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An algebra of differential operators and generating functions on the set of univalent functions

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Abstract

With a method close to that of Kirillov [4], we define sequences of vector fields on the set of univalent functions and we construct systems of partial differential equations which have the sequence of the Faber polynomials (F_n) as a solution. Through the Faber polynomials and Grunsky coefficients, we obtain the generating functions for some of the sequences of vector fields. © 2002 Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

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1. Introduction

In the first part, we consider the function

$$g(z) = z + b_1 + \sum_{n=1}^{\infty} b_{n+1} \frac{1}{z^n} \tag{1.1}$$

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the coefficients $(b_1, b_2, \ldots, b_n, \ldots)$ are in the subset $\mathcal{M}$ of C^N such that $g(z)$ is univalent outside the unit disc.

$$z\frac{g'(z)}{g(z)} = \sum_{n=0}^{\infty} F_n(b_1, b_2, b_3, \ldots, b_n)\frac{1}{z^n} \tag{1.2}$$

where $F_n(b_1, b_2, b_3, \ldots, b_n)$ is a homogeneous polynomial of degree n, the variables b_1, $b_2, \ldots, b_n$ have respective weight, 1 for b_1, 2 for b_2, 3 for $b_3, \ldots, n$ for b_n. We have (see for example [3])

$$\begin{aligned} &F_0 = 1, && F_3 = -b_1^3 + 3b_2b_1 - 3b_3,\\ &F_1 = -b_1, && F_4 = b_1^4 - 4b_1^2b_2 + 4b_1b_3 + 2b_2^2 - 4b_4.\\ &F_2 = b_1^2 - 2b_2, && \end{aligned}$$

On the submanifold $\mathcal{M}$, we define the partial differential operators $(Z_k)_{k\geqslant 1}$, the variables are $b_1, b_2, \ldots, b_n, \ldots,$ and $\partial_n = \frac{\partial}{\partial b_n}$ denotes the partial derivative with respect to the nth variable b_n,

$$Z_k = -\sum_{n=1}^{\infty} n\, b_n \partial_{n+k-1}. \tag{1.3}$$

For a function $\phi(z, b_1, b_2, \ldots, b_n, \ldots)$, we consider the system of partial differential equations, $Z_k\phi = \frac{\partial}{\partial z}\phi$. We find that the sequence (F_n) of the Faber polynomials is a solution of an infinite system of partial differential equations involving the (Z_k). We show how to calculate the Faber polynomials from the coefficients in the asymptotic expansion of the function $(\frac{g(z)}{z})^p$ where p is an integer,

$$\frac{g(z)^p}{z^p} = 1 + \sum_{n\geqslant 1} K_n^p \frac{1}{z^n} \tag{1.4}$$

in the notation K_n^p, n and p are indices. We define generalized Faber polynomials (H_j^k), $j \geqslant 0$, $k \in Z$, associated to g by

$$z\frac{g'(z)}{g(z)}\left(\frac{g(z)}{z}\right)^k = 1 + \sum_{j\geqslant 1} H_j^{k-j}\frac{1}{z^j} \tag{1.5}$$

and generalized Faber polynomials (F_j^k), $j \geqslant 0$, $k \in Z$, associated to the univalent function $f(z) = z(1 + b_1z + b_2z^2 + \cdots + b_nz^n + \cdots)$ by

$$z\frac{f'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^k = 1 - \sum_{j\geqslant 1} F_j^{k+j} z^j = 1 + (k+1)b_1z + \cdots. \tag{1.6}$$

When $\frac{g(z)}{z} = zf(\frac{1}{z})$, then $K_j^k = \frac{1}{2}(H_j^{k-j} - F_j^{k+j})$ and

$$F_n^k = -\frac{k}{k-2n}H_n^{k-2n} \quad \text{and} \quad F_n^{n+k} = -\left(1 + \frac{n}{k}\right)K_n^k. \tag{1.7}$$

With an iteration procedure, we obtain homogeneous fractions which are solutions of a system of partial differential equations involving the $(Z_k)_{k\geqslant 1}$.

The sequence of Faber polynomials and the other polynomials (K_n^p), (F_n^j), (H_n^p) can be obtained as the solution of many systems of partial differential equations with an infinite number of variables. For example (see [1,4]), with the embedding map

$$f(z) = z\left(1 + \sum_{n\geqslant 1} b_n z^n\right) \to (b_1, b_2, \ldots, b_n, \ldots)$$

in the submanifold $\mathcal{M}$ of C^N, we put $\partial_n = \frac{\partial}{\partial b_n}$ and we associate to $L_k f(z) = z^{1+k} f'(z)$, where $k \geqslant 1$, the vector fields

$$L_k = \partial_k + \sum_{n=1}^{\infty} (n+1) b_n \partial_{n+k}. \tag{1.8}$$

We calculate the polynomials $(F_k)_{k\geqslant 0}$ and (F_j^k) with $L_p(F_j^k) = 0$ for $j \leqslant p$, and $L_p(F_{p+1}^k) = -k$, $L_p(F_j^k) = k F_{j-p-1}^{k-p-1}$ for $j \geqslant p+2$.

In the second part, we study the generating functions associated to the sequence of vector fields found by Kirillov in [4]. Let $f(z) = z(1 + \sum_{n\geqslant 1} c_n z^n)$ be a univalent function on the unit disc, and let $g(z) = b_0 z + b_1 + \frac{b_2}{z} + \cdots + \frac{b_n}{z^{n-1}} + \cdots$ be a univalent function outside the unit disc. With the action of vector fields on the set of the diffeomorphisms of the circle $\mathrm{Diff}(S^1)$, and a variational approach on the equation $f \circ \gamma = g$, where γ is a diffeomorphism of the circle, Kirillov [4] obtained the following vectors fields on the set of univalent functions

$$L_{-p} f(z) = \frac{f(z)^2}{2\mathrm{i}\pi} \int_{\partial D} \frac{t^2 f'(t)^2}{f(t)^2} \frac{1}{f(t) - f(z)} \frac{\mathrm{d}t}{t^{p+1}} = \phi_p(z) + z^{1-p} f'(z). \tag{1.9}$$

The term $\phi_p(z)$ comes from the residue at $t = 0$ for the contour integral (1.9) and vanishes if $p < 0$. In that case $L_k f(z) = z^{1+k} f'(z) = L_k[f(z)]$, where $k \geqslant 1$ and L_k is given by (1.8). We assume $p \geqslant 0$. We have $\phi_p(z) = N_p(f(z))$ and $(N_p(w))_{p\geqslant 0}$ is given by the generating function

$$\phi_f(\xi, w) = \sum_{p\geqslant 0} N_p(w) \xi^p = \frac{\xi^2 f'(\xi)^2}{f(\xi)^2} \frac{w^2}{f(\xi) - w}. \tag{1.10}$$

In this work, we obtain (1.10) with the Faber polynomials of $h(z) = 1/f(\frac{1}{z})$. Moreover,

$$L_{-p} f(z) = \sum_{n=1}^{\infty} A_n^p z^{n+1}. \tag{1.11}$$

In [1], the (A_n^p) are given in terms of the polynomials K_n^j and P_n^k such that

$$\frac{f(z)^j}{z^j} = \sum_{n\geqslant 0} K_n^j z^n \quad \text{and} \quad z^2 \left(\frac{f'(z)}{f(z)}\right)^2 \left(\frac{f(z)}{z}\right)^k = 1 + \sum_{n\geqslant 1} P_n^{n+k} z^n. \tag{1.12}$$

In the present paper, we obtain the generating function for the (A_n^p) in two different manners,

$$\theta_f(u,v)=\sum_{k\geqslant 1}\sum_{p\geqslant 0}A_k^p u^p v^{k+1}=\frac{u^2 f'(u)^2}{f(u)^2}\frac{f(v)^2}{[f(u)-f(v)]}+\frac{v^2 f'(v)}{v-u}. \tag{1.13}$$

We deduce (1.13) from (1.10) or in Part II, Section 2, we prove (1.13) writing the A_n^p in terms of the Grunsky coeffients β_{jk} of $h(z)$, given by (see [3,6])

$$K(u,v)=\log\frac{\frac{1}{f(u)}-\frac{1}{f(v)}}{\frac{1}{u}-\frac{1}{v}}=-\sum_{n\geqslant 1}\sum_{k\geqslant 1}\frac{1}{n}\beta_{nk}u^n v^k. \tag{1.14}$$

Section 3, Part II, is a further step towards the classification of Faber type polynomials which was started in Part I. We express the Grunsky coefficients β_{kj} of $h(z)$ in terms of the polynomials K_n^j with generating function (1.12), or in terms of the P_j^k, the following identity (1.15) is a factorization of the symmetric matrix $k\beta_{jk}$ into a product of two infinite matrices

$$\frac{1}{k}\beta_{kj}=\sum_{n=1}^{k}\frac{1}{n}K_{k-n}^n K_{j+n}^{-n}. \tag{1.15}$$

The expression of the (A_n^p) found in [1] yields

$$\begin{aligned}&K_{k+p}^{1-p}+\sum_{j=1}^{p-1}P_j^p K_{k+p-j}^{j+1-p}\\&\quad=\beta_{p-1,k+1}+2c_1\beta_{p-2,k+1}+\cdots+(p-1)c_{p-2}\beta_{1,k+1}.\end{aligned} \tag{1.16}$$

Let the Schwarzian derivative of f, $S_f(z)=(\frac{f''}{f'})'-\frac{1}{2}(\frac{f''}{f'})^2$ and the asymptotic expansion $z^2S_f(z)=6\sum_{n\geqslant 0}\mathcal{P}_n z^n$. Since $\frac{\partial^2}{\partial u\partial v}|_{u=v}K(u,v)=-\frac{1}{6}S_f(u)$, see [2], we obtain the Neretin polynomials $(\mathcal{P}_n)_{n\geqslant 0}$ in terms of the (β_{jk}),

$$\mathcal{P}_n=\beta_{n-1,1}+2\beta_{n-2,2}+3\beta_{n-3,3}+\cdots+(n-1)\beta_{1,n-1}. \tag{1.17}$$

Then, as in [4] and [1], the set of functions univalent inside the unit disc is embedded into the submanifold $\mathcal{M}$ in C^N, via the map

$$f(z)=z\left(1+\sum_{n\geqslant 1}c_n z^n\right)\to(c_1,c_2,\ldots,c_n,\ldots).$$

For $k\geqslant 1$, $\partial_k=\frac{\partial}{\partial c_k}$. On $\mathcal{M}\in C^N$, we consider the partial differential operators (1.8) and (1.3). We have $\partial_n L_j=L_j\partial_n+(n+1)\partial_{n+j}$. We express ∂_k in terms of the $(L_p)_{p\geqslant k}$ with the generating function $\frac{1}{f'(z)}=1+\sum_{n\geqslant 1}B_n z^n$. We put $L_0=\sum_{n\geqslant 1}nc_n\partial_n$ and for $p\geqslant 1$, following [4] and [1], we associate to (1.9) the operator $L_{-p}=\sum_{n\geqslant 1}A_n^p\partial_n$. We obtain L_{-p} in terms of the $(L_j)_{j\geqslant 1}$.

In the third part, with the function $g(z) = b_0 z + b_1 + \frac{b_2}{z} + \cdots + \frac{b_n}{z^{n-1}} + \cdots$, with $b_0 \neq 0$, we consider on the subset of $(b_0, b_1, \ldots, b_n, \ldots)$ such that $g(z)$ is univalent outside the unit disc, the vector fields

$$R_{-p} = b_0 \frac{\partial}{\partial b_p} - b_2 \frac{\partial}{\partial b_{p+2}} - \cdots - (n-1) b_n \frac{\partial}{\partial b_{n+p}} - \cdots \quad \text{if } k \geqslant 0. \tag{1.18}$$

The (R_p) are obtained in [4] from right invariant vector fields on $\mathrm{Diff}(S^1)$. We relate the $(Z_k)_{k\geqslant 1}$ of (1.3) to the (R_{-k}). It permits to define Z_k for negative k.

Part I

I.1. The differential operators $(Z_k)_{k\geqslant 1}$ and the Faber polynomials

Let

$$h(z) = \frac{1}{z} g(z) = 1 + \frac{b_1}{z} + \sum_{n\geqslant 1} b_n \frac{1}{z^n} \quad \text{and} \quad h'(z) = -\sum_{n\geqslant 1} n\, b_n \frac{1}{z^{n+1}}. \tag{I.1.1}$$

We consider $h(z)$ as a function $h(z) = \phi(z, b_1, b_2, \ldots, b_n, \ldots)$ of the infinite number of variables $z, b_1, b_2, \ldots, b_n, \ldots$. In this way, we have

$$\frac{\partial}{\partial b_1}[h(z)] = \frac{1}{z}, \ \ldots \ \frac{\partial}{\partial b_n}[h(z)] = \frac{1}{z^n}, \ldots. \tag{I.1.2}$$

Thus $zh'(z) = -\sum_{n=1}^{\infty} n b_n \frac{\partial}{\partial b_n}(h(z)) = Z_1[h(z)]$ with $Z_1 = -\sum_{n\geqslant 1} n b_n \frac{\partial}{\partial b_n}$ and the function $h(z) = \phi(z, b_1, b_2, \ldots, b_n, \ldots)$ satisfies the partial differential equation $z\frac{\partial}{\partial z}\phi = Z_1 \phi$. In the same manner, since $\frac{1}{z^{k-2}} h'(z) = -\sum_{n\geqslant 1} n \frac{b_n}{z^{k+n-1}}$, we obtain for $k \geqslant 1$, the partial differential equation

$$\frac{1}{z^{k-2}} \frac{\partial}{\partial z}[h(z)] = Z_k[h(z)] \tag{I.1.3}$$

where (see (1.3))

$$Z_k = -\sum_{n=1}^{\infty} n\, b_n \partial_{n+k-1}. \tag{I.1.4}$$

We have $[Z_1, Z_2] = -Z_2, \ldots, [Z_k, Z_{k+r}] = -r Z_{2k+r-1}$. The operators $(Z_k)_{k\geqslant 1}$ defined above are different from the $(L_k)_{k\geqslant 1}$ of (1.8). We do not know a priori if the $(Z_k)_{k\geqslant 1}$ in (1.3) come from an ε-perturbation on $\mathrm{Diff}(S^1)$ as it appears in [4] for the (L_k).

Lemma. *Let the function $\psi(z, b_1, b_2, b_3, \ldots, b_n, \ldots) = \sum_{s\geqslant 0} P_s(b_1, b_2, b_3, \ldots, b_n, \ldots) \frac{1}{z^s}$, we have for any $k \geqslant 1$, $Z_k(\frac{\partial}{\partial z}\psi) = \frac{\partial}{\partial z} Z_k(\psi)$. In particular, $Z_k[h'(z)] = (Z_k[h(z)])'$ where $' = \frac{\partial}{\partial z}$ and for the derivative of order p, we denote $^{(p)} = \frac{\partial^p}{\partial z^p}$,*

$$Z_k\big[h^{(p)}(z)\big] = \big(Z_k[h(z)]\big)^{(p)}. \tag{I.1.5}$$

The proof is straightforward.

Theorem. *The Faber polynomials* (F_n) *defined by* (1.2) *are solutions of the system of partial differential equations*

$$\begin{aligned}
&Z_1F_n = -nF_n,\\
&\begin{cases} Z_2F_0 = 0,\\ Z_2F_1 = 1 - F_0 = 0,\\ Z_2F_n = -nF_{n-1} & \textit{for } n \geqslant 2,\end{cases}\\
&\begin{cases} Z_kF_n = 0 & \textit{for } n \leqslant k-2,\\ Z_kF_{k-1} = (k-1)(1-F_0) = 0,\\ Z_kF_n = -nF_{n-k+1} & \textit{for } n > k-1.\end{cases}
\end{aligned} \tag{I.1.6}$$

The equation $Z_1F_n = -nF_n$ shows that the polynomial F_n is homogeneous of degree n and the variables $b_1, b_2, b_3, \ldots, b_n$ have respectively the weight 1 for b_1, 2 for b_2, 3 for $b_3, \ldots, n$ for b_n.

The coefficient of b_1^n in F_n is $(-1)^n$. With the operators $Z_k,\ k \leqslant n$, and with (I.1.6), we calculate the homogeneous polynomial F_n from the $F_k,\ k < n$. For example, F_5 is homogeneous of degree 5 and $Z_5F_5 = -5F_1,\ Z_4F_5 = -5F_2,\ Z_3F_5 = -5F_3,\ Z_2F_5 = -5F_4$. We find

$$\begin{aligned}
F_5 &= -b_1^5 + 5b_1^3b_2 - 5b_1^2b_3 - 5b_1b_2^2 + 5b_2b_3 + 5b_1b_4 - 5b_5,\\
F_6 &= b_1^6 + 3b_3^2 + 6b_1^3b_3 - 12b_1b_2b_3 - 6b_1^4b_2 - 2b_2^3\\
&\quad + 9b_1^2b_2^2 + 6b_1b_5 + 6b_2b_4 - 6b_1^2b_4 - 6b_6.
\end{aligned}$$

Proof of (I.1.6). From $h(z) = \frac{1}{z}g(z)$, we have $\frac{h'}{h} = -\frac{1}{z} + \frac{g'}{g}$ and

$$z\frac{h'}{h} = -1 + \sum_{n\geqslant 0} F_n\frac{1}{z^n} = \frac{F_1}{z} + \sum_{n\geqslant 2} F_n\frac{1}{z^n}. \tag{I.1.7}$$

Since Z_k is a derivation, with (I.1.5), and then using (I.1.3) to replace $Z_k[h(z)]$, we obtain

$$\begin{aligned}
Z_k\left[z\frac{h'(z)}{h(z)}\right] &= z\frac{Z_k[h'(z)]}{h(z)} - z\frac{h'(z)Z_k[h(z)]}{h(z)^2} = z\left(\frac{Z_k[h(z)]}{h(z)}\right)'\\
&= z\left(\frac{1}{z^{k-2}}\frac{h'(z)}{h(z)}\right)'.
\end{aligned} \tag{I.1.8}$$

We replace the expression $z\frac{h'(z)}{h(z)}$ by its expansion given by (I.1.7),

$$\begin{aligned}
\sum_{n\geqslant 0} Z_k(F_n)\frac{1}{z^n} &= -z\left(\frac{1}{z^{k-1}}\right)' + z\sum_{n\geqslant 0} F_n\left(\frac{1}{z^{n+k-1}}\right)'\\
&= \frac{k-1}{z^{k-1}} - \sum_{n\geqslant 0}(k-1+n)F_n\frac{1}{z^{n+k-1}}.
\end{aligned} \tag{I.1.9}$$

By identifying in (I.1.9) the coefficients of $\frac{1}{z^p}$, we obtain (I.1.6).

With the same method, and from the identity $Z_k(\phi^p) = p\phi^{p-1}L_k(\phi)$, we deduce

Lemma. *Consider* $v = (z\frac{h'}{h})^p = \sum_{n\geqslant p} H_n \frac{1}{z^n}$ *for any integer* p*, then*

$$Z_k(v) = z^{(k-1)(p-1)+1}\left(\frac{v}{z^{p(k-1)}}\right)'.$$

Moreover $Z_k(H_n) = -(n+(p-1)(k-1))H_{n-k+1}$ *for* $n \geqslant p+k-1$ *and if* $n \leqslant p+k-2$, $Z_k(H_n) = 0$.

The Faber polynomials in (1.2), can be obtained from the following (K_n^p). *For any integer* $p \in Z$*, let* $\frac{g(z)^p}{z^p} = 1 + \sum_{n\geqslant 1} K_n^p \frac{1}{z^n}$ *where in the notation* K_n^p*,* n *and* p *are indices,* (see (1.4)). From (I.1.3), we have

$$Z_k\left(\frac{g(z)^p}{z^p}\right) = \frac{1}{z^{k-2}}\left(\frac{g(z)^p}{z^p}\right)'. \tag{I.1.10}$$

K_n^p, $n \geqslant 1$, $p \in Z$, is a homogeneous polynomial in the variables $b_1, b_2, \ldots, b_n$ where b_1 has weight 1, b_2 weight $2, \ldots$. Moreover $Z_k(K_n^p) = -(n-k+1)K_{n-k+1}^p$ for $n \geqslant k$ and $Z_k(K_n^p) = 0$ for $n < k$. This permits to obtain the K_n^p. The coefficient of b_1^n in the polynomial K_n^p is $\frac{p!}{(p-n)!n!}b_1^n$. Remark that the notation $\frac{p!}{(p-n)!n!} = \frac{(p-n+1)(p-n+2)\cdots p}{n!}$ extends to any $p \in Z$. The other coefficients in K_n^p are calculated with the $(Z_p)_{p\geqslant 2}$. For example, $K_2^p = \frac{p(p-1)}{2}b_1^2 + \alpha b_2$ and $Z_2(K_2^p) = -K_1^p$ with $K_1^p = pb_1$, $Z_2 = -b_1\frac{\partial}{\partial b_2}$. This gives the value of α and $K_2^p = \frac{p(p-1)}{2}b_1^2 + pb_2$. In [1] (A.1.6), the K_n^p are obtained with the operators L_k, see (1.8).

$$\begin{aligned}
K_3^p &= p(p-1)b_1b_2 + pb_3 + \frac{p(p-1)(p-2)}{3!}b_1^3,\\
K_4^p &= p(p-1)b_1b_3 + pb_4 + \frac{p(p-1)}{2}b_2^2 + \frac{p(p-1)(p-2)}{2}b_1^2b_2\\
&\quad + \frac{p!}{(p-4)!4!}b_1^4,\\
K_n^p &= \frac{p!}{(p-n)!n!}b_1^n + \frac{p!}{(p-n+1)!(n-2)!}b_1^{n-2}b_2 + \frac{p!}{(p-n+2)!(n-3)!}b_1^{n-3}b_3\\
&\quad + \frac{p!}{(p-n+3)!(n-4)!}b_1^{n-4}\left[b_4 + \frac{p-n+3}{2}b_2^2\right]\\
&\quad + \frac{p!}{(p-n+4)!(n-5)!}b_1^{n-5}\left[b_5 + (p-n+4)b_2b_3\right] + \sum_{j\geqslant 6} b_1^{n-j}V_j,
\end{aligned}$$

where V_j with $6 \leqslant j \leqslant n$, is a homogeneous polynomial of degree j in the variables $b_2, \ldots, b_n$.

I.2. Generalized Faber polynomials

We assume that $g(z)$ is given by (1.1). We have $\frac{g(z)}{z} = zf(\frac{1}{z})$ where

$$f(z) = z\left(1 + \sum_{n\geqslant 1} b_n z^n\right). \tag{I.2.1}$$

Since

$$\frac{z}{g(z)} = 1 - \frac{b_1}{z} + \frac{(b_1^2 - b_2)}{z^2} + \cdots \quad \text{and} \quad z\frac{g'(z)}{g(z)} = 2 - \frac{1}{z}\frac{f'(\frac{1}{z})}{f(\frac{1}{z})} \tag{I.2.2}$$

we deduce $u\frac{f'(u)}{f(u)} = 1 + b_1 u + \cdots = 1 - \sum_{n=1}^{\infty} F_n(b_1, b_2, \ldots, b_n)u^n$. We put

$$u\frac{f'(u)}{f(u)}\left(\frac{f(u)}{u}\right)^k = 1 - \sum_{j\geqslant 1} F_j^{k+j} u^j = 1 + (k+1)b_1 u + \cdots \tag{I.2.3}$$

then $F_j^j = F_j$ where F_j is the jth Faber polynomial. On the other hand, if we take $k = 1$ in (I.2.3), we find $F_j^{1+j} = -(j+1)\,b_j$. With the same polynomials F_j^k, we have, for any $k \in Z$,

$$\left(\frac{f(u)}{u}\right)^k = 1 - \sum_{j\geqslant 1} F_j^k f(u)^j. \tag{I.2.4}$$

Definition. We call the polynomials (F_j^k), the generalized polynomials associated to the function $f(z) = z(1 + \sum_{n\geqslant 1} b_n z^n)$.

When $k = -1$ in (I.2.4), we obtain

$$u = f(u) - \sum_{j\geqslant 1} F_j^{-1} f(u)^{j+1}. \tag{I.2.5}$$

In particular, if $w = f(u)$ where f is univalent, we have the asymptotic expansion of the inverse map f^{-1},

$$\begin{aligned} u = f^{-1}(w) = w - \sum_{j\geqslant 1} F_j^{-1} w^{j+1} &= w + b_1 w^2 + \left(b_2 - 2b_1^2\right)w^3 \\ &\quad + \left(5b_1^3 - 5b_1 b_2 + b_3\right)w^4 + \cdots. \end{aligned} \tag{I.2.6}$$

We rewrite (I.2.4) with $k \in Z$ as $u^k = f(u)^k - \sum_{j\geqslant 1} F_j^{-k} f(u)^{j+k}$, or

$$\left(f^{-1}(w)\right)^k = w^k - \sum_{j\geqslant 1} F_j^{-k} w^{j+k} = w^k + kb_1 w^{k+1} + \cdots.$$

The homogeneous polynomials (F_j^k) can be calculated with (I.2.3) and the $(L_p)_{p\geqslant 0}$ as in [1] (A.3.3)–(A.6.5): We have $L_p[f(z)] = z^{p+2} f'(z)$. Let $\phi(z) = z\frac{f'(z)}{f(z)}(\frac{f(z)}{z})^k$. By commuting $\frac{\partial}{\partial z}$ and L_p, we get $L_p[\phi(z)] = z^{p+2}\phi'(z) + (p+1+k)z^{p+1}\phi(z)$. From

(I.2.3), $L_p[\phi(z)] = -\sum_{j\geqslant 1} L_p(F_j^{k+j})z^j$. With the identification of the coefficients in the asymptotic expansions (I.2.3), we obtain (compare with (I.1.6)) $L_p(F_j^k) = 0$ for $j \leqslant p$, and $L_p(F_{p+1}^k) = -k$, $L_p(F_j^k) = kF_{j-p-1}^{k-p-1}$ for $j \geqslant p+2$. This gives $F_1^k = -kb_1$,

$$F_2^k = \frac{k(3-k)}{2}b_1^2 - kb_2, \qquad F_3^k = \frac{k(k-5)(4-k)}{6}b_1^3 + k(4-k)b_1b_2 - kb_3,$$

$$\begin{aligned} F_4^k &= \frac{k(5-k)(k-6)(k-7)}{4!}b_1^4 + \frac{k(5-k)(k-6)}{2}b_2b_1^2 + k(5-k)b_1b_3 \\ &\quad + \frac{k(5-k)}{2}b_2^2 - kb_4, \end{aligned}$$

$$\begin{aligned} F_5^k &= \frac{k(6-k)(k-7)(k-8)(k-9)}{5!}b_1^5 + \frac{k(k-7)(6-k)(k-8)}{3!}b_1^3b_2 \\ &\quad + \frac{k(k-7)(6-k)}{2}b_1^2b_3 + \frac{k(6-k)(k-7)}{2}b_1b_2^2 \\ &\quad + k(6-k)b_2b_3 + k(6-k)b_1b_4 - kb_5, \end{aligned}$$

$$\begin{aligned} F_6^k &= -\frac{(k-7)!k}{6!(k-12)!}b_1^6 - \frac{(k-7)!k}{4!(k-11)!}b_1^4b_2 - \frac{(k-7)!k}{3!(k-10)!}b_1^3b_3 + \frac{k(7-k)}{2}b_3^2 \\ &\quad + k(7-k)\big[(k-8)b_2b_3 + b_5\big]b_1 - \frac{(k-7)!k}{2!(k-9)!}\left[\frac{(k-9)}{2}b_2^2 + b_4\right]b_1^2 \\ &\quad + \frac{k(7-k)(k-8)}{6}b_2^3 + k(7-k)b_2b_4 - kb_6, \end{aligned}$$

$$\begin{aligned} F_7^k &= -\frac{(k-8)!k}{7!(k-14)!}b_1^7 - \frac{(k-8)!k}{5!(k-13)!}b_1^5b_2 - \frac{(k-8)!k}{4!(k-12)!}b_1^4b_3 \\ &\quad - \frac{(k-8)!k}{3!(k-11)!}\left[\frac{(k-11)}{2}b_2^2 + b_4\right]b_1^3 - \frac{(k-8)!k}{2!(k-10)!}\big[(k-10)b_2b_3 + b_5\big]b_1^2 \\ &\quad + k(8-k)\big[(k-9)b_2b_4 + b_6\big]b_1 + \frac{k(8-k)(k-9)}{2}\left[\frac{k-10}{3}b_2^3 + b_3^2\right]b_1 \\ &\quad + \frac{k(8-k)(k-9)}{2}b_2^2b_3 + k(8-k)b_3b_4 + k(8-k)b_2b_5 - kb_7. \end{aligned}$$

Like in (I.2.3)–(I.2.4), we find that

$$\left(\frac{g(z)}{z}\right)^k = 1 + \sum_{j\geqslant 1} H_j^k \frac{1}{g(z)^j} \quad \text{and} \quad z\frac{g'(z)}{g(z)}\left(\frac{g(z)}{z}\right)^k = 1 + \sum_{j\geqslant 1} H_j^{k-j}\frac{1}{z^j} \qquad \text{(I.2.7)}$$

with the same polynomials H_j^k, $k \in Z$ and $j \geqslant 1$. Because of (I.2.2), the polynomials $H_j^{-j} = F_j$ are the Faber polynomials.

Definition. We call (H_j^k) the generalized Faber polynomials associated $g(z)$.

We have $H_j^{1-j} = -(j-1)\,b_j$ and $H_k^0 = 0$ for any $k \geqslant 1$. As in (I.2.5), we put $k = -1$ in the first equation of (I.2.7), this gives $z = g(z) + \sum_{j\geqslant 1} H_j^{-1} g(z)^{1-j}$ and if $w = g(z)$, then

$$z = g^{-1}(w) = w + \sum_{j\geqslant 1} H_j^{-1} w^{1-j}.$$

We have

$$H_1^k = kb_1, \quad H_2^k = \frac{k(k+1)}{2} b_1^2 + kb_2,$$

$$H_3^k = \frac{k(k+1)(k+2)}{6} b_1^3 + k(k+2) b_1 b_2 + kb_3,$$

$$H_4^k = \frac{k(k+1)(k+2)(k+3)}{4!} b_1^4 + \frac{k(k+2)(k+3)}{2} b_2 b_1^2 + k(k+3) b_1 b_3 + \frac{k(k+3)}{2} b_2^2 + kb_3.$$

We can obtain the H_j^k as follows: Let $h(z) = \frac{g(z)}{z}$. Then $h(z)^k = 1 + \sum_{n\geqslant 1} K_n^k \frac{1}{z^n}$. Since

$$z\frac{g'(z)}{g(z)}\left(\frac{g(z)}{z}\right)^k = h(z)^k + zh(z)^{k-1}h'(z) = 1 + \sum_{n\geqslant 1}\left(1 - \frac{n}{k}\right) K_n^k \frac{1}{z^n},$$

we deduce from (I.2.7) that

$$H_n^{k-n} = \left(1 - \frac{n}{k}\right) K_n^k. \tag{I.2.8}$$

Proposition. (See (1.7).) *We have* $F_n^k = -\frac{k}{k-2n} H_n^{k-2n}$ *and* $F_n^{n+k} = -(1 + \frac{n}{k}) K_n^k$.

Proof. $h(z) = \frac{f(1/z)}{1/z} = \frac{f(u)}{u}$ where $u = 1/z$. Thus

$$z\frac{f'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^k = z^{1-k} f'(z) f(z)^{k-1} = \left[h\left(\frac{1}{z}\right)\right]^k + z\frac{1}{k}\frac{\partial}{\partial z}\left[\left(h\left(\frac{1}{z}\right)\right)^k\right]. \tag{I.2.9}$$

Since $[h(1/z)]^k = 1 + \sum_{n\geqslant 1} K_n^k z^n$, we replace in (I.2.9). If we put $k = n$ in (1.7), we find again $F_n^n = H_n^{-n} = F_n$.

Corollary.

$$\sum_{p=1}^{n-1} (n-p) F_{n-p}^{-k} F_p^n = K_n^{-k} + F_n^{-k}. \tag{I.2.10}$$

Proof. We have

$$\left(\frac{u}{f(u)}\right)^k = 1 + \sum_{n\geqslant 1} K_n^{-k} u^n \quad \text{and} \quad \left(\frac{u}{f(u)}\right)^k = 1 - \sum_{j\geqslant 1} F_j^{-k} \frac{f(u)^j}{u^j} u^j.$$

We replace $\frac{f(u)^j}{u^j} = 1 + \sum_{n\geqslant 1} K_n^j u^n$. We identify the coefficients of equal powers of u and we replace K_n^j by its expression (1.7) to obtain (I.2.10). If we put $k = -n$ in (I.2.10), it gives $K_n^n + F_n^n = \sum_{p=1}^{n-1} \frac{n-p}{n} F_p^n F_{n-p}^n$. In particular

$$F_2^2 + K_2^2 = \frac{1}{2}\big(F_1^2\big)^2, \qquad F_3^3 + K_3^3 = F_1^3 F_2^3,$$
$$F_4^4 + K_4^4 = \frac{1}{2}\big(F_2^4\big)^2 + F_1^4 F_3^4 + \cdots.$$

I.3. Homogeneous fractions of the $(b_i)_{i\geqslant 1}$

Let $u(z) = 1 - z\frac{g'(z)}{g(z)} = -z\frac{h'(z)}{h(z)}$, then $1 - z\frac{u'}{u} = z\frac{h'}{h} - z\frac{h''}{h'}$. From (I.1.8), $Z_k(u) = z(\frac{1}{z^{k-1}}u)'$. Since $Z_k(z\frac{u'}{u}) = z(\frac{Z_k(u)}{u})'$, we obtain

Theorem. *Let $v = 1 + z\frac{u'}{u} = \sum_{n\geqslant 1} G_n \frac{1}{z^n}$, then*

$$Z_k(v) = z\left(\frac{1}{z^{k-1}}v\right)' + \frac{k(k-1)}{z^{k-1}}$$

and

$$Z_1 G_n = -nG_n \quad \text{and for } k > 1, \quad \begin{array}{ll} Z_k(G_n) = 0 & \text{if } n \leqslant k-2 \\ Z_k(G_{k-1}) = k(k-1) & \\ Z_k(G_n) = -nG_{n-k+1} & \text{if } n \geqslant k. \end{array}$$

For $n \geqslant 1$, the relation $Z_1 G_n = -nG_n$ implies that G_n is homogeneous of degree n in the variables $b_1, b_2, \ldots, b_n$; $G_n = \frac{1}{b_1^n} \times \mathcal{Q}_n$ where $\mathcal{Q}_n$ is a homogeneous polynomial of degree $2n$ in the variables $b_1, b_2, \ldots, b_n$, if $b_1 \neq 0$.

$$G_1 = \frac{F_2}{b_1} = b_1 - 2\frac{b_2}{b_1}, \qquad G_2 = \frac{2F_3}{b_1} + \frac{F_2^2}{b_1^2} = \frac{1}{b_1^2}\big(-b_1^4 + 2b_2 b_1^2 - 6b_3 b_1 + 4b_2^2\big).$$

Part II

In the following, $f(z)$ is a univalent function inside the unit disc,

$$f(z) = z\left(1 + \sum_{n\geqslant 1} c_n z^n\right).$$

Let $\chi(z)$ be a function and assume that the expansion $\chi(f^{-1}(u)) = \sum_{n\in Z} \alpha_n u^n$ converges, α_n is obtained with the contour integral $\alpha_n = \frac{1}{2\mathrm{i}\pi}\int \chi(f^{-1}(u))u^{-n}\frac{\mathrm{d}u}{u}$. We put $u = f(z)$ in the two last expressions and we obtain the expansion of $\chi(z)$ in sum of powers of $f(z)$,

$$\chi(z) = \sum_{j\in Z} f(z)^j \frac{1}{2\mathrm{i}\pi}\int \frac{uf'(u)}{f(u)} f(u)^{-j} \chi(u)\frac{\mathrm{d}u}{u}.$$

In this formula, instead of $\chi(z)$, let $z\chi(z)f'(z)$, it gives

$$z\chi(z)f'(z) = \sum_{j\in Z} f(z)^{1-j}\frac{1}{2\mathrm{i}\pi}\int \frac{u^2 f'(u)^2}{f(u)^2} f(u)^j \chi(u)\frac{\mathrm{d}u}{u} = S_1 + S_2$$

with

$$S_1 = \sum_{j\geqslant 0} f(z)^{1-j}\frac{1}{2\mathrm{i}\pi}\int \frac{u^2 f'(u)^2}{f(u)^2} f(u)^j \chi(u)\frac{\mathrm{d}u}{u},$$

$$S_2 = \sum_{j\geqslant 0} f(z)^{2+j}\frac{1}{2\mathrm{i}\pi}\int \frac{u^2 f'(u)^2}{f(u)^2} f(u)^{-1-j} \chi(u)\frac{\mathrm{d}u}{u} = L_\chi[f(z)].$$

If $\chi(z) = z^{-p}$, where p is a positive integer $p \geqslant 0$, S_1 reduces to a finite sum of powers of $f(z)$ and is equal to

$$\frac{f(z)^{1-p}}{2\mathrm{i}\pi}\int \frac{u^2 f'(u)^2}{f(u)^2}\frac{(f(z)^{p+1} - f(u)^{p+1})}{(f(z) - f(u))}\frac{\mathrm{d}u}{u^{p+1}} = -N_p\big(f(z)\big).$$

From this expression, we see that the generating function for the $N_p(w)$ is given by (1.10). In the next section, we find (1.10) through Faber polynomials.

With the embedding $f(z) = z(1 + \sum_{n\geqslant 1} c_n z^n) \to (c_1, c_2, \ldots, c_n, \ldots)$ in the submanifold $\mathcal{M}$ of C^N, if $\chi(z) = z^{-p}$, $S_2 = L_{-p}[f(z)]$, and expanding $f(z)^{2+j}$ in powers of z in S_2, we find $L_{-p} = \sum_{n\geqslant 1} A_n^p \partial_n$ where $\partial_n = \frac{\partial}{\partial c_n}$.

II.1. The vector fields L_{-p}

Theorem. *If $p \geqslant 0$, there exists a function $N_p(w)$ such that*

$$z^{1-p} f'(z) + N_p\big(f(z)\big) = \sum_{n=1}^{\infty} A_n^p z^{n+1}. \tag{II.1.1}$$

The A_n^k are homogeneous polynomials in the variables $(c_n)_{n\geqslant 1}$, $N_p(w)$ is calculated with (1.10). *Let $\phi_p(z) = N_p(f(z))$, then*

$$\sum_{p\geqslant 0}\phi_p(z)\xi^p = \frac{\xi^2 f'(\xi)^2}{f(\xi)^2}\frac{f(z)^2}{(f(\xi) - f(z))}$$

and

$$\frac{f(z)^2}{2\mathrm{i}\pi}\int \frac{\xi^2 f'(\xi)^2}{f(\xi)^2}\frac{1}{(f(\xi) - f(z))}\frac{\mathrm{d}\xi}{\xi^{p+1}} = \phi_p(z) + z^{1-p} f'(z). \tag{II.1.2}$$

The polynomials A_k^p defined by (II.1.1), *are given by* (1.13).

We have $N_0(w) = -w$, $N_1(w) = -1 - 2c_1 w$, $N_2(w) = -\frac{1}{w} - 3c_1 - (4c_2 - c_1^2)w$, $N_3(w) = -\frac{1}{w^2} - 4c_1\frac{1}{w} - (c_1^2 + 5c_2) - P_3^3 w, \ldots$. Let $N_p'(w) = \frac{\mathrm{d}}{\mathrm{d}w}N_p(w)$ be the derivative of N_p. From (II.1.1), we deduce that for any $j \geqslant 1$,

$$\int N_j'\big(f(z)\big)\frac{\mathrm{d}z}{z} + \int \frac{f"(z)}{f'(z)}\frac{\mathrm{d}z}{z^j} = 0 \quad \text{and}$$
$$\int \frac{N_j(f(z))}{f(z)}\frac{\mathrm{d}z}{z} + \int \frac{f'(z)}{f(z)}\frac{\mathrm{d}z}{z^j} = 0. \tag{II.1.3}$$

Proof of (II.1.1)–(1.10)–(II.1.2). Consider the function

$$\begin{aligned} h(z) = \frac{1}{f(\frac{1}{z})} &= z - c_1 + \big(c_1^2 - c_2\big)\frac{1}{z} + \big(2c_1c_2 - c_3 - c_1^3\big)\frac{1}{z^2} \\ &\quad + \big(2c_1c_3 - c_4 + c_2^2 - 3c_1^2c_2 + c_1^4\big)\frac{1}{z^3} + \cdots \\ &= z + \sum_{n=0}^{\infty} b_{n+1}\frac{1}{z^n} \end{aligned} \tag{II.1.4}$$

we have (see [5] and [6]) $\frac{\xi h'(\xi)}{h(\xi)-w} = \sum_{n=0}^{\infty} F_n(w)\xi^{-n}$ where $F_n(w)$ are the Faber polynomials associated to the function h. In terms of f,

$$\psi(z,w) = \frac{zf'(z)}{f(z) - wf(z)^2} = 1 + \sum_{n\geqslant 1} F_n(w)z^n, \tag{II.1.5}$$

$F_1(w) = w + c_1$, $F_2(w) = w^2 + 2c_1 w + 2c_2 - c_1^2$, $F_3(w) = w^3 + 3c_1w^2 + 3c_2w + c_1^3 - 3c_1c_2 + 3c_3$. If we take the derivative of (II.1.5) with respect to w and then integrate with respect to z, we obtain

$$\frac{f(z)}{(1 - wf(z))} = \sum_{n\geqslant 1} F_n'(w)\frac{z^n}{n}, \tag{II.1.6}$$

$F_1'(w) = 1$, $\frac{1}{2}F_2'(w) = w + c_1$, $\frac{1}{3}F_3'(w) = (w + c_1)^2 - (c_1^2 - c_2)(w + c_1)$, $\frac{1}{4}F_4'(w) = (w + c_1)^3 - 2(c_1^2 - c_2)(w + c_1) - (2c_1c_3 - c_4 + c_2^2 - 3c_1^2c_2 + c_1^4)$. Moreover

$$F_n\big(h(z)\big) = z^n + \sum_{k=1}^{\infty} \beta_{nk} z^{-k}, \tag{II.1.7}$$

where the β_{nk} are the Grunsky coefficients of h. Thus

$$F_n\left(\frac{1}{f(z)}\right) = z^{-n} + \sum_{k=1}^{\infty} \beta_{nk} z^k. \tag{II.1.8}$$

We rewrite (II.1.8) as $z^{-n} = F_n(\frac{1}{f(z)}) - \sum_{k=1}^{\infty} \beta_{nk} z^k$. On the other hand, if $p > 1$,

$$\begin{aligned} z^{1-p} f'(z) &= z^{1-p}(1 + 2c_1 z + 3c_2 z^2 + \cdots + (n+1)c_n z^n + \cdots \\ &= \frac{1}{z^{p-1}} + \frac{2c_1}{z^{p-2}} + \frac{3c_2}{z^{p-3}} + \cdots + \frac{(p-1)c_{p-2}}{z} + pc_{p-1} \\ &\quad + (p+1)c_p z + \sum_{k\geqslant 1}(p+k+1)c_{p+k} z^{k+1}. \end{aligned} \tag{II.1.9}$$

We replace the negative powers of z by their expressions given by (II.1.8). We obtain

$$\begin{aligned} z^{1-p} f'(z) &= F_{p-1}\left(\frac{1}{f(z)}\right) + 2c_1 F_{p-2}\left(\frac{1}{f(z)}\right) + 3c_2 F_{p-3}\left(\frac{1}{f(z)}\right) + \cdots \\ &\quad + (p-1)c_{p-2} F_1\left(\frac{1}{f(z)}\right) + pc_{p-1} + (p+1)c_p z \\ &\quad - \left[\beta_{p-1,1} + 2c_1\beta_{p-2,1} + \cdots + (p-1)c_{p-2}\beta_{1,1}\right] z \\ &\quad + \sum_{k\geqslant 1}\left[(p+k+1)c_{p+k} - [\beta_{p-1,k+1} + 2c_1\beta_{p-2,k+1} + \cdots \right. \\ &\quad \left. + (p-1)c_{p-2}\beta_{1,k+1}]\right] z^{k+1}. \end{aligned} \tag{II.1.10}$$

For $p \geqslant 2$, we consider

$$\begin{aligned} T_{p-1}(w) &= F_{p-1}(w) + 2c_1 F_{p-2}(w) + 3c_2 F_{p-3}(w) + \cdots \\ &\quad + (p-1)c_{p-2} F_1(w) + pc_{p-1}. \end{aligned} \tag{II.1.11}$$

From the expansion of $f'(z)$ and (II.1.5), we obtain the generating function

$$\frac{zf'(z)^2}{f(z) - wf(z)^2} = 1 + \sum_{n\geqslant 1} T_n(w) z^n, \tag{II.1.12}$$

$T_0(w) = 1$, $T_1(w) = w + 3c_1$, $T_2(w) = w^2 + 4c_1 w + (c_1^2 + 5c_2)$, $T_3(w) = w^3 + 5c_1 w^2 + (4c_1^2 + 6c_2)w + P_3^4 + \cdots$. With (1.12), we obtain

$$T_p(w) = \sum_{j=0}^{p} P_{p-j}^{p+1} w^j.$$

We write (II.1.10) as

$$\begin{aligned} z^{1-p} f'(z) - T_{p-1}\left(\frac{1}{f(z)}\right) &= \sum_{k\geqslant 1} B_{k-1}^p z^k \\ &= (p+1)c_p z - \left[\beta_{p-1,1} + 2c_1\beta_{p-2,1} + \cdots \right. \\ &\quad \left. + (p-1)c_{p-2}\beta_{1,1}\right] z + \sum_{k\geqslant 1} B_k^p z^{k+1} \end{aligned} \tag{II.1.13}$$

with

$$\begin{aligned} B_k^p &= (p+k+1)c_{p+k} \\ &\quad - \left[\beta_{p-1,k+1} + 2c_1\beta_{p-2,k+1} + \cdots + (p-1)c_{p-2}\beta_{1,k+1}\right]. \end{aligned} \tag{II.1.14}$$

This gives, for $p \geqslant 1$,

$$z^{1-p} f'(z) - T_{p-1}\left(\frac{1}{f(z)}\right) - P_p^p f(z) = \sum_{k \geqslant 1} A_k^p z^{k+1}, \tag{II.1.15}$$

where $P_1^1 = 2c_1$ and if $p > 1$,

$$P_p^p = -\left[\beta_{p-1,1} + 2c_1\beta_{p-2,1} + \cdots + (p-1)c_{p-2}\beta_{1,1}\right] + (p+1)c_p \qquad (\text{II}.1.16)_1$$

and for $p \geqslant 1$, $k \geqslant 1$, we put $A_k^p = B_k^p - P_p^p c_k$. With the convention $c_0 = 1$, $B_0^p = P_p^p$, we have $A_0^p = 0$. In [1], we obtained the generating function of the (P_p^p),

$$\frac{z^2 f'(z)^2}{f(z)^2} = 1 + \sum_{p \geqslant 1} P_p^p z^p. \qquad (\text{II}.1.16)_2$$

In fact, since $\frac{1}{f(z)} = \frac{1}{z} + \sum_{n=0}^{\infty} b_{n+1} z^n$ and $b_{n+1} = \beta_{1n} = \frac{1}{n}\beta_{n1}$, by taking the derivative with respect to z, we obtain

$$\frac{f'(z)}{f(z)^2} = \frac{1}{z^2} - \sum_{n=1}^{\infty} \beta_{n1} z^{n-1}$$

thus $z^2 \frac{f'(z)^2}{f(z)^2} = f'(z) - (\sum_{n=1}^{\infty} \beta_{n1} z^{n-1}) f'(z)$ and $(\text{II}.1.16)_2$ implies $(\text{II}.1.16)_1$. From (II.1.11)–(II.1.15)–(II.1.16), we obtain the generating function of

$$M_p(w) = -T_{p-1}(w) - P_p^p \frac{1}{w}. \qquad (\text{II}.1.17)_1$$

We put $M_0(w) = -\frac{1}{w}$, then

$$\sum_{p \geqslant 0} M_p(w)\xi^p = -\frac{\xi^2 f'(\xi)^2}{f(\xi)^2} \frac{1}{w(1 - wf(\xi))} \qquad (\text{II}.1.17)_2$$

and from (II.1.14), we see that

$$z^{1-p} f'(z) + M_p\left(\frac{1}{f(z)}\right) = \sum_{n=1}^{\infty} A_n^p z^{n+1}.$$

We obtain (II.1.1)–(II.1.10) with $N_p(w) = M_p(\frac{1}{w})$.

Since $\phi_p(z)$ is the coefficient of ξ^p in the asymptotic expansion of the function $\frac{\xi^2 f'(\xi)^2}{f(\xi)^2} \frac{f(z)^2}{(f(\xi) - f(z))}$, we obtain (II.1.2). We deduce (1.13) from (II.1.1) and (1.10). This ends the proof of the theorem. If we divide (II.1.12) by $(\text{II}.1.17)_2$, we obtain $\frac{\xi}{f(\xi)} \sum_{p \geqslant 0} T_p(w)\xi^p = -\sum_{n \geqslant 0} w M_n(w)\xi^n$. Since $\frac{f(\xi)}{\xi} = 1 + \sum_{n \geqslant 1} c_n \xi^n$, this yields a relation between the polynomials M_p and T_p. Combining with $(\text{II}.1.17)_1$, it gives

$$\begin{aligned} T_p(w) &= w\left(T_{p-1}(w) + c_1 T_{p-2}(w) + \cdots + c_{p-1} T_0(w)\right) \\ &\quad + P_p^p + c_1 P_{p-1}^{p-1} + \cdots + c_p. \end{aligned} \tag{II.1.18}$$

In Section 2, we give a direct proof of (1.13).

II.2. The generating function for the (A_n^p)

The polynomials A_k^p *defined by* (II.1.1) *can be calculated with* (1.13).

Proof. Taking the derivative of (1.14) with respect to u yields

$$\psi\left(u, \frac{1}{f(v)}\right) - \frac{v}{v-u} = \frac{uf'(u)}{f(u)} \frac{f(v)}{f(v)-f(u)} - \frac{v}{v-u} = \sum_{n\geqslant 1}\sum_{k\geqslant 1} \beta_{nk} u^n v^k. \tag{II.2.1}$$

We multiply (II.2.1) by $f'(u)$,

$$\frac{uf'(u)^2}{f(u)} \frac{f(v)}{f(v)-f(u)} - \frac{vf'(u)}{v-u} = \sum_{n\geqslant 1}\sum_{k\geqslant 1} \big[\beta_{nk} + 2c_1\beta_{n-1,k} + \cdots + nc_{n-1}\beta_{1,k}\big] u^n v^k. \tag{II.2.2}$$

The (B_k^p) are given by (II.1.14) for $p > 1$ and $B_k^1 = (k+2)c_{k+1}$. We rewrite (II.2.2) as

$$\sum_{k\geqslant 0}\sum_{p\geqslant 2} \big(B_k^p - (p+k+1)c_{p+k}\big) u^{p-1} v^{k+1} = -\frac{uf'(u)^2}{f(u)} \frac{f(v)}{f(v)-f(u)} + \frac{vf'(u)}{v-u}. \tag{II.2.3}$$

Since $B_k^1 = (k+2)c_{k+1}$, we can write the sum in (II.2.3), starting from $p = 1$. On the other hand,

$$\sum_{k\geqslant 0}\sum_{p\geqslant 1} (p+k+1)c_{p+k} u^{p-1} v^{k+1} = -\frac{vf'(u)}{v-u} + \frac{vf'(v)}{v-u}.$$

Thus

$$\gamma(u,v) = \sum_{k\geqslant 0}\sum_{p\geqslant 1} B_k^p u^p v^{k+1} = -\frac{u^2 f'(u)^2}{f(u)} \frac{f(v)}{f(v)-f(u)} + \frac{uvf'(v)}{v-u}. \tag{II.2.4}$$

Since

$$A_k^p = B_k^p - P_p^p c_k \tag{II.2.5}$$

and $A_0^p = 0$, we have

$$\sum_{k\geqslant 1}\sum_{p\geqslant 1} A_k^p u^p v^{k+1} = \frac{u^2 f'(u)^2}{f(u)^2} \frac{f(v)^2}{f(u)-f(v)} + f(v) + \frac{uv}{v-u} f'(v).$$

We divide by v. Since

$$\sum_{k\geqslant 1}\sum_{p\geqslant 0} A_k^p u^p v^k = \sum_{k\geqslant 1}\sum_{p\geqslant 1} A_k^p u^p v^k + \sum_{k\geqslant 1} A_k^0 v^k$$

and

$$\sum_{k\geqslant 1} A_k^0 v^k = \sum_{k\geqslant 0} k c_k v^k = f'(v) - \frac{f(v)}{v},$$

we obtain (1.13).

II.3. Identities between polynomials

From (1.12), we obtain that for any $p \neq 0$, and $k \neq 0$,

$$P_n^{n+p+k} = \sum_{j=0}^{n} \frac{(j+p)(n-j+k)}{kp} K_j^p K_{n-j}^k. \tag{II.3.1}$$

In the following, we prove more identities between the polynomials.

Proof of (1.15). With (II.1.6), $F'_{k+1}(w) = (k+1)\sum_{n=0}^{k} K_{k-n}^{n+1} w^n$. Thus

$$F_k(w) = F_k(0) + \sum_{n=1}^{k} \frac{k}{n} K_{k-n}^n w^n. \tag{II.3.2}$$

We replace w by $\frac{1}{f(z)}$, with (II.1.7), we obtain (1.15). The factorization can also be obtained from (II.2.1). We have

$$K_2^1 = c_2, \qquad K_3^{-1} = 2c_1c_2 - c_3 - c_1^3, \qquad K_1^{-2} = 2c_1,$$

$$K_4^{-2} = 6c_1c_3 - 2c_4 + 3c_2^2 - 12c_1^2c_2 + 5c_1^4,$$

$$K_5^{-3} = 12c_2c_3 + 12c_1c_4 - 3c_5 - 30c_1c_2^2 - 30c_1^2c_3 + 60c_1^3c_2 - 21c_1^5$$

and with (1.15), it gives $\beta_{3,2} = 3K_2^1K_3^{-1} + \frac{3}{2}K_1^{-2}K_4^{-2} + K_5^{-3}$. In the same way,

$$K_1^1 = c_1, \qquad K_4^{-1} = 2c_1c_3 - c_4 + c_2^2 - 3c_1^2c_2 + c_1^4,$$

$$K_5^{-2} = 6c_2c_3 + 6c_1c_4 - 2c_5 - 12c_1c_2^2 - 12c_1^2c_3 + 20c_1^3c_2 - 6c_1^5$$

and $\beta_{2,3} = 2K_1^1K_4^{-1} + K_5^{-2}$.

Proof of (1.17). We deduce (1.17) from (II.2.5)–(II.1.14) and (A.4.8) in [1]. It can also be proved as follows.

Let $S_f(z) = (f''/f')' - \frac{1}{2}(f''/f')^2$ and $\phi(u) = 1/f(u)$, then

$$\frac{f'(u)f'(v)}{(f(u)-f(v))^2} = \frac{\phi'(u)\phi'(v)}{(\phi(u)-\phi(v))^2} \quad \text{and} \quad S_\phi(u) = S_f(u).$$

Thus (see [3], p. 64),

$$\frac{f'(u)f'(v)}{(f(u)-f(v))^2} - \frac{1}{(u-v)^2} = \frac{\partial^2}{\partial u \partial v} \log \frac{\frac{1}{f(u)} - \frac{1}{f(v)}}{u-v}$$

and

$$\frac{\partial^2}{\partial u\partial v}\bigg|_{u=v}\log\frac{\frac{1}{f(u)}-\frac{1}{f(v)}}{u-v}=-\frac{1}{6}S_f(u)=-\sum_{n\geqslant 1,k\geqslant 1}k\beta_{nk}u^{n+k-2}. \tag{II.3.3}$$

Theorem. *Let $z^2S_f(z)=6\sum_{n\geqslant 0}\mathcal{P}_nz^n$ be the expansion of the Schwarzian derivative. We have $P_0=0$, $P_1=0$, and the $(P_n)_{n\geqslant 2}$ are given by* (1.17) *in terms of the coefficients in* (1.14)*, $\mathcal{P}_2=\beta_{11}=c_1^2-c_2$, $\mathcal{P}_3=\beta_{21}+2\beta_{12}=8c_1c_2-4c_3-4c_1^3$, $\mathcal{P}_4=\beta_{31}+2\beta_{22}+3\beta_{13}=12c_1^4-34c_1^2c_2+12c_2^2+20c_1c_3-10c_4,\ldots$.*

Let $b_{nk}=k\beta_{nk}$, then (b_{nk}) is a symmetric matrix, moreover $\mathcal{P}_n=\operatorname{trace}(A_nB_n)$ where A_n is the square matrix of order $n-1$ such that $a_{n-1,1}=a_{n-2,2}=a_{n-j,j}=1$ and all the other coefficients are zero and $B_n=(b_{ij})_{1\leqslant i\leqslant n-1,1\leqslant j\leqslant n-1}$.

II.4. The operators $(\partial_k)_{k\geqslant 1}$ and $(L_{-k})_{k\geqslant 0}$ in terms of $(L_k)_{k>0}$

We identify the set of univalent functions with the submanifold $\mathcal{M}$ in C^N via the map $z(1+\sum_{n\geqslant 1}c_nz^n)\to(c_1,c_2,\ldots,c_n,\ldots)$. For $k\geqslant 1$, L_k is given by (1.8) and (A_n^k) by (II.1.1) or (1.13). We put

$$L_0=\sum_{n\geqslant 1}nc_n\partial_n \quad \text{and} \quad \text{for } k\geqslant 1 \quad L_{-k}=\sum_{n\geqslant 1}A_n^k\partial_n. \tag{II.4.1}$$

In the following, we relate the $(\partial_k)_{k\geqslant 1}$ and $(L_{-k})_{k\geqslant 0}$ to the $(L_k)_{k>0}$ and study the action of these operators on some of the generating functions.

Proposition. *For $k\geqslant 1$,*

$$\partial_k=L_k-2c_1L_{k+1}+(4c_1^2-3c_2)L_{k+2}+\cdots+B_nL_{k+n}+\cdots \tag{II.4.2}$$

where the $(B_n)_{n\geqslant 1}$ are independent of k.

$$\frac{1}{f'(z)}=1+\sum_{n\geqslant 1}B_nz^n. \tag{II.4.3}$$

Proof. We verify (II.4.3) on $f(z)$. We have $L_k[f(z)]=z^{k+1}f'(z)$. Since

$$\partial_k[f(z)]=z^{k+1} \quad \text{and} \quad \partial_k[f'(z)]=(k+1)z^k \tag{II.4.4}$$

we have to prove $z^{k+1}=z^{k+1}f'(z)-2c_1z^{k+2}f'(z)+\cdots+B_nz^{k+n}f'(z)$. We divide by z^{k+1} and we obtain (II.4.3).

Consider the generating functions (see (II.1.12) and (II.1.5)),

$$\psi(z,w)=\frac{zf'(z)}{f(z)-wf(z)^2}=1+\sum_{n\geqslant 1}F_n(w)z^n$$

and

$$\frac{zf'(z)^2}{f(z)-wf(z)^2}=1+\sum_{n\geqslant 1}T_n(w)z^n.$$

The polynomials $(T_p)_{p\geqslant 0}$ are expressed in (II.1.11) in terms of the $(F_p)_{p\geqslant 0}$. Conversely, with (II.4.3), we have

$$F_0 = T_0, \qquad F_1 = T_1 - 2c_1T_0, \qquad F_2 = T_2 - 2c_1T_1 + \left(4c_1^2 - 3c_2\right)T_0, \ldots,$$
$$F_p = T_p - 2c_1T_{p-1} + \left(4c_1^2 - 3c_2\right)T_{p-2} + \cdots + B_kT_{p-k} + \cdots + B_pT_0. \tag{II.4.5}$$

From the generating functions, we obtain the action of (L_k) *on the polynomials*. Let $k \geqslant 1$, we have $L_k(T_p) = (k+p+1)T_{p-k}$ if $p \geqslant k$ and $L_k(T_p) = 0$ if $p < k$. From

$$L_k\left(\frac{1}{f'(z)}\right) = z^{k+1}\left(\frac{1}{f'(z)}\right)' - (k+1)z^k\frac{1}{f'(z)},$$

we obtain $L_k(B_p) = (p-2k-1)B_{p-k}$ if $k \leqslant p$ and $L_k(B_p) = 0$ if $k > p$. From (II.1.5), we deduce that $\psi(z,w)$ satisfies

$$L_k[\psi(z,w)] = z^{k+1}\frac{\partial\psi}{\partial z}\psi(z,w) + kz^k\psi(z,w), \tag{II.4.6}$$

$L_k(F_n) = nF_{n-k}$ if $n \geqslant k$ and $L_k(F_n) = 0$ if $n < k$. We obtain $L_k(\beta_{n,p})$ with (II.4.1) and (II.1.5). From

$$L_k\left[\psi\left(z, \frac{1}{f(v)}\right)\right]$$
$$= z^{k+1}\frac{\partial}{\partial z}\psi\left(z, \frac{1}{f(v)})\right) + v^{p+1}\frac{\partial\psi}{\partial v} + kz^k\psi\left(z, \frac{1}{f(v)}\right), \tag{II.4.7}$$

we deduce $L_k(\beta_{n,p}) = n\beta_{n-k,p} + (p-k)\beta_{n,p-k}$ if $k \leqslant p-1$, $k \leqslant n-1$ and the formula extends for any $n \geqslant 1$, $p \geqslant 1$ with the convention $\beta_{mn} = 0$ if $m \leqslant 0$ or $n \leqslant 0$.

Lemma. *Let* (*see* (1.10))

$$\phi_f(u,w) = \frac{u^2f'(u)^2}{f(u)^2} \times \frac{w^2}{f(u)-w} = \sum_{p\geqslant 0} N_p(w)u^p, \tag{II.4.8}$$

then for $k \geqslant 1$,

$$L_k[\phi_f(u,w)] = 2ku^k\phi_f(u,w) + u^{k+1}\frac{\partial\phi_f}{\partial u}(u,w), \tag{II.4.9}$$

$$\partial_k[\phi_f(u,w)] = \left[2u\frac{\partial}{\partial u}\left(\frac{u^k}{f'(u)}\right)\right]\phi_f(u,w) + \frac{u^{k+1}}{f'(u)}\frac{\partial\phi_f}{\partial u}(u,w). \tag{II.4.10}$$

Proof. The proof of (II.4.9) is straightforward, then (II.4.9) and (II.4.2) imply that

$$\partial_k[\phi_f(u,w)] = \left[2ku^k\frac{1}{f'(u)} + 2u^{k+1}\left(\frac{1}{f'(u)}\right)'\right]\phi_f(u,w) + \frac{u^{k+1}}{f'(u)}\frac{\partial\phi_f}{\partial u}(u,w).$$

Corollary. $L_k(N_p) = (k+p)N_{p-k}$ *if* $p \geqslant k$ *and* $L_k(N_p) = 0$ *if* $p < k$. *Let* B_n *as in* (II.2.3), *then if* $p \geqslant k$,

$$\partial_k N_p = \sum_{j=0}^{p-k}(2p-j)B_{p-k-j}N_j(w). \tag{II.4.11}$$

Remark. Let (see (1.13))

$$\sum_{k\geqslant 1}\sum_{p\geqslant 0} A_k^p u^p v^{k+1} = \phi_f\big(u, f(v)\big) + \frac{v^2 f'(v)}{v-u} = \theta_f(u,v),$$

then

$$\theta_f(u,v)|_{v=u} = \frac{3}{2}u^2 f''(u) - 2u^2\frac{f'u)^2}{f(u)} + 2uf'(u), \tag{II.4.12}$$

$$\left.\frac{\partial\theta_f}{\partial v}\right|_{v=u} = \left[\frac{1}{6}u^2 S_f(u) - \frac{u^2 f'(u)^2}{f(u)^2}\right] f'(u) + \frac{1}{2}\big(u^2 f'(u)\big)'', \tag{II.4.13}$$

$$\begin{aligned}\left.\frac{\partial\theta_f}{\partial u}\right|_{u=v} &= \left[-\frac{1}{6}v^2 S_f(v) + \frac{3v^2 f'(v)^2}{f(v)^2}\right] f'(v) + \big(v^2 f'(v)\big)'' \\ &\quad - \left[1 + 4v\frac{f'(v)}{f(v)}\right] \times \big(vf'(v)\big)'.\end{aligned}$$

In the same manner, consider (see (II.4.4))

$$\gamma_f(u,v) = \frac{u^2 f'(u)^2}{f(u)} \times \frac{f(v)}{f(u)-f(v)} + \frac{uvf'(v)}{v-u} = \sum_{k\geqslant 0}\sum_{p\geqslant 1} B_k^p u^p v^{k+1}$$

then

$$\gamma_f(u,v)|_{v=u} = \frac{3}{2}u^2 f''(u) - u^2\frac{f'u)^2}{f(u)} + uf'(u), \tag{II.4.14}$$

$$\left.\frac{\partial\gamma_f}{\partial v}\right|_{v=u} = \frac{1}{6}u^2 S_f(u) f'(u) + \frac{u}{2}\big(uf'(u)\big)''. \tag{II.4.15}$$

Lemma. *If $p \geqslant 1$, for any k such that $k > p$, $\partial_k A_k^p$ is independent of k and we have*

$$\partial_k A_k^p = -\frac{1}{2\mathrm{i}\pi}\int \frac{f''}{f'}\frac{\mathrm{d}z}{z^p}. \tag{II.4.16}$$

If $k \leqslant p$,

$$\partial_k A_k^p = -\frac{1}{2\mathrm{i}\pi}\int \frac{f''(z)}{f'(z)}\frac{\mathrm{d}z}{z^p} + \frac{1}{2\mathrm{i}\pi}\int (\partial_k N_p)\big(f(z)\big)\frac{\mathrm{d}z}{z^{k+2}}. \tag{II.4.17}$$

Proof. From (II.1.1), we have

$$(1+k)z^{1-p+k} + N_p'\big(f(z)\big)z^{k+1} + (\partial_k N_p)\big(f(z)\big) = \sum_{n=1}^{\infty}(\partial_k A_n^p)z^{n+1}.$$

With (II.1.3), it gives (II.4.17). Since $\partial_k N_p = 0$ if $k > p$, we obtain (II.4.16).

With (II.4.2), we obtain L_{-k}, $k>0$, in terms of the $(L_j)_{j\geqslant 1}$.

Proposition. *Let* $L_{-k}=\sum_{j\geqslant 1} D_j^k L_j$, *then* $g_k(z)=\sum_{j\geqslant 1} D_j^k z^{j+1}$ *satisfies*

$$g_k(z)=\frac{L_{-k}[f(z)]}{f'(z)}=z^{1-k}+\frac{\phi_k(z)}{f'(z)},$$

$$L_{-k}\left[\frac{f''(z)}{f'(z)}\right]=g_k''(z)+g_k'(z)\frac{f''(z)}{f'(z)}+g_k(z)\left(\frac{f''(z)}{f'(z)}\right)',$$

$$L_{-k}\big[S(f)(z)\big]=g_k''(z)+2g_k'(z)S(f)(z)+g_k(z)\big(S(f)(z)\big)'.$$

Part III

III.1. The vector fields R_p

In [1], we related the asymptotic expansion of $z^2(\frac{f'(z)}{f(z)})^2(\frac{f(z)}{z})^k$ in sum of powers of z to that of $z\frac{f'(z)}{f(z)}(\frac{f(z)}{z})^k$ in sum of powers of $f(z)$. In the same way, for $g(z)$ with expansion (1.1), we have

$$z\frac{g'(z)}{g(z)}\left(\frac{g(z)}{z}\right)^k=1+\sum_{j\geqslant 1}V_j^k\frac{1}{g(z)^j}\quad\text{and}$$
$$\left(z\frac{g'(z)}{g(z)}\right)^2\left(\frac{g(z)}{z}\right)^k=1+\sum_{j\geqslant 1}V_j^{k-j}\frac{1}{z^j}. \tag{III.1.1}$$

The coefficients V_j^k are obtained with contour integrals. The first equation in (III.1.1) gives

$$z^{1+k}g'(z)=g(z)^{1+k}+\sum_{1\leqslant j\leqslant k-1}V_j^{-k}g(z)^{1+k-j}+\sum_{j\geqslant k}V_j^{-k}g(z)^{1+k-j}. \tag{III.1.2}$$

Let $g(z)$ be a univalent function outside the unit disc

$$g(z)=b_0z+b_1+\frac{b_2}{z}+\cdots+\frac{b_n}{z^{n-1}}+\cdots. \tag{III.1.3}$$

We assume $b_0\neq 0$. To g, it corresponds the sequence $(b_0,b_1,\ldots,b_n,\ldots)\in C^N$.

$$\frac{\partial}{\partial b_0}[g(z)]=z,\quad \frac{\partial}{\partial b_1}[g(z)]=1,\quad \ldots\quad \frac{\partial}{\partial b_n}[g(z)]=\frac{1}{z^{n-1}},\quad \ldots. \tag{III.1.4}$$

Thus, if $k\geqslant 0$, we have $z^{1-k}g'(z)=R_{-k}[g(z)]$ where

$$R_{-k}=b_0\frac{\partial}{\partial b_k}-b_2\frac{\partial}{\partial b_{k+2}}-\cdots-(n-1)b_n\frac{\partial}{\partial b_{n+k}}. \tag{III.1.5}$$

Consider the expansion

$$\chi(z)=\sum_{j\in Z}g(z)^j\frac{1}{2\mathrm{i}\pi}\int\frac{ug'(u)}{g(u)}g(u)^{-j}\chi(u)\frac{\mathrm{d}u}{u}$$

and then replace $\chi(z)$ by $z\chi(z)g'(z)$, compare with Part II,

$$z\chi(z)g'(z) = \sum_{j\geqslant 0} g(z)^{1-j}\frac{1}{2i\pi}\int \frac{u^2g'(u)^2}{g(u)^2}g(u)^j\chi(u)\frac{du}{u} + \sum_{j\geqslant 1} g(z)^{1+j}\frac{1}{2i\pi}\int \frac{u^2g'(u)^2}{g(u)^2}g(u)^{-j}\chi(u)\frac{du}{u} = S_1 + S_2. \quad \text{(III.1.6)}$$

The first sum,

$$S_1 = R_\chi\big(g(z)\big) = \sum_{j\geqslant 0} g(z)^{1-j}\frac{1}{2i\pi}\int \frac{u^2g'(u)^2}{g(u)^2}g(u)^j\chi(u)\frac{du}{u}$$

can be expanded in powers of z^k, $k \leqslant 1$. Assume that $\chi(z) = z^p$, where $p \in Z$, in that case, there exists a unique holomorphic vector field R_p on $\mathcal{M} \subset C^N$, satisfying $R_p[g(z)] = R_\chi(g(z))$ where

$$S_1 = R_p(g(z)) = \frac{g(z)^2}{2i\pi}\int \frac{u^2g'(u)^2}{g(u)^2}\frac{u^p}{g(z)-g(u)}\frac{du}{u}.$$

If $\chi(z) = z^{-k}$, with $k \geqslant 0$, then S_2 vanishes and $R_{-k}[g(z)] = z^{1-k}g'(z)$.
If $\chi(z) = z^k$ with $k \geqslant 1$, $S_2 = J_k(g(z)) = \sum_{j=1}^k \alpha_j g(z)^{1+j}$ and J_k is a polynomial.

$$R_k\big(g(z)\big) = z^{1+k}g'(z) - J_k\big(g(z)\big) = E_0^k z + E_1^k + \sum_{n\geqslant 1}\frac{E_{n+1}^k}{z^n}. \quad \text{(III.1.7)}$$

Thus $R_k(g(z)) = R_k[g(z)]$ with the partial differential operator $R_k = \sum_{n\geqslant 0} E_n^k \frac{\partial}{\partial b_n}$.

Proposition. *The polynomials J_k are given by*

$$\frac{u^2g'(u)^2}{g(u)^2}\frac{w^2}{(g(u)-w)} = \sum_{k\geqslant 1} J_k(w)\frac{1}{u^k}. \quad \text{(III.1.8)}$$

The (E_n^k) are given by

$$\sum_{k\geqslant 1}\sum_{n\geqslant 0} E_n^k z^{-n}u^{-k} = -\frac{zg'(z)}{z-u} - \frac{u^2g'(u)^2}{g(u)^2}\times\frac{g(z)^2}{z(g(u)-g(z))}. \quad \text{(III.1.9)}$$

Compare (III.1.9) with (1.13). With (III.1.8), we obtain

$$J_1(w) = \frac{w^2}{b_0},$$
$$J_2(w) = \frac{w^3}{b_0^2} - \frac{3b_1}{b_0^2}w^2,$$
$$J_3(w) = \frac{w^4}{b_0^3} - \frac{4b_1}{b_0^3}w^3 + \frac{(6b_1^2 - 5b_0b_2)}{b_0^3}w^2, \ldots.$$

Proposition. *Let* $(Z_k)_{k\geqslant 1}$ *the operators defined by* (1.3). *For* $k\geqslant 1$,

$$R_{1-k}-Z_k=b_0\frac{\partial}{\partial b_{k-1}}+b_1\frac{\partial}{\partial b_k}+\cdots+b_n\frac{\partial}{\partial b_{n+k-1}}. \tag{III.1.10}$$

For any $k\in Z$, *let*

$$\widetilde{Z}_k(g(z))=\frac{g(z)^2}{2\mathrm{i}\pi}\int\frac{ug'(u)}{g(u)}\frac{u^k}{(g(z)-g(u))}\frac{\mathrm{d}u}{u}. \tag{III.1.11}$$

There exists a unique holomorphic vector field $\widetilde{Z}_k=\sum_{j\geqslant 0}W_j^k\frac{\partial}{\partial b_j}$ *on* $\mathcal{M}$ *such that* $\widetilde{Z}_k[g(z)]=\widetilde{Z}_k(g(z))$ *is given by* (II.1.11). *For* $k\geqslant 0$, *we have* $Z_{1+k}=R_{-k}-\widetilde{Z}_{-k}$ *where* $\widetilde{Z}_{-k}$ *satisfies* $\widetilde{Z}_{-k}(g(z))=z^{-k}g(z)$.

If $k\geqslant 1$, *then*

$$\widetilde{Z}_k[g(z)]=z^kg(z)-\nu_k\big(g(z)\big) \tag{III.1.11$_1$}$$

ν_k *is a polynomial function,*

$$\sum_{k\geqslant 1}\nu_k(w)\frac{1}{\xi^k}=w^2\frac{\xi g'(\xi)}{g(\xi)}\times\frac{1}{g(\xi)-w}. \tag{III.1.12}$$

Proof. Let $h(z)=\frac{g(z)}{z}=b_0+\frac{b_1}{z}+\cdots+\frac{b_n}{z^n}+\cdots$, we assume that $b_0\neq 0$. Then

$$\frac{\partial}{\partial b_0}h(z)=1,\qquad \frac{\partial}{\partial b_1}h(z)=\frac{1}{z},\qquad \dots\qquad \frac{\partial}{\partial b_n}h(z)=\frac{1}{z^n}.$$

For $k\geqslant 1$, $z^{2-k}h'(z)=Z_k[h(z)]=\frac{1}{z}Z_k[g(z)]$, where Z_k is given by (1.3). On the other hand, we have $R_{-k}[g(z)]=z^{1-k}g'(z)$ and $h'(z)=\frac{g'(z)}{z}-\frac{g(z)}{z^2}$, we deduce $[R_{1-k}-Z_k](g(z))=z^{1-k}g(z)$, this proves (III.1.10).

For (III.1.11), let

$$z^{2+k}h'(z)=\sum_{j\in Z}g(z)^j\frac{1}{2\mathrm{i}\pi}\int\frac{ug'(u)}{g(u)}g(u)^{-j}u^{2+k}h'(u)\frac{\mathrm{d}u}{u}.$$

Since $h'(u)=\frac{g'(u)}{u}-\frac{g(u)}{u^2}$, we obtain $z^{2+k}h'(z)=R_k[g(z)]+J_k(g(z))-S$ with

$$\begin{aligned}S=&\sum_{j\geqslant 0}g(z)^{1-j}\frac{1}{2\mathrm{i}\pi}\int\frac{ug'(u)}{g(u)}g(u)^ju^k\frac{\mathrm{d}u}{u}\\&+\sum_{j\geqslant 1}g(z)^{1+j}\frac{1}{2\mathrm{i}\pi}\int\frac{ug'(u)}{g(u)}g(u)^{-j}u^k\frac{\mathrm{d}u}{u}.\end{aligned}$$

Let $S=S_1+S_2$. For any $k\in Z$, $S_1=\widetilde{Z}_k(g(z))=\sum_{j\geqslant 0}W_j^kz^{1-j}$ is given by (III.1.11). We put $\widetilde{Z}_k=\sum_{j\geqslant 0}W_j^k\frac{\partial}{\partial b_j}$.

If $k\leqslant 0$, S_2 vanishes and the integrand (III.1.11) has a pole only at $u=z$, we have $\widetilde{Z}_k(g(z))=z^kg(z)$. Thus, for $k\geqslant 1$, we have $\widetilde{Z}_{1-k}=R_{1-k}-Z_k$ and $\widetilde{Z}_{1-k}$ is given by

(III.1.10). If $k \geqslant 1$, the integrand (III.1.11) has a pole at $u = z$ and a pole at $u = 0$, the residue at $u = 0$ is given by

$$S_2 = \frac{g(z)^2}{2i\pi} \int \frac{ug'(u)}{g(u)} \frac{u^k}{g(u)^k} \times \frac{g(u)^k - g(z)^k}{g(u) - g(z)} \frac{du}{u} = \sum_{j=1}^{k} \alpha_j g(z)^{1+j} = \nu_k\big(g(z)\big)$$

and the polynomials ν_k are obtained with (III.1.12).

Remark. With (III.1.12) and $(\text{III.1.11})_1$, it is easy to verify that the $(W_j^k)_{k\geqslant 1}$ in $\widetilde{Z}_k$ satisfy

$$\lambda(\xi, z) = \sum_{j\geqslant 0} \sum_{k\geqslant 1} W_j^k z^{1-j} \xi^{-k} = \frac{zg(z)}{\xi - z} - \frac{\xi g'(\xi)}{g(\xi)} \times \frac{g(z)^2}{g(\xi) - g(z)}.$$

Moreover, let $S_g(\xi)$ the Schwarzian derivative of g, then

$$\left.\frac{\partial\lambda}{\partial z}\right|_{z=\xi} = -\xi g(\xi)\left[\frac{1}{6} S_g(\xi) - \frac{g'(\xi)^2}{g(\xi)^2}\right] - \frac{1}{2}\xi g''(\xi) - g'(\xi).$$

III.2. The Faber polynomials $(G_n)_{n\geqslant 0}$ of $g(z)$. The $G_n(f(z))_{n\geqslant 0}$

Let $g(z)$ be given by (III.1.3), its derivative $g'(z) = b_0 - \frac{b_2}{z^2} - \cdots - \frac{nb_{n+1}}{z^{n+1}} - \cdots$ and

$$\frac{zg'(z)}{g(z) - w} = 1 + \sum_{n\geqslant 1} G_n(w) \frac{1}{z^n},$$

then

$$G_n\big(g(z)\big) = z^n + \sum_{k\geqslant 1} \gamma_{nk} \frac{1}{z^k}. \tag{III.2.1}$$

Let $f(z) = z(1 + \sum_{n\geqslant 1} c_n z^n)$, we have $G_n(f(z)) = G_n(0) + \sum_{k\geqslant 1} p_{kn} z^k$. *The matrix $P = (p_{kn})$ is given by the generating function*

$$\frac{\xi g'(\xi)}{g(\xi) - f(z)} = 1 + \sum_{n\geqslant 1} G_n(0)\xi^{-n} + \sum_{n\geqslant 1} \sum_{i\geqslant 1} p_{in} z^i \xi^{-n}. \tag{III.2.2}$$

Let $\overline{P}$ the matrix complex conjugated of P and P^t the transposed matrix, then the coefficients of the matrix $P\overline{P}^t$ are given by

$$\frac{1}{2i\pi} \int_{\partial D} \left|\frac{\xi g'(\xi)}{g(\xi) - f(z)}\right|^2 \frac{d\xi}{\xi} = \sum_{i,k} \big(P\overline{P}^t\big)_{ki} \bar{z}^i z^k.$$

The proof is straightforward.

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