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The truncated Hamburger matrix moment problems in the nondegenerate and degenerate cases, and matrix continued fractions [1]

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Abstract

The present paper deals simultaneously with the nondegenerate and degenerate truncated Hamburger matrix moment problems in a unified way based on the use of the Schur algorithm involving matrix continued fractions. A full analysis of them together with a relative matrix moment problem on the real axis is given. With the help of the correspondence between the moment problem on the real axis and the Nevanlinna–Pick (NP) interpolation, the solutions of the nontangential NP interpolation in the Nevanlinna class are derived as an application. © 1998 Elsevier Science Inc. All rights reserved.

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1. Introduction and preliminaries

The natural matrix version of the classical Hamburger moment problem (see, e.g., [2]) consists in finding a bounded Hermitian measure $\sigma(u)$ $(-\infty < u < +\infty)$ such that

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$$S_k = \int_{-\infty}^{+\infty} u^k \,\mathrm{d}\sigma(u), \quad k = 0, 1, \ldots. \tag{1.1}$$

Of particular importance are the questions of solvability, of the number of solutions, and of the construction of all solutions if they exist.

In the present paper we give a careful treatment of the truncated Hamburger matrix moment problem (TH problem), where one asks to describe the solutions $\sigma(u)$ $(-\infty < u < +\infty)$ having precribed matrix moments $S_0, S_1, \ldots, S_{2n}$ only. A variation on the problem with a more natural mathematical solution is where one asks only for an inequality "$\leqslant$" in the moment condition for $k = 2n$. Also very different is the theory for the case where one prescribes the moments up to an odd index $k = 2n + 1$. The present paper deals with the case of maximum k equal to $2n$ only and primarily with equality on the last moment condition.

The subject now has a long history and has given rise to important applications in many branches of analysis and others. In the scalar case: $p = 1$, the reader may consult to the fundamental books [2,19,21] for details of numerous moment problems. In the matrix case, the TH problem for operators was first studied systematically by Ando [4], who gave the conditions for the problem to be solvable. In the nondegenerate case, Kovalishina [17] has presented a method for the solution of the Hamburger moment problem (1.1) based on the use of the fundamental matrix inequality and the analytic J-theory. Bolotnikov [6] treated the solution of the TH problem for the degenerate case by means of the Schur's stepwise algorithm. (Unfortunately, one of his basic results, Theorem 1.1 of [6], is untrue for the TH problem even for the scalar case.) Further on, the TH problem can be solved using a number of other approaches, e.g. reproducing kernels method [12,13], methods based on operator theory [2,18], or on realization theory of matrix-valued functions [5]. There has also been recent work of Kheifets [16], where the equality versus inequality issue is analyzed via a different approach (embedding the problem in the so-called abstract interpolation problem which identifies solutions of the TH problem with unitary colligation extensions of a given isometric colligation directly constructible from the data of the problem).

In the present paper we present a common method of solving simultaneously the nondegenerate and degenerate truncated Hamburger matrix moment problems based on the use of a matrix version of the Schur algorithm involving matrix continued fractions, suitably adapted to the present framework. Some new results and more explicit version of known results for the TH problem and for its variation mentioned above are given. The notion of the nonnegatively extendable Hermitian-block Hankel matrix plays a key role in our investigations. It turns out that the TH problem has a solution if and only if the corresponding block-Hankel matrix is nonnegatively extendable, i.e., if and only if the block-

Hankel matrix can be enlarged to a block-Hankel matrix of larger size (one more block-row and -column) which is still nonnegative definite. As the main result, the general solution $\sigma(u)$ to the TH problem is formulated in both an algorithm form and a more closed form (Theorems 3.4 and 4.12). With the help of the one-to-one correspondence between the moment problem and the Nevanlinna–Pick (NP) interpolation problem [10] the solution of the NP problem in the class of the Nevanlinna matrix-valued functions is acheived as an application.

Although the concern of this paper is mainly the "truncated" moment problems, our descriptions of the solutions in terms of matrix continued fractions are complete to the degree that transition to the infinite moment problems becomes quite transparent.

This work could reveal certain interesting connections with other problems in analysis in addition to the interpolation problem of NP type, for example to the work of Adamjan–Arov–Krein [1] on the reduction of the Nehari problem to the contractive completion problem for infinite, contractive block-Hankel matrices, where the issue is to extend an infinite block-Hankel matrix by one more block-row while maintaining the property of being a contraction. In the Nehari problem case, contractivity of the original matrix guarantees the extendability property. This presents an interesting theory parallel to the theory of the TH problem. As was mentioned above, the corresponding phenomenon for the TH problem situation fails – nonnegativity of the original block-Hankel matrix in general does not guarantee that it admits an extension of a larger block-Hankel matrix which is still nonnegative. This illustrates the subtlely of the TH problem compared to a seemingly analogous Nehari problem.

To introduce the discussion, we devote Section 2 to establish the various auxiliary propositions useful for our investigations later on. The criteria of existence and uniqueness of the solutions to the TH problem, and a unified description of the solutions to the TH problems in the nondegenerate and degenerate cases are settled in Section 3. As a consequence, the solution of the matrix moment problem on the real axis is also derived with little additional effort. In Section 4 it is shown how the descriptions of the solutions to the moment problems in terms of matrix continued fractions are interlaced with other existing ones in terms of linear fractional transformations. Some properties of nonnegatively extendable Hermitian block-Hankel matrices are given as well. Section 5 traces out an application of our results on moment problems to the solution of the NP problem in the class of the Nevanlinna matrix-valued functions with the help of the correspondence between the moment problem on the real axis and that NP interpolation.

The following notation and conventions will be used throughout the paper. All matrices and vectors are assumed to be complex. The positive semidefiniteness (definiteness) of a Hermitian matrix A is denoted by $A \geqslant 0$ ($A > 0$). By π denote the open upper-half plane, its subset $\{z \mid \epsilon \leqslant \arg z \leqslant \pi - \epsilon,\ \epsilon \in (0, \pi/2)\}$

by π_c. For a square matrix B, the symbol B^D denotes its Drazin inverse, i.e., the unique solution of the equations: $XBX = X, BX = XB$, and $B^k = XB^{k+1}$, $k = \text{index}(B)$. For a polynomial matrix $A(\lambda) = \sum_{k=0}^{m} A_k\lambda^k, A_k \in M_p(\mathbb{C})$, the symbol $\hat{A}(\lambda)$ designates the polynomial matrix $\sum_{k=0}^{m} A_k^*\lambda^k$. The notation $\frac{B}{A}$ stands for BA^{-1} if A, B are $p \times p$ matrices and A is nonsingular, and the following notation for the matrix continued fraction:

$$\frac{B_1}{A_1} \pm \frac{B_2}{A_2} \pm \frac{B_3}{A_3} \pm \cdots \pm \frac{B_k}{A_k} \pm \cdots,$$

in which all fractions are assumed to be meaningful. A $\mathbb{C}^{p\times p}$-valued function $\sigma(u)$ defined on the real axis is called a *Hermitian measure* if it is nondecreasing, i.e., $\sigma(\lambda) - \sigma(\mu) \geqslant 0$ for all $\lambda > \mu$. Without being pointed out explicitly, all Hermitian measures in integral formulas of this paper are assumed to be bounded: $\int_{-\infty}^{+\infty} \operatorname{tr} \, \mathrm{d}\sigma(u) < +\infty$. A $\mathbb{C}^{p\times p}$-valued function $F(\lambda)$ is of the *Nevanlinna class* $\mathcal{N}_p$ if it is analytic in π^- and such that

$$\frac{F(\lambda) - F(\lambda)^*}{2i} \geqslant 0, \quad \lambda \in \pi^+.$$

Each $F(\lambda) \in \mathcal{N}_p$ can be continued onto the open lower-half plane Im $\lambda < 0$ by reflection: $F(\lambda) = F(\bar{\lambda})^*$, Im $\lambda < 0$, and it admits an integral representation

$$F(\lambda) = \alpha\lambda + \beta + \int_{-\infty}^{+\infty} \frac{1 + u\lambda}{u - \lambda} \mathrm{d}\tau(u), \quad \text{Im } \lambda \neq 0, \tag{1.2}$$

where $\alpha \geqslant 0$, $\beta = \beta^*$, $\tau(u)$ is suitable Hermitian measure. The collection of functions $F(\lambda) \in \mathcal{N}_p$, represented in the form

$$F(\lambda) = \int_{-\infty}^{+\infty} \frac{1}{u - \lambda} \mathrm{d}\sigma(u) \tag{1.3}$$

forms a subclass $\mathcal{N}_p^0$ of the class $\mathcal{N}_p$, where $\sigma(u)$ is as before. This subclass $\mathcal{N}_p^0$ can be characterized intrinsically: in order that $F(\lambda) \in \mathcal{N}_p$ belongs to $\mathcal{N}_p^0$, it is necessary and sufficient that

$$\sup_{y>0} \|yF(iy)\| < +\infty, \tag{1.4}$$

where $\|\cdot\|$ is a certain matrix norm on $\mathbb{C}^{p\times p}$ (see [2] for the scalar case. The extension to the matrix-valued functions causes no difficulty).

2. Some important lemmata

We shall establish some auxiliary propositions which are useful later on.

To begin with, we formulate the TH problem on the real axis. Given a sequence of $p \times p$ Hermitian matrices, $S_0, S_1, \ldots, S_{2n}$, we seek a Hermitian measure $\sigma(u)$ $(-\infty < u < +\infty)$ such that

$$S_k = \int_{-\infty}^{+\infty} u^k \, \mathrm{d}\sigma(u), \quad k = 0, 1, \ldots, 2n. \tag{2.1}$$

It is required to find conditions for a solution and for a unique solution $\sigma(u)$ to exist, and to describe the solutions when these conditions are met.

The well-known Hamburger–Nevanlinna theorem shows the close relation of the TH problem (2.1) to the following problem: Given a sequence of $p \times p$ Hermitian matrices, $S_0, S_1, \ldots, S_{2n}$, it is required to describe all $F(\lambda) \in \mathcal{N}_p$ with a given asymptotical expansion of the form

$$F(\lambda) = -\frac{S_0}{\lambda} - \frac{S_1}{\lambda^2} - \cdots - \frac{S_{2n}}{\lambda^{2n-1}} - R_{2n+1}(\lambda), \tag{2.2}$$

where $R_{2n+1}(\lambda) = \mathrm{o}(\lambda^{-2n-1})$ as $\lambda \to \infty$ uniformly in each sector π_ε.

In what follows, all other asymptotic expansions will be assumed to be applicable in the same range of sectors π_ε as also will the symbol $\mathrm{o}(\lambda^{-m})$.

Lemma 2.1 (Hamburger–Nevanlinna) [2,17]. *If $\sigma(u)$ $(-\infty < u < +\infty)$ is a solution to the TH problem (2.1), then there exists $F(\lambda) \in \mathcal{N}_p$*

$$F(\lambda) = \int_{-\infty}^{+\infty} \frac{1}{u - \lambda} \mathrm{d}\sigma(u) \tag{2.3}$$

for which

$$\lim_{\lambda \to \infty} \lambda^{2n+1} \left[F(\lambda) + \frac{S_0}{\lambda} + \frac{S_1}{\lambda^2} + \cdots + \frac{S_{2n-1}}{\lambda^{2n}} \right] = -S_{2n} \tag{2.4}$$

uniformly in each π_ε. Conversely, if (2.4) holds, at least for $\lambda = \mathrm{i}y$ $(y \to +\infty)$, for some $F(\lambda) \in \mathcal{N}_p$, then $F(\lambda)$ has the representation (2.3), where $\sigma(u)$ has $2n+1$ moments $S_0, \ldots, S_{2n}$.

Lemma 2.2. *Suppose that $F(\lambda) \in \mathcal{N}_p$ admits the representation (1.2). If either $\alpha > 0$ or $\int_{-\infty}^{+\infty} \mathrm{d}\tau(u) > 0$, then $-F^{-1}(z)$ exists in π^+ and belongs to the class $\mathcal{N}_p$.*

Proof. Observe that for each $\lambda = a + b\mathrm{i} \in \pi^+$ $(a, b \in \mathbb{R}, b > 0)$,

$$\operatorname{Im} F(\lambda) = \alpha b + \int_{-\infty}^{+\infty} \frac{b^2(1+u^2)}{(u-a)^2 + b^2} \mathrm{d}\tau(u).$$

If $\alpha > 0$, Im $F(\lambda) > 0$, $\forall \lambda \in \pi^+$. If $\int_{-\infty}^{+\infty} \mathrm{d}\sigma(u) > 0, \int_{-\delta}^{\delta} \mathrm{d}\sigma(u) > 0$ for a certain $\delta > 0$, and therefore

$$\operatorname{Im} F(\lambda) \geqslant \frac{b^2}{(\delta + |a|)^2 + b^2} \int_{-\delta}^{\delta} \mathrm{d}\tau(u) > 0.$$

On the other hand, we have that $|\det F(\lambda)| = |\det(\mathrm{i}F(\lambda))| \geqslant \det \operatorname{Im} F(\lambda) > 0$ if $\alpha > 0$ or $\int_{-\infty}^{+\infty} \mathrm{d}\tau(u) > 0$, and therefore $-F^{-1}(\lambda)$ exists and is analytic in π^+ under the hypothesis. Finally, the fact that $\operatorname{Im}(-F^{-1}(\lambda)) \geqslant 0, \forall \lambda \in \pi^+$ follows from the relation

$$\operatorname{Im}\,(-F^{-1}(\lambda)) = F^{-1}(\lambda) \operatorname{Im} F(\lambda) F^{-1}(\lambda)^* \geqslant 0, \quad \lambda \in \pi^+.$$

Hence, $-F^{-1}(\lambda) \in \mathcal{N}_p$. $\square$

Lemma 2.3. *Suppose that the Hermitian block-Hankel matrix* $\Gamma_k = (S_{i+j})_{i,j=0}^k$, *is Hermitian nonnegative*: $\Gamma_k \geqslant 0$, *where each* S_k *of order p is Hermitian. If* $S_0 = 0$, *then* $S_1 = \cdots = S_{2k-1} = 0$; *if* $S_0 \neq 0$ *is singular, then there exists a certain nonsingular T such that*

$$T^* S_i T = \begin{bmatrix} \tilde{S}_i & 0 \\ 0 & 0 \end{bmatrix}, \quad i = 0, 1, \ldots, 2k-1, \qquad T^* S_{2k} T = \begin{bmatrix} \tilde{S}_{2k} & 0 \\ 0 & D \end{bmatrix}, \qquad (2.5)$$

where $\tilde{S}_0 > 0$, $D \geqslant 0$, *and all* $\tilde{S}_i$ *are of the same order.*

We remark that there exists a misstatement in [6] (p. 1255) in constructing Schur's stepwise algorithm. He asserted that if det $S_0 = 0$ then from $\Gamma_k \geqslant 0$ there follows the existence of a unitary matrix U such that

$$U^* S_i U = \begin{bmatrix} \tilde{S}_i & 0 \\ 0 & 0 \end{bmatrix}, \quad i = 0, 1, \ldots, 2k; \ \tilde{S}_0 > 0.$$

This, however, is not correct, as an example,

$$\Gamma_2 = \begin{bmatrix} 0 & 0 \\ 0 & I_2 \end{bmatrix} \geqslant 0.$$

The proof of Lemma 2.3. If $S_0 = 0$ then $S_1 = \cdots = S_{2k-1} = 0$ evidently. If $S_0 \neq 0$ is singular, then there exists a certain unitary matrix V such that

$$V^* S_0 V = \begin{bmatrix} \tilde{S}_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{S}_0 > 0.$$

From the fact $(I_p \otimes V)^* \Gamma_k (I_p \otimes V) \geqslant 0$, where $A \otimes B$ stands for the tensor product of a pair of matrices A and B, we obtain that

$$V^*S_iV = \begin{bmatrix} \tilde{S}_i & 0 \\ 0 & 0 \end{bmatrix}, \quad i = 0, \ldots, 2k-1,$$

where all $\tilde{S}_i$ have the same order as that of $\tilde{S}_0$. Let us write $V^*S_{2k}V$ in the block form

$$V^*S_{2k}V = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix} \geqslant 0,$$

where order A_{11} = order $\tilde{S}_0$. We distinguish three cases. In the case $A_{22} > 0$, we have

$$P = \begin{bmatrix} I & 0 \\ -A_{22}^{-1}A_{12}^* & I \end{bmatrix}$$

such that

$$P^*V^*S_{2k}VP = \begin{bmatrix} \tilde{S}_{2k} & 0 \\ 0 & A_{22} \end{bmatrix}$$

in which $\tilde{S}_{2k} = A_{11} - A_{12}A_{22}^{-1}A_{12}^*$. In the case $A_{22} = 0$, then $A_{12} = 0$, and thus

$$V^*S_iV = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix}.$$

In the case when $A_{22} \neq 0$ is singular, there exists a unitary matrix $\tilde{V}$ such that

$$\tilde{V}^*A_{22}\tilde{V} = \begin{bmatrix} \tilde{A}_{22} & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{A}_{22} > 0,$$

and thus

$$\begin{bmatrix} I & 0 \\ 0 & \tilde{V}^* \end{bmatrix} V^*S_{2k}V \begin{bmatrix} I & 0 \\ 0 & \tilde{V} \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & 0 \\ \tilde{A}_{12}^* & \tilde{A}_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{A}_{22} > 0.$$

So, like the first case, there exists a nonsingular matrix P such that

$$P^* \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & 0 \\ \tilde{A}_{12}^* & \tilde{A}_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} P = \begin{bmatrix} \tilde{S}_{2k} & & \\ & \tilde{A}_{22} & 0 \\ & 0 & 0 \end{bmatrix},$$

where order $\tilde{S}_{22}$ = order A_{11}. Thus, we complete the proof by setting $T = VP$ in the first two cases and

$$T = V \begin{bmatrix} I & 0 \\ 0 & \tilde{V} \end{bmatrix} P$$

in the last case. $\square$

The key role of the TH Problem is played by the following notion.

Definition 2.4. A Hermitian block-Hankel matrix $\Gamma_k = (S_{i+j})_{i,j=0}^{k}$, is called nonnegatively extendable (n.e., for short) if there exist Hermitian matrices S_{2k+1}, S_{2k+2} of order p such that $\Gamma_{k+1} = (S_{i+j})_{i,j=0}^{k+1} \geqslant 0$.

It is clear that $\Gamma_k > 0$ implies the nonnegative extendability of Γ_k, and that the latter in turn implies $\Gamma_k \geqslant 0$, but each of the converse propositions is not true, for instance,

$$\Gamma_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \Gamma_2 = \begin{bmatrix} 0 & 0 \\ 0 & I_2 \end{bmatrix}.$$

In the scalar case: $p = 1, \Gamma_k$ is n.e., if and only if either $\Gamma_k > 0$ or $\Gamma_k \geqslant 0$ is singular and proper, i.e., the leading principal submatrix of Γ_k of order $m = \operatorname{rank} \Gamma_k$ is Hermitian positive (see [8] for details).

In the Sections 4 and 5 we shall characterize the nonnegative extendability of a Hermitian block-Hankel matrix (see Theorem 3.9 and Corollary 4.10).

In the case when Γ_k is n.e., from Lemma 2.3 we obtain

Corollary 2.5. *Suppose Γ_k is n.e. If $S_0 = 0$, then $\Gamma_k = 0$. If $S_0 \neq 0$, there exists a unitary matrix U such that*

$$U^* S_k U = \begin{bmatrix} \tilde{S}_i & 0 \\ 0 & 0 \end{bmatrix}, \quad i = 0, \ldots, 2k, \tag{2.6}$$

in which $\tilde{S}_0 > 0$ and all $\tilde{S}_k$ have the same order as that of $\tilde{S}_0$.

Lemma 2.6. *Suppose that $F(\lambda) \in \mathcal{N}_p$ has the asymptotical expansion (2.2), $n \geqslant 1$. Then either $S_0 = 0$ and $F(\lambda) \equiv 0$ or $S_0 \neq 0$ and*

$$F(\lambda) = -\frac{S_0}{\lambda I_P - S_0^{\mathrm{D}} S_1 + S_0^{\mathrm{D}} F_1(\lambda)}, \tag{2.7}$$

where $F_1(\lambda) \in \mathcal{N}_p$, and

$$F_1(\lambda) = -\frac{S_0^{(1)}}{\lambda} - \frac{S_1^{(1)}}{\lambda^2} - \cdots - \frac{S_{2n-2}^{(1)}}{\lambda^{2n-1}} + \mathrm{o}(\lambda^{-2n+1}) \tag{2.8}$$

in which $S_0^{(1)}, \ldots, S_{2n-2}^{(1)}$ are defined by the formula:

$$\Gamma_{n-1}^{(1)} = [S_{i-j}^{(1)}]_{i,j=0}^{n-1} = \begin{bmatrix} S_0 & & 0 \\ & \ddots & \\ 0 & & S_0 \end{bmatrix} \begin{bmatrix} S_0 & & 0 \\ \vdots & \ddots & \\ S_{n-1} & \cdots & S_0 \end{bmatrix}^{\mathrm{D}}$$

$$\times \left\{ [S_{i+j}]_{i,j=1}^{n} - \begin{bmatrix} S_1 \\ \vdots \\ S_n \end{bmatrix} S_0^{\mathrm{D}} [S_1, \ldots, S_n] \right\}$$

$$\times \begin{bmatrix} S_0 & \cdots & S_{n-1} \\ & \ddots & \vdots \\ 0 & & S_0 \end{bmatrix}^{\mathrm{D}} \begin{bmatrix} S_0 & & 0 \\ & \ddots & \\ 0 & & S_0 \end{bmatrix}. \tag{2.9}$$

Proof. In the case of $S_0 = 0, F(\lambda) \equiv 0$ follows from Lemma 2.1. In the case of $S_0 \neq 0$, we may make the assumption that $S_0 > 0$ throughout our proof (for if $S_0 \geqslant 0$ is nonzero and singular, by Lemma 2.1, then there exist $\sigma(u)$ such that $S_i = \int_{-\infty}^{+\infty} u^i \, \mathrm{d}\sigma(u) (i = 0, \ldots, 2n)$ and a unitary matrix U such that

$$U^* S_0 U = \begin{bmatrix} \tilde{S}_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{S}_0 > 0.$$

Thus

$$U^* \mathrm{d}\sigma(u) U = \begin{bmatrix} \mathrm{d}\tilde{\sigma}(u) & 0 \\ 0 & 0 \end{bmatrix},$$

so that

$$U^* S_i U = \begin{bmatrix} \tilde{S}_i & 0 \\ 0 & 0 \end{bmatrix}, \quad i = 0, 1, \ldots, 2n,$$

where $\tilde{S}_i = \int_{-\infty}^{+\infty} u^i \mathrm{d}\tilde{\sigma}(u)$ of order equal to rank $\tilde{S}_0$). Then by Lemma 2.1 again, there exists a Hermitian measure $\sigma(u) (-\infty < u < \infty)$ such that

$$F(\lambda) = \int_{-\infty}^{-\infty} \frac{1}{u - \lambda} \mathrm{d}\sigma(u)$$

and

$$S_0 = \int_{-\infty}^{+\infty} \mathrm{d}\sigma(u) > 0,$$

whence $-F^{-1}(\lambda) \in \mathcal{N}_p$ by Lemma 2.2. Suppose now that $-F^{-1}(\lambda)$ permits the representation (1.2). It is readily seen that (2.7) holds with

$$\begin{aligned} F_1(\lambda) &= -S_0 F^{-1}(\lambda) S_0 - S_0 \lambda + S_1 \\ &= (S_0 \alpha S_0 - S_0)\lambda + (S_0 \beta S_0 + S_1) + \int_{-\infty}^{+\infty} \frac{1 + u\lambda}{u - \lambda} \mathrm{d}(S_0 \tau(u) S_0). \end{aligned} \tag{2.10}$$

In view of the fact that

$$\alpha = \lim_{y\to+\infty} \frac{-F^{-1}(\mathrm{i}y)}{\mathrm{i}y} = \lim_{y\to+\infty} \frac{-I_p}{\mathrm{i}yF(\mathrm{i}y)} = S_0^{-1},$$

$F_1(\lambda)$ has the form

$$F_1(\lambda) = (S_0\beta S_0 + S_1) + \int_{-\infty}^{+\infty} \frac{1+u\lambda}{u-\lambda}\,\mathrm{d}(S_0\tau(u)S_0),$$

and therefore $F_1(\lambda) \in \mathcal{N}_p$, since $S_0\beta S_0 + S_1$ is Hermitian.

On the other hand, from (2.10), $F_1(z) \in \mathcal{N}_p$ can be rewritten in the form

$$\begin{aligned} F_1(\lambda) &= -S_0[\lambda^{-1}F^{-1}(\lambda)][S_0\lambda + F(\lambda)\lambda^2 - F(\lambda)S_0^{-1}S_1\lambda] \\ &= -S_0\left[\lambda\left(F(\lambda) + \frac{S_0}{\lambda} + \cdots + \frac{S_{2n}}{\lambda^{2n-1}}\right) - S_0 - \frac{S_1}{\lambda} - \cdots - \frac{S_{2n}}{\lambda^{2n}}\right]^{-1} \\ &\quad \times \left[S_0\lambda + \lambda^2\left(F(\lambda) + \frac{S_0}{\lambda} + \cdots + \frac{S_{2n}}{\lambda^{2n+1}}\right) - S_0\lambda - S_1\right. \\ &\quad - \cdots - \frac{S_{2n}}{\lambda^{2n-1}} - \left(F(\lambda) + \frac{S_0}{\lambda} + \cdots + \frac{S_{2n}}{\lambda^{2n+1}}\right) \\ &\quad \left.\times S_0^{-1}S_1\lambda + S_1 + \frac{S_1S_0^{-1}S_1}{\lambda} + \cdots + \frac{S_{2n}S_0^{-1}S_1}{\lambda^{2n}}\right]. \end{aligned}$$

Thus $F_1(\lambda)$ has an asymptotic expansion of the form (2.8) in which $S_0^{(1)}, \ldots, S_{2n-1}^{(1)}$ are determined by $S_0, \ldots, S_{2n}$. To complete the proof, it remains only to verify the formula (2.9). Let

$$\Phi(\lambda) = -\frac{S_0}{\lambda} - \frac{S_1}{\lambda^2} - \cdots - \frac{S_{2n}}{\lambda^{2n-1}}. \tag{2.11}$$

We will find the first $2n-1$ coefficients of the Laurent expansion of $\Phi_1(\lambda) = -S_0\Phi^{-1}(\lambda)\ S_0 - S_0\lambda + S_1$ at infinity. In the remainding part of the proof, we always assume $\Gamma_n > 0$ (since in the case when $\Gamma_n \geqslant 0$ is singular, one may choose a certain Hermitian block-Hankel matrix $\tilde{\Gamma}_n > 0$ and let $\Gamma_n(\epsilon) = \Gamma_n + \epsilon\tilde{\Gamma}_n, \forall\epsilon > 0$, and prove that (2.9) holds for $\Gamma_n(\epsilon)$ and $[S_{i+j}^{(1)}(\epsilon)]_{i,j=0}^{n-1}$. Then go to the limit as $\epsilon \to 0$). Then, it follows from [15] that there exists a pair of polynomial matrices $A(\lambda) = I_p\lambda^{n+1} + \sum_{i=0}^{n} A_i\lambda^i$ and $B(\lambda) = \sum_{i=0}^{n} B_i\lambda^i$ such that

$$A^{-1}(\lambda)B(\lambda) = \widehat{B}(\lambda)\widehat{A}^{-1}(\lambda), \tag{2.12}$$

where for a polynomial matrix $D(\lambda) = \sum_{i=0}^{n} D_i\lambda^i$, $\hat{D}(\lambda)$ denotes the corresponding polynomial matrix $\sum_{i=0}^{n} D_i^*\lambda^i$, and the Laurent expansion of $A^{-1}(\lambda)B(\lambda)$ at infinity has the form

$$A^{-1}(\lambda)B(\lambda) = -\frac{S_0}{\lambda} - \frac{S_1}{\lambda^2} - \cdots - \frac{S_{2n}}{\lambda^{2n+1}} + \mathrm{o}(\lambda^{-2n-1}). \tag{2.13}$$

Moreover, the generalized Bezoutian of the quadruple $(A(\lambda), B(\lambda); \widehat{B}(\lambda), \widehat{A}(\lambda))$ Bez $(A, B; \widehat{B}, \widehat{A})$ can be written in the form [3]

$$\mathrm{Bez}(A, B; \hat{B}, \hat{A} = S(A)^* \Gamma_n S(A), \tag{2.14}$$

in which

$$S(A) = \begin{bmatrix} A_1 & \cdots & A_n & I_p \\ \vdots & \iddots & \iddots & \\ A_n & \iddots & & \\ I_p & & & \end{bmatrix}. \tag{2.15}$$

Let $C(\lambda), R(\lambda)$ be the polynomial matrices such that

$$A(\lambda) = B(\lambda)C(\lambda) + R(\lambda), \quad \deg R(\lambda) < \deg B(\lambda), \tag{2.16}$$

where deg $R(\lambda)$ stands for the formal degree of the polynomial matrix $R(\lambda)$. Then $[A^{-1}(\lambda)B(\lambda)]^{-1} = B^{-1}(\lambda)A(\lambda) = C(\lambda) + B^{-1}(\lambda)R(\lambda)$, so that, by the argument given above,

$$\begin{aligned} B^{-1}(\lambda)R(\lambda) &= \widehat{R}(\lambda)\widehat{B}^{-1}(\lambda) \\ &= -\frac{S_0^{-1}S_0^{(1)}S_0^{-1}}{\lambda} - \frac{S_0^{-1}S_1^{(1)}S_0^{-1}}{\lambda^2} - \cdots - \frac{S_0^{-1}S_{2n-2}^{(1)}S_0^{-1}}{\lambda^{2n-1}} + o(\lambda^{-2n-1}). \end{aligned} \tag{2.17}$$

Thanks to the relation

$$\begin{aligned} \mathrm{Bez}(A, B; \widehat{B}, \widehat{A}) &= \mathrm{Bez}(BC + R, B; \widehat{B}, \widehat{C}\widehat{B} + \widehat{R}) \\ &= \mathrm{Bez}(BC, B; \widehat{B}, \widehat{C}\widehat{B}) + \begin{bmatrix} \mathrm{Bez}(R, B; \widehat{B}, \widehat{R}) & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

we obtain from (2.14) that

$$\begin{bmatrix} I_p & 0 & \cdots & -B_0 S_0^{-1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & -B_{n-1}S_0^{-1} \\ 0 & & & I_p \end{bmatrix} S(A)^* \Gamma_n S(A) \begin{bmatrix} I_p & & & 0 \\ 0 & \ddots & & \\ \vdots & \ddots & \ddots & \\ -S_0^{-1}B_0^* & \cdots & -S_0^{-1}B_{n-1}^* & I_p \end{bmatrix}$$

$$= \begin{bmatrix} S(B)^* \begin{bmatrix} S_0 & & \\ & \ddots & \\ & & S_0 \end{bmatrix}^{-1} \Gamma_{n-1}^{(1)} \begin{bmatrix} S_0 & & \\ & \ddots & \\ & & S_0 \end{bmatrix}^{-1} S(B) & 0 \\ 0 & S_0 \end{bmatrix}.$$

Multiplying by

$$T = \operatorname{diag}\left[\begin{bmatrix} S_0 & & 0 \\ \vdots & \ddots & \\ S_{n-1} & \cdots & S_0 \end{bmatrix}, I_p\right]\begin{bmatrix} S(B)^* & 0 \\ 0 & I_p \end{bmatrix}^{-1}$$

on the left, by T^* on the right, and then computing the existing result, we have

$$\begin{bmatrix} -S_1S_0^{-1} & I_p & & 0 \\ \vdots & & \ddots & \\ -S_nS_0^{-1} & & & I_p \\ I_p & 0 & \cdots & 0 \end{bmatrix} \Gamma_n \begin{bmatrix} -S_1S_0^{-1} & \cdots & -S_nS_0^{-1} & I_p \\ I_p & & & 0 \\ & \ddots & & \vdots \\ 0 & & I_p & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} S_0 & & 0 \\ \vdots & \ddots & \\ S_{n-1} & \cdots & S_0 \end{bmatrix} (I_n \otimes S_0)^{-1}\Gamma_{n-1}^{(1)}(I_n \otimes S_0)^{-1} \begin{bmatrix} S_0 & \cdots & S_{n-1} \\ & \ddots & \vdots \\ & & S_0 \end{bmatrix} & 0 \\ 0 & S_0 \end{bmatrix},$$

whence

$$[S_{i+j}]_{i,j=1}^n - \begin{bmatrix} S_1 \\ \vdots \\ S_n \end{bmatrix} S_0^{-1}[S_0, \dots, S_n] = \begin{bmatrix} S_0 & & 0 \\ \vdots & \ddots & \\ S_{n-1} & \cdots & S_0 \end{bmatrix} \begin{bmatrix} S_0 & & \\ & \ddots & \\ & & S_0 \end{bmatrix}^{-1} \Gamma_{n-1}^{(1)}$$

$$\times \begin{bmatrix} S_0 & & \\ & \ddots & \\ & & S_0 \end{bmatrix}^{-1} \begin{bmatrix} S_0 & \cdots & S_{n-1} \\ & \ddots & \vdots \\ & & S_0 \end{bmatrix}.$$

Hence, the formula (2.9) follows from the last equation. □

Lemma 2.7. *Suppose that $n \geqslant 1$ and Γ_n is n.e. and that $F_1(\lambda) \in \mathcal{N}_p$ has the asymptotic expansion (2.13), where $S_0^{(1)}, \dots, S_{2n-2}^{(1)}$ are defined by Eq. (2.9). Let $F(\lambda)$ be of the form (2.7). Then $\Gamma_{n-1}^{(1)} = [S_{i+j}^{(1)}]_{i,j=0}^{n-1}$ is also n.e., and $F(\lambda) \in \mathcal{N}_p$, which has the asymptotic expansion (2.2).*

Proof. Suppose that Γ_n is n.e. ($n \geqslant 1$). By Corollary 2.5, we may assume that $S_0 > 0$. Then by definition, $(S_{i+j})_{i,j=0}^{n+1} \geqslant 0$ for a suitable pair of Hermitian matrices S_{2n+1} and S_{2n+2}. Define $S_{2n-1}^{(1)}$ and $S_{2n}^{(1)}$ via the formula

$$[S_{i+j}^{(1)}]_{i,j=0}^{n} = (I_{n+1} \otimes S_0) \begin{bmatrix} S_0 & & & 0 \\ \vdots & \ddots & & \\ S_{n-1} & \cdots & S_0 & \\ S_n & \cdots & S_1 & S_0 \end{bmatrix}^{-1}$$
$$\times \left([S_{i+j}]_{i,j=1}^{n} - \begin{bmatrix} S_1 \\ \vdots \\ S_{n+1} \end{bmatrix} S_0^{-1} [S_1, \cdots, S_{n+1}] \right)$$
$$\times \begin{bmatrix} S_0 & \cdots & S_{n-1} & S_n \\ & \ddots & \vdots & \vdots \\ & & S_0 & S_1 \\ & & & S_0 \end{bmatrix}^{-1} (I_{n+1} \otimes S_0).$$

Then obviously $[S_{i+j}^{(1)}]_{i,j=0}^{n-1} \geqslant 0$, so that $\Gamma_n^{(1)}$ is n.e. On the other hand, it is easy to prove in much the same way as that used in Lemma 2.6 that $F(\lambda) \in \mathcal{N}_p$ and it has an asymptotic expansion of the form

$$F(\lambda) = -\frac{S_0}{\lambda} - \frac{S_1}{\lambda^2} - \frac{S_2'}{\lambda^3} - \cdots - \frac{S_{2n}'}{\lambda^{2n+1}} + o(\lambda^{-2n-1}).$$

Then Eq. (2.9) holds with S_i' $(i \geqslant 2)$ in place of S_i $(i \geqslant 2)$ therein, and therefore $S_i = S_i'(i \geqslant 2)$, noting that if S_0 and S_1 are fixed, then $S_2, \ldots, S_{2n}$ are uniquely determined by $S_0^{(1)}, \ldots, S_{2n-2}^{(1)}$ via the formula (2.9). □

3. The solution of the TH problem

In this section, the nondegenerate and degenerate TH problems are treated in a unified way of constructing matrix continued fractions. Some results on the matrix moment problem on the real axis are given as well with little additional effort.

The following theorem gives a criterion of existence of the solutions to the TH problem in terms of the notion of n.e., which is a natural generalization of the corresponding theorem in the scalar case (see, e.g., [21], Theorem 2.3, pp. 31–32; [8], Theorem 3.6). See [4], Theorem 1, for the same result in the operator case.

Theorem 3.1. *The TH problem (2.1) is solvable if and only if $\Gamma_n = (S_{i+j})_{i,j=0}^{n}$ is n.e.*

Proof. Assume that the problem (2.1) has a solution $\sigma(u)$. Then the problem (2.2) also has a solution $F(\lambda)$ by Lemma 2.1. It is obvious that

$\Gamma_n = (S_{i+j})_{i,j=0}^n = [\int_{-\infty}^{+\infty} u^{i+j}\, \mathrm{d}\sigma(u)]_{i,j=0}^n \geqslant 0$. Applying Lemma 2.6 successively, one can derive that

$$F(\lambda) = -\frac{S_0}{\lambda I_p - S_0^{\mathrm{D}} S_1} - \frac{S_0^{\mathrm{D}} S_0^{(1)}}{\lambda I_p - S_0^{(1)\mathrm{D}} S_1^{(1)}} - \cdots - \frac{S_0^{(n-3)\mathrm{D}} S_0^{(n-2)}}{\lambda I_p - S_0^{(n-2)\mathrm{D}} S_1^{(n-2)}}$$

$$- \frac{S_0^{(n-2)\mathrm{D}} S_0^{(n-1)}}{\lambda I_p - S_0^{(n-1)\mathrm{D}} S_1^{(n-1)} + S_0^{(n-1)\mathrm{D}} \Phi(\lambda)} \tag{3.1}$$

for a certain $\Phi(\lambda) = -S_0^{(n)}/\lambda + \mathrm{o}(\lambda^{-1}) \in \mathcal{N}_p$, where $S_0^{(1)}, S_1^{(1)}, \ldots, S_0^{(n-1)}, S_0^{(n)}$ are uniquely determined by Γ_n. Let $\tilde{F}(\lambda)$ denote the matrix-valued function $F(\lambda)$ given in Eq. (3.1) but with $\Phi(\lambda)$ replaced by $\tilde{\Phi}(\lambda) = -S_0^{(n)}/\lambda \in \mathcal{N}_p$. Obviously, by Lemma 2.7, $\tilde{F}(\lambda)$ is also a solution to the problem (2.2), which is rational. Suppose now that the Laurent expansion of $\tilde{F}(\lambda)$ at infinity has the form

$$\tilde{F}(\lambda) = -\frac{S_0}{\lambda} - \frac{S_1}{\lambda^2} - \frac{S_2}{\lambda^3} - \cdots \frac{S_{2n}}{\lambda^{2n+1}} - \frac{S_{2n+1}}{\lambda^{2n+2}} - \frac{S_{2n+2}}{\lambda^{2n+3}} + \mathrm{o}(\lambda^{-2n-3}).$$

Then Lemma 2.1 implies that there exists a Hermitian measure $\tau(u)$ such that

$$S_k = \int_{-\infty}^{+\infty} u^k\, \mathrm{d}\tau(u), \quad k = 0, 1, \ldots, 2n+2,$$

so that

$$\Gamma_{n+1} = [S_{i+j}]_{i,j=0}^{n+1} = \left[\int_{-\infty}^{+\infty} u^{i+j}\, \mathrm{d}\tau(u)\right]_{i,j=0}^{n+1} \geqslant 0.$$

Hence, Γ_n is n.e.

Conversely, if Γ_n is n.e., we may assume $S_0 > 0$ by Corollary 2.5. Let

$$F(\lambda) = -\frac{S_0}{\lambda} - \frac{S_1}{\lambda^2} - \frac{S_2}{\lambda^3} - \cdots - \frac{S_{2n}}{\lambda^{2n+1}} + \mathrm{o}(\lambda^{-2n-1}).$$

Then we have

$$F(\lambda) = -\frac{S_0}{\lambda I_p - S_0^{-1} S_1 + S_0^{-1} F_1(\lambda)} \tag{3.2}$$

in which

$$F_1(\lambda) = -\frac{S_0^{(1)}}{\lambda} - \frac{S_1^{(1)}}{\lambda^2} - \cdots - \frac{S_{2n-2}^{(1)}}{\lambda^{2n-1}} + \mathrm{o}(\lambda^{-2n+1}) \tag{3.3}$$

with coefficients $S_0^{(1)}, \ldots, S_{2n-2}^{(1)}$ defined by (2.9). By Lemma 2.7, $\Gamma_{n-1}^{(1)}$ is n.e. Also, $F(\lambda) \in \mathcal{N}_p$ is equivalent to the fact $F_1(\lambda) \in \mathcal{N}_p$ by the proof of Lemma 2.6. For $F_1(\lambda)$, we may assume $S_0^{(1)} > 0$, and put

$$F_1(\lambda) = -\frac{S_0^{(1)}}{\lambda I_p - S_0^{(1)-1} S_1^{(1)} + S_0^{(1)-1} F_2(\lambda)}$$

in which

$$F_2(\lambda) = -\frac{S_0^{(2)}}{\lambda} - \frac{S_1^{(2)}}{\lambda^2} - \cdots - \frac{S_{2n-4}^{(2)}}{\lambda^{2n-3}} + \mathrm{o}(\lambda^{-2n+3})$$

with coefficient $S_0^{(2)}, \dots, S_{2n-4}^{(2)}$ defined by $\Gamma_{n-1}^{(1)}$ via a formula similar to (2.9). The further continuation of this procedure is evident. Hence, in order to find a solution $F(\lambda)$ to the problem (2.2), it is only required to find a matrix-valued function $F_n(\lambda)$, which belongs to $\mathcal{N}_p$ and has the asymptotic expansion $F_n(\lambda) = -S_0^{(n)}/\lambda + \mathrm{o}(\lambda^{-1}), S_0^{(n)} \geqslant 0$. It is clear that such a function $F_n(\lambda)$ exists, and can be taken as $F_n(\lambda) = -S_0^{(n)}/\lambda$. Thus the problem (2.2) (and therefore the TH problem) is solvable. □

It is noteworthy that in [6] the author concluded that $\Gamma_n \geqslant 0$ was a sufficient and necessary condition for a solution of the TH problem to exist. This, however, is not true even for the scalar case [8]. But it can be verified that $\Gamma_n \geqslant 0$ is actually a criterion of solvability for the matrix moment problem on the real axis of finding a Hermitian measure $\sigma(u)(-\infty < u < +\infty)$ such that

$$S_k = \int_{-\infty}^{+\infty} u^k \, \mathrm{d}\sigma(u), \quad i = 0, 1, \dots, 2n-1; S_{2n} \geqslant \int_{-\infty}^{+\infty} u^{2n} \, \mathrm{d}\sigma(u), \tag{3.4}$$

where $S_0, \dots, S_{2n}$ are given $p \times p$ Hermitian matrices. (See [2] for scalar case.)

Theorem 3.2. *The moment problem (3.4) is solvable if and only if* $\Gamma_n = [S_{i+j}]_{i,j=0}^n \geqslant 0$.

Proof. Suppose that the problem (3.4) has a solution $\sigma(u)$. Let

$$\tilde{S}_{2n} = \int_{-\infty}^{+\infty} u^{2n} \, \mathrm{d}\sigma(u),$$

then $0 \leqslant \tilde{S}_{2n} \leqslant S_{2n}$. Therefore

$$\Gamma_n = \left[\int_{-\infty}^{+\infty} u^{i+j} \, \mathrm{d}\sigma(u) \right]_{i,j=0}^n + \mathrm{diag}[0, \dots, 0, S_{2n} - \tilde{S}_{2n}] \geqslant 0. \tag{3.5}$$

Suppose conversely that $\Gamma_n \geqslant 0$. Now put

$$F(\lambda) = -\frac{S_0}{\lambda I_p - S_0^{\mathrm{D}} S_1} - \frac{S_0^{\mathrm{D}} S_0^{(1)}}{\lambda I_p - S_0^{(1)\mathrm{D}} S_1^{(1)}} - \cdots - \frac{S_0^{(n-3)\mathrm{D}} S_0^{(n-2)}}{\lambda I_p - S_0^{(n-2)\mathrm{D}} S_1^{(n-2)}}$$

$$- \frac{S_0^{(n-2)\mathrm{D}} S_0^{(n-1)}}{\lambda I_p S_0^{(n-1)\mathrm{D}} S_1^{(n-1)} + S_0^{(n-1)\mathrm{D}} \Phi(\lambda)},$$

in which $\Phi(\lambda) = -\alpha/\lambda \in \mathcal{N}_p$ $(0 \leqslant \alpha \leqslant S_0^{(n)})$ and $S_0^{(1)}, S_1^{(1)}, \ldots, S_0^{(n-1)}, S_1^{(n-1)}, S_0^{(n)}$ are defined by the formula (2.9) successively. By using Lemma 2.7 repeatedly, we have that $F(\lambda) \in \mathcal{N}_p$, which is rational, and has Laurent expansion at infinity

$$F(\lambda) = -\frac{S_0}{\lambda} - \frac{S_1}{\lambda^2} - \frac{S_2}{\lambda^3} - \cdots - \frac{S_{2n-1}}{\lambda^{2n}} - \frac{\tilde{S}_{2n}}{\lambda^{2n+1}} + \mathrm{o}(\lambda^{-2n-1}).$$

Since $0 \leqslant \alpha \leqslant S_0^{(n)}$, we conclude from Eq. (2.9) that $0 \leqslant \tilde{S}_{2n} \leqslant S_{2n}$. Hence, by Lemma 2.1, there exists a certain Hermitian measure $\sigma(u)$ with moments $S_0, \ldots, S_{2n-1}, \tilde{S}_{2n}$, that is, $\sigma(u)$ is a solution to the problem (3.4). □

As a consequence of Theorem 3.2 we have a useful conclusion which is actually implied by Ando in [4], Corollary 3.

Corollary 3.3 (Ando [4]). *If $\Gamma_n = [S_{i+j}]_{i,j=0}^n \geqslant 0$, then there exists a $C \geqslant 0$ of order p such that Γ_n can be written as a certain sum of two nonnegative Hermitian block-Hankel matrices:*

$$\Gamma_n = \begin{bmatrix} S_1 & \cdots & S_n \\ \vdots & & \vdots \\ S_n & \cdots & S_{2n} - C \end{bmatrix} + \begin{bmatrix} 0 & & 0 \\ & \ddots & \\ 0 & & C \end{bmatrix}, \tag{3.6}$$

where the first matrix on the right side of the last equality is n.e.

Note that in the scalar case: $p = 1$, if $\Gamma_n \geqslant 0$ is singular, the form (3.6), alias its quasidirect decomposition, is uniquely defined by Γ_n itself (see [8], Lemma 2.7), and the singularity of Γ_n always leads to a unique solution to the problem (3.4), which has a finite number of points of increase.

Although, the singularity of Γ_n in the matrix case will not always lead to uniqueness, but to reduction of the dimension of the solutions (see Theorem 4.5). We will consider these questions in the following results. To begin with we present the explicit forms of the solutions to the TH problem and to the moment problem (3.4), which follow from Lemma 2.1 and Theorems 3.2 and 3.3 at once.

Theorem 3.4. *The general solution $\sigma(u)$ to the TH problem is representable as a matrix continued fraction*

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}\sigma(u)}{u-\lambda} = -\frac{S_0}{\lambda I_p - S_0^{\mathrm{D}} S_1} - \frac{S_0^{\mathrm{D}} S_0^{(1)}}{\lambda I_p - S_0^{(1)\mathrm{D}} S_1^{(1)}} - \cdots - \frac{S_0^{(n-3)\mathrm{D}} S_0^{(n-2)}}{\lambda I_p - S_0^{(n-2)\mathrm{D}} S_1^{(n-2)}}$$
$$- \frac{S_0^{(n-2)\mathrm{D}} S_0^{(n-1)}}{\lambda I_p - S_0^{(n-1)\mathrm{D}} S_1^{(n-1)} + S_0^{(n-1)\mathrm{D}} \Phi(\lambda)}, \tag{3.7}$$

where $\Phi(\lambda) = \int_{-\infty}^{+\infty} 1/(u-\lambda)\mathrm{d}\tau(u)$ *and* $S_0^{(n)} = \int_{-\infty}^{+\infty} \mathrm{d}\tau(u)$ *for a Hermitian measure* $\tau(u)$.

(To determine the Hermitian measure $\sigma(u)$ from the function $\Phi(\lambda)$ one can use the Stieltjes–Perron inversion formula in the matrix case.)

It is easy to see that in the case when Γ_n is n.e., the TH problem has infinitely many solutions provided $S_0^{(n)} \neq 0$ (see the proof of Theorem 4.1 for details).

Corollary 3.5. *The TH problem has a unique solution* $\sigma(u)$ *if and only if* Γ_n *is n.e., and* $S_0^{(n)} = 0$. *In this case, the unique solution* $\sigma(u)$ *has a finite number of points of increase such that Eq. (3.7) holds.*

Proof. Follows from Theorems 3.1 and 3.4 and the Stieltjes–Perron inversion formula. □

Theorems 3.1 and 3.4, and Corollary 3.5 support that a great amount of all information about the TH problem is contained in the matrix Γ_n and the matrix continued fraction (3.7).

Theorem 3.6. *The general solution to the moment problem (3.4) is representable as a matrix continued fraction given as in (3.7), where* $\Phi(\lambda) = \int_{-\infty}^{+\infty} 1/(u-\lambda)\mathrm{d}\tau(u)$ *but only* $S_0^{(n)} \geqslant \int_{-\infty}^{+\infty} \mathrm{d}\tau(u)$ *for a Hermitian measure* $\tau(u)$.

In the next section, we will generalize a theorem of Kovalishina [17] and use it to rewrite the aforementioned general solution to the problem (3.4) as a linear fractional transformation of an arbitrary Nevanlinna pair.

Corollary 3.7. *The moment problem (3.4) has a unique solution* $\sigma(u)$ *if and only if* $\Gamma_n \geqslant 0$ *and* $S_0^{(n)} = 0$. *In this case, the unique solution* $\sigma(u)$ *has a finite number of points of increase.*

Proof. Follows from Theorems 3.2 and 3.5, and the Stieltjes–Perron inversion formula. □

Corollary 3.8. *If* $\Gamma_n \geqslant 0$ *and* $S_0^{(n)} = 0$, *then* Γ_n *has only a trivial decomposition of the form (3.6).*

Proof. Follows from Corollaries 3.3, 3.5 and 3.7. □

Note that under the assumption of Corollary 3.8, Γ_n is always singular (see Theorem 3.9).

Theorem 3.9. *If Γ_n is n.e., then there exists a nonsingular matrix P of order $(n+1)p$ such that*

$$P\Gamma_n P^* = \begin{bmatrix} S_0^{(0)} & & & \\ & S_0^{(1)} & & \\ & & \ddots & \\ & & & S_0^{(n)} \end{bmatrix},$$

$$R(S_0^{(n)}) \subseteq R(S_0^{(n-1)}) \subseteq \cdots \subseteq R(S_0^{(0)}), \tag{3.8}$$

where $R(A)$ stands for the column space of a matrix A.

Proof. Set $\Gamma_n^{(0)} = \Gamma_n$. From (2.9) we obtain that

$$\begin{bmatrix} S_0^{(0)} & 0 \\ 0 & \Gamma_{n-1}^{(1)} \end{bmatrix} = \begin{bmatrix} I_P & & & 0 \\ & S_0^{(0)} & & \\ & & \ddots & \\ 0 & & & S_0^{(0)} \end{bmatrix} \begin{bmatrix} I_P & & & \\ & S_0^{(0)} & & \\ & \vdots & \ddots & \\ & S_{n-1}^{(0)} & \cdots & S_0^{(0)} \end{bmatrix}^{\mathrm{D}}$$
$$\times \begin{bmatrix} I_p & & & \\ -S_1 S_0^{\mathrm{D}} & I_p & & \\ \vdots & & \ddots & \\ -S_n S_0^{\mathrm{D}} & 0 & \cdots & I_p \end{bmatrix} \Gamma_n^{(0)} \begin{bmatrix} I_p & -S_0^{\mathrm{D}} S_1 & \cdots & -S_0^{\mathrm{D}} S_n \\ & I_p & & 0 \\ & & \ddots & \vdots \\ & & & I_p \end{bmatrix}$$
$$\times \begin{bmatrix} I_p & & & \\ & S_0^{(0)} & \cdots & S_{n-1}^{(0)} \\ & & \ddots & \vdots \\ & & & S_0^{(0)} \end{bmatrix}^{\mathrm{D}} \begin{bmatrix} I_p & & & 0 \\ & S_0^{(0)} & & \\ & & \ddots & \\ 0 & & & S_0^{(0)} \end{bmatrix}.$$

By using the operation similar to that given in the last equation on $\Gamma_{n-1}^{(1)}$, we obtain a formula similar to the last one with $S_0^{(0)}, \Gamma_{n-1}^{(1)}$ replaced by $S_0^{(1)}, \Gamma_{n-2}^{(2)}$ respectively, so that

$$\begin{bmatrix} S_0^{(0)} & & \\ & S_0^{(1)} & \\ & & \Gamma_{n-2}^{(2)} \end{bmatrix}$$

$$= \begin{bmatrix} I_p & & & & \\ & I_p & & & \\ & & S_0^{(1)} & & \\ & & & \ddots & \\ & & & & S_0^{(1)} \end{bmatrix} \begin{bmatrix} I_p & & & & 0 \\ & I_p & & & \\ & & S_0^{(1)} & & \\ & & \vdots & \ddots & \\ & & S_{n-1}^{(1)} & \cdots & S_0^{(1)} \end{bmatrix}^{\mathrm{D}}$$

$$\times \begin{bmatrix} I_p & & & & \\ & I_p & & & \\ & -S_1^{(1)} S_0^{(1)\mathrm{D}} & I_p & & \\ & \vdots & & \ddots & \\ & -S_{n-1}^{(1)} S_0^{(1)\mathrm{D}} & 0 & \cdots & I_p \end{bmatrix} \begin{bmatrix} S_0^{(0)} & \\ & \Gamma_{n-1}^{(1)} \end{bmatrix}$$

$$\begin{bmatrix} I_p & & & & \\ & I_p & & & \\ & -S_1^{(1)} S_0^{(1)\mathrm{D}} & I_p & & \\ & \vdots & & \ddots & \\ & -S_{n-1}^{(1)\mathrm{D}} S_0^{(1)\mathrm{D}} & 0 & \cdots & I_p \end{bmatrix}^*$$

$$\times \begin{bmatrix} I_p & & & & \\ & I_p & & & \\ & & S_0^{(1)} & \cdots & S_{n-1}^{(1)} \\ & & & \ddots & \vdots \\ & & & & S_0^{(1)} \end{bmatrix}^{\mathrm{D}} \begin{bmatrix} I_p & & & & \\ & I_p & & & \\ & & S_0^{(1)} & & \\ & & & \ddots & \\ & & & & S_0^{(1)} \end{bmatrix}.$$

The further continuation of the procedure is evident. After n such steps we get to a block diagonal matrix $\mathrm{diag}[S_0^{(0)}, S_0^{(1)}, \ldots, S_0^{(n)}]$ which coincides with $P\Gamma_n P^*$, where P is a certain nonsingular matrix of order $(n+1)p$. The second assertion follows directly from the nonnegative extendability of $\Gamma_{n-k}^{(k)} (k = 0, 1, \ldots, n-1)$. $\square$

We point out that Theorem 3.9 is not true for the case of $\Gamma_n \geqslant 0$. As an example,

$$\Gamma_2 = \begin{bmatrix} 0 & 0 \\ 0 & I_2 \end{bmatrix}, \qquad \mathrm{R}(S_0^{(1)}) = C^2, \qquad \mathrm{R}(S_0^{(0)}) = \{0\}.$$

Also, by the proof of Theorem 3.9 we can take

$$P_n = \prod_{i=0}^{n-1} \begin{bmatrix} I_p & & & & & \\ & \ddots & & & & \\ & & I_p & & & \\ & & -S_1^{(i)}S_0^{(i)\mathrm{D}} & I_p & & \\ & & \vdots & \ddots & \ddots & \\ & & -S_{n-i}^{(i)}S_0^{(i)\mathrm{D}} & \cdots & -S_1^{(i)}S_0^{(i)\mathrm{D}} & I_p \end{bmatrix}$$

$$\times \begin{bmatrix} I_p & & & & \\ & \ddots & & & \\ & & I_p & & \\ & & -S_1^{(i)}S_0^{(i)\mathrm{D}} & I_p & \\ & & \vdots & & \ddots \\ & & -S_{n-i}^{(i)}S_0^{(i)\mathrm{D}} & 0 & \cdots & I_p \end{bmatrix} \tag{3.9}$$

as the matrix P given in Theorem 3.9, which can be considered as the product of a number of elementary operations on block rows without exchange of block rows.

Therefore, we have the following

Corollary 3.10. *If Γ_n is n.e., there exists a certain principal submatrix, of order equal to rank of Γ_n, which is Hermitian positive, and is located at the left upper corner as it possibly can, that is, the ith row (column) of it is located at the jth block row (column) of $\Gamma_n, j \leqslant i$.*

Note that in the scalar case: $p = 1$, if Γ_n is n.e., then the leading principal submatrix of Γ_n, of order equal to rank of Γ_n is always Hermitian positive, and vice versa.

The following result characterizes the class of Hermitian block-Hankel matrices which are n.e.

Theorem 3.11. *Γ_n is n.e., if and only if $\Gamma_n \geqslant 0$ and $R(\tilde{S}_0^{(n)}) \subseteq R(S_0^{n-1})$, where $\tilde{S}_0^{(n)}$ is defined via $P_n\Gamma_nP_n^* = \mathrm{diag}[S_0^{(0)}, \ldots, S_0^{(n-1)}, \tilde{S}_0^{(n)}]$ in which P_n is as in (3.9).*

Proof. The "only if" part follows immediately from Theorem 3.9 the "if" part can be derived in much the same way as that used in Theorem 3.1. □

It is worth to point out that we deal only with the TH problem of the form (2.1) in the present paper. As for another type of the TH problem of finding a Hermitian measure $\sigma(u)$ subject to

$$S_i = \int_{-\infty}^{+\infty} u_i \mathrm{d}\sigma(u), \quad i = 0, 1, \dots, 2n-1, \tag{3.10}$$

one may also give its criteria of solvability and uniqueness in a similar way to that used in the TH problem (2.1), and will be omitted here. (The change, however, from the TH problem (2.1) to (3.10) seems to be not a simple matter, and some details need to be discussed rather carefully.)

4. The relation between the solution forms of matrix continued fractions and of linear fractional transformations

In this section we will consider an intrinsic relation between the descriptions of the solutions to the moment problems (2.1) and (3.4) in terms of matrix continued fractions and in terms of the general and popular way of linear fractional transformations.

A well-known theorem of Kovalishina ([17], Theorem 3) are modified and extended to some extent.

Starting from Theorem 3.4, we can readily show the close relation of the solution form (3.7) of matrix continued fractions of the TH problem to the solution form of linear fractional transformations [17]. Note that here a uniform description of the solutions is given in nondegenerate case ($\Gamma_n > 0$) or not ($\Gamma_n \geqslant 0$ is singular and n.e.).

Theorem 4.1. *The general solution to the TH problem (2.1) is representable as a linear fractional transformation of an arbitrary $G(\lambda) \in \mathcal{N}_p$ such that $\lim_{\lambda\to\infty} G(\lambda)/\lambda = 0$ uniformly in each $\pi_\in$:*

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}\sigma(u)}{u-\lambda} = \frac{\theta_{11}(\lambda)G(\lambda) + \theta_{12}(\lambda)}{\theta_{21}(\lambda)G(\lambda) + \theta_{22}(\lambda)}, \tag{4.1}$$

whose coefficient matrix is of the decomposable form

$$\Theta(\lambda) = \begin{bmatrix} \theta_{11}(\lambda) & \theta_{12}(\lambda) \\ \theta_{21}(\lambda) & \theta_{22}(\lambda) \end{bmatrix} = \prod_{i=0}^{n} \begin{bmatrix} 0 & S_0^{(i)} \\ S_0^{(i)\mathrm{D}} & \lambda I_p - S_0^{(i)\mathrm{D}} S_1^{(i)} \end{bmatrix}, \quad S_1^{(n)} = 0. \tag{4.2}$$

In the degenerate case, if

$$U^* S_0^{(n)} U = \begin{bmatrix} \tilde{S}_0^{(n)} & 0 \\ 0 & 0_r \end{bmatrix}, \quad \tilde{S}_0^{(n)} > 0, \ 0 < r \leqslant p,$$

for a certain unitary matrix U, then

$$U^* G(\lambda) U = \begin{bmatrix} \widehat{G}(\lambda) & 0 \\ 0 & 0_r \end{bmatrix}.$$

Further, the problem has only a solution $\sigma(u)$ if and only if Γ_n is n.e. and $S_0^{(n)} = 0$. In this case, the unique solution $\sigma(u)$ is such that (4.1) holds with $G(\lambda) \equiv 0$.

Proof. Observe that in order to prove the theorem, it is sufficient to show the following assertion: if $\Phi(\lambda) = \int_{-\infty}^{\infty} (u-\lambda)^{-1}\, \mathrm{d}\tau(u)$, where $\int_{-\infty}^{\infty} \mathrm{d}\tau(u) = S_0^{(n)}$, it can be rewritten as the form

$$\Phi(\lambda) = \frac{S_0^{(n)}}{\lambda I_p + S_0^{(n)D} G(\lambda)}, \tag{4.3}$$

where $G(\lambda)$ is the parameter function described in the theorem, and vice versa.

As before, we may assume that $S_0^{(n)} > 0$. Suppose that $\Phi(\lambda)$ admits a integral representation $\Phi(\lambda) = \int_{-\infty}^{+\infty} (u-\lambda)^{-1}\, \mathrm{d}\tau(u), \int_{-\infty}^{+\infty} \mathrm{d}\tau(u) = S_0^{(n)}$. Then, by Lemma 2.2, $\Phi^{-1}(\lambda)$ exists and $-\Phi^{-1}(\lambda) \in \mathcal{N}_p$. On the other hand,

$$\lim_{y\to\infty} \frac{-\Phi^{-1}(\mathrm{i}y)}{\mathrm{i}y} = \lim_{y\to\infty} \frac{-I_p}{\mathrm{i}y\Phi(\mathrm{i}y)} = S_0^{(n)-1},$$

so that $-\Phi^{-1}(\lambda)$ permits an integral representation of the form

$$-\Phi^{-1}(\lambda) = S_0^{(n)-1}\lambda + \tilde{\beta} + \int_{-\infty}^{-\infty} \frac{(1+u\lambda)}{(u-\lambda)} \mathrm{d}\tilde{\tau}(u), \quad \tilde{\beta} = \tilde{\beta}^*.$$

We have in turn

$$\Phi(\lambda) = -\frac{S_0^{(n)}}{\lambda I_p + S_0^{(n)-1} S_0^{(n)} (\tilde{\beta} + \int_{-\infty}^{-\infty} (1+u\lambda)/(u-\lambda) \mathrm{d}\tilde{\tau}(u)) S_0^{(n)}},$$

which coincides with Eq. (4.3) with $G(\lambda) = S_0^{(n)}(\tilde{\beta} + \int_{-\infty}^{+\infty} (1+u\lambda)/(u-\lambda)\, \mathrm{d}\tilde{\tau}(u)) S_0^{(n)}$.

Conversely, if $\Phi(\lambda)$ is of the form (4.3), then $\Phi(\lambda) \in \mathcal{N}_p$ and $\lim_{y\to\infty} - \mathrm{i}y\Phi(\mathrm{i}y) = S_0^{(n)}$, so that $\Phi(\lambda) \in \mathcal{N}_p^0$ and $\Phi(\lambda) = \int_{-\infty}^{+\infty} (u-\lambda)^{-1}\, \mathrm{d}\tau(u)$ for a certain $\tau(u)$ satisfying $\int_{-\infty}^{+\infty} \mathrm{d}\tau(u) = S_0^{(n)}$, as desired. The rest is plain. $\square$

Note that in the case of $\Gamma_n > 0$, let

$$J = \begin{bmatrix} 0 & \mathrm{i}I_p \\ -\mathrm{i}I_p & 0 \end{bmatrix},$$

then, by direct verification, each factor on the right side of Eq. (4.2) (and therefore the matrix $\Theta(\lambda)$) is a matrix-valued function which is J-expanding in π^+, J-unitary on the real axis, of full rank, and has a pole of order one at $\lambda = \infty$.

The correspondence defined by Eq. (4.1) between Hermitian measures $\sigma(u)$ and functions $G(\lambda)$ with the properties given in Theorem 4.1 is one-to-one.

Before parametrizing the general solution to the moment problem (3.4) in terms of a linear fractional transformation, we need a result which is an extension, to some extent, of a well-known theorem of [17] (Theorem 3).

Theorem 4.2. *If a matrix-valued function $W(\lambda)$ holomorphic in π^+ satisfying the fundamental matrix inequality:*

$$\begin{bmatrix} S_0 & S_1 & \cdots & S_n & \vdots & W(\lambda) \\ S_1 & S_2 & \cdots & S_{n+1} & \vdots & \lambda[W\lambda) + S_0/\lambda] \\ \cdots & \cdots & \cdots & \cdots & \vdots & \cdots \\ S_n & S_{n+1} & \cdots & S_{2n} & \vdots & \lambda^n[W(\lambda) + S_0/\lambda + \cdots + S_{n-1}/\lambda^n] \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & * & & \vdots & \frac{W(\lambda)-W(\lambda)^*}{\lambda-\bar{\lambda}} \end{bmatrix} \geqslant 0,$$

$$\lambda \in \pi, \qquad (\mathrm{FMI}(\mathscr{H}))$$

then $W(\lambda)$ belongs to $\mathcal{N}_p$ and has the asymptotic representation

$$\lim_{\lambda\to\infty} \lambda^{2n+1}\left[W(\lambda) + \frac{S_0}{\lambda} + \cdots + \frac{S_{2n-1}}{\lambda^{2n}}\right] = -\tilde{S}_{2n},$$

where $0 \leqslant \tilde{S}_{2n} \leqslant S_{2n}$. Conversely, if $W(\lambda) \in \mathcal{N}_p$ has the asymptotic representation

$$\lim_{\lambda\to\infty} \lambda^{2n+1}\left[W(\lambda) + \frac{S_0}{\lambda} + \cdots + \frac{S_{2n-1}}{\lambda^{2n}}\right] = -S_{2n},$$

then it satisfies FMI($\mathscr{H}$).

Proof. The second part of the theorem has been proved by Kovalishina in [17]. To verify the first part, let us begin with the problem in the case of $n = 0$. Then FMI($\mathscr{H}$) is reduced to

$$\begin{bmatrix} S_0 & W(\lambda) \\ W(\lambda)^* & \frac{W(\lambda)-W(\lambda)^*}{\lambda-\bar{\lambda}} \end{bmatrix} \geqslant 0, \quad \lambda \in \pi^+. \tag{4.4}$$

Letting $\lambda = \mathrm{i}y$ $(y > 0)$ and multiplying by $T = \mathrm{diag}[I_p, -\mathrm{i}yI_p]$ on the right and by T^* on the left, from the last matrix inequality we obtain

$$\begin{bmatrix} S_0 & -\mathrm{i}yW(\mathrm{i}y) \\ (-\mathrm{i}yW(\mathrm{i}y))^* & \mathrm{Re}[-\mathrm{i}yW(\mathrm{i}y)] \end{bmatrix} \geqslant 0, \quad y > 0, \tag{4.5}$$

so that

$$(S_0)_{kk}\mathrm{Re}[-\mathrm{i}yW(\mathrm{i}y)]_{kl} - |-\mathrm{i}yW(\mathrm{i}y)_{kl}|^2 \geqslant 0, \quad 1 \leqslant k, l \leqslant p,$$

whence

$$|-\mathrm{i}yW(\mathrm{i}y)_{kl}| \leqslant (S_0)_{kk}, \quad 1 \leqslant k, l \leqslant p,$$

where $(A)_{kl}$ denotes the (k, l) entry of a matrix A. Thus, $\sup_{y \geqslant 1}\|\mathrm{i}yW(\mathrm{i}y)\| < +\infty$ for a certain matrix norm $\|\cdot\|$ on $M_p(\mathbb{C})$, and therefore, by (1.4), $W(\lambda) = \int_{-\infty}^{+\infty}(u-\lambda)^{-1}\,\mathrm{d}\sigma(u)$ which has the asymptotic representation: $\lim_{\lambda \to \infty} -\lambda W(\lambda) = \int_{-\infty}^{+\infty}\mathrm{d}\sigma(u)$ by Lemma 2.1. Going to the limit as $y \to +\infty$, from (4.5) we obtain

$$\begin{bmatrix} S_0 & \int_{-\infty}^{+\infty}\mathrm{d}\sigma(u) \\ \int_{-\infty}^{+\infty}\mathrm{d}\sigma(u) & \int_{-\infty}^{+\infty}\mathrm{d}\sigma(u) \end{bmatrix} \geqslant 0,$$

which is equivalent to

$$S_0 - \int_{-\infty}^{+\infty}\mathrm{d}\sigma(u)\left[\int_{-\infty}^{+\infty}\mathrm{d}\sigma(u)\right]^{\mathrm{D}}\int_{-\infty}^{+\infty}\mathrm{d}\sigma(u) = S_0 - \int_{-\infty}^{+\infty}\mathrm{d}\sigma(u) \geqslant 0,$$

since index $[\int_{-\infty}^{+\infty}\mathrm{d}\sigma(u)] = 1$ obviously, that is, $\int_{-\infty}^{+\infty}\mathrm{d}\sigma(u) \leqslant S_0$.

Next, we consider FMI($\mathscr{H}$) in the case of $n \geqslant 1$. If we multiply FMI($\mathscr{H}$), step by step, on the left by

$$T_i = \begin{bmatrix} I_{(n+1)p} & \vdots & 0 \\ \cdots\ \cdots\ \cdots\ \cdots & \cdots & \cdots \\ 0, \ldots, I_p, 0, \ldots, 0 & \vdots & \bar{\lambda}I_p \end{bmatrix}, \quad i = 1, \ldots, n,$$

and on the left by T_i^*, and applying the similar argument to that given in [17], we obtain

$$\begin{bmatrix} S_0 & S_1 & \cdots & S_n & \vdots & \lambda^n[W(\lambda)+S_0/\lambda+\cdots+S_{n-1}/\lambda^n] \\ S_1 & S_2 & \cdots & S_{n+1} & \vdots & \lambda^{n+1}[W(\lambda)+S_0/\lambda+\cdots+S_n/\lambda^{n+1}] \\ \cdots & \cdots & \cdots & \cdots & \vdots & \cdots \\ S_n & S_{n+1} & \cdots & S_{2n} & \vdots & E(\lambda) \\ \cdots & \cdots & \cdots & \cdots & \vdots & \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots \\ & & * & & \vdots & \frac{E(\lambda)-E(\lambda)^*}{\lambda-\bar{\lambda}} \end{bmatrix} \geqslant 0,$$

$\lambda \in \pi^+$,

in which

$$E(\lambda) = \lambda^{2n}\left[W(\lambda) + \frac{S_0}{\lambda} + \cdots + \frac{S_{2n-1}}{\lambda^{2n}}\right],$$

and therefore, in particular,

$$\begin{bmatrix} S_{2n} & E(\lambda) \\ E(\lambda)^* & \frac{E(\lambda)-E(\lambda)^*}{\lambda-\bar{\lambda}} \end{bmatrix} \geqslant 0, \quad \lambda \in \pi^+.$$

From the result proved for the case of $n = 0$, it follows that $E(\lambda) \in \mathcal{N}_p$ has the asymptotic representation: $\lim_{\lambda\to\infty} - \lambda E(\lambda) = \tilde{S}_{2n} \leqslant S_{2n}$, where $\tilde{S}_{2n} \geqslant 0$, that is,

$$\lim_{\lambda\to\infty} - \lambda^{2n+1}\left[W(\lambda) + \frac{S_0}{\lambda} + \cdots + \frac{S_{2n-1}}{\lambda^{2n}}\right] = \tilde{S}_{2n} \leqslant S_{2n}.$$

This completes the proof. □

Thanks to Lemma 2.1, it is easy to see that Theorem 4.2 is equivalent to the following result, which is at first considered in [20] (Theorem 2.10) in the non-degenerate case.

Theorem 4.3. *The moment problem (3.4) is equivalent to* FMI($\mathscr{H}$). *Further, the relation between their solutions can be formulated as*

$$W(\lambda) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\sigma(u)}{u-\lambda}, \tag{4.6}$$

where $\sigma(u)$, $W(\lambda)$ *are the solutions to the problem (3.4) and to* FMI($\mathscr{H}$), *respectively.*

It is known ([17], p. 448) that the general solution to FMI($\mathscr{H}$) in the case when $n = 0$ and $S_0 > 0$ can be described as a linear fractional transformation of an arbitrary Nevanlinna pair

$$\begin{bmatrix} p(\lambda) \\ q(\lambda) \end{bmatrix}:$$

$$W(\lambda) = \frac{-q(\lambda)}{p(\lambda) + \lambda S_0^{-1} q(\lambda)}, \tag{4.7}$$

which can be rewritten as

$$W(\lambda) = \frac{-S_0[S_0^{-1} q(\lambda) S_0^{-1}]}{S_0^{-1}[S_0 p(\lambda) S_0^{-1}] + \lambda [S_0^{-1} q(\lambda) S_0^{-1}]}.$$

Let $\tilde{p}(\lambda) = S_0 p(\lambda) S_0^{-1}$ and $\tilde{q}(\lambda) = S_0^{-1} q(\lambda) S_0^{-1}$, then

$$\begin{bmatrix} \tilde{p}(\lambda) \\ \tilde{q}(\lambda) \end{bmatrix}$$

is also an arbitrary Nevanlinna pair, if

$$\begin{bmatrix} p(\lambda) \\ q(\lambda) \end{bmatrix}$$

is, and vice versa. Then we have an equivalent form of Eq. (4.7)

$$W(\lambda) = \frac{-S_0 \tilde{q}(\lambda)}{S_0^{-1} \tilde{p}(\lambda) + \lambda \tilde{q}(\lambda)}. \tag{4.8}$$

Thus, we have proved the following lemma.

Lemma 4.4. *Let $S_0 > 0$. Then the general solution $\sigma(u)$ to the moment problem (3.4) for the case of $n = 0$ is representable as a linear fractional transformation of an arbitrary Nevanlinna pair*

$$\begin{bmatrix} p(\lambda) \\ q(\lambda) \end{bmatrix}:$$

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}\sigma(u)}{u - \lambda} = \frac{-S_0 q(\lambda)}{S_0^{-1} p(\lambda) + \lambda q(\lambda)} \tag{4.9}$$

with coefficient matrix

$$\begin{bmatrix} 0 & -S_0 \\ S_0^{-1} & \lambda I_p \end{bmatrix}. \tag{4.10}$$

Note that let

$$J = \begin{bmatrix} 0 & \mathrm{i} I_p \\ -\mathrm{i} I_p & 0 \end{bmatrix},$$

then, as before, the matrix valued function (4.10) is J-expanding in π^+, J-unitary on the real axis, of full rank, and has a pole of order one at ∞.

Also, if $S_0 \geqslant 0$ is singular, it is not difficult to prove that Lemma 4.4 is valid as well with S_0^{D} in place of S_0^{-1} therein and that if

$$U^* S_0 U = \begin{bmatrix} \tilde{S}_0 & 0 \\ 0 & 0_r \end{bmatrix}, \quad \tilde{S}_0 > 0,$$

then

$$U^* p(\lambda) U = \begin{bmatrix} \tilde{p}(\lambda) & 0 \\ 0 & 0_r \end{bmatrix} \quad \text{and} \quad U^* q(\lambda) U = \begin{bmatrix} \tilde{q}(\lambda) & 0 \\ 0 & I_r \end{bmatrix}, \qquad 0 < r \leqslant p,$$

where

$$\begin{bmatrix} \tilde{p}(\lambda) \\ \tilde{q}(\lambda) \end{bmatrix}$$

is also an arbitrary Nevanlinna pair with reduced dimension.

Combining Theorem 3.6 with Lemma 4.4, we have

Theorem 4.5. *The general solution to the moment problem (3.4) is representable as a linear fractional transformation of an arbitrary Nevanlinna pair*

$$\begin{bmatrix} p(\lambda) \\ q(\lambda) \end{bmatrix}: \quad \int_{-\infty}^{+\infty} \frac{\mathrm{d}\sigma(u)}{u - \lambda} = \frac{\theta_{11}(\lambda) p(\lambda) + \theta_{12}(\lambda) q(\lambda)}{\theta_{21}(\lambda) p(\lambda) + \theta_{22}(\lambda) q(\lambda)}, \tag{4.11}$$

whose coefficient matrix is of the decomposable form

$$\Theta(\lambda) = \begin{bmatrix} \theta_{11}(\lambda) & \theta_{12}(\lambda) \\ \theta_{21}(\lambda) & \theta_{22}(\lambda) \end{bmatrix} = \prod_{i=0}^{n} \begin{bmatrix} 0 & -S_0^{(i)} \\ S_0^{(i)\mathrm{D}} & \lambda I_p - S_0^{(i)\mathrm{D}} S_1^{(i)} \end{bmatrix}, \quad S_1^{(n)} = 0. \tag{4.12}$$

In the degenerate case ($\Gamma_n \geqslant 0$ is singular), if

$$U^* S^{(n)} U = \begin{bmatrix} \widehat{S}_0^{(n)} & 0 \\ 0 & 0_r \end{bmatrix}, \quad \widehat{S}_0^{(n)} > 0, 0 < r \leqslant p,$$

then

$$U^* p(\lambda) U = \begin{bmatrix} \hat{p}(\lambda) & 0 \\ 0 & 0_r \end{bmatrix} \quad \text{and} \quad U^* q(\lambda) U = \begin{bmatrix} \hat{q}(\lambda) & 0 \\ 0 & I_r \end{bmatrix},$$

where

$$\begin{bmatrix} \hat{p}(\lambda) \\ \hat{q}(\lambda) \end{bmatrix}$$

is also an arbitrary Nevanlinna pair. Further, if and only if $S_0^{(n)} = 0$, *the problem (3.4) has only one solution* $\sigma(u)$, *which corresponds to (4.11) with* $p(\lambda) = 0$ *and* $q(\lambda) = I_p$.

It is worth to point out that in the nondegenerate case: $\Gamma_n > 0$, the matrix decomposition of the form (4.12) is essentially nothing other than a superposition of elementary linear fractional transformations which has been presented by Kovalishina [17] as a Blaschke–Potopov product in the so-called Schur's stepwise process. Thus this leads to the solution of the full moment problem consisting of finding a Hermitian measure $\sigma(u)$ subject to $S_k = \int_{-\infty}^{+\infty} \mathrm{d}\sigma(u), k = 0, 1, \ldots$, to an infinite product of the form

$$\prod_{i=0}^{\curvearrowright \infty} \begin{bmatrix} 0 & -S_0^{(i)} \\ S_0^{(i)-1} & \lambda I_p - S_0^{(i)\mathrm{D}} S_1^{(i)} \end{bmatrix}, \tag{4.13}$$

each factor of which is J-expanding in π^+, J-unitary on the real axis, of full rank and has only a pole of order one at $\lambda = \infty$.

In the degenerate case ($\Gamma_n \geqslant 0$ is singular), Theorem 4.5 coincides essentially with Theorem 1.1 of [6] (pp. 25–27) but with "the moment problem 3.4" in place of "the problem H" therein. Consequently, this also leads to the solution of the full moment problem mentioned above under the condition $\Gamma_n \geqslant 0, n = 0, 1, \ldots$. We will omit this investigation here.

As a rule, the independent parameters in Eq. (4.11) are the equivalence classes of Nevanlinna pairs: different pairs lead to the same measure $\sigma(u)$ if and only if they are equivalent to each other.

Our next object is to consider the close relation of Theorems 3.4 and 3.6 to the corresponding results given in [11,9] based on the use of the theory of orthogonal polynomial matrices in the nondegenerate case, which receive further development suitable to the degenerate case.

Defined the recurrence relations under the condition $\Gamma_n \geqslant 0$:

$$\begin{aligned} M_{i+1}(\lambda) &= M_i(\lambda)[\lambda I_p - S_0^{(i)\mathrm{D}} S_1^{(i)}] - M_{i-1}(\lambda) S_0^{(i-1)\mathrm{D}} S_1^{(i)}, \\ N_{i+1}(\lambda) &= N_i(\lambda)[\lambda I_p - S_0^{(i)\mathrm{D}} S_1^{(i)}] - N_{i-1}(\lambda) S_0^{(i-1)\mathrm{D}} S_1^{(i)}, \end{aligned} \tag{4.14}$$

for $i = 1, \ldots, n$, with initial conditions:

$$\begin{aligned} M_0(\lambda) &= I_p, & M_1(\lambda) &= \lambda I_p - S_0^{(0)\mathrm{D}} S_1^{(i)}, \\ N_0(\lambda) &= 0, & N_1(\lambda) &= S_0^{(0)}, \end{aligned}$$

where $S_0^{(0)}, S_1^{(0)}, \ldots, S_0^{(n-1)}, S_1^{(n-1)}, S_0^{(n)}$ are defined as in Section 3. Thus

$$
\begin{aligned}
F_1(\lambda) &= -\frac{S_0^{(0)}}{\lambda I_P - S_0^{(0)\mathrm{D}} S_1^{(0)}} = -\frac{N_1(\lambda)}{M_1(\lambda)}, \\
F_2(\lambda) &= -\frac{S_0^{(0)}}{\lambda I_P - S_0^{(0)\mathrm{D}} S_1^{(0)} - \dfrac{S_0^{(0)\mathrm{D}} S_0^{(1)}}{\lambda I_P - S_0^{(1)\mathrm{D}} S_1^{(1)}}} = -\frac{N_2(\lambda)}{M_2(\lambda)}, \\
&\cdots \quad \cdots \quad \cdots \\
F_{n-1}(\lambda) &= -\frac{N_{n-1}(\lambda)}{M_{n-1}(\lambda)},
\end{aligned}
\tag{4.15}
$$

which can be considered as the truncated parts of the matrix continued fraction given in Eq. (3.7). We now show that if Γ_n is n.e., the matrix sequence of $M_0(\lambda), \dots, M_n(\lambda)$ defined by Eq. (4.14) has a generalized orthogonal property with respect to the sequence $S_0, \dots, S_{2n}$ in the sense that

$$
\int_{-\infty}^{+\infty} M_i^*(u)\, \mathrm{d}\sigma(u) M_j(u)
= \begin{cases} 0, & i \neq j, \\ S_{2i} - [S_i, \dots, S_{2i-1}] \Gamma_{i-1}^{\mathrm{D}} [S_i, \dots, S_{2i-1}]^*, & i = j \geqslant 1 \\ S_0, & i = j = 0, \end{cases}
\tag{4.16}
$$

where $\sigma(u)$ is a solution to the TH problem (2.1). (In the case of $\Gamma_n > 0$, Eq. (4.16) means that $\{M_i(\lambda)\}$ is a sequence of right orthogonal matrices associated with the Hermitian measure $\sigma(u)$ (or the sequence of $S_0, \dots, S_{2n}$) [11].)

Indeed, assume $i \geqslant j$ and $i \geqslant 1$ (for if $i = j = 0$, Eq. (4.16) holds obviously), and rewrite $M_i(\lambda)$ and $M_j(\lambda)$ in the form

$$
\begin{aligned}
M_i(\lambda) &= M_{i0} + M_{i1}\lambda + \cdots + M_{i,i-1}\lambda^{i-1} + I_p \lambda^i, \\
M_j(\lambda) &= M_{j0} + M_{j1}\lambda + \cdots + M_{j,j-1}\lambda^{j-1} + I_p \lambda^j.
\end{aligned}
\tag{4.17}
$$

Then

$$
\int_{-\infty}^{+\infty} M_i^*(u)\, \mathrm{d}\sigma(u) M_j(u)
= [M_{i0}^*, \dots, M_{i,i-1}^*, I_p] \Gamma_i [M_{j0}^*, \dots, M_{j,j-1}^*, I_p, 0, \dots, 0]^*.
\tag{4.18}
$$

On the other hand, it is not difficult to verify that the Laurent expansion of the rational matrix $N_i(\lambda)/M_i(\lambda)$ at $\lambda = \infty$ admits the form

$$
\frac{N_i(\lambda)}{M_i(\lambda)} = \frac{S_0}{\lambda} + \cdots + \frac{S_{2i-1}}{\lambda^{2i}} + \frac{\tilde{S}_{2i}}{\lambda^{2i+1}} + \mathrm{o}(\lambda^{-2i-1}),
$$

so that

$$[M_{i0}^*, \dots, M_{i,i-1}^*, I_p]\begin{bmatrix} S_0 & S_1 & \cdots & S_i \\ S_1 & S_2 & \cdots & S_{i+1} \\ \cdots & \cdots & \cdots & \cdots \\ S_i & S_{i+1} & \cdots & \tilde{S}_{2i} \end{bmatrix} = [0, \dots, 0].$$

or equivalently

$$[M_{i0}^*, \dots, M_{i,i-1}^*]\Gamma_{i-1} = -[S_i, \dots, S_{2i-1}],$$
$$\tilde{S}_{2i} = -M_{i0}^* S_i - \cdots - M_{i,i-1}^* S_{2i-1}.$$

Thus, since $\Gamma_{i-1}\Gamma_{i-1}^{\mathrm{D}} = \Gamma_{i-1}^{\mathrm{D}}\Gamma_{i-1}$ is the orthogonal projection onto $R(\Gamma_{i-1})$ we have

$$-[S_i, \dots, S_{2,i-1}] = -[S_i, \dots, S_{2i-1}]\Gamma_{i-1}^{\mathrm{D}}\,\Gamma_{i-1},$$

and therefore

$$S_{2i} - \tilde{S}_{2i} = S_{2i} - [S_i, \dots, S_{2i-1}]\Gamma_{i-1}^{\mathrm{D}}[S_i, \dots, S_{2i-1}]^*. \tag{4.19}$$

Then

$$\int_{-\infty}^{+\infty} M_i^*(u)\,\mathrm{d}\sigma(u)M_j(u) = [0, \dots, 0, S_{2i} - \tilde{S}_{2i}][M_{j0}^*, \dots, M_{j,j-1}^*, I_p, 0, \dots, 0]^*.$$
$$= \begin{cases} 0, & i \neq j, \\ S_{2i} - \tilde{S}_{2i} & i = j \geqslant 1, \end{cases}$$

as needed thanks to (4.19).

Hence we have shown the following result.

Lemma 4.6. *Let Γ_n be n.e. and $M_i(\lambda)$ $(i = 0, 1, \dots, n)$ be defined by (4.14). Then $\{M_i(\lambda)\}_{i=0}^n$ has the generalized orthonogal property with respect to the sequence $S_0, \dots, S_{2n}$ in the sense given in (4.16).*

Note that the matrix $S_{2i} - \tilde{S}_{2i}$ given in Eq. (4.19) is as a general rule called the generalized Schur complement of Γ_{i-1} in Γ_i, denoted by $\Gamma_i/\Gamma_{i-1}, i = 1, \dots, n$.

As a consequence of Lemma 4.6, we have

Corollary 4.7. *Let Γ_n be n.e. and $M_i(\lambda)$ $(i = 0, 1, \dots, n)$ be defined by (4.14) or (4.17). Then*

$$\begin{bmatrix} M_0(\lambda) \\ M_1(\lambda) \\ \vdots \\ M_n(\lambda) \end{bmatrix} = P_n \begin{bmatrix} I_p \\ \lambda I_p \\ \vdots \\ \lambda^n I_p \end{bmatrix}, \tag{4.20}$$

in which P_n is as in (3.9).

Proof. Write the lower block triangular matrix P_n in the form

$$P_n = \begin{bmatrix} I_p & & & 0 \\ P_{10} & I_p & & \\ \vdots & \ddots & \ddots & \\ P_{n0} & \dots & P_{n,n-1} & I_p \end{bmatrix}.$$

Then Eq. (4.20) is equivalent to

$$M_n = \begin{bmatrix} I_p & & & 0 \\ M_{10} & I_p & & \\ \vdots & \ddots & \ddots & \\ M_{n0} & \dots & M_{n,n-1} & I_p \end{bmatrix} = P_n. \tag{4.21}$$

But, by Lemma 4.6, $\int_{-\infty}^{+\infty} M_i^*(u)\,\mathrm{d}\sigma(u)M_j(u) = \hat{S}_0^{(i)}\delta_{ij}$, where $\hat{S}_0^{(0)} = S_0$ and $\hat{S}_0^{(i)} = \Gamma_i/\Gamma_{i-1}$, $i \geqslant 1$, and therefore

$$M_n\Gamma_n M_n^* = \begin{bmatrix} \hat{S}_0^{(0)} & & & 0 \\ & \hat{S}_0^{(1)} & & \\ & & \ddots & \\ 0 & & & \hat{S}_0^{(n)} \end{bmatrix}.$$

Now put

$$P_n^{-1}M_n = \begin{bmatrix} I_p & & & 0 \\ C_{10} & I_p & & \\ \vdots & \ddots & \ddots & \\ C_{n0} & \dots & C_{n,n-1} & I_p \end{bmatrix} = C_n.$$

To prove Eq. (4.12), we need only to verify $C_{ij} = 0$ $(0 \leqslant j < i \leqslant n)$. But this follows from Theorem 3.9 and the following equality:

$$C_n \begin{bmatrix} S_0^{(0)} & & & 0 \\ & S_0^{(1)} & & \\ & & \ddots & \\ 0 & & & S_0^{(n)} \end{bmatrix} C_n^* = \begin{bmatrix} \hat{S}_0^{(0)} & & & 0 \\ & \hat{S}_0^{(1)} & & \\ & & \ddots & \\ 0 & & & \hat{S}_0^{(n)} \end{bmatrix}.$$

This completes the proof. □

As a useful by-product of the proof of Corollary 4.7, we obtain

$$\hat{S}_0^{(0)} = S_0^{(0)}, \qquad \hat{S}_0^{(i)} = \Gamma_i/\Gamma_{i-1} = S_0^{(i)}, \quad i = 1, \dots, n. \tag{4.22}$$

Thus, Lemma 4.6 can be reformulated as follows.

Lemma 4.8. *Let Γ_n be n.e. and $M_i(\lambda)$ $(i = 0, 1, \dots, n)$ be defined by (4.14). Then $\{M_i(\lambda)\}_{i=0}^{n}$ has the generalized orthogonal property with respect to the sequence $S_0, \dots, S_{2n}$ in the sense that*

$$\int_{-\infty}^{+\infty} M_i^*(u)\, \mathrm{d}\sigma(u) M_j(u) = S_0^{(i)} \delta_{ij}, \tag{4.16'}$$

where $\sigma(u)$ is a solution to the TH problem (2.1).

As for the moment problem (3.4), we have a result little short of Eq. (4.16').

Lemma 4.9. *Let $\Gamma_n \geqslant 0$ and $M_i(\lambda)\,(i = 0, 1, \dots, n)$ be as before. Then*

$$\int_{-\infty}^{+\infty} M_i^*(u)\, \mathrm{d}\sigma(u) M_j(u) \begin{cases} \leqslant \tilde{S}_0^{(n)}, & i = j = n, \\ = S_0^{(i)} \delta_{ij}, & \text{otherwise}, \end{cases}$$

where $\sigma(u)$ is a solution to the moment problem (3.4), and $\tilde{S}_0^{(n)}$ is as in Theorem 3.11.

Thanks to Theorems 3.4 and 3.6, and letting $\Gamma_{n-1}/\Gamma_{n-2} = S_0$ if $n = 1; \Gamma_{n-1}/\Gamma_{n-2} = I_p$, if $n \leqslant 0$, Lemma 4.9 together with Theorem 3.11 leads to the following.

Corollary 4.10. *Γ_n is n.e. if and only if $\Gamma_n \geqslant 0$ and*

$$R(\Gamma_n/\Gamma_{n-1}) \subset R(\Gamma_{n-1}/\Gamma_{n-2}). \tag{4.23}$$

Corollary 4.11. *The TH problem (2.1) (or the moment problem (3.4)) has at most one solution if and only if*

$$R(\Gamma_n/\Gamma_{n-1}) \perp R(\Gamma_{n-1}/\Gamma_{n-2}). \tag{4.24}$$

Proof. Write

$$\Gamma_n/\Gamma_{n-1} = [I_p - S_0^{(n-1)}S_0^{(n-1)\mathrm{D}}]\Gamma_n/\Gamma_{n-1} + S_0^{(n-1)}S_0^{(n-1)\mathrm{D}}\Gamma_n/\Gamma_{n-1}.$$

It is easy to check that $S_0^{(n)} = S_0^{(n-1)}S_0^{(n-1)\mathrm{D}}\Gamma_n/\Gamma_{(n-1)}S_0^{(n-1)\mathrm{D}}S_0^{(n-1)}$. In view of the facts $(S_0^{(n-1)\mathrm{D}})^{\mathrm{D}} = S_0^{(n-1)}$ and $S_0^{(n-1)} = \Gamma_{n-1}/\Gamma_{n-2}, R((I_p - S_0^{(n-1)}S_0^{(n-1)\mathrm{D}})\Gamma_n/\Gamma_{n-1}) \perp R(S_0^{(n)})$. If now (4.24) holds, then $(S_0^{(n-1)})^2 S_0^{(n-1)\mathrm{D}}\Gamma_n/\Gamma_{n-1} = S_0^{(n-1)}\Gamma_n/\Gamma_{n-1} = 0$, so that

$$\begin{aligned} S_0^{(n-1)\mathrm{D}}S_0^{(n-1)}\Gamma_n/\Gamma_{n-1}S_0^{(n-1)\mathrm{D}}S_0^{(n-1)} &= S_0^{(n-1)}S_0^{(n-1)\mathrm{D}}\Gamma_n/\Gamma_{n-1}S_0^{(n-1)\mathrm{D}}S_0^{(n-1)} \\ &= S_0^{(n)} = 0. \end{aligned}$$

Applying Corollary 3.5 or 3.7, we obtain that the problem (2.1) or (3.4) has at most one solution. Conversely, if

$$S_0^{(n)} = S_0^{(n-1)\mathrm{D}}S_0^{(n-1)}\Gamma_n/\Gamma_{n-1}S_0^{(n-1)\mathrm{D}}S_0^{(n-1)} = 0,$$

then $S_0^{(n-1)\mathrm{D}}(\Gamma_n/\Gamma_{n-1})^{1/2} = 0$, and therefore $(S_0^{(n-1)})^2 S_0^{(n-1)\mathrm{D}}\Gamma_n/\Gamma_{n-1} = 0$. Hence Eq. (4.24) holds, as claimed. □

Note that Ando ([4], Section 3) has settled the extension theorems for a bounded positive operator.

In conclusion, thanks to Theorems 3.4 and 3.6 together with Lemmata 4.8 and 4.9, we may derive the general solution to the TH problem or to the moment problem (3.4) in terms of a linear fractional transformation based on the use of the orthogonal polynomial matrices $M_0(\lambda), \ldots, M_n(\lambda)$, and the corresponding polynomial matrices of the second kind, $N_0(\lambda), \ldots, N_n(\lambda)$ (see [11,9] for the nondegenerate case).

Theorem 4.12. *The general solution $\sigma(u)$ to the TH problem (2.1) (The moment problem (3.4) resp.) is representable as a lineal fractional transformation*

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}\sigma(u)}{u-\lambda} = -\frac{N_n(\lambda) + N_{n-1}(\lambda)S_0^{(n-1)\mathrm{D}}\Phi(\lambda)}{M_n(\lambda) + M_{n-1}(\lambda)S_0^{(n-1)\mathrm{D}}\Phi(\lambda)}, \tag{4.25}$$

where $\Phi(\lambda) = \int_{-\infty}^{+\infty} \mathrm{d}\sigma(u)/(u-\lambda)$ for an arbitrary Hermitian measure $\sigma(u)$ satisfying $\int_{-\infty}^{+\infty} \mathrm{d}\tau(u) = S_0^{(n)}$ ($\int_{-\infty}^{+\infty} \mathrm{d}\tau(u) \leqslant S_0^{(n)}$, resp.)x.

Note that the correspondence defined by Eq. (4.25) between Hermitian measures $\sigma(u)$ and function $\Phi(\lambda)$ with the aforementioned properties is one-to-one.

In the case when Γ_n is n.e., the general solution $\sigma(u)$ to the TH problem of the form (4.25) can be rewritten, by Eq. (4.3), in the form

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}\sigma(u)}{u-\lambda} = -\frac{N_n(\lambda)[\lambda I_p + S_0^{(n)\mathrm{D}} G(\lambda)] - N_{n-1}(\lambda) S_0^{(n-1)\mathrm{D}} S_0^{(n)}}{M_n(\lambda)[\lambda I_p + S_0^{(n)\mathrm{D}} G(\lambda)] - M_{n-1}(\lambda) S_0^{(n-1)\mathrm{D}} S_0^{(n)}}, \tag{4.26}$$

where $G(\lambda) \in \mathcal{N}_p$ is anarbitrary matrix-valued function such that $\lim_{\lambda\to\infty} G(\lambda)/\lambda = 0$. In particular, we may set $S_1^{(n)} = 0$, so that $M_{n+1}(\lambda)$ and $N_{n+1}(\lambda)$ make sense by Eq. (4.14). Then Eq. (4.26) has the further form

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}\sigma(u)}{u-\lambda} = -\frac{N_{n+1}(\lambda) + N_n(\lambda) S_0^{(n)\mathrm{D}} G(\lambda)}{M_{n+1}(\lambda) + M_n(\lambda) S_0^{(n)\mathrm{D}} G(\lambda)}, \quad S_1^{(n)} = 0. \tag{4.26'}$$

Also, in the case of $\Gamma_n \geqslant 0$, the general solution $\sigma(u)$ to the moment problem (3.4) can be rewritten, by Lemma 4.3 and the remark about it, in the form

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}\sigma(u)}{u-\lambda} = -\frac{N_n(\lambda) S_0^{(n)\mathrm{D}} p(\lambda) + [\lambda N_n(\lambda) - N_{n-1}(\lambda) S_0^{(n-1)\mathrm{D}} S_0^{(n)}] q(\lambda)}{M_n(\lambda) S_0^{(n)\mathrm{D}} p(\lambda) + [\lambda M_n(\lambda) - M_{n-1}(\lambda) S_0^{(n-1)\mathrm{D}} S_0^{(n)}] q(\lambda)}, \tag{4.27}$$

where

$$\begin{bmatrix} p(\lambda) \\ q(\lambda) \end{bmatrix}$$

is an arbitrary Nevanlinna pair of the form formulated as in Theorem 4.5.

The correspondence defined by Eq. (4.27) between $\sigma(u)$ and all equivalence classes of Nevanlinna pairs

$$\begin{bmatrix} p(\lambda) \\ q(\lambda) \end{bmatrix}$$

is one-to-one. Analogous to (4.26′), (4.27) has the further form

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}\sigma(u)}{u-\lambda} = -\frac{N_n(\lambda) S_0^{(n)\mathrm{D}} p(\lambda) + N_{n+1}(\lambda) q(\lambda)}{M_n(\lambda) S_0^{(n)\mathrm{D}} p(\lambda) + M_{n+1}(\lambda) q(\lambda)}, \quad S_1^{(n)} = 0. \tag{4.27'}$$

5. The Nevanlinna–Pick interpolation with multiple nodes in the Nevanlinna class $\mathcal{N}_p$

As an application of the aforementioned results in this paper, in this section, we consider the nontangential Nevanlinna–Pick interpolation (NP) problem with multiple nodes in the class $\mathcal{N}_p$. (See [5,12,14] and references therein for more information.)

The problem is as follows. Given $\lambda_1, \ldots, \lambda_\theta \in \pi^+$, which are distinct, with multiplicities $\tau_1, \ldots, \tau_\theta$, respectively, $n = \sum_{j=1}^{\theta} \tau_j$, and n matrices of order p,

C_{ik} $(i = 1, \dots, \theta, k = 0, 1, \dots, \tau_i - 1)$, it is required to find the conditions for a $F(\lambda) \in \mathcal{N}_p$ (or for only one $F(\lambda) \in \mathcal{N}_p$) to exist subject to

$$\frac{1}{k!} F^{(k)}(\lambda_i) = C_{ik}, \quad i = 1, \dots, \theta, \; k = 0, 1 \dots, \tau_i - 1, \tag{5.1a}$$

and to describe all the solutions if these conditions are met.

It is known [8–10] that to the NP problem (5.1a) there corresponds a unique Hermitian block vector $(S_0, \dots, S_{2n-2}), S_j = S_j^*$, order $S_j = p$ such that the NP problem (5.1a) is equivalent to a certain matrix moment problem on the real axis associated with that vector $(S_0, \dots, S_{2n-2})$:

$$S_k = \int_{-\infty}^{+\infty} u^k \, \mathrm{d}\sigma(u), \quad k = 0, 1, \dots, 2n - 3, \tag{5.2}$$

$$S_{n-2} \geqslant \int_{-\infty}^{+\infty} u^{2n-2} \, \mathrm{d}\sigma(u).$$

Moreover, if $\sigma(u)$ is a solution to the moment problem (5.2), then

$$F(\lambda) = \Omega(\lambda) - \int_{-\infty}^{+\infty} \frac{\mathrm{d}\sigma(u)}{u - \lambda} A(\lambda) \tag{5.3}$$

is a solution to that NP problem, and vice versa, where $\Omega(\lambda)$ is the (unique) polynomial matrix of degree $2n - 1$ at most, subject only to Eq. (5.1a) and

$$\frac{1}{k!} F^k(\overline{\lambda_i}) = C_{ik}^*, \quad i = 1, \dots, \theta, \; k = 0, 1, \dots, \tau_i - 1, \tag{5.1b}$$

and

$$A(\lambda) = \prod_{i=1}^{\theta} (\lambda - \lambda_i)^{\tau_i} (\lambda - \overline{\lambda_i})^{\tau_i} \tag{5.4}$$

That Hermitian block vector $(S_0, \dots, S_{2n-2})$ is called the Hankel block-vector of the NP problem (5.1a) and plays a key role in the NP problem. It is also known [8,9] that the Hankel block-vector can be found from the Laurent expension of $\Omega(\lambda)/A(\lambda)$ at $\lambda = \infty$:

$$\frac{\Omega(\lambda)}{A(\lambda)} = \frac{S_0}{\lambda} + \frac{S_1}{\lambda^2} + \cdots + \frac{S_{2n-2}}{\lambda^{2n-1}} + \cdots \tag{5.5}$$

Thanks to the one-to-one correspondence defined by Eq. (5.3) between $\sigma(u)$ and $F(\lambda)$, the moment problem enables one not only to find the criteria of existence and uniqueness for the solutions to that NP problem but also to describe all the solutions in the nondegenerate case or not, and so on.

By now we are in such a more advantageous position that we are able to tackle seriously that NP problem with little additional effort. The following assertions are all at hand.

Theorem 5.1 ([20]). *The NP problem (5.1a) has a solution if and only if $\Gamma_{n-1} = [S_{i+j}]_{i,j=0}^{n-1} \geqslant 0$, where $(S_0, \dots, S_{2n-2})$ is the Hankel block-vector of that NP problem.*

Theorem 5.2. *The general solution $F(\lambda)$ to the NP problem (5.1a) is representable as a linear fractional transformation of an arbitrary Nevanlinna pair*

$$\begin{bmatrix} p(\lambda) \\ q(\lambda) \end{bmatrix}:$$

$$F(\lambda) = \frac{\alpha(\lambda)p(\lambda) + \beta(\lambda)q(\lambda)}{\gamma(\lambda)p(\lambda) + \delta(\lambda)q(\lambda)} \tag{5.6}$$

whose coefficient matrix is of the decomposable form

$$\begin{bmatrix} \alpha(\lambda) & \beta(\lambda) \\ \gamma(\lambda) & \delta(\lambda) \end{bmatrix} = \begin{bmatrix} A(\lambda)I_p & \Omega(\lambda) \\ 0 & I_p \end{bmatrix} \prod_{i=0}^{\curvearrowleft\, n-1} \begin{bmatrix} 0 & -S_0^{(i)} \\ S_0^{(i)\mathrm{D}} & \lambda I_p - S_0^{(i)\mathrm{D}} S_1^{(i)} \end{bmatrix}, \quad S_1^{(n-1)} = 0,$$

where $S_0^{(0)}, S_0^{(1)}, \dots, S_0^{(n-1)}$ are defined as before from Γ_{n-1}.

In the degenerate case ($\Gamma_{n-1} \geqslant 0$ is singular), if

$$U^* S_0^{(n-1)} U = \begin{bmatrix} \hat{S}_0^{(n-1)} & 0 \\ 0 & 0_r \end{bmatrix}, \quad \hat{S}_0^{(n-1)} > 0, 0 < r \leqslant p,$$

then

$$U^* p(\lambda) U = \begin{bmatrix} \hat{p}(\lambda) & 0 \\ 0 & 0_r \end{bmatrix} \quad \text{and} \quad U^* q(\lambda) U = \begin{bmatrix} \hat{q}(\lambda) & 0 \\ 0 & I_r \end{bmatrix},$$

where

$$\begin{bmatrix} \hat{p}(\lambda) \\ \hat{q}(\lambda) \end{bmatrix}$$

is also an arbitrary Nevanlinna pair with reduced dimension. Further, the NP problem (5.1a) has only one solution, if and only if $\Gamma_{n-1} \geqslant 0$ and $S_0^{(n-1)} = 0$. In this case, the unique solution $F(\lambda)$ is rational, of the form

$$F(\lambda) = \frac{\beta(\lambda)}{\delta(\lambda)}. \tag{5.7}$$

Corollary 5.3. *Let $n \geqslant 3$. Then the NP problem (5.1a) has only one solution $F(\lambda)$, if and only if $\Gamma_{n-1} \geqslant 0$ and $R(\Gamma_{n-1}/\Gamma_{n-2}) \perp R(\Gamma_{n-2}/\Gamma_{n-3})$, where $\Gamma_k = [S_{i+j}]_{i,j=0}^{k}$.*

Starting from Theorem 4.12, we have

Theorem 5.4. *The general solution $F(\lambda)$ to the NP problem (5.1a) is representable as a linear fractional transformation*

$$F(\lambda) = \Omega(\lambda) - \frac{N_{n-1}(\lambda) + N_{n-2}(\lambda)S_0^{(n-2)\mathrm{D}}\Phi(\lambda)}{M_{n-1}(\lambda) + M_{n-2}(\lambda)S_0^{(n-2)\mathrm{D}}\Phi(\lambda)}A(\lambda), \tag{5.8}$$

where $\Phi(\lambda) = \int_{-\infty}^{+\infty} 1/(u-z)\mathrm{d}\tau(u)$ for an arbitrary Hermitian measure $\tau(u)$ satisfying

$$\int_{-\infty}^{+\infty} \mathrm{d}\tau(u) \leqslant S_0^{(n-1)}.$$

The NP problem has only a solution if and only if $\Gamma_{n-1} \geqslant 0$ and $S_0^{(n-1)} = 0$. In this case, the unique solution has the form

$$F(\lambda) = \Omega(\lambda) - \frac{N_{n-1}(\lambda)}{M_{n-1}(\lambda)}A(\lambda). \tag{5.9}$$

In the scalar case, $\Gamma_{n-1} \geqslant 0$ and $S_0^{(n-1)} = 0$, if and only if $\Gamma_{n-1} \geqslant 0$ is singular. Thus, the singularity of Γ_{n-1} will always lead to uniqueness of the solutions to the NP problem (5.1a).

Note that Eq. (5.8) be rewritten in another form by means of Eq. (4.27) (or Eq. (4.27′), if we set $S_1^{(n-1)} = 0$).

Additional Note. While this article was in the course of the final manuscript the authors discovered that there is a paper by Bolotnikov [7] where the degenerate Hamburger matrix moment problem (which coincides with the moment problem (3.4) here) and extensions of nonnegative block-Hankel matrices are considered via a different appraoch following the Potapov's method of the fundamental matrix inequality [17]. That paper appears to have several points in common with our paper (see Lemma 2.10 and Theorem 4.6 of [7]), and by the way the misstatement in [6] referred to by us here is corrected there.

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