An algebra of differential operators and generating functions on the set of univalent functions

Hélène Airault \textsuperscript{a}, Jiagang Ren \textsuperscript{b}

\textsuperscript{a} INSSET, Université de Picardie Jules Verne, 48 rue Raspail, 02100 Saint-Quentin, (Aisne), Laboratoire CNRS FRE 2270, LAMFA (Amiens), France

\textsuperscript{b} Department of Mathematics, Huazhong University of Science and Technology, Wuhan, Hubei 430074, PR China

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Abstract

With a method close to that of Kirillov [4], we define sequences of vector fields on the set of univalent functions and we construct systems of partial differential equations which have the sequence of the Faber polynomials \((F_n)\) as a solution. Through the Faber polynomials and Grunsky coefficients, we obtain the generating functions for some of the sequences of vector fields. © 2002 Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

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1. Introduction

In the first part, we consider the function

\[ g(z) = z + b_1 + \sum_{n=1}^{\infty} \frac{b_{n+1}}{z^n} \quad (1.1) \]
the coefficients \((b_1, b_2, \ldots, b_n, \ldots)\) are in the subset \(\mathcal{M}\) of \(\mathbb{C}^N\) such that \(g(z)\) is univalent outside the unit disc. 
\[
z \frac{g'(z)}{g(z)} = \sum_{n=0}^{\infty} F_n(b_1, b_2, b_3, \ldots, b_n) \frac{1}{z^n} \tag{1.2}
\]
where \(F_n(b_1, b_2, b_3, \ldots, b_n)\) is a homogeneous polynomial of degree \(n\), the variables \(b_1, b_2, \ldots, b_n\) have respective weight, 1 for \(b_1\), 2 for \(b_2\), 3 for \(b_3\), \ldots, \(n\) for \(b_n\). We have (see for example [3])
\[
F_0 = 1, \quad F_3 = -b_1^3 + 3b_2b_1 - 3b_3, \\
F_1 = -b_1, \quad F_4 = b_1^4 - 4b_2b_1^3 + 4b_1b_3 + 2b_2^2 - 4b_4. \\
F_2 = b_2^2 - 2b_2.
\]
On the submanifold \(\mathcal{M}\), we define the partial differential operators \((Z_k)_{k \geq 1}\), the variables are \(b_1, b_2, \ldots, b_n, \ldots\), and \(\partial_n = \frac{\partial}{\partial b_n}\) denotes the partial derivative with respect to the \(n\)th variable \(b_n\),
\[
Z_k = -\sum_{n=1}^{\infty} n b_n \partial_n + k - 1. \tag{1.3}
\]
For a function \(\phi(z, b_1, b_2, \ldots, b_n, \ldots)\), we consider the system of partial differential equations,
\[
Z_k \phi = \frac{\partial}{\partial z} \phi.
\]
We find that the sequence \((F_n)\) of the Faber polynomials is a solution of an infinite system of partial differential equations involving the \((Z_k)\). We show how to calculate the Faber polynomials from the coefficients in the asymptotic expansion of the function \((\frac{g(z)}{z})^p\) where \(p\) is an integer,
\[
\frac{g(z)^p}{z^p} = 1 + \sum_{n \geq 1} K_n^p \frac{1}{z^n} \tag{1.4}
\]
in the notation \(K_n^p\), \(n\) and \(p\) are indices. We define generalized Faber polynomials \((H_j^k)\), \(j \geq 0, k \in \mathbb{Z}\), associated to \(g\) by
\[
z \frac{g'(z)}{g(z)} \left( \frac{g(z)}{z} \right)^k = 1 + \sum_{j \geq 1} H_j^{k-j} \frac{1}{z^j} \tag{1.5}
\]
and generalized Faber polynomials \((F_j^k)\), \(j \geq 0, k \in \mathbb{Z}\), associated to the univalent function \(f(z) = z(1 + b_1 z + b_2 z^2 + \cdots + b_n z^n + \cdots)\) by
\[
z \frac{f'(z)}{f(z)} \left( \frac{f(z)}{z} \right)^k = 1 - \sum_{j \geq 1} F_j^{k+j} z^j = 1 + (k+1)b_1 z + \cdots. \tag{1.6}
\]
When \(\frac{\partial}{\partial z} = z f(\frac{1}{z})\), then \(K_j^k = \frac{1}{\pi} (H_j^{k-j} - F_j^{k+j})\) and
\[
F_n^k = -\frac{k}{k-2n} H_n^{k-2n} \quad \text{and} \quad F_n^{n+k} = -\left(1 + \frac{n}{k}\right) K_n^k. \tag{1.7}
\]
With an iteration procedure, we obtain homogeneous fractions which are solutions of a system of partial differential equations involving the \((Z_k)_{k \geq 1}\).
The sequence of Faber polynomials and the other polynomials \((K^p_n)\), \((F^j_n)\), \((H^p_n)\) can be obtained as the solution of many systems of partial differential equations with an infinite number of variables. For example (see [1,4]), with the embedding map

\[
f(z) = z \left(1 + \sum_{n \geq 1} b_n z^n\right) \rightarrow (b_1, b_2, \ldots, b_n, \ldots)
\]

in the submanifold \(\mathcal{M}\) of \(\mathbb{C}^N\), we put \(\partial_n = \frac{\partial}{\partial b_n}\) and we associate to \(L_k f(z) = z^{1+k} f'(z)\), where \(k \geq 1\), the vector fields

\[
L_k = \partial_k + \sum_{n=1}^{\infty} (n+1)b_n \partial_{n+k}.
\] (1.8)

We calculate the polynomials \((F_k)_{k \geq 0}\) and \((F^j_n)\) with \(L_p(F^j_n) = 0\) for \(j \leq p\), and \(L_p(F^k_{p+1}) = -k, L_p(F^k_{j}) = k j^{-p-1}\) for \(j \geq p + 2\).

In the second part, we study the generating functions associated to the sequence of vector fields found by Kirillov in [4]. Let \(f(z) = z(1 + \sum_{n \geq 1} c_n z^n)\) be a univalent function on the unit disc, and let \(g(z) = b_0 z + b_1 + b_2 z + \cdots + b_n z^n + \cdots\) be a univalent function outside the unit disc. With the action of vector fields on the set of the diffeomorphisms of the circle \(\text{Diff}(S^1)\), and a variational approach on the equation \(f \circ \gamma = g\), where \(\gamma\) is a diffeomorphism of the circle, Kirillov [4] obtained the following vectors fields on the set of univalent functions

\[
L_{-p} f(z) \equiv \frac{\partial_r}{r} \left[\frac{f(z)^2}{f'(z)} \right]^2 \frac{1}{f(t) - f(z)} \int_{\partial D} \frac{dt}{t^{p+1}} = \phi_p(z) + z^{1-p} f'(z).
\] (1.9)

The term \(\phi_p(z)\) comes from the residue at \(t = 0\) for the contour integral (1.9) and vanishes if \(p < 0\). In that case \(L_k f(z) = z^{1+k} f'(z) = L_k[f(z)]\), where \(k \geq 1\) and \(L_k\) is given by (1.8). We assume \(p \geq 0\). We have \(\phi_p(z) = N_p(f(z))\) and \((N_p(w))_{p \geq 0}\) is given by the generating function

\[
\phi_f(\xi, w) = \sum_{p \geq 0} N_p(w) \xi^p = \frac{\xi^2 f'(\xi)^2}{f(\xi)^2} \left(\frac{w^2}{f(\xi) - w}\right).
\] (1.10)

In this work, we obtain (1.10) with the Faber polynomials of \(h(z) = 1/f(\frac{1}{z})\). Moreover,

\[
L_{-p} f(z) = \sum_{n=1}^{\infty} A^p_n z^{n+1}.
\] (1.11)

In [1], the \((A^p_n)\) are given in terms of the polynomials \(K^j_n\) and \(P^k_n\) such that

\[
\frac{f(z)}{z^j} = \sum_{n \geq 0} K^j_n z^n \quad \text{and} \quad z^2 \left(\frac{f'(z)}{f(z)}\right)^2 \left(\frac{f(z)}{z}\right)^k = 1 + \sum_{n \geq 1} P^{n+k}_n z^n.
\] (1.12)
In the present paper, we obtain the generating function for the \( (A^n_p) \) in two different manners,

\[
\theta_f(u, v) = \sum_{k \geq 1} \sum_{p \geq 0} A^n_p u^p v^{k+1} = \frac{u^2 f'(u)^2}{f(u)^2} \frac{f(v)^2}{[f(u) - f(v)]} + \frac{v^2 f'(v)}{v - u}.
\]  

(1.13)

We deduce (1.13) from (1.10) or in Part II, Section 2, we prove (1.13) writing the \( A^n_p \) in terms of the Grunsky coefficients \( \beta_{jk} \) of \( h(z) \), given by (see [3,6])

\[
K(u, v) = \log \frac{1}{f(u) - f(v)} = \sum_{n \geq 1} \frac{1}{n} \beta_{nk} u^k v^k.
\]  

(1.14)

Section 3, Part II, is a further step towards the classification of Faber type polynomials which was started in Part I. We express the Grunsky coefficients \( \beta_{kj} \) of \( h(z) \) in terms of the polynomials \( K_j^n \) with generating function (1.12), or in terms of the \( P^k_j \), the following identity (1.15) is a factorization of the symmetric matrix \( k \beta_{jk} \) into a product of two infinite matrices

\[
\frac{1}{k} \beta_{kj} = \sum_{n=1}^{k} \frac{1}{n} K^n_{k-n} K^n_{j+n}.
\]  

(1.15)

The expression of the \( (A^n_p) \) found in [1] yields

\[
K^{1-p}_{k+p} + \sum_{j=1}^{p-1} P^p_j K^{j+1-p}_{k+p-j} = \beta_{p-1,k+1} + 2c_1 \beta_{p-2,k+1} + \cdots + (p-1)c_{p-2} \beta_{1,k+1}.
\]  

(1.16)

Let the Schwarzian derivative of \( f \), \( S_f(z) = (f^2)' - \frac{1}{2} (f')^2 \) and the asymptotic expansion

\[
\mathcal{P}_n = 6 \sum_{n \geq 0} \mathcal{P}_n z^n.
\]  

Since \( \frac{d^2}{dz^2} \bigg|_{z=0} S_f(u, v) = -\frac{1}{6} S_f(u) \), see [2], we obtain the Neretin polynomials \( (\mathcal{P}_n)_{n \geq 0} \) in terms of the \( (\beta_{jk}) \),

\[
\mathcal{P}_n = \beta_{n-1,1} + 2 \beta_{n-2,2} + 3 \beta_{n-3,3} + \cdots + (n-1) \beta_{1,n-1}.
\]  

(1.17)

Then, as in [4] and [1], the set of functions univalent inside the unit disc is embedded into the submanifold \( \mathcal{M} \) in \( C^N \), via the map

\[
f(z) = z \left( 1 + \sum_{n \geq 1} c_n z^n \right) \rightarrow (c_1, c_2, \ldots, c_n, \ldots).
\]

For \( k \geq 1 \), \( \delta_k = \frac{\partial}{\partial z^k} \). On \( \mathcal{M} \subset C^N \), we consider the partial differential operators (1.8) and (1.3). We have \( \delta_n L_j = L_j \delta_n + (n+1) \delta_{n+j} \). We express \( \delta_k \) in terms of the \( (L_p)_p \geq k \) with the generating function \( \frac{1}{f(z)} = 1 + \sum_{n \geq 1} B_n z^n \). We put \( L_0 = \sum_{n \geq 1} n c_n \delta_n \) and for \( p \geq 1 \), following [4] and [1], we associate to (1.9) the operator \( L_p = \sum_{n \geq 1} A^n_p \delta_n \). We obtain \( L_p \) in terms of the \( (L_j)_{j \geq 1} \).
In the third part, with the function $g(z) = b_0z + b_1 + \frac{b_2}{z} + \cdots + \frac{b_n}{z^n} + \cdots$, with $b_0 \neq 0$, we consider on the subset of $(b_0, b_1, \ldots, b_n, \ldots)$ such that $g(z)$ is univalent outside the unit disc, the vector fields

$$R_p = b_0 \frac{\partial}{\partial b_p} - b_2 \frac{\partial}{\partial b_{p+2}} - \cdots - (n-1)b_n \frac{\partial}{\partial b_{n+p}} - \cdots \quad \text{if } k \geq 0. \quad (1.18)$$

The $(R_p)$ are obtained in [4] from right invariant vector fields on $\text{Diff}(S^1)$. We relate the $(Z_k)_{k \geq 1}$ of (1.3) to the $(R_{-k})$. It permits to define $Z_k$ for negative $k$.

**Part I**

**I.1. The differential operators $(Z_k)_{k \geq 1}$ and the Faber polynomials**

Let

$$h(z) = \frac{1}{z} g(z) = 1 + \frac{b_1}{z} + \sum_{n \geq 1} b_n \frac{1}{z^n} \quad \text{and} \quad h'(z) = - \sum_{n \geq 1} n b_n \frac{1}{z^{n+1}}. \quad (I.1.1)$$

We consider $h(z)$ as a function $h(z) = \phi(z, b_1, b_2, \ldots, b_n, \ldots)$ of the infinite number of variables $z, b_1, b_2, \ldots, b_n, \ldots$. In this way, we have

$$\frac{\partial}{\partial b_1}[h(z)] = \frac{1}{z}, \ldots \frac{\partial}{\partial b_n}[h(z)] = \frac{1}{z^n}, \ldots. \quad (I.1.2)$$

Thus $zh'(z) = - \sum_{n=1}^{\infty} n b_n \frac{\partial}{\partial b_n}(h(z)) = Z_1[h(z)]$ with $Z_1 = - \sum_{n=1}^{\infty} n b_n \frac{\partial}{\partial b_n}$ and the function $h(z) = \phi(z, b_1, b_2, \ldots, b_n, \ldots)$ satisfies the partial differential equation $\frac{\partial}{\partial z} \phi = Z_1 \phi$. In the same manner, since $\frac{1}{z^k-2} h'(z) = - \sum_{n=1}^{\infty} n b_n \frac{b_n}{z^k z^{n-1}}$, we obtain for $k \geq 1$, the partial differential equation

$$\frac{1}{z^{k-2}} \frac{\partial}{\partial z}[h(z)] = Z_k[h(z)] \quad (I.1.3)$$

where (see (1.3))

$$Z_k = - \sum_{n=1}^{\infty} n b_n \partial_{n+k-1}. \quad (I.1.4)$$

We have $[Z_1, Z_2] = - Z_2, \ldots, [Z_k, Z_{k+r}] = - r Z_{2k+r-1}$. The operators $(Z_k)_{k \geq 1}$ defined above are different from the $(L_k)_{k \geq 1}$ of (1.8). We do not know a priori if the $(Z_k)_{k \geq 1}$ in (1.3) come from an $\epsilon$-perturbation on $\text{Diff}(S^1)$ as it appears in [4] for the $(L_k)$.

**Lemma.** Let the function $\psi(z, b_1, b_2, b_3, \ldots, b_n, \ldots) = \sum_{x \geq 0} P_x(b_1, b_2, b_3, \ldots, b_n, \ldots) \frac{1}{z^x}$, we have for any $k \geq 1$, $Z_k(\frac{\partial}{\partial z} \psi) = \psi \psi \frac{1}{z^k} Z_k(\psi)$. In particular, $Z_k[h'(z)] = (Z_k[h(z)])'$ where $'$ means $\frac{1}{z^x} \frac{\partial}{\partial z}$ and for the derivative of order $p$, we denote $(p) = \frac{\partial^p}{\partial z^p}$.

$$Z_k[h^{(p)}(z)] = (Z_k[h(z)])^{(p)}. \quad (1.1.5)$$
The proof is straightforward.

**Theorem.** The Faber polynomials \( (F_n) \) defined by (1.2) are solutions of the system of partial differential equations

\[
Z_1 F_n = -n F_n, \\
Z_{k+1} F_k = Z_k F_{k-1} + (k-1)(1 - F_0) = 0, \\
Z_k F_1 = -n F_{n-1}, \quad \text{for } n \geq 2, \\
Z_k F_n = 0, \quad \text{for } n \leq k - 2, \\
Z_k F_n = -n F_{n-k+1}, \quad \text{for } n > k - 1.
\]

(I.1.6)

The equation \( Z_1 F_n = -n F_n \) shows that the polynomial \( F_n \) is homogeneous of degree \( n \) and the variables \( b_1, b_2, b_3, \ldots, b_n \) have respectively the weight 1 for \( b_1 \), 2 for \( b_2 \), 3 for \( b_3, \ldots, n \) for \( b_n \).

The coefficient of \( b_n^k \) in \( F_n \) is \((-1)^k\). With the operators \( Z_k \), \( k \leq n \), and with (I.1.6), we calculate the homogeneous polynomial \( F_n \) from the \( F_k \), \( k < n \). For example, \( F_5 \) is homogeneous of degree 5 and \( Z_5 F_5 = -5 F_1, Z_4 F_5 = -5 F_2, Z_3 F_5 = -5 F_3, Z_2 F_5 = -5 F_4 \). We find

\[
F_5 = -b_1^5 + 5 b_1^3 b_2 - 5 b_1^2 b_3 - 5 b_1 b_2^2 + 5 b_2 b_3 + 5 b_1 b_4 - 5 b_5, \\
F_6 = b_1^6 + 3 b_1^4 b_2 + 6 b_1^3 b_3 - 12 b_1^2 b_2 b_3 - 6 b_1 b_2^2 - 2 b_2^3 + 9 b_1^2 b_2^2 + 6 b_1 b_5 + 6 b_2 b_4 - 6 b_1^2 b_4 - 6 b_6.
\]

**Proof of (I.1.6).** From \( h(z) = \frac{1}{z} g(z) \), we have \( \frac{h'}{h} = -\frac{1}{z} + \frac{z}{g} \) and

\[
\frac{h'}{h} = -1 + \sum_{n \geq 0} F_n \frac{1}{z^n} = -1 + \sum_{n \geq 2} F_n \frac{1}{z^n}.
\]

(I.1.7)

Since \( Z_k \) is a derivation, with (I.1.5), and then using (I.1.3) to replace \( Z_k [h(z)] \), we obtain

\[
Z_k \left[ \frac{h'(z)}{h(z)} \right] = z \frac{Z_k [h'(z)]}{h(z)} - \frac{h'(z)}{h(z)^2} Z_k [h(z)] = z \left( \frac{1}{z^{k-2}} \frac{h'(z)}{h(z)} \right).
\]

(I.1.8)

We replace the expression \( z \frac{h'(z)}{h(z)} \) by its expansion given by (I.1.7),

\[
\sum_{n \geq 0} Z_k (F_n) \frac{1}{z^n} = -z \left( \frac{1}{z^{k-1}} \right)' + z \sum_{n \geq 0} F_n \frac{1}{z^{n+k-1}} = -\frac{k - 1}{z^{k-1}} - \sum_{n \geq 0} b_n (k - 1 + n) F_n \frac{1}{z^{n+k-1}}.
\]

(I.1.9)

By identifying in (I.1.9) the coefficients of \( \frac{1}{z^r} \), we obtain (I.1.6).
With the same method, and from the identity \( Z_k(\phi^p) = p\phi^{p-1}L_k(\phi) \), we deduce

**Lemma.** Consider \( v = \left(\frac{z^k}{z^p}\right)^p = \sum_{n \geq p} H_n \frac{1}{z^n} \) for any integer \( p \), then

\[
Z_k(v) = z^{k-1}\left(\frac{v}{z^{p-(k-1)}}\right)^{p-1}.
\]

Moreover \( Z_k(H_n) = -(n + (p - 1)(k - 1))H_{n-k} \) for \( n \geq p + k - 1 \) and if \( n \leq p + k - 2 \), \( Z_k(H_n) = 0 \).

The Faber polynomials in (1.2), can be obtained from the following \( (K^p_n) \). For any integer \( p \in \mathbb{Z} \), let \( F(z)^n = 1 + \sum_{n \geq 1} K^p_n \frac{1}{z^n} \) where in the notation \( K^p_n \), \( n \) and \( p \) are indices, (see (1.4)). From (1.1.3), we have

\[
Z_k \left( \frac{g(z)^p}{z^p} \right) = \frac{1}{z^{k-2}} \left( \frac{g(z)^p}{z^p} \right)^{p-1}.
\] (1.1.10)

\( K^p_n \), \( n \geq 1 \), \( p \in \mathbb{Z} \), is a homogeneous polynomial in the variables \( b_1, b_2, \ldots, b_n \) where \( b_1 \) has weight 1, \( b_2 \) weight 2, \ldots. Moreover \( Z_k(K^p_n) = -(n - k + 1)K^p_{n-k+1} \) for \( n \geq k \) and \( Z_k(K^p_n) = 0 \) for \( n < k \). This permits to obtain the \( K^p_n \). The coefficient of \( b_{1}^{n} \) in the polynomial \( K^p_n \) is \( \frac{1}{(p-n)!n!}b_{1}^{n} \). Remark that the notation \( \frac{1}{m!} = \frac{(p-n+1)(p-n+2)\cdots p}{n!} \) extends to any \( p \in \mathbb{Z} \). The other coefficients in \( K^p_n \) are calculated with the \( (Z_p)_{p \geq 2} \). For example, \( K^p_1 = \frac{p(p-1)}{2}b_{1}^{2} + ab_{2} \) and \( Z_2(K^p_1) = -K^p_1 \) with \( K^p_0 = pb_{1} \), \( Z_2 = -b_{1} \frac{\partial}{\partial b_{2}} \). This gives the value of \( a \) and \( K^p_1 = \frac{p(p-1)}{2}b_{1}^{2} + pb_{2} \). In [1] (A.1.6), the \( K^p_n \) are obtained with the operators \( L_k \), see (1.8).

\[
K^p_3 = p(p-1)b_{1}b_{2} + pb_{3} + \frac{p(p-1)(p-2)}{3!}b_{1}^{3},
\]

\[
K^p_4 = p(p-1)b_{1}b_{3} + pb_{4} + \frac{p(p-1)}{2}b_{1}^{2}b_{2} + \frac{p(p-1)(p-2)}{2}b_{2}^{2}b_{1}^{2} + \frac{p!}{(p-4)!4!}b_{1}^{4},
\]

\[
K^p_n = \frac{p!}{(p-n)!n!}b_{1}^{n} + \frac{p!}{(p-n+1)!(n-2)!}b_{1}^{n-2}b_{2} + \frac{p!}{(p-n+2)!(n-3)!}b_{1}^{n-3}b_{3} + \frac{p!}{(p-n+3)!(n-4)!}b_{1}^{n-4}b_{4} + \frac{p!}{(p-n+4)!(n-5)!}b_{1}^{n-5}b_{5} + \sum_{j \geq 6} b_{1}^{n-j}V_{j},
\]

where \( V_{j} \), with \( 6 \leq j \leq n \), is a homogeneous polynomial of degree \( j \) in the variables \( b_{2}, \ldots, b_{n} \).
I.2. Generalized Faber polynomials

We assume that $g(z)$ is given by (1.1). We have

$$f(z) = z \left( 1 + \sum_{n \geq 1} b_n z^n \right). \quad \text{(I.2.1)}$$

Since

$$\frac{z}{g(z)} = 1 - \frac{b_1}{z} + \frac{(b_1^2 - b_2)}{z^2} + \cdots \quad \text{and} \quad \frac{g'(z)}{g(z)} = 2 - \frac{1}{z} \frac{f'(\frac{1}{z})}{f(\frac{1}{z})} \quad \text{(I.2.2)}$$

we deduce

$$u f'(u) \frac{f(u)}{u} = 1 + \sum_{n \geq 1} F_k(b_1, b_2, \ldots, b_n) u^n \quad \text{(I.2.3)}$$

then $F_j \equiv F_j$ where $F_j$ is the $j$th Faber polynomial. On the other hand, if we take $k = 1$ in (I.2.3), we find $F_1^{1+j} = -(j+1)b_j$. With the same polynomials $F_k$, we have, for any $k \in \mathbb{Z}$,

$$\left( \frac{f(u)}{u} \right)^k = 1 - \sum_{j \geq 1} F_j^k f(u)^j. \quad \text{(I.2.4)}$$

**Definition.** We call the polynomials $(F_k)$, the generalized polynomials associated to the function $f(z) = z(1 + \sum_{n \geq 1} b_n z^n)$.

When $k = -1$ in (I.2.4), we obtain

$$u = f(u) - \sum_{j \geq 1} F_j^{-1} f(u)^{j+1}. \quad \text{(I.2.5)}$$

In particular, if $w = f(u)$ where $f$ is univalent, we have the asymptotic expansion of the inverse map $f^{-1}$,

$$u = f^{-1}(w) = w - \sum_{j \geq 1} F_j^{-1} w^{j+1} = w + b_1 w^2 + (b_2 - 2b_1^2) w^3 \quad \text{and} \quad z + \frac{(b_1^2 - b_2)}{z^2} + \cdots \quad \text{(I.2.6)}$$

We rewrite (I.2.4) with $k \in \mathbb{Z}$ as

$$u^k = f(u)^k - \sum_{j \geq 1} F_j^{-k} f(u)^{j+k}, \quad \text{or} \quad \left( f^{-1}(w) \right)^k = w^k - \sum_{j \geq 1} F_j^{-k} w^{j+k} = w^k + kb_1 w^{k+1} + \cdots$$

The homogeneous polynomials $(F_j^k)$ can be calculated with (I.2.3) and the $(L_p)_{p \geq 0}$ as in [1] (A.3.3)–(A.6.5): We have $L_p[f(z)] = z^{p+2} \phi'(z)$. Let $\phi(z) = z \frac{f'(z)}{f(z)} (\frac{f(z)}{z})^k$. By commuting $\frac{\partial}{\partial z}$ and $L_p$, we get $L_p[\phi(z)] = z^{p+2} \phi'(z) + (p + 1)z^{p+1} \phi(z)$. From
(1.2.3), \( L_p[\phi(z)] = - \sum_{j \geq 1} L_p(F_j^k z^j) \). With the identification of the coefficients in the asymptotic expansions (1.2.3), we obtain (compare with (1.1.6)) \( L_p(F_j^k) = 0 \) for \( j \leq p \), and \( L_p(F_{p+1}^k) = -k, L_p(F_{j-p}^k) = kF_{j-p-1}^k \) for \( j \geq p + 1 \). This gives \( F_1^k = -kb_1 \),

\[
F_2^k = \frac{k(3-k)}{2}b_1^2 - kb_2, \quad F_3^k = \frac{k(k-5)(4-k)}{6}b_1^3 + k(4-k)b_1b_2 - kb_3,
\]

\[
F_4^k = k(5-k)(k-6)(k-7)b_1^4 + \frac{k(5-k)(k-6)}{2}b_2b_1^2 + k(5-k)b_1b_3
+ \frac{k(5-k)}{2}b_2^2 - kb_4,
\]

\[
F_5^k = \frac{k(6-k)(k-7)(k-8)(k-9)}{5!}b_1^5 + \frac{k(k-7)(6-k)(k-8)}{3!}b_1^3b_2
+ \frac{k(6-k)(k-7)}{2}b_1^2b_3 + k(6-k)b_1b_4 - kb_5,
\]

\[
F_6^k = -\frac{(k-7)k}{6!(k-12)!}b_1^6 + \frac{(k-7)k}{4!(k-11)!}b_1^4b_2 - \frac{(k-7)k}{3!(k-10)!}b_1^3b_3 + \frac{(7-k)k}{2}b_1^2b_4
+ \frac{(7-k)(k-8)}{6}b_2^2 + k(7-k)b_2b_4 - kb_6,
\]

\[
F_7^k = -\frac{(k-8)k}{7!(k-14)!}b_1^7 + \frac{(k-8)k}{5!(k-13)!}b_1^5b_2 - \frac{(k-8)k}{4!(k-12)!}b_1^4b_3
- \frac{(k-8)k}{3!(k-11)!}b_1^3b_4 + \frac{(k-8)k}{2!(k-10)!}b_1^2b_5 + \frac{(k-8)(k-9)}{2}b_1b_6
+ \frac{k(8-k)(k-9)}{3}b_2^2b_3 + k(8-k)b_3b_4 + k(8-k)b_2b_5 - kb_7.
\]

Like in (1.2.3)–(1.2.4), we find that

\[
\left( \frac{g(z)}{z} \right)^k = 1 + \sum_{j \geq 1} H_j^k \frac{1}{g(z)^j} \quad \text{and} \quad \frac{g'(z)}{g(z)} \left( \frac{g(z)}{z} \right)^k = 1 + \sum_{j \geq 1} H_j^{k-j} \frac{1}{z^j}
\]

(1.2.7) with the same polynomials \( H_j^k \), \( k \in \mathbb{Z} \) and \( j \geq 1 \). Because of (1.2.2), the polynomials \( H_j^{-j} = F_j \) are the Faber polynomials.

**Definition.** We call \( H_j^k \) the generalized Faber polynomials associated \( g(z) \).
We have $H_{j-1}^1 = -(j-1)b_j$ and $H_k^0 = 0$ for any $k \geq 1$. As in (1.2.5), we put $k = -1$ in the first equation of (1.2.7), this gives $z = g(z) + \sum_{j \geq 1} H_{j-1}^1 g(z)^{j-1}$ and if $w = g(z)$, then

$$z = g^{-1}(w) = w + \sum_{j \geq 1} H_{j-1}^1 w^{-j}.$$ 

We have

$$H_1^k = kb_1, \quad H_2^k = \frac{k(k+1)}{2} b_1^2 + kb_2,$$

$$H_3^k = \frac{k(k+1)(k+2)}{6} b_1^3 + k(k+2)b_1b_2 + kb_3,$$

$$H_4^k = \frac{k(k+1)(k+2)(k+3)}{4!} b_1^4 + \frac{k(k+1)(k+3)}{2} b_2^2 b_1^2 + k(k+3)b_1b_3$$

$$+ \frac{k(k+3)}{2} b_2^2 + kb_3.$$ 

We can obtain the $H_j^k$ as follows: Let $h(z) = g(z) z$. Then $h(z)^k = 1 + \sum_{n \geq 1} K_n^k \frac{1}{z^n}$. Since

$$z g'(z) \left( \frac{g(z)}{z} \right)^k = h(z)^k + zh(z)^{k-1} h'(z) = 1 + \sum_{n \geq 1} \left( 1 - \frac{n}{k} \right) K_n^k \frac{1}{z^n},$$

we deduce from (I.2.7) that

$$H_{k-n}^k = \left( 1 - \frac{n}{k} \right) K_n^k. \quad (1.2.8)$$

**Proposition.** (See (1.7).) We have $F_n^k = -\frac{k}{k-2n} H_{n-2n}^k$ and $F_{n+k}^n = -(1 + \frac{n}{k}) K_n^k$.

**Proof.** $h(z) = \frac{f(1/z)}{1/z} = \frac{f(a)}{u}$ where $u = 1/z$. Thus

$$z \frac{g'(z)}{g(z)} \left( \frac{f(z)}{z} \right)^k = z^{-k} f'(z) f(z)^{k-1} = \left[ \frac{1}{z} \right]^k + z \frac{1}{k} \frac{\partial}{\partial z} \left[ \frac{1}{z} \right]^k. \quad (1.2.9)$$

Since $[h(1/z)]^k = 1 + \sum_{n \geq 1} K_n^k z^n$, we replace in (1.2.9). If we put $k = n$ in (1.7), we find again $F_n^k = H_n^n = F_n$.

**Corollary.**

$$\sum_{n=1}^{n-1} (n+p) n^{-k} p F_{n-p}^n = K_{n-k} F_n^k + F_n^k. \quad (1.2.10)$$

**Proof.** We have

$$\left( \frac{u}{f(u)} \right)^k = 1 + \sum_{n \geq 1} K_n^k u^n \quad \text{and} \quad \left( \frac{u}{f(u)} \right)^k = 1 + \sum_{j \geq 1} F_j^k \frac{f(u)^j}{u^j}.$$
We replace \( \frac{f(u)}{u} = 1 + \sum_{n \geq 1} K_n^u u^n \). We identify the coefficients of equal powers of \( u \) and we replace \( K_n^u \) by its expression (1.7) to obtain (1.2.10). If we put \( k = -n \) in (1.2.10), it gives \( K_n^u + F_n^u = \sum_{p=1}^{n-1} \frac{n-p}{n} F_p^u F_{n-p}^u \). In particular

\[
F_2^2 + K_2^2 = \frac{1}{2} (F_1^2)^2, \quad F_3^3 + K_3^3 = F_1^3 F_2^3.
\]

\[
F_4^4 + K_4^4 = \frac{1}{2} (F_2^2)^2 + F_1^4 F_3^3 + \cdots.
\]

1.3. Homogeneous fractions of the \( (b_i)_{i \geq 1} \)

Let \( u(z) = 1 - z \frac{g'(z)}{g(z)} = -z \frac{k'(z)}{k(z)} \), then \( 1 - z \frac{u'}{u} = z \frac{k'}{k} - z \frac{k''}{k} \). From (1.1.8), \( Z_k(u) = z \left( \frac{1}{z^{k'} u} \right) - Z_k'(u) \). Since \( Z_k(\frac{z}{u}) = z \left( \frac{Z_k(u)}{u} \right) \), we obtain

**Theorem.** Let \( v = 1 + z \frac{u'}{u} = \sum_{n \geq 1} G_n \frac{1}{z^n} \), then

\[
Z_k(v) = z \left( \frac{1}{z^{k'-1}} v \right)^k + \frac{k(k-1)}{z^{k-1}}
\]

and

\[
Z_k G_n = -n G_n \quad \text{and for } k > 1, \quad Z_k(G_n) = 0 \quad \text{if } n \leq k-2
\]

\[
Z_k(G_{k-1}) = k(k-1) \quad \text{if } n \geq k.
\]

For \( n \geq 1 \), the relation \( Z_k G_n = -n G_n \) implies that \( G_n \) is homogeneous of degree \( n \) in the variables \( b_1, b_2, \ldots, b_n \); \( G_n = \frac{1}{n!} \times Q_n \) where \( Q_n \) is a homogeneous polynomial of degree \( 2n \) in the variables \( b_1, b_2, \ldots, b_n \), if \( b_1 \neq 0 \).

\[
G_1 = \frac{F_2}{b_1} = b_1 - 2 \frac{b_2}{b_1}, \quad G_2 = \frac{2 F_3}{b_1} + \frac{F_2^2}{b_1^2} = \frac{1}{b_1^2} \left( -b_1^4 + 2 b_2 b_1^2 - b_3 b_1 + 4 b_2^2 \right).
\]

**Part II**

In the following, \( f(z) \) is a univalent function inside the unit disc,

\[
f(z) = z \left( 1 + \sum_{n \geq 1} c_n z^n \right).
\]

Let \( \chi(z) \) be a function and assume that the expansion \( \chi \left( f^{-1}(u) \right) = \sum_{n \in \mathbb{Z}} \alpha_n u^n \) converges, \( \alpha_n \) is obtained with the contour integral \( \alpha_n = \frac{1}{2\pi i} \int \chi \left( f^{-1}(u) \right) u^{-n} \frac{du}{u} \). We put \( u = f(z) \) in the two last expressions and we obtain the expansion of \( \chi(z) \) in sum of powers of \( f(z) \),

\[
\chi(z) = \sum_{j \in \mathbb{Z}} f(z)^j \frac{1}{2\pi i} \int \frac{u f'(u)}{f(u)} f(u)^{-j} \chi(u) \frac{du}{u}.
\]
In this formula, instead of \( \chi(z) \), let \( z \chi(z) f'(z) \), it gives

\[
z \chi(z) f'(z) = \sum_{j \in \mathbb{Z}} f(z)^{1-j} \frac{1}{2i\pi} \int \frac{u^2 f'(u)^2}{f(u)^2} f(u)^j \chi(u) \frac{du}{u} = S_1 + S_2
\]

with

\[
S_1 = \sum_{j \geq 0} f(z)^{1-j} \frac{1}{2i\pi} \int \frac{u^2 f'(u)^2}{f(u)^2} f(u)^j \chi(u) \frac{du}{u},
\]

\[
S_2 = \sum_{j \geq 0} f(z)^{2+j} \frac{1}{2i\pi} \int \frac{u^2 f'(u)^2}{f(u)^2} f(u)^{-1-j} \chi(u) \frac{du}{u} = L \chi[f(z)].
\]

If \( \chi(z) = z^p \), where \( p \) is a positive integer \( p \geq 0 \), \( S_1 \) reduces to a finite sum of powers of \( f(z) \) and is equal to

\[
\frac{f(z)^{1-p}}{2i\pi} \int \frac{u^2 f'(u)^2}{f(u)^2} \left( \frac{f(z)^{p+1} - f(u)^{p+1}}{f(z) - f(u)} \right) \frac{du}{u^{p+1}} = -N_p(f(z)).
\]

From this expression, we see that the generating function for the \( N_p(w) \) is given by (1.10). In the next section, we find (1.10) through Faber polynomials.

With the embedding \( f(z) = z(1 + \sum_{n \geq 1} c_n z^n) \to (c_1, c_2, \ldots, c_n, \ldots) \) in the submanifold \( \mathcal{M} \) of \( C^N \), if \( \chi(z) = z^{-p} \), \( S_2 = L_{-p} \{ f(z) \} \), and expanding \( f(z)^{2+j} \) in powers of \( z \) in \( S_2 \), we find \( L_{-p} = \sum_{n \geq 1} A_n^p \partial_n \) where \( \partial_n = \frac{\partial}{\partial c_n} \).

II.1. The vector fields \( L_{-p} \)

**Theorem.** If \( p \geq 0 \), there exists a function \( N_p(w) \) such that

\[
z^{1-p} f'(z) + N_p(f(z)) = \sum_{n=1}^{\infty} A_n^p z^{n+1}.
\]

(II.1.1)

The \( A_n^p \) are homogeneous polynomials in the variables \( (c_n)_{n \geq 1} \). \( N_p(w) \) is calculated with (1.10). Let \( \phi_p(z) = N_p(f(z)) \), then

\[
\sum_{p \geq 0} \phi_p(z) \xi^p = \frac{\xi^2 f'(\xi)^2}{f(\xi)^2} \frac{f(z)^2}{(f(\xi) - f(z))}
\]

and

\[
\frac{f(z)^2}{2i\pi} \int \frac{\xi^2 f'(\xi)^2}{f(\xi)^2} \frac{1}{(f(\xi) - f(z))} \frac{d\xi}{\xi^{p+1}} = \phi_p(z) + z^{1-p} f'(z).
\]

(II.1.2)

The polynomials \( A_n^p \) defined by (II.1.1), are given by (1.13).
We have \( N_0(w) = -w, N_1(w) = -1 - 2c_1 w, N_2(w) = -\frac{1}{w} - 3c_1 - (4c_2 - c_1^2), N_3(w) = -\frac{1}{w} - 4c_1 \frac{1}{w} - (c_1^2 + 5c_2) - P_3^1 w, \ldots \). Let \( N'_p(w) = \frac{d}{dw} N_p(w) \) be the derivative of \( N_p \). From (II.1.1), we deduce that for any \( j \geq 1 \),

\[
\int \frac{N'_j(f(z))}{f(z)} \frac{dz}{z} + \int \frac{f''(z)}{f'(z)} \frac{dz}{z^j} = 0 \quad \text{and} \quad \int \frac{N'_j(f(z))}{f(z)} \frac{dz}{z} + \int \frac{f''(z)}{f(z)} \frac{dz}{z^j} = 0. \tag{II.1.3}
\]

**Proof of (II.1.1)–(1.10)–(II.1.2).** Consider the function

\[
h(z) = \frac{1}{f(\frac{z}{z^j})} = z - c_1 + \left( c_1^2 - c_2 \right) \frac{1}{z} + \left( 2c_1 c_2 - c_3^3 \right) \frac{1}{z^2} + \left( 2c_1 c_3 - c_4 + c_2 - 3c_2 c_2 + c_1^4 \right) \frac{1}{z^3} + \cdots
\]

\[
= z + \sum_{n=0}^{\infty} b_{n+1} \frac{1}{z^n} \tag{II.1.4}
\]

we have (see [5] and [6]) \( \frac{\xi^k(w)}{\Pi(1-w)} = \sum_{n=0}^{\infty} F_n(w) \xi^{-n} \) where \( F_n(w) \) are the Faber polynomials associated to the function \( h \). In terms of \( f \),

\[
\psi(z, w) = \frac{\xi f'(z)}{f(z) - w f'(z)} = 1 + \sum_{n \geq 1} F_n(w) z^n, \tag{II.1.5}
\]

\( F_1(w) = w + c_1, F_2(w) = w^2 + 2c_1 w + 2c_2 - c_1^2, F_3(w) = w^3 + 3c_1 w^2 + 3c_2 w + c_3^3 - 3c_1 c_2 + 3c_3 \). If we take the derivative of (II.1.5) with respect to \( w \) and then integrate with respect to \( z \), we obtain

\[
\frac{f(z)}{(1-wf(z))} = \sum_{n \geq 1} F'_n(w) \frac{z^n}{n}, \tag{II.1.6}
\]

\( F'_1(w) = 1, F'_2(w) = w + c_1, F'_3(w) = (w + c_1)^2 - (c_1^2 - c_2)(w + c_1), F'_4(w) = (w + c_1)^3 - 2(c_1^2 - c_2)(w + c_1) - (2c_1 c_3 - c_4 + c_2 - 3c_2 c_2 + c_1^4) \). Moreover

\[
F_n(h(z)) = z^n + \sum_{k=1}^{\infty} \beta_{nk} z^{-k}, \tag{II.1.7}
\]

where the \( \beta_{nk} \) are the Grunsky coefficients of \( h \). Thus

\[
F_n \left( \frac{1}{f(z)} \right) = z^{-n} + \sum_{k=1}^{\infty} \beta_{nk} z^k. \tag{II.1.8}
\]
We rewrite (II.1.8) as $z^{-n} = F_p \left( \frac{1}{f(z)} \right) - \sum_{k=1}^{\infty} \beta_{nk} z^k$. On the other hand, if $p > 1$,

$$z^{-p} f'(z) = z^{-1-p}(1 + 2c_1 z + 3c_2 z^2 + \cdots + (n+1)c_n z^n + \cdots)$$

$$= \frac{1}{z^{p-1}} + \frac{2c_1}{z^{p-2}} + \frac{3c_2}{z^{p-3}} + \cdots + \frac{(p-1)c_{p-2}}{z} + p c_{p-1}$$

$$+ (p+1)c_p z + \sum_{k \geq 1} (p+k+1)c_{p+k} z^{k+1}. \quad (II.1.9)$$

We replace the negative powers of $z$ by their expressions given by (II.1.8). We obtain

$$z^{-p} f'(z) = F_{p-1} \left( \frac{1}{f(z)} \right) + 2c_1 F_{p-2} \left( \frac{1}{f(z)} \right) + 3c_2 F_{p-3} \left( \frac{1}{f(z)} \right) + \cdots$$

$$+ (p-1)c_{p-2} F_1 \left( \frac{1}{f(z)} \right) + p c_{p-1} + (p+1)c_p z$$

$$- \left[ \beta_{p-1,1} + 2c_1 \beta_{p-2,1} + \cdots + (p-1)c_{p-2} \beta_{1,1} \right] z$$

$$+ \sum_{k \geq 1} \left[ (p+k+1)c_{p+k} - \beta_{p-1,k+1} + 2c_1 \beta_{p-2,k+1} + \cdots \right] z^{k+1}.$$ \quad (II.1.10)

For $p \geq 2$, we consider

$$T_{p-1}(w) = F_{p-1}(w) + 2c_1 F_{p-2}(w) + 3c_2 F_{p-3}(w) + \cdots$$

$$+ (p-1)c_{p-2} F_1(w) + p c_{p-1}. \quad (II.1.11)$$

From the expansion of $f'(z)$ and (II.1.5), we obtain the generating function

$$\frac{zf'(z)^2}{f(z) - w f'(z)^2} = 1 + \sum_{n \geq 1} T_n(w) z^n, \quad (II.1.12)$$

$T_0(w) = 1$, $T_1(w) = w + 3c_1$, $T_2(w) = w^2 + 4c_1 w + (c_1^2 + 5c_2)$, $T_3(w) = w^3 + 5c_1 w^2 + (4c_1^2 + 6c_2) w + P_3 + \cdots$. With (1.12), we obtain

$$T_p(w) = \sum_{j=0}^{p} p_{p-j} w^j.$$

We write (II.1.10) as

$$z^{-p} f'(z) - T_{p-1} \left( \frac{1}{f(z)} \right) = \sum_{k \geq 1} B_{k-1} z^k$$

$$= (p+1)c_p z - \left[ \beta_{p-1,1} + 2c_1 \beta_{p-2,1} + \cdots \right] z + \sum_{k \geq 1} B_k z^{k+1}.$$ \quad (II.1.13)

with

$$B_k = (p+k+1)c_{p+k}$$

$$- \left[ \beta_{p-1,k+1} + 2c_1 \beta_{p-2,k+1} + \cdots + (p-1)c_{p-2} \beta_{1,k+1} \right]. \quad (II.1.14)$$
This gives, for $p \geq 1$,
\[
z^{1-p} f'(z) - T_{p-1} \left( \frac{1}{f(z)} \right) - P_p^p f(z) = \sum_{k \geq 1} A_p^p z^{k+1}, \tag{II.1.15}
\]
where $P_1^1 = 2c_1$ and if $p > 1$,
\[
P_p^p = -[\beta_{p-1,1} + 2c_1\beta_{p-2,1} + \cdots + (p-1)c_{p-2}\beta_{1,1}] + (p+1)c_p \tag{II.1.16}_1
\]
and for $p \geq 1$, $k \geq 1$, we put $A_p^k = B_p^k - P_p^p c_k$. With the convention $c_0 = 1$, $B_0^0 = P_p^p$, we have $A_0^p = 0$. In [1], we obtained the generating function of the $(P_p^p)$,
\[
\frac{z^2 f'(z)^2}{f(z)^2} = 1 + \sum_{p \neq 1} P_p^p c_p. \tag{II.1.16}_2
\]
In fact, since $\frac{1}{f(z)} = \frac{1}{z} + \sum_{n=0}^{\infty} b_{n+1} z^n$ and $b_{n+1} = \beta_{1n} = \frac{1}{n} \beta_{11}$, by taking the derivative with respect to $z$, we obtain
\[
f'(z)f(z) = \frac{1}{z^2} - \sum_{n=1}^{\infty} \beta_{1n} z^{n-1}
\]
thus $\frac{z^2 f'(z)^2}{f(z)^2} = f'(z) - (\sum_{n=1}^{\infty} \beta_{1n} z^{n-1}) f'(z)$ and (II.1.16) implies (II.1.16)_1. From (II.1.11)–(II.1.15)–(II.1.16), we obtain the generating function of $M_p(w) = - T_{p-1}(w) - P_p^p \frac{1}{w}$. We put $M_0(w) = - \frac{1}{w}$, then
\[
\sum_{p \geq 0} M_p(w) \xi^p = - \sum_{p \geq 0} w M_p(w) \xi^p. \tag{II.1.17}_1
\]
and from (II.1.14), we see that
\[
z^{1-p} f'(z) + M_p \left( \frac{1}{f(z)} \right) = \sum_{n=1}^{\infty} A_n^p z^{n+1}.
\]
We obtain (II.1.1)–(II.1.10) with $N_p(w) = M_p \left( \frac{1}{w} \right)$.

Since $\phi_p(z)$ is the coefficient of $\xi^p$ in the asymptotic expansion of the function $\frac{z^2 f'(z)^2}{f(z)^2}$, we obtain (II.1.2). We deduce (1.13) from (II.1.1) and (1.10). This ends the proof of the theorem. If we divide (II.1.12) by (II.1.17)_2, we obtain
\[
\frac{1}{f(\xi)} \sum_{p \geq 0} T_p(w) \xi^p = - \sum_{p \geq 0} w M_p(w) \xi^p. \tag{II.1.17}_2
\]
This yields a relation between the polynomials $M_p$ and $T_p$. Combining with (II.1.17)_1, it gives
\[
T_p(w) = w (T_{p-1}(w) + c_1 T_{p-2}(w) + \cdots + c_{p-1} T_0(w))
+ P_p^p + c_1 P_{p-1}^p + \cdots + c_p. \quad \tag{II.1.18}
\]
In Section 2, we give a direct proof of (1.13).
II.2. The generating function for the \( (A_p^n) \)

The polynomials \( A_p^n \) defined by (II.1.1) can be calculated with (1.13).

**Proof.** Taking the derivative of (1.14) with respect to \( u \) yields

\[
\psi(u, \frac{1}{f(v)}) - \frac{v}{v-u} = \frac{uf'(u)}{f(u)} \frac{f(v)}{f(v) - f(u)} - \frac{v}{v-u} = \sum_{n \geq 1} \sum_{k \geq 1} \beta_{nk} u^n v^k. \tag{II.2.1}
\]

We multiply (II.2.1) by \( f'(u) \),

\[
\psi(u, \frac{1}{f(v)}) - \frac{v}{v-u} = \frac{uf'(u)}{f(u)} \frac{f(v)}{f(v) - f(u)} - \frac{v}{v-u} = \sum_{n \geq 1} \sum_{k \geq 1} \left[ \beta_{nk} + 2c_1 \beta_{n-1,k} + \cdots + nc_{n-1} \beta_{1,k} \right] u^n v^k. \tag{II.2.2}
\]

The \( (B_p^k) \) are given by (II.1.14) for \( p > 1 \) and \( B_1^1 = (k + 2)c_{k+1} \). We rewrite (II.2.2) as

\[
\sum_{k \geq 0} \sum_{p \geq 2} (B_p^k - (p + k + 1)c_{p+1}) u^{p-1} v^{k+1} = -f'(u) \frac{f'(u)}{f(u) (f(v) - f(u))} + \frac{vf'(u)}{v-u}. \tag{II.2.3}
\]

Since \( B_1^1 = (k + 2)c_{k+1} \), we can write the sum in (II.2.3), starting from \( p = 1 \). On the other hand,

\[
\sum_{k \geq 0} \sum_{p \geq 1} (p + k + 1)c_{p+k} u^{p-1} v^{k+1} = -\frac{vf'(u)}{v-u} + \frac{vf'(u)}{v-u}.
\]

Thus

\[
\gamma(u, v) = \sum_{k \geq 0} \sum_{p \geq 1} B_p^k u^p v^{k+1} = -\frac{u^2 f'(u)^2}{f(u)} \frac{f(v)}{f(v) - f(u)} + \frac{uv f'(v)}{v-u}. \tag{II.2.4}
\]

Since

\[
A_p^0 = B_p^0 - P_p^0 c_k \tag{II.2.5}
\]

and \( A_0^p = 0 \), we have

\[
\sum_{k \geq 1} \sum_{p \geq 1} A_p^0 u^p v^{k+1} = \frac{u^2 f'(u)^2}{f(u)^2} \frac{f(v)^2}{f(u) - f(v)} + f(v) + \frac{uv f'(v)}{v-u}. \]

We divide by \( v \). Since

\[
\sum_{k \geq 1} \sum_{p \geq 0} A_p^0 u^p v^k = \sum_{k \geq 1} A_k^0 v^k + \sum_{k \geq 1} A_k^0 v^k
\]
and
\[ \sum_{k \geq 1} A_k^0 v^k = \sum_{k \geq 0} k c_k v^k = f'(v) - \frac{f(v)}{v}, \]
we obtain (1.13).

II.3. Identities between polynomials

From (1.12), we obtain that for any \( p \neq 0 \) and \( k \neq 0 \),
\[ p_n^{p+k} = \sum_{j=0}^{n} \frac{(j+p)(n-j+k)}{k p} K_j K_{n-j}^p, \quad (II.3.1) \]
In the following, we prove more identities between the polynomials.

**Proof of (1.15).** With (II.1.6),
\[ F_k(w) = F_k(0) + \sum_{n=1}^{k} k n K_{k-n} w^n. \quad (II.3.2) \]
We replace \( w \) by \( \frac{1}{f(z)} \), with (II.1.7), we obtain (1.15). The factorization can also be obtained from (II.2.1). We have
\[ K_1^1 = c_2, \quad K_3^{-1} = 2c_1 c_2 - c_3 - c_1^3, \quad K_4^{-2} = 2c_1, \]
\[ K_5^{-3} = 12c_2c_3 + 12c_1c_4 - 3c_5 - 30c_1c_2^2 - 30c_1^2c_3 + 60c_1^3c_2 - 21c_1^5 \]
and with (1.15), it gives \( \beta_{3,2} = 3K_2 K_3^{-1} + \frac{3}{2} K_4^{-2} K_5^{-3} \). In the same way,
\[ K_1^1 = c_1, \quad K_4^{-1} = 2c_1 c_3 - c_4 + c_2^3 - 3c_1^2c_2 + c_1^4, \]
\[ K_5^{-2} = 6c_2c_3 + 6c_1c_4 - 2c_5 - 12c_1c_2^2 - 12c_1^2c_3 + 20c_1^3c_2 - 6c_1^5 \]
and \( \beta_{2,3} = 2K_1 K_4^{-1} + K_5^{-2} \).

**Proof of (1.17).** We deduce (1.17) from (II.2.5)–(II.1.14) and (A.4.8) in [1]. It can also be proved as follows.

Let \( S_f(z) = (f''/f')' - \frac{4}{z} (f''/f')^2 \) and \( \phi(u) = 1/f(u) \), then
\[ \frac{f'(u) f'(v)}{(f(u) - f(v))^2} = \frac{\phi'(u) \phi'(v)}{(\phi(u) - \phi(v))^2} \quad \text{and} \quad S\phi(u) = S_f(u). \]
Thus (see [3], p. 64),
\[ \frac{f'(u) f'(v)}{(f(u) - f(v))^2} - \frac{1}{(u-v)^2} = \frac{\partial^2}{\partial u \partial v} \log \frac{1}{f(u) - f(v)} \cdot \frac{1}{u-v} \]
and
\[ \frac{\partial^2}{\partial u \partial v} \log \frac{f(u)}{f(v)} = -\frac{1}{6} S_f(u) = - \sum_{n \geq 1, k \geq 1} k \beta_{nk} u^{n+k-2}. \]  

(II.3.3)

**Theorem.** Let \( z^2 S_f(z) = 6 \sum_{n \geq 0} \mathcal{P}_n z^n \) be the expansion of the Schwarzian derivative. We have \( P_0 = 0, P_1 = 0, \) and the \( (P_n)_{n \geq 2} \) are given by (1.17) in terms of the coefficients in (1.14). \( \mathcal{P}_2 = \beta_{11} = c_1^2 - c_2, \mathcal{P}_3 = \beta_{21} + 2 \beta_{12} = 8c_3c_2 - 4c_3 - 4c_1^3, \mathcal{P}_4 = \beta_{31} + 2 \beta_{22} + 3 \beta_{13} = 12c_4 - 34c_3c_2 + 12c_2^2 + 20c_1c_3 - 10c_4, \ldots \)

Let \( b_{nk} = k \beta_{nk}, \) then \( (b_{nk}) \) is a symmetric matrix, moreover \( \mathcal{P}_n = \text{trace}(A_n B_n) \) where \( A_n \) is the square matrix of order \( n - 1 \) such that \( a_{n-1, 1} = a_{n-2, 2} = a_{n-j, j} = 1 \) and all the other coefficients are zero and \( B_n = (b_{ij})_{1 \leq i \leq n-1, 1 \leq j \leq n-1}. \)

**II.4. The operators \((\partial_k)_{k \geq 1}\) and \((L_{-k})_{k \geq 0}\) in terms of \((L_k)_{k \geq 0}\)**

We identify the set of univalent functions with the submanifold \( \mathcal{M} \) in \( C^n \) via the map \( z(1 + \sum_{n \geq 1} c_n z^n) \rightarrow (c_1, c_2, \ldots, c_n, \ldots). \) For \( k \geq 1, \) \( L_k \) is given by (1.8) and \((\lambda_n^k)\) by (II.1.1) or (1.13). We put
\[ L_0 = \sum_{n \geq 1} n c_n \partial_n \quad \text{and for} \quad k \geq 1 \quad L_{-k} = \sum_{n \geq 1} A_n^k \partial_n. \]  

(II.4.1)

In the following, we relate the \((\partial_k)_{k \geq 1}\) and \((L_{-k})_{k \geq 0}\) to the \((L_k)_{k \geq 0}\) and study the action of these operators on some of the generating functions.

**Proposition.** For \( k \geq 1, \)
\[ \partial_k L_k - 2c_1 L_k + (4c_1^2 - 3c_2)L_k + 2 + B_n L_k + \cdots \]  

(II.4.2)

where the \((B_n)_{n \geq 1}\) are independent of \( k. \)

\[ \frac{1}{f'(z)} = 1 + \sum_{n \geq 1} B_n z^n. \]  

(II.4.3)

**Proof.** We verify (II.4.3) on \( f(z) \). We have \( L_k \{ f(z) \} = z^{k+1} f'(z). \) Since
\[ \partial_k \{ f(z) \} = z^{k+1} \quad \text{and} \quad \partial_k \{ f'(z) \} = (k + 1)z^k \]  

(II.4.4)

we have to prove \( z^{k+1} = z^{k+1} f'(z) - 2c_1 z^{k+2} f'(z) + \cdots + B_n z^{k+n} f'(z). \) We divide by \( z^{k+1} \) and we obtain (II.4.3).

Consider the generating functions (see (II.1.12) and (II.1.5)),
\[ \psi(z, w) = \frac{zf'(z)}{f(z) - w f(z)} = 1 + \sum_{n \geq 1} F_n(w) z^n \]

and
\[ \frac{zf'(z)^2}{f(z) - w f(z)^2} = 1 + \sum_{n \geq 1} T_n(w) z^n. \]
The polynomials \( (T_p)_p \geq 0 \) are expressed in (II.1.11) in terms of the \((F_p)_p \geq 0\). Conversely, with (II.4.3), we have

\[
\begin{align*}
F_0 &= T_0, & F_1 &= T_1 - 2c_1T_0, & F_2 &= T_2 - 2c_1T_1 + (4c_1^2 - 3c_2)T_0, & \ldots, \\
F_p &= T_p - 2c_1T_{p-1} + (4c_1^2 - 3c_2)T_{p-2} + \cdots + B_kT_{p-k} + \cdots + B_pT_0.
\end{align*}
\] (II.4.5)

From the generating functions, we obtain the action of \((L_k)\) on the polynomials. Let \(k \geq 1\), we have \(L_k(T_p) = (k + p + 1)T_{p-k}\) if \(p \geq k\) and \(L_k(T_p) = 0\) if \(p < k\). From

\[
L_k \left( \frac{1}{f'(z)} \right) = \frac{z^{k+1}}{f'(z)} - (k+1)z^k \frac{1}{f'(z)},
\]

we obtain \(L_k(B_p) = (p + k)B_{p-k}\) if \(p \geq k\) and \(L_k(B_p) = 0\) if \(k < p\). From (II.1.5), we deduce that \(\psi(z, w)\) satisfies

\[
L_k[\psi(z, w)] = z^{k+1} (\frac{\partial \psi}{\partial z}) (z, \frac{1}{f(v)}) + k z^k \psi(z, \frac{1}{f(v)}),
\] (II.4.6)

and equals \(n \beta_{n-k}\) if \(n \geq k\) and \(L_k(F_n) = 0\) if \(n < k\). We obtain \(L_k(\beta_{n,p})\) with (II.4.1) and (II.1.5). From

\[
L_k \left[ \psi \left( z, \frac{1}{f(v)} \right) \right] = \frac{z^{k+1}}{f'(v)} (\frac{\partial \psi}{\partial z}) (z, \frac{1}{f(v)}) + \frac{z^k}{f'(v)} \psi(z, \frac{1}{f(v)}),
\] (II.4.7)

we deduce \(L_k(\beta_{n,p}) = n \beta_{n-k,p} + (p - k) \beta_{n-p,k}\) if \(k \leq p - 1\), \(k \leq n - 1\) and the formula extends for any \(n \geq 1\), \(p \geq 1\) with the convention \(\beta_{m0} = 0\) if \(m \leq 0\) or \(n \leq 0\).

**Lemma.** Let (see (1.10))

\[
\phi_f(u, w) = \frac{u^2 f'(u)^2}{f(u)^2} \times \frac{w^2}{f(u) - w} = \sum_{p \geq 0} N_p(w)u^p,
\] (II.4.8)

then for \(k \geq 1\),

\[
L_k[\phi_f(u, w)] = 2ku^k \phi_f(u, w) + u^{k+1} \frac{\partial \phi_f}{\partial u}(u, w),
\] (II.4.9)

\[
\partial_k[\phi_f(u, w)] = \left[ 2u \frac{\partial}{\partial u} \left( \frac{u^k}{f'(u)} \right) \right] \phi_f(u, w) + \frac{u^{k+1}}{f'(u)} \frac{\partial \phi_f}{\partial u}(u, w).
\] (II.4.10)

**Proof.** The proof of (II.4.9) is straightforward, then (II.4.9) and (II.4.2) imply that

\[
\partial_k[\phi_f(u, w)] = \left[ 2ku^k \frac{1}{f'(u)} + 2u^{k+1} \frac{1}{f'(u)} \right] \phi_f(u, w) + \frac{u^{k+1}}{f'(u)} \frac{\partial \phi_f}{\partial u}(u, w).
\]

**Corollary.** \(L_k(N_p) = (k + p)N_{p-k}\) if \(p \geq k\) and \(L_k(N_p) = 0\) if \(p < k\). Let \(B_n\) as in (II.2.3), then if \(p \geq k\),

\[
\partial_k N_p = \sum_{j=0}^{p-k} (2p - j) B_{p-k-j} N_j(w).
\] (II.4.11)
Remark. Let (see (1.13))
\[
\sum_{k \geq 1} \sum_{p \geq 0} A^p_k u^p v^{k+1} = \phi_f(u, f(v)) + \frac{v^2 f'(v)}{v - u} = \theta_f(u, v),
\]
then
\[
\theta_f(u, v)_{|v=u} = \frac{3}{2} u^2 f''(u) - 2u^2 \frac{f'(u)^2}{f(u)} + 2uf'(u), \quad \text{(II.4.12)}
\]
\[
\left. \frac{\partial \theta_f}{\partial v} \right|_{v=u} = \left[ \frac{1}{6} u^2 S_f(u) - \frac{u^2 f'(u)^2}{f(u)^2} \right] f'(u) + \frac{1}{2} \left( u^2 f'(u) \right)'' \quad \text{and} \quad \left. \frac{\partial \theta_f}{\partial u} \right|_{v=u} = \left[ -\frac{1}{6} v^2 S_f(v) + \frac{3v^2 f'(v)^2}{f(v)^2} \right] f'(v) + \left( v^2 f'(v) \right)''
\]

In the same manner, consider (see (II.4.4))
\[
\gamma_f(u, v) = \frac{u^2 f'(u)^2}{f(u)} \times \frac{f(v)}{f(u) - f(v)} + \frac{uvf'(v)}{v - u} = \sum_{k \geq 0} \sum_{p \geq 1} B^p_k u^p v^{k+1}
\]
then
\[
\gamma_f(u, v)_{|v=u} = \frac{3}{2} u^2 f''(u) - u^2 \frac{f'(u)^2}{f(u)} + uf'(u), \quad \text{(II.4.14)}
\]
\[
\left. \frac{\partial \gamma_f}{\partial v} \right|_{v=u} = \frac{1}{6} u^2 S_f(u) f'(u) + \frac{u}{2} \left( u f'(u) \right)'' \quad \text{and} \quad \left. \frac{\partial \gamma_f}{\partial u} \right|_{v=u} = \left[ -\frac{1}{6} v^2 S_f(v) + \frac{3v^2 f'(v)^2}{f(v)^2} \right] f'(v) + \left( v^2 f'(v) \right)''
\]

Lemma. If \( p \geq 1 \), for any \( k \) such that \( k > p \), \( \partial_k A^p_k \) is independent of \( k \) and we have
\[
\partial_k A^p_k = -\frac{1}{2i\pi} \int \frac{f''(z)}{f'(z)^p} \frac{dz}{z^{p+k}}. \quad \text{(II.4.16)}
\]
If \( k \leq p \),
\[
\partial_k A^p_k = -\frac{1}{2i\pi} \int \frac{f''(z)}{f'(z)^p} \frac{dz}{z^{p+k}} + \frac{1}{2i\pi} \int (\partial_k N_p)(f(z)) \frac{dz}{z^{k+2}}. \quad \text{(II.4.17)}
\]

Proof. From (II.1.1), we have
\[
(1 + k)z^{1-p+k} + N'_p(f(z))z^{k+1} + (\partial_k N_p)(f(z)) = \sum_{n=1}^{\infty} (\partial_k A^p_n)z^{n+1}.
\]
With (II.1.3), it gives (II.4.17). Since \( \partial_k N_p = 0 \) if \( k > p \), we obtain (II.4.16).
With (II.4.2), we obtain \( L_{-k}, k > 0 \), in terms of the \((L_j)_{j \geq 1}\).

**Proposition.** Let \( L_{-k} = \sum_{j \geq 1} D^k_j L_j \), then \( g_k(z) = \sum_{j \geq 1} D^k_j z^{j+1} \) satisfies

\[
L_{-k} f(z) = g_k(z) + \sum_{j \geq 1} D^k_j z^{j+1} \left( f(z) ight) ^j,
\]

\[
L_{-k} g_k(z) = g_k(z) + \sum_{j \geq 1} D^k_j z^{j+1} \left( g_k(z) S(f)(z) - f(z) g_k(z) S(f)(z) \right) ^j.
\]

**Part III**

**III.1. The vector fields \( R_p \)**

In [1], we related the asymptotic expansion of \( z^2 \left( \frac{f'(z)}{f(z)} \right)^2 \left( \frac{L(z)}{z} \right)^k \) in sum of powers of \( z \) to that of \( z \left( \frac{f'(z)}{f(z)} \right)^2 \left( \frac{L(z)}{z} \right)^k \) in sum of powers of \( f(z) \). In the same way, for \( g(z) \) with expansion (1.1), we have

\[
z g'(z) g(z) \left( \frac{g(z)}{z} \right) ^k = 1 + \sum_{j \geq 1} V_k^j \frac{1}{g(z)^j} \quad \text{and} \quad \left( z g'(z) g(z) \right) ^k = 1 + \sum_{j \geq 1} V_{k-j}^j \frac{1}{z^j}.
\]

(III.1.1)

The coefficients \( V_k^j \) are obtained with contour integrals. The first equation in (III.1.1) gives

\[
z^{1+k} g'(z) g(z) \left( \frac{g(z)}{z} \right) ^k = g(z)^{1+k} + \sum_{1 \leq j \leq k-1} V_j^{k-1} g(z)^{1+k-j} + \sum_{j \geq k} V_j^{k-j} g(z)^{1+k-j}.
\]

(III.1.2)

Let \( g(z) \) be a univalent function outside the unit disc

\[
g(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n + \cdots
\]

(III.1.3)

We assume \( b_0 \neq 0 \). To \( g \), it corresponds the sequence \((b_0, b_1, \ldots , b_n, \ldots) \in C^N \).

\[
\frac{\partial}{\partial b_0} [g(z)] = z, \quad \frac{\partial}{\partial b_1} [g(z)] = 1, \quad \ldots \quad \frac{\partial}{\partial b_n} [g(z)] = \frac{1}{z^{n-1}}, \quad \ldots
\]

(III.1.4)

Thus, if \( k \geq 0 \), we have \( z^{1-k} g'(z) = R_{-k}[g(z)] \) where

\[
R_{-k} = b_0 \frac{\partial}{\partial b_k} - b_2 \frac{\partial}{\partial b_{k+2}} - \cdots - (n-1) b_n \frac{\partial}{\partial b_{n+k}}.
\]

(III.1.5)

Consider the expansion

\[
\chi(z) = \sum_{j \in Z} g(z)^j \frac{1}{2i\pi} \int \frac{u g'(u)}{g(u)} g(u)^{-j} \chi(u) \frac{du}{u}
\]
and then replace \( \chi(z) \) by \( z \chi(z) g(z) \), compare with Part II,

\[
z \chi(z) g'(z) = \sum_{j \geq 0} g(z)^{1-j} \frac{1}{2\pi i} \int \frac{u^2 g'(u)^2}{g(u)^2} \frac{du}{u} - \int \frac{u^2 g'(u)^2}{g(u)^2} \frac{du}{u} = S_1 + S_2. \tag{III.1.6}
\]

The first sum,

\[
S_1 = R_\chi(g(z)) = \sum_{j \geq 0} g(z)^{1-j} \frac{1}{2\pi i} \int \frac{u^2 g'(u)^2}{g(u)^2} \frac{du}{u}
\]

can be expanded in powers of \( z^k \), \( k \leq 1 \). Assume that \( \chi(z) = z^p \), where \( p \in \mathbb{Z} \), in that case, there exists a unique holomorphic vector field \( R_p \) on \( \mathcal{M} \subset \mathbb{C}^N \), satisfying

\[
R_p [g(z)] = R_\chi(g(z))
\]

If \( \chi(z) = z^{-k} \), with \( k \geq 0 \), then \( S_2 \) vanishes and \( R_{-k}[g(z)] = z^{1-k} g'(z) \).

If \( \chi(z) = z^k \) with \( k \geq 1 \), \( S_2 = R_k(g(z)) = \sum_{j=1}^k a_j g(z)^{1+j} \) and \( J_k \) is a polynomial.

\[
R_k(g(z)) = z^{1+k} g'(z) - J_k(g(z)) = E_0^k z + E_1^k + \sum_{n \geq 1} \frac{E_{n+1}^k}{z^n}.
\tag{III.1.7}
\]

Thus \( R_k(g(z)) = R_k[g(z)] \) with the partial differential operator \( R_k = \sum_{n \geq 0} E_n^k \frac{\partial}{\partial n} \).

**Proposition.** The polynomials \( J_k \) are given by

\[
\sum_{k \geq 1} \sum_{n \geq 0} E_n^k z^{-n} u^{-k} = \frac{z g'(z)}{z - u} - \frac{u^2 g'(u)^2}{g(u)^2} \times \frac{g(z)^2}{z(g(u) - g(z))}. \tag{III.1.9}
\]

Compare (III.1.9) with (1.13). With (III.1.8), we obtain

\[
J_1(u) = \frac{w^2}{b_0},
\]

\[
J_2(u) = \frac{w^3}{b_0^2} - \frac{3b_1}{b_0^2} w^2,
\]

\[
J_3(u) = \frac{w^4}{b_0^3} - \frac{4b_1}{b_0^3} w^3 + \frac{(6b_1^2 - 5b_0 b_2)}{b_0^3} w^2, \ldots
\]
Proposition. Let \((Z_k)_{k \geq 1}\) the operators defined by (1.3). For \(k \geq 1\),
\[
R_{1-k} - Z_k = b_0 \frac{\partial}{\partial b_{k-1}} + b_1 \frac{\partial}{\partial b_k} + \cdots + b_n \frac{\partial}{\partial b_{n+k-1}}. \tag{III.1.10}
\]

For any \(k \in \mathbb{Z}\), let
\[
Z_k(g(z)) = \frac{g(z)^2}{2i\pi} \int \frac{u g'(u)}{g(u)} \frac{u^k}{(g(z) - g(u))} \, du. \tag{III.1.11}
\]

There exists a unique holomorphic vector field \(\tilde{Z}_k = \sum_{j \geq 0} W_j \frac{\partial}{\partial b_j}\) on \(M\) such that \(\tilde{Z}_k \tilde{g}(z) = \tilde{Z}_k g(z)\) is given by (II.1.11). For \(k \geq 0\), we have \(Z_{1+k} = R_{-k} - \tilde{Z}_{-k}\) where \(\tilde{Z}_{-k}\) satisfies \(\tilde{Z}_{-k} g(z) = z^{-k} g(z)\).

If \(k \geq 1\), then
\[
\tilde{Z}_k(g(z)) = z^k g(z) - v_k(g(z)) \tag{III.1.11}_1
\]

\(v_k\) is a polynomial function,
\[
\sum_{k \geq 1} v_k(u) \frac{1}{z^k} = u^2 \frac{g'(\xi)}{g(\xi)} \times \frac{1}{g(\xi) - u}. \tag{III.1.12}
\]

Proof. Let \(h(z) = \frac{g(z)}{z} = b_0 + b_1 \frac{1}{z} + \cdots + b_n \frac{1}{z^n} + \cdots\), we assume that \(b_0 \neq 0\). Then
\[
\frac{\partial}{\partial b_0} h(z) = 1, \quad \frac{\partial}{\partial b_1} h(z) = \frac{1}{z}, \quad \cdots \quad \frac{\partial}{\partial b_n} h(z) = \frac{1}{z^n}.
\]

For \(k \geq 1\), \(z^{2-k} h'(z) = Z_k[h(z)] = z^{1-k} Z_k g(z)\), where \(Z_k\) is given by (1.3). On the other hand, we have \(R_{-k}[g(z)] = z^{1-k} g(z)\) and \(h'(z) = \frac{g'(z)}{z} - \frac{g(z)}{z^2}\), we deduce \([R_{1-k} - Z_k][g(z)] = z^{1-k} g(z)\), this proves (III.1.10).

For (III.1.11), let
\[
z^{2+k} h'(z) = \sum_{j \geq 0} g(z)^{1+j} \frac{1}{2i\pi} \int \frac{u g'(u)}{g(u)} g(u)^{j-1} u^{2+k} h'(u) \, du.
\]

Since \(h'(u) = \frac{g'(u)}{u} - \frac{g(u)}{u^2}\), we obtain \(z^{2+k} h'(z) = R_k[g(z)] + J_k(g(z)) - S\) with
\[
S = \sum_{j \geq 0} g(z)^{1+j} \frac{1}{2i\pi} \int \frac{u g'(u)}{g(u)} g(u)^j u^k \frac{du}{u}
\]
\[
+ \sum_{j \geq 1} g(z)^{1+j} \frac{1}{2i\pi} \int \frac{u g'(u)}{g(u)} g(u)^j u^{k-1} \, du.
\]

Let \(S = S_1 + S_2\). For any \(k \in \mathbb{Z}\), \(S_1 = \tilde{Z}_k(g(z)) = \sum_{j \geq 0} W_j z^{1+j}\) is given by (III.1.11).

We put \(\tilde{Z}_k = \sum_{j \geq 0} W_j \frac{\partial}{\partial b_j}\).

If \(k \leq 0\), \(S_2\) vanishes and the integrand (III.1.11) has a pole only at \(u = z\), we have \(\tilde{Z}_k(g(z)) = z^k g(z)\). Thus, for \(k \geq 1\), we have \(\tilde{Z}_{1-k} = R_{1-k} - Z_k\) and \(\tilde{Z}_{1-k}\) is given by
(III.1.10). If \( k \geq 1 \), the integrand (III.1.11) has a pole at \( u = z \) and a pole at \( u = 0 \), the residue at \( u = 0 \) is given by

\[
S_2 = \frac{g(z)^2}{2i\pi} \int \frac{u g'(u)}{g(u)} \frac{u^k}{g(u)^k} \times \frac{g(u)^k - g(z)^k}{g(u) - g(z)} \frac{du}{u} = \sum_{j=1}^{k} \alpha_j g(z)^{1+j} = v_k(g(z))
\]

and the polynomials \( v_k \) are obtained with (III.1.12).

**Remark.** With (III.1.12) and (III.1.11), it is easy to verify that the \( (W_k^j)_{k \geq 1} \) in \( \tilde{Z}_k \) satisfy

\[
\lambda(\xi, z) = \sum_{j \geq 0} \sum_{k \geq 1} W_k^j z^{1-j} \xi^{-k} = \frac{z g(z)}{\xi - z} - \frac{\xi g'(\xi)}{g(\xi)} \times \frac{g(z)^2}{g(\xi) - g(z)}.
\]

Moreover, let \( S_g(\xi) \) the Schwarzian derivative of \( g \), then

\[
\frac{\partial \lambda}{\partial z} \bigg|_{z=\xi} = -\xi g'(\xi) \left[ \frac{1}{6} S_g(\xi) - \frac{g'(\xi)^2}{g(\xi)^2} \right] - \frac{1}{2} \xi g''(\xi) - g'(\xi).
\]

**III.2. The Faber polynomials \( (G_n)_{n \geq 0} \) of \( g(z) \).** The \( G_n(f(z))_{n \geq 0} \)

Let \( g(z) \) be given by (III.1.3), its derivative \( g'(z) = b_0 - \frac{b_2}{z^2} - \cdots - \frac{b_n}{z^n} - \cdots \) and

\[
\frac{z g'(z)}{g(z) - w} = 1 + \sum_{n \geq 1} G_n(w) \frac{1}{z^n},
\]

then

\[
G_n(g(z)) = z^n + \sum_{k \geq 1} \gamma_{nk} \frac{1}{z^k}. \tag{III.2.1}
\]

Let \( f(z) = z(1 + \sum_{n \geq 1} c_n z^n) \), we have \( G_n(f(z)) = G_n(0) + \sum_{k \geq 1} p_{nk} z^k \). The matrix \( P = (p_{nk}) \) is given by the generating function

\[
\frac{\xi g'(\xi)}{g(\xi) - f(z)} = 1 + \sum_{n \geq 1} G_n(0) \xi^{-n} + \sum_{n \geq 1} \sum_{i \geq 1} p_{in} z^i \xi^{-n}. \tag{III.2.2}
\]

Let \( \overline{P} \) the matrix complex conjugated of \( P \) and \( P' \) the transposed matrix, then the coefficients of the matrix \( P \overline{P}' \) are given by

\[
\frac{1}{2i\pi} \int_{\partial D} \left| \frac{\xi g'(\xi)}{g(\xi) - f(z)} \right|^2 \frac{d\xi}{\xi} = \sum_{i,k} (P \overline{P}')_{ki} z^k.
\]

The proof is straightforward.
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References