ and ψ are two such, then their G -sum is $(\varphi +_G \psi)(x) = G(\varphi(x), \psi(x))$. Using the additive \mathbb{Z} -valued valuation v_x on the ambient ring $k[[x]]$ and the shape of $G(x, y)$ as $x + y + (\text{higher terms})$, we see that if $v_x(\psi) > v_x(\varphi)$, then $v_x(\varphi +_G \psi) = v_x(\varphi)$; for similar reasons, if $v_x(\varphi) = v_x(\psi)$, then $v_x(\varphi - \psi) = v_x(\varphi -_G \psi)$. Note that the corresponding statement about the two kinds of addition does not hold.

We will need to make reference to the closeness of a series $u(x)$ to the identity series x , and in our context it is most appropriate to use the nonstandard measure of proximity $w(u) = v_x(u(x) - x)$. This instead of $v_x(\frac{u(x)}{x} - 1)$, which is usually more natural.

As we have said above, the ring of k -endomorphisms of G is denoted $\text{End}_k(G)$; the group of its invertible elements is denoted $\text{Aut}_k(G)$. Because of our assumption that k is sufficiently large, we will drop the subscript. Much of the time, we will be wanting to consider the ring and its units as abstract topological ring and group, so we denote them \mathbf{E} and \mathbf{A} , respectively; then $\mathbf{A} = \mathbf{E}^*$. We will also be dealing with the (noncommutative) fraction field of \mathbf{E} , $\mathbf{D} = \mathbf{E} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. We will have occasion later to use the fact that there are algebraic extensions K of \mathbb{Q}_p such that $\mathbf{D} \otimes_{\mathbb{Q}_p} K \cong M_h(K)$, the ring of h -by- h matrices over K .

If $f(x) \in k[[x]]$, we will denote its n -fold iterate $f^{\circ n}$, but when f is in $\text{End}(G)$ and is considered as element of \mathbf{E} , then this iterate will be denoted simply f^n .

We refer the reader in search of a readable introduction to formal groups to the classic text [2].

2. Information from a single automorphism

How much can you tell about a formal group G in characteristic $p > 0$ by looking at one of its automorphisms $u(x)$? In the case of nontorsion automorphisms, there is one piece of information that comes fairly easily and cheaply: by looking at how fast certain iterates of u approach the identity, we can determine the height of G . Let us explain this by going into the arithmetic of \mathbf{E} a little more deeply.

The division ring \mathbf{D} has on it an additive valuation extending the canonical \mathbb{Z} -valued v_p for which $v_p(p) = 1$. Namely, for $z \in \mathbf{D}$, $V(z)$ is defined to be $v_p(\text{redn}(z))/h$, where $\text{redn}: \mathbf{D} \rightarrow \mathbb{Q}_p$ is the reduced norm. Recall that this multiplicative homomorphism is actually defined for any central simple finite-dimensional K algebra A , and has the property that if $\dim_K A = h^2$ and F is a commutative field with $K \subset F \subset A$ and of degree h over K , then the reduced norm agrees on F with the ordinary F -over- K norm of Galois theory. This is not a definition of the reduced norm, but it is a sufficient description for our purposes. In case A is the ring of matrices over K , the reduced norm is the determinant. One shows easily that $V: \mathbf{D} \rightarrow \frac{1}{h}\mathbb{Z} \cup \{\infty\}$ is an additive valuation in the usual sense: $V(zz') = V(z) + V(z')$ and $V(z + z') \geq \min(V(z), V(z'))$.

Although it represents a break in the presentation, it is most convenient to insert here a mention of the commutator of two elements a and b of \mathbf{E} , even of \mathbf{D} . We define $[a, b]$ to be $ab - ba$. Then as may be easily verified, the commutator is \mathbb{Z}_p -bilinear, $V([a, b]) \geq V(a) + V(b)$, and if $V(a)$ and $V(b)$ both are positive, then $V([1 + a, 1 + b]) \geq V(a) + V(b)$. A little less obvious is the following:

Proposition 1. *If $a, b \in \mathbf{D}$, then $V([a^{n+1}, b]) \geq V([a, b]) + nV(a)$. Moreover, if $V(a) > \frac{1}{p-1}$, then $V([(1 + a)^p, b]) = 1 + V([a, b]) = 1 + V([1 + a, b])$.*

The first statement follows from the fact that

$$[a^{n+1}, b] = \sum_{i=0}^n a^i [a, b] a^{n-i}.$$

The second follows from the fact that in the expansion of $[(1 + a)^p, b]$ coming from the binomial expansion, the first term is zero, while the V -value of the second term is definitely less than that of any of the others.

The hypothesis that h is the height of G is exactly that $v_x([p]_G) = p^h$. Since $v_x(\varphi \circ \psi) = v_x(\varphi)v_x(\psi)$, we conclude that for $z \in \text{End}(G)$, $V(z) = \frac{1}{h} \log_p(v_x(z))$. So we have:

Proposition 2. *Let u be an automorphism of G , and call the identity automorphism $\mathbf{i}(x) = x$. Then*

$$w(u) = v_x(u -_G \mathbf{i}) = p^{hV(u-1)},$$

where in the rightmost member of the display, u is thought of as an element of \mathbf{E} .

Any element of \mathbf{E} sits in a commutative subfield, of residue field extension degree dividing h , so that a unit u^{p^h-1} will always be congruent to 1 modulo the maximal ideal of \mathbf{E} . But once a quantity is this close to the identity, we understand fully how rapidly its successive p th powers approach 1, just from the expansion of $(1 + \varepsilon)^p$:

Observation 3. If F is an algebraic extension of \mathbb{Q}_p and $z \in F$ such that $v_p(z - 1) > 0$, then:

$$v_p(z^p - 1) \begin{cases} = pv(z - 1) & \text{if } v(z - 1) < \frac{1}{p-1}, \\ \geq p/(p - 1) & \text{if } v(z - 1) = \frac{1}{p-1}, \\ = 1 + v(z - 1) & \text{if } v(z - 1) > \frac{1}{p-1}. \end{cases}$$

By combining 2 and 3, we conclude:

Theorem 4. *Let u be an automorphism of the height- h formal group G , with $u(x) \equiv x \pmod{x^2}$. Then $w(u^{\circ p})$ depends on $w(u)$ in the following way:*

$$w(u^{\circ p}) \begin{cases} = w(u)^p & \text{if } w(u) < p^{h/(p-1)}, \\ \geq p^{h/(p-1)} & \text{if } w(u) = p^{h/(p-1)}, \\ = p^h w(u) & \text{if } w(u) > p^{h/(p-1)}. \end{cases}$$

The doubt inherent in the second rule is unavoidable: the only way to get around it would be to add to the hypotheses an additional statement about the proximity of u to a p th root of unity. In any event, the directly observable quantity $w(u)$ and the perfectly computable corresponding quantities for the p -power iterates of u exhibit the height of G , since p^h is the stable value of $w(u^{\circ p^{m+1}})/w(u^{\circ p^m})$ as $m \rightarrow \infty$.

Definition 5. When an automorphism u of G satisfies the condition that $w(u) > p^{h/(p-1)}$, we say that u is in the stable range.

3. Series that commute with an automorphism

The result of this section answers almost completely the question of how a general k -series can commute with a G -automorphism. The only question remaining open is whether a nonendomorphoric series can commute with a p -power torsion automorphism, and we do not have even partial results there.

Theorem 6. *Let G be a formal group of finite height, defined over a field k of characteristic $p > 0$, with a nontorsion automorphism u . If $\psi(x) \in k[[x]]$ with $u \circ \psi = \psi \circ u$, then $\psi \in \text{End}(G)$.*

This theorem is new and of interest only in the case that $h > 1$, since in the height-one case, all endomorphisms are in \mathbb{Z}_p , and Theorem 7 of [9] shows that a series commuting with a nontorsion element of $\mathbb{Z}_p^* \subset \text{Aut}(G)$ is an endomorphism. The proof requires a few new concepts and lemmas, which we now present. The first remark to make is that since any two formal groups of the same height are isomorphic over any algebraically closed field containing k , it suffices to prove 6 for a single formal group G of each height. The most important standardization that we choose is of the kind mentioned in Section 3.3.2 of [9], where the formal group $G \in \mathbb{F}_p[[x, y]]$ satisfies

$$G(x, y) = x + y + G_0(x^{p^{h-1}}, y^{p^{h-1}}) \quad \text{and} \quad [p]_G(x) = x^{p^h} + \dots,$$

but for our purposes, all we will be using of the first condition is the fact that the two partial derivatives of G are constant 1. To construct a height- h formal group satisfying both conditions, one may start with, say, the p -typical logarithm $x + \sum_{m>0} p^{-m} x^{p^{mh}}$ to get a formal group law in characteristic zero, reduce it to get a group law F with $[p]_F(x) = x^{p^h}$, and then apply Lemma 23 of [9] to get an \mathbb{F}_p -isomorphic formal group G of the form $x + y + G_0(x^p, y^p)$, for which the p -endomorphism will still be x^{p^h} . For details of the construction of a formal group from a logarithm in characteristic zero, refer to [3]. Our formal group has one other property that we will be needing, and that is that for every $\alpha \in \mathbb{F}_{p^h}$ and every r , there is an endomorphism whose lowest monomial is αx^{p^r} . This is a general fact about formal groups defined over \mathbb{F}_p , not dependent on our construction of G . From the fact that $[p]_G(x) = x^{p^h}$, it follows that all endomorphisms of G are defined over \mathbb{F}_{p^h} .

A consequence of this standardization of G is that if an automorphism u has $u'(0) = 1$, then $u'(x) = 1$, constant. For, $u - G\mathbf{i}$ is an endomorphism without first-degree term, so that it is a power series of the form $\gamma(x^p)$, and hence $u = G(x, \gamma(x^p))$, whose derivative is 1.

For uniformity, we will denote series not known to be endomorphisms by Greek letters, endomorphisms by Latin. In particular, we will call the Frobenius $\mathbf{f}(x) = x^p$, an endomorphism because of our assumption that G is defined over the prime field. For any $\rho \in k[[x]]$, we have $\mathbf{f}^{\circ r} \circ \rho = \rho^{(p^r)} \circ \mathbf{f}^{\circ r}$, where the exponent on ρ is the method we use to denote that each coefficient of ρ has been raised to the p^r -power.

Definition 7. For series φ and ψ in $xk[[x]]$, the G -commutator of the two, $[\varphi, \psi]$, is $\varphi \circ \psi - G\psi \circ \varphi$.

Since the two compositions have the same initial degree, it follows that $v_x([\varphi, \psi]) = v_x(\varphi \circ \psi - \psi \circ \varphi)$.

Lemma 8. *Let u and ψ be commuting k -series, with u a nontorsion element of $\text{Aut}(G)$. Then $\psi'(0) \in \mathbb{F}_{p^h}$.*

This proposition appears in [9], but we repeat it here for completeness' sake. Starting with a general nontorsion u , we replace u by a suitable iterate that is in the stable range, i.e., $w(u) >$

$p^{h/(p-1)}$, so that u falls into the last case of Theorem 4. Throughout the rest of the proof of Theorem 6, we will assume that u has this property.

Now, for $u(x) \equiv x + \lambda x^{p^r} \pmod{x^{p^r+1}}$, where $r > h/(p - 1)$ and $\lambda \neq 0$, we also have $u^{\circ p}(x) \equiv x + \lambda' x^{p^{r+h}} \pmod{x^{p^{r+h}+1}}$. Write $\psi(x) \equiv \alpha x \pmod{x^2}$, and perform the two compositions of u and ψ to get the congruences

$$\begin{aligned} \psi(u(x)) &\equiv \psi(x) + \alpha \lambda x^{p^r}, \\ u(\psi(x)) &\equiv \psi(x) + \lambda(\alpha x)^{p^r}, \end{aligned}$$

both modulo (x^{p^r+1}) , which gives us the equality $\lambda \alpha = \lambda \alpha^{p^r}$, so that $\alpha \in \mathbb{F}_{p^r}$. The same argument applied to $u^{\circ p}$ and ψ shows that $\alpha \in \mathbb{F}_{p^{r+h}}$, and these two facts imply that $\alpha \in \mathbb{F}_{p^h}$.

The G -commutator does not behave as well as one might like: in particular, it is not bilinear, nor even biadditive, although one checks easily that if g is an endomorphism, $[g, \rho]$ is \mathbb{Z}_p -linear in ρ . As a result, under G -addition, the set of all series ρ such that $[g, \rho] \in \text{End}(G)$ is a left- \mathbb{Z}_p -module containing both $\text{End}(G)$ and all series commuting with g .

Since $\text{End}(G)$ is topologically closed in $xk[[x]]$, for any $\psi(x) \in xk[[x]]$ there is an endomorphism g whose distance from ψ is least—if $\psi \notin \text{End}(G)$, there will be many. So, assuming that ψ is not an endomorphism, we will choose g and δ such that $\psi = g +_G \delta$ and $v_x(\delta)$ is maximum. From Lemma 8 it follows that $v_x(\delta) > 1$, since we may G -subtract from ψ any endomorphism with the same first-degree monomial.

Indeed, to prove Theorem 6 our aim will be to show that the lowest monomial in δ is of the form αx^{p^r} with $\alpha \in \mathbb{F}_{p^h}$, for if so, we would be able to subtract an endomorphism with the same leading monomial from δ to get a higher v_x -value.

First we show that the initial degree of δ is a power of p . To do this, we first write $\delta(x) = \Delta(x^{p^r})$, with $\Delta'(x) \neq 0$. Now consider the G -endomorphism $z = [u, \delta]$, for which we have:

$$\begin{aligned} z &= u \circ \delta -_G \delta \circ u \\ &= u \circ \Delta \circ \mathbf{f}^{\circ r} -_G \Delta \circ \mathbf{f}^{\circ r} \circ u \\ &= u \circ \Delta \circ \mathbf{f}^{\circ r} -_G \Delta \circ u^{(p^r)} \circ \mathbf{f}^{\circ r} \\ &= (u \circ \Delta -_G \Delta \circ u^{(p^r)}) \circ \mathbf{f}^{\circ r}, \end{aligned}$$

which at the very least shows that $z = z_0 \circ \mathbf{f}^{\circ r}$, for an endomorphism $z_0 = u \circ \Delta -_G \Delta \circ u^{(p^r)}$. Recalling that $u(x) = x + \beta(x^p)$ and that $G(x, y) = x + y + G_0(x^p, y^p)$, we take the equality

$$z_0 +_G \Delta \circ u^{(p^r)} = u \circ \Delta \tag{A}$$

and conclude, up to p th powers, that $z_0 + \Delta = \Delta$. That is, z_0 involves only p th powers, and so its derivative is zero. Differentiating equation (A) and taking account of the shape of u and G , we see that $\Delta' \circ u^{(p^r)} = \Delta'$. But since $u^{(p^r)}$ is a nontorsion invertible series, Δ' must be constant, nonzero by construction, so that the initial degree of δ is indeed p^r , and as we have observed, $r > 0$.

It will take a bit more work than above to show that the coefficient of x^{p^r} in δ is in \mathbb{F}_{p^h} . Let $u = x + \lambda x^{p^m} + \dots$ and, as above, $\delta = \alpha x^{p^r} + \dots$. From the definition of δ as fitting into the equation $\psi = g +_G \delta$ and the hypothesis that $[u, \psi] = 0$, we conclude that $[u, \delta] = [g, u]$, which

is the commutator of two endomorphisms in just the sense that we mentioned in Section 2. Now let us look at the first terms of $[\delta, u]$, modulo degree $1 + p^{r+m}$:

$$\delta(u(x)) \equiv \Delta((x + \lambda x^{p^m})^{p^r}) = \Delta(x^{p^r}) + \alpha \lambda^{p^r} x^{p^{r+m}}$$

while in the other direction, $u(\delta(x)) \equiv \delta(x) + \lambda \alpha^{p^m} x^{p^{r+m}}$, so that the initial degree of $[\delta, u]$ is at least p^{r+m} , and the coefficient in that degree is $\alpha \lambda^{p^r} - \lambda \alpha^{p^m}$. From Proposition 1, we know that the initial degree of $[u^{\circ p}, g] = [\delta, u^{\circ p}]$ is p^h times the initial degree of $[u, g]$ and thus at least p^{r+m+h} . Let us write $u = \mathbf{i} + {}_G a$, where, in accordance with the notation we have established, a is an endomorphism whose first nonzero term is λx^{p^m} . Using the binomial expansion and our assumption on the closeness of u to the identity, we get the congruence $u^{\circ p} \equiv \mathbf{i} + {}_G a \circ [p]_G \pmod{x^{1+p^{m+h}}}$; thus $[\delta, u^{\circ p}] = [u^{\circ p}, g] \equiv [\mathbf{i} + {}_G a \circ [p]_G, g] = [a, g] \circ [p]_G = [\mathbf{i} + {}_G a, g] \circ [p]_G = [\delta, u] \circ [p]_G$, where the congruence is modulo degree $1 + p^{r+m+h}$. This shows that the coefficient of $x^{p^{r+m+h}}$ in $[\delta, u^{\circ p}]$ is also equal to $\alpha \lambda^{p^r} - \lambda \alpha^{p^m}$. On the other hand using the fact that $u^{\circ p}(x) \equiv x + \lambda x^{p^{m+h}} \pmod{x^{1+p^{m+h}}}$, which again uses our special normalization of G , the same computation as before shows that the coefficient of $x^{p^{r+m+h}}$ in $[\delta, u^{\circ p}]$ is $\alpha \lambda^{p^r} - \lambda \alpha^{p^{m+h}}$. The upshot is the equation

$$\alpha \lambda^{p^r} - \lambda \alpha^{p^{m+h}} = \alpha \lambda^{p^r} - \lambda \alpha^{p^m},$$

and as we have seen, this shows that $\alpha \in \mathbb{F}_{p^h}$, and so finishes the proof of Theorem 6.

4. Series that normalize the automorphism group

In this section we will be working with a series $\psi(x) \in k[[x]]$ such that for every $u \in \text{Aut}(G)$, we have $\psi \circ u \circ \psi^{-1} \in \text{Aut}(G)$. It may be of interest to note that conjugation by such a series leaves $\text{End}(G)$ invariant as well. Indeed, the center of $\text{Aut}(G)$ will be invariant, so we let $g \in \text{End}(G)$ and $u = [a]_G$ for $a \in \mathbb{Z}_p^*$, making u a nontorsion central automorphism of G . Then $\psi \circ u \circ \psi^{-1} = [a']_G$, a nontorsion central automorphism which commutes with $\psi \circ g \circ \psi^{-1}$. Since this last commutes with a nontorsion element of $\mathbb{Z}_p^* \subset \text{Aut}(G)$, it is a G -endomorphism. Our aim in this section is to prove:

Theorem 9. *Let G be a formal group of finite height defined over a field k of characteristic $p > 0$, k being large enough for all endomorphisms of G to be defined over k . Then $\text{Aut}_k(G)$ is its own normalizer in $k[[x]]^\circ$.*

An equivalent statement is:

Equivalent Formulation 9.1. *Under the same hypotheses as Theorem 9, if F is a formal group over k with $\text{Aut}_k(F) = \text{Aut}_k(G)$, then the two series $G(x, y)$ and $F(x, y)$ are the same.*

Every ψ in the normalizer of $\text{Aut}_k(G)$ corresponds to a formal group $G^\psi = \psi(G(\psi^{-1}x, \psi^{-1}y))$ such that $\text{Aut}_k(G) = \text{Aut}_k(G^\psi)$. Conversely, suppose F is another formal group over k such that $\text{Aut}_k(G) = \text{Aut}_k(F)$. Theorem 4 insures G and F have the same height, and so they are isomorphic over the algebraic closure of k ; any such isomorphism would normalize $\text{Aut}_k(G)$.

If the height of the formal group G is 1, then Theorem 9.1 can be painlessly proven via [5] as follows. In the notation of pp. 59–60 of [5],

$$e([1 + p]_G) = \lim_{n \rightarrow \infty} \left(\frac{(p - 1)v_x \left(\frac{[(1+p)^{p^n}]_G - 1}{x} \right)}{p^{n+1}} \right) = \begin{cases} p - 1 & \text{if } p > 2, \\ 2 & \text{if } p = 2. \end{cases}$$

Let $A = \{[(1 + p)^z]_G : z \in \mathbb{Z}_p\}$. By Théorème 5.9 of [5], we know that the separable normalizer of A is an extension of a finite group of order dividing $e([1 + p]_G)$, by the group A . But $\text{Aut}(G)$ is such a group, obviously contained in this separable normalizer. We will therefore concern ourselves only with the case $h > 1$.

For most of the rest of this section, we will be considering the ring $\text{End}(G)$ abstractly, and so will denote it, its group of units, and its fraction field by the letters \mathbf{E} , \mathbf{A} , and \mathbf{D} , as in section 1. The operation θ of conjugation by the series ψ that normalizes \mathbf{A} in the ambient group $k[[x]]^\circ$ induces an isometric isomorphism of the group \mathbf{A} , and consequently the corresponding action $\bar{\theta}$ on the Lie algebra of \mathbf{A} , namely on \mathbf{D} with the Lie bracket $zw - wz$, is a \mathbb{Q}_p -Lie-algebra automorphism.

More than that: the kernel of the reduced norm in \mathbf{D} is contained in \mathbf{A} , and thus is a subgroup of \mathbf{A} , which we call \mathbf{A}_0 . We call \mathbf{A}' the commutator subgroup of \mathbf{A} , so that we have $\mathbf{A} \supset \mathbf{A}_0 \supset \mathbf{A}'$, and these two algebraic subgroups of \mathbf{A} , being both of dimension $h^2 - 1$, have the same Lie algebra, namely the kernel of the trace from \mathbf{D} to \mathbb{Q}_p , which we will denote \mathbf{D}_0 . The operation θ leaves \mathbf{A}' invariant, so induces an automorphism $\bar{\theta}_0$ of the \mathbb{Q}_p -Lie algebra \mathbf{D}_0 . Upon extension of the base from \mathbb{Q}_p to an algebraically closed field Ω , \mathbf{D}_0 becomes isomorphic to $\mathfrak{sl}_h(\Omega)$ and \mathbf{D} becomes isomorphic to $M_h(\Omega)$. Now we know (for instance from Theorem 5, p. 283 of [4]) that any automorphism of the Ω -Lie algebra $\mathfrak{sl}_h(\Omega)$ is of the form $z \mapsto AzA^{-1}$ for some $A \in \text{GL}_h(\Omega)$ or (in case $h > 2$) of the form $z \mapsto -Az^tA^{-1}$, where z^t denotes the transpose of z considered as a matrix. The identity component of the automorphism group is thus of index two in the full group of automorphisms, and in particular, the square of any automorphism of $\mathfrak{sl}_h(\Omega)$ is in this identity component.

Now for an element y of \mathbf{D} , the specification that for every $z \in \mathbf{D}_0$, $\bar{\theta}_0(z)y = yz$ amounts to a set of \mathbb{Q}_p -linear conditions on y . If these linear conditions have a nonzero solution after extension of the base to Ω , they already do so over \mathbb{Q}_p . We therefore are ready to show:

Proposition 10. *Let G be a formal group of finite height $h > 1$ over a field k such that all endomorphisms of G are defined over k . If $\psi(x) \in k[[x]]^\circ$ and ψ normalizes $\text{Aut}(G)$, then in case $h = 2$, $\psi \in \text{Aut}(G)$; otherwise, $\psi^{\circ 2} \in \text{Aut}(G)$.*

Let φ be ψ or $\psi^{\circ 2}$ according as h is equal to or greater than 2, and represent by the letter θ the action of conjugation by φ on \mathbf{A} , so that by the remarks preceding the statement, there is a nonzero $y \in \mathbf{D}$ for which $\theta(z) = yzy^{-1}$ for all $z \in \mathbf{D}_0$, the kernel of the trace from \mathbf{D} to \mathbb{Q}_p . Since p commutes with all elements of \mathbf{D} , we may take $y \in \mathbf{E}$.

Now consider the mapping $z \mapsto y^{-1}\theta(z)y$, which is identity on the Lie algebra \mathbf{D}_0 so that the automorphism of \mathbf{A} given by $a \mapsto y^{-1}\theta(a)y$ is identity in a neighborhood of 1 in \mathbf{A}_0 . Retranslating this fact to a statement about series, we see that there is a G -endomorphism g such that for any automorphism u with trivial reduced norm and for which u is sufficiently close to the identity series, $\varphi \circ u \circ \varphi^{-1} \circ g = g \circ u$. That is, $\varphi^{-1} \circ g$ commutes with the nontorsion automorphism u . But Theorem 6 now applies, showing that $\varphi^{-1} \circ g$ is an endomorphism of G . Using the fact

that g is a nonzero endomorphism, we see that φ^{-1} is an endomorphism, and an automorphism because invertible.

The only task remaining is to show that if $h > 2$, then not only $\psi^{\circ 2}$ but the series ψ of Proposition 10 is actually an automorphism. We should say here that according to a communication from G. Prasad [8], which appeals to Proposition 3, p. 226 of his paper [7], any automorphism of \mathbf{A}_0 is a Lie-group automorphism, and so is inner. Appeal to Theorem 6 in the way we have already done would prove our desired result directly. In order to make our presentation as self-contained as possible, however, we will continue in the vein already established.

Lemma 11. *Let ψ be in the normalizer in $k[[x]]^\circ$ of $\text{Aut}(G)$, where G is a formal group of finite height $h > 1$ defined over k . Then $\psi'(0) \in \mathbb{F}_{p^h}$.*

In case ψ is already an automorphism, the conclusion is true; so the only case of concern is the one where $\psi^{\circ 2}$ is an automorphism but ψ is not. We will refer to such series as *exceptional*. Conjugation by ψ will leave stable the subgroup $1 + p\mathbb{Z}_p$ of the center \mathbb{Z}_p^* of $\text{Aut}(G)$. But the automorphism group of $1 + p\mathbb{Z}_p$ is \mathbb{Z}_p^* , so that the only involutory automorphism of $1 + p\mathbb{Z}_p$ is $u \mapsto u^{-1}$. Thus we have $\psi \circ u \circ \psi^{-1} = u^{-1}$ for every u in the subgroup of $\text{Aut}(G)$ corresponding to $1 + p\mathbb{Z}_p$. Suppose now that u is in the stable range (automatic if $p > 2$), and that $u(x) \equiv x + \lambda x^{p^{mh}} \pmod{x^{1+p^{mh}}}$ for some m and some nonzero λ . Then $u^{-1}(x) \equiv x - \lambda x^{p^{mh}} \pmod{x^{1+p^{mh}}}$ and if we call $\alpha = \psi'(0)$ we have, in a way that is by now very familiar:

$$\begin{aligned} \psi(u(x)) &\equiv \psi(x) + \alpha \lambda x^{p^{mh}}, \\ u^{-1}(\psi(x)) &\equiv \psi(x) - \lambda(\alpha x)^{p^{mh}}, \end{aligned}$$

both congruences modulo $(x^{1+p^{mh}})$. So $\alpha = -\alpha^{p^{mh}}$, and by using the same argument with $u^{\circ p}$ instead of u , we get $\alpha = -\alpha^{p^{(m+1)h}}$, again implying that $\alpha \in \mathbb{F}_{p^h}$.

From the proof of Lemma 11, specifically $\alpha = -\alpha^{p^{mh}}$, we see that if $p > 2$ then in fact $\alpha = 0$. This proves Theorem 9 for odd primes.

The case of characteristic two is rather more difficult to handle by these methods.

5. Nonexistence of exceptional series in characteristic two

An exceptional series is a $\psi(x) \in k[[x]]^\circ$ that normalizes $\text{Aut}(G)$ but is not itself an automorphism. We know that in the normalizer, the group $\text{Aut}(G)$ is a subgroup of index at most two, and that if ψ is exceptional, then $\psi'(0) \in \mathbb{F}_{p^h}$. Consider an exceptional series ψ : it commutes with the automorphism $\psi^{\circ 2}$ yet is itself not an automorphism, a contradiction to Theorem 6 unless $\psi^{\circ 2}$ is torsion. Thus ψ itself must be a torsion series.

We now specialize to the case that $p = 2$. If ψ is exceptional and of period $2^n m$ with m odd, then by replacing ψ by $\psi^{\circ m}$, we will get an exceptional series of 2-power period, and with first degree coefficient equal to 1. We will call such a series *2-exceptional*. Let us note first what the period of such a series can be.

Lemma 12. *The period of a 2-exceptional series is greater than 4.*

Let us show first that an exceptional series ψ cannot be an involution. If it were, then conjugation by ψ would induce a nontrivial involution $\bar{\theta}$ on the Lie algebra \mathbf{D}_0 ; if this has so much as a single eigenvalue equal to 1, we can get elements of \mathbf{A}_0 close to the identity and commuting with ψ , so that by Theorem 6, ψ would be an automorphism, contrary to our assumption of exceptionality. So all eigenvalues would be -1 , which means that the involution $\bar{\theta}$ is the negative of the identity. But this is not a Lie-algebra homomorphism, since \mathbf{D}_0 is not commutative.

The proof that ψ can not be of period 4 is similar, since $\psi^{\circ 2}$ would be an involutory automorphism of G . But there is only one such, namely $[-1]_G$, which induces identity on the Lie algebra \mathbf{D}_0 . This means again that $\bar{\theta}$ is an involutory automorphism of \mathbf{D}_0 ; again its only possible eigenvalues are ± 1 , and the rest of the argument is the same.

Lemma 13. *If $\psi(x)$ is a 2-exceptional series for the formal group G of height h , then h is even, and $\psi(x) \equiv x + \lambda x^r \pmod{x^{r+1}}$, with $\lambda \neq 0$ and $r < 2^{h/2}$.*

Since ψ is torsion of period at least 8, some even iterate of ψ is an automorphism g of G of period four. Such an automorphism generates a commutative field extension of \mathbb{Q}_2 of degree 2, so that $2|h$. But we know that $v_2(g - 1) = 1/2$, so that $w(g) = 2^{h/2}$, in the notation of Section 2. Since a 2-power iterate of ψ is equal to g , we must have $w(\psi) < w(g)$.

Lemma 14. *Let ψ be a 2-exceptional series for the formal group G of height $h > 1$. Then $\psi(x) = x + {}_G\Delta(x^{2^m})$ for a series Δ with $\Delta'(x) = \Delta'(0) \neq 0$, and $m \neq 0$.*

We assume at the outset, as we may, that G is parametrized so that $G(x, y) = x + y + G_0(x^{2^{h-1}}, y^{2^{h-1}})$, with $G_0(x, y) \in \mathbb{F}_2[[x, y]]$. Now write $\delta = \psi - {}_G\mathbf{i}$ and $\delta(x) = \Delta(x^{2^m})$ with $\Delta'(x) \neq 0$ and $m > 0$. From the preceding lemma, we know that $m < h/2$. Let u be a nontrivial G -automorphism in $1 + 4\mathbb{Z}_2$, and set $v_2(u - 1) = r$. Calling $z = u - u^{-1}$, we then have $v_2(z) = r + 1$. Now use the equation $\psi \circ u = u^{-1} \circ \psi$, substituting $\psi = \mathbf{i} + {}_G\delta$. So we have:

$$\begin{aligned} u(x) + {}_G\Delta(u(x)^{2^m}) &= u^{-1}(x) + {}_G u^{-1}(\Delta(x^{2^m})), \\ z(x) = g(x^{2^{h(r+1)}}) &= u^{-1}(\Delta(x^{2^m})) - {}_G\Delta(u(x^{2^m})), \end{aligned}$$

using the fact that $u \in \mathbb{F}_2[[x]]$. Since $m < h(r + 1)$, we may write $g(x^{2^t}) = u^{-1} \circ \Delta - {}_G\Delta \circ u$ with t some positive number. In particular, the left-hand side of this equation has derivative zero, while the right has derivative $\Delta' - \Delta' \circ u$, because the special parametrization of G guarantees that $u'(x) = 1$. It follows that Δ' is a constant, nonzero by construction.

From this lemma, we see that the first term in $\psi(x)$ after the linear one will be of 2-power degree. What remains is to show that the coefficient of this next monomial is in \mathbb{F}_{2^h} .

Lemma 15. *Let $\psi(x) \equiv x + cx^{2^m} \pmod{x^{1+2^m}}$, a 2-exceptional series for the formal group G of height $h > 1$. Then $c \in \mathbb{F}_{2^h}$.*

The preceding lemmas have shown that a 2-exceptional series does necessarily have the specified form, and that $m < h/2$. We now take any nontrivial u in $1 + 4\mathbb{Z}_2 \subset \text{Aut}(G)$, and write $v_2(u - 1) = r \geq 2$, so that the first terms of $u(x)$ are $x + x^{2^r}$. This u is in the stable range. From the equation $\psi \circ u = u^{-1} \circ \psi$ we get $\psi \circ u = u^{\circ(-2)} \circ u \circ \psi$; in other words, using the congruence $u^{\circ 2} \equiv x \pmod{x^{2^{(r+1)h}}}$, we can conclude that ψ and u commute modulo degree $2^{(r+1)h}$.

Using the notation of the preceding lemma, where $\psi = \mathbf{i} + {}_G\delta$ and $\delta(x) = \Delta(x^{2^m})$, we conclude that u and δ commute modulo degree $2^{(r+1)h}$, while u and Δ commute modulo degree $2^{(r+1)h-m}$. Now we have the two congruences $u(\Delta(x)) \equiv \Delta(x) + c^{2^{rh}}x^{2^{rh}}$ and $\Delta(u(x)) \equiv \Delta(x) + \Delta'(0)x^{2^{rh}}$, both modulo degree $2^{(r+1)h-m}$. Since $\Delta'(0) = c$ and $m < h/2$, we derive the relation $c = c^{2^{rh}}$. By using the same argument with $u^{\circ 2}$ instead of u , we get $c = c^{2^{2(r+1)h}}$, which tells us that $c \in \mathbb{F}_{2^h}$. This concludes the proof of the lemma.

It now requires only a few words to complete the proof of Theorem 9 in case the characteristic is 2. If there were any exceptional series, there would be one that was closest to the identity, necessarily 2-exceptional, of the form $\psi(x) = x + cx^{2^m} + \dots$. But there is also an automorphism U that has the same first two monomials, and $U^{-1} \circ \psi$ would be an exceptional series closer to the identity than ψ , which gives the necessary contradiction.

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