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On the number of empty convex quadrilaterals of a finite set in the plane[☆]

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Abstract

Let P be a set of n points in the plane, no three collinear. A convex polygon of P is called empty if no point of P lies in its interior. An empty partition of P is a partition of P into empty convex polygons. Let k be a positive integer and $N_k^\pi(P)$ be the number of empty convex k-gons in an empty partition π of P. Define $g_k(P) =: \max\{N_k^\pi(P) : \pi$ is an empty partition of $P\}$, $G_k(n) =: \min\{g_k(P) : |P| = n\}$. We mainly study the case of $k = 4$ and get the result that $G_4(n) \geq \lfloor \frac{9n}{38} \rfloor$. For specified $n = 21 \times 2^{k-1} - 4$ $(k \geq 1)$, we obtain the better bound $G_4(n) \geq \lfloor \frac{5n-1}{21} \rfloor$.

Keywords: Convex position; Convex hull; Convex partition; Empty convex polygon; Empty partition

1. Introduction

Let P be a set of n points in general position in the plane, that is, with no three points collinear. If P is partitioned into k subsets $S_1, S_2, \ldots, S_k$ such that each S_i $(i = 1, 2, \ldots, k)$ is the vertex set of a convex polygon, then the partition obtained is called a convex partition of P. A subset of a finite set of points in the plane is called an empty convex polygon if it forms the set of vertices of a convex polygon whose interior contains no point of the set. A convex partition of P is called a disjoint partition if $CH(S_i) \cap CH(S_j) = \emptyset$ for any pair of indices i, j, where CH denotes the convex hull, and it is called an empty partition if each $CH(S_i)$ is an empty convex polygon of P. Given a point set P, let $f(P)$ denote the minimum number of disjoint convex polygons over all disjoint partitions of P, and $g(P)$ the minimum number of empty convex polygons over all empty partitions of P. Define $F(n) =: \max\{f(P) : |P| = n\}$ and $G(n) =: \max\{g(P) : |P| = n\}$. In 1996 M. Urabe found some lower bounds and upper bounds for $F(n)$ and $G(n)$ (see [1]). In 2003 R. Ding, K. Hosono, M. Urabe and C. Xu improved the bounds for $G(n)$ (see [2]).

In 2001 K. Hosono and M. Urabe considered the problem of the number of disjoint empty convex k-gons in a planar point set for a fixed k; they mainly discussed the case of $k = 4$ (see [3]). In this work, we remove the restriction

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of disjointness and consider a related problem: how many empty convex k-gons can be constructed in a planar point set for a fixed k? Mainly we study the case of $k = 4$ as well.

Let k be a positive integer, $\Pi_k^\pi(P)$ be the number of disjoint convex k-gons in a disjoint partition π of P, and $N_k^\pi(P)$ be the number of empty convex k-gons in an empty partition π of P. Define

$$f_k(P) =: \max\left\{\prod_k^\pi(P) : \pi \text{ is a disjoint partition of } P\right\}.$$

$$g_k(P) =: \max\{N_k^\pi(P) : \pi \text{ is an empty partition of } P\}.$$

$$F_k(n) =: \min\{f_k(P) : |P| = n\}.$$

$$G_k(n) =: \min\{g_k(P) : |P| = n\}.$$

Since a set of disjoint convex k-gons is a set of empty convex k-gons, we have

Lemma 1. $G_k(n) \geq F_k(n)$.

K. Hosono and M. Urabe proved:

Lemma 2 ([3]). $F_4(9) = 2$ *and* $F_4(n) \geq \lfloor \frac{5n}{22} \rfloor$.

In this work, we obtain the following results:

Lemma 3. $G_4(5) = 1$.

Lemma 4. $G_4(9) = 2$.

Lemma 5. $G_4(13) = 3$.

Lemma 6. $G_4(17) = 4$.

By using these lemmas we show that for a set of 38 points we can construct nine empty convex quadrilaterals and so we obtain:

Theorem 7. $G_4(n) \geq \lfloor \frac{9n}{38} \rfloor$.

Moreover, we get the following better bound for a specified integer n:

Theorem 8. $G_4(n) \geq \lfloor \frac{5n-1}{21} \rfloor$, $n = 21 \times 2^{k-1} - 4$ $(k \geq 1)$.

In the proof, we make use of the following result proved by K. Hosono and M. Urabe:

Lemma 9 ([3]). *For any set of* $2m + 4$ *points in the plane, no three collinear, we can divide the plane into three disjoint convex regions such that one contains a convex quadrilateral and the others contain m points each, where m is a positive integer.*

2. Proofs

First, we need the following definitions and notation. For a given point set P, a convex region R is called empty, denoted by $R \cong \emptyset$, if its interior contains no point of P. We call the interior region of the angular domain in the plane determined by the points a, b and c a convex cone, denoted by $C(a; b, c)$, if a is the apex and both b and c are on the boundary of the angular domain such that $\angle bac$ is acute. If $C(a; b, c)$ contains some points of a given point set P, then we call the point $q \in P \cap C(a; b, c)$ the attack point, denoted by $A(a; b, c)$, if $C(a; b, q) \cong \emptyset$. The subset of P on the boundary of $CH(P)$ is denoted by $V(P) = \{v_1, v_2, \ldots, v_t\}$ with the order anticlockwise. The interior points of P are the points of P that are not on the boundary of $CH(P)$, and the set of interior points of P is denoted by Q. We use the notation $\overline{ab}$ to refer to the line segment between a and b, and ab to refer to the extended straight line associated with two points a and b.

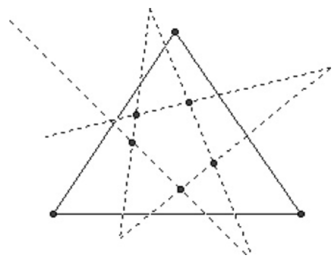

Fig. 1.

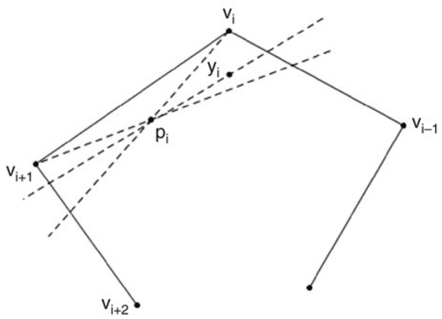

Fig. 2.

Proof of Lemma 3. The result is obvious by the fact that $F_4(5) = 1$ (see [3]). □

Proof of Lemma 4. We can find an eight-point set P with $g_4(P) = 1$ (see Fig. 1); thus $G_4(8) = 1$ holds, and hence $G_4(9) \leq 2$. On the other hand, we have $G_4(9) \geq F_4(9) = 2$ by Lemmas 1 and 2. So $G_4(9) = 2$. □

Proof of Lemma 5. To prove $G_4(13) = 3$ it suffices to prove $g_4(P) \geq 3$ for any finite set P in general position in the plane with $|P| = 13$.

If $|V(P)| \geq 7$, that is, if $CH(P)$ is a k-gon ($k \geq 7$), then there exists an extended straight line l associated with an edge of $CH(Q)$ such that l separates a convex i-gon ($i \geq 4$) from the remaining at most nine points. By Lemma 4, we therefore find three empty convex quadrilaterals, that is, $g_4(P) \geq 3$.

In the following proof, we assume that:

($*$) *There does not exist the extended straight line l associated with an edge of $CH(Q)$ which separates an i-gon ($i \geq 4$) from the remaining points of P.*

If there exists a triangle, say $\triangle v_{i-1}v_iv_{i+1}$, determined by three adjacent points of $V(P)$, which is empty, then the empty quadrilateral $v_{i-1}v_iv_{i+1}v$ is separated from the remaining nine points, where v is the point nearest to $\overline{v_{i-1}v_{i+1}}$ of the remaining points of P. So we may assume that no such triangle is empty. Let $p_i = A(v_i; v_{i+1}, v_{i+2})$, where we identify indices modulo t which is the number of vertices of $CH(P)$. Obviously, the convex cone $C(v_i; v_{i+1}, p_i)$ is empty. If $C(v_{i+1}; v_i, p_i)$ is not empty, let $y_i = A(p_i; v_i, v_{i-1})$; then the line segment $\overline{p_iy_i}$ is an edge of $CH(Q)$ and the extended straight line p_iy_i separates the empty convex quadrilateral $v_iv_{i+1}p_iy_i$ from the remaining points of P, which is contrary to ($*$) (see Fig. 2). So we may assume that $C(v_{i+1}; v_i, p_i) \cong \emptyset$.

Next we show that $p_i \neq p_j$ for any pair of indices $i \neq j$ ($i, j = 1, 2, \ldots, t$). If not, there must exist some i such that $p_i = p_{i+1}$; then $C(v_i; v_{i+1}, p_i) \cup C(v_{i+1}; v_{i+2}, p_i) \cong \emptyset$ holds. Let $z = A(p_i; v_i, v_{i-1})$; then the line segment $\overline{p_iz}$ is an edge of $CH(Q)$ and the extended straight line p_iz separates the empty convex quadrilateral $v_iv_{i+1}p_iz$ from the other points of P, which is contrary to ($*$) (see Fig. 3). Thus we may assume:

($**$) $C(v_i; v_{i+1}, p_i) \cup C(v_{i+1}; v_i, p_i) \cong \emptyset$, *and $p_i \neq p_j$ for any pair of indices $i \neq j$ ($i, j = 1, 2, \ldots, t$).*

Now we discuss the remaining cases.

Case 1: $|V(P)| = 6$ or $|V(P)| = 5$. Consider the interior of the triangular domain T_i determined by $v_{i-1}p_{i-1}$, $v_{i+1}p_i$ and $\overline{p_ip_{i-1}}$ for every i. By ($**$), there exists some $T_i \cong \emptyset$. Denote the quadrilateral corresponding to T_i by $Q_i = p_ip_{i-1}v_{i-1}v_{i+1}$ (see Fig. 4). If $Q_i \cong \emptyset$, then the remaining points of P are contained in $C(v_i; p_i, p_{i-1})$; there

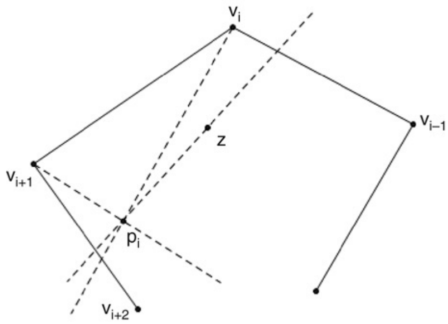

Fig. 3.

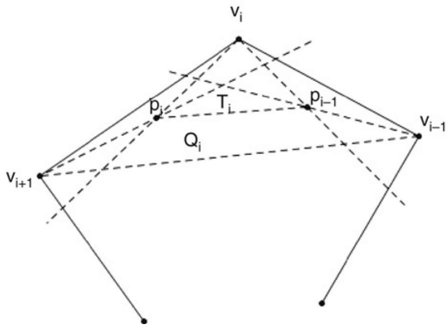

Fig. 4.

exist three empty convex quadrilaterals by Lemma 4, that is, $g_4(P) \geq 3$. If Q_i is not empty, then $v_i p_i v p_{i-1}$ is an empty convex quadrilateral disjoint from the other nine points, where v is the point nearest to $\overline{p_i p_{i-1}}$ in Q_i, so we can find three empty convex quadrilaterals by Lemma 4.

Case 2: $|V(P)| = 4$. There exists some T_i, say T_1, which contains at most one point by $(**)$. If $T_1 \cong \emptyset$, we are done by the same argument as for Case 1. Next we discuss case of T_1 containing one point q.

Let $p = A(v_2; p_1, q)$; then $v_1 p_1 p A(p; v_1, q)$ is an empty convex quadrilateral disjoint from the remaining nine points of P, and we obtain the result $g_4(P) \geq 3$ by Lemma 4. Thus we may assume that $C(v_2; p_1, q) \cong \emptyset$. Let $q_1 = A(v_2; q, p_4)$. Considering the positions of q_1 we have two subcases:

(1) $q_1 \in C(v_1; p_1, q)$. Then the empty convex quadrilateral $v_1 p_1 q_1 q$ is separated from the other nine points; $g_4(P) \geq 3$ holds.
(2) $q_1 \in C(v_1; q, p_4)$.

(a) $C(q; v_2, q_1) \cong \emptyset$. Then $v_2 q_1 q p_1$ is an empty convex quadrilateral disjoint from the other nine points; we get the conclusion $g_4(P) \geq 3$.
(b) $C(q; v_2, q_1)$ is not empty.

(i) $C(p_4; p_1, q_1)$ is not empty. Let $q_2 = A(p_4; p_1, q_1)$. Then the two empty quadrilaterals $v_1 q q_2 p_4$ and $v_2 p_1 q_1 A(q_1; v_2, p_2)$ are separated from the other five points; there exist three empty convex quadrilaterals by Lemma 3, that is, $g_4(P) \geq 3$.
(ii) $C(p_4; p_1, q_1) \cong \emptyset$.

(A) $q_1 \in C(p_1; p_4, v_4)$; let $q_3 = A(q_1; p_4, v_4)$. If $q_3 = v_4$, then $v_1 q q_1 p_4$ is an empty convex quadrilateral disjoint from the other nine points. Otherwise, $v_2 p_1 q A(q_1; v_2, p_2)$ and $v_1 q_1 q_3 p_4$ are two empty quadrilaterals disjoint from the other five points.
(B) q_1 is not in $C(p_1; p_4, v_4)$. Let $q_4 = A(q_1; p_4, v_4)$.

(I) $q_4 \neq v_4$. Then the two empty quadrilaterals $v_2 p_1 q A(q_1; v_2, p_2)$ and $v_1 q_1 q_4 p_4$ are separated from the other five points.

(II) $q_4 = v_4$. Let $q_5 = A(q_1; v_4, p_3)$.

If q_5 is on the opposite side of v_1q_1 to v_4, then the empty convex quadrilateral $q_1q_5v_4p_4$ is disjoint from the other nine points. Next we discuss the case of q_5 on the same side of v_1q_1 as v_4.

(α) q_5 is on the same side of $\overline{v_2v_4}$ as v_1.

- $C(v_4; q_1, q_5) \cong \emptyset$. Then the two empty quadrilaterals $v_2p_1qA(q_1; v_2, p_2)$ and $v_1q_1q_5p_4$ are separated from the other five points.
- $C(v_4; q_1, q_5)$ is not empty. Let $q_6 = A(v_4; q_1, q_5)$.

$\diamond$ $q_6 \in C(v_1; p_1, q)$. Then $v_1p_1q_6q$ and $q_1p_4v_4q_5$ are two empty quadrilaterals disjoint from the other five points.

$\diamond$ $q_6 \in C(v_1; q, q_1)$.

If $C(q; v_2, q_6) \cong \emptyset$, the empty quadrilateral $v_2p_1qq_6$ is separated from the other nine points. If $C(q; v_2, q_6)$ is not empty but $C(q_6; p_1, v_2)$ is empty, then $v_2p_1q_6A(q_6; v_2, p_2)$ and $v_1qq_1p_4$ are two empty quadrilaterals disjoint from the other five points. If both $C(q; v_2, q_6)$ and $C(q_6; p_1, v_2)$ are not empty, let $q_7 = A(q_6; p_1, v_2)$. If $q_7 \in C(v_1; p_1, q)$, then the two empty quadrilaterals $v_1p_1q_7q$ and $v_4p_4q_1q_6$ are separated from the remaining five points. If $q_7 \in C(v_1; q, q_6)$, then the two empty quadrilaterals $v_1qq_7q_6$ and $p_1q_1v_4p_4$ are separated from the remaining five points.

$\diamond$ $q_6 \in C(v_1; q_1, q_5)$. Then $v_2p_1qA(q_1; v_2, p_2)$ and $v_1q_1q_6p_4$ are two empty quadrilaterals disjoint from the other five points.

(β) q_5 is on the opposite side of $\overline{v_2v_4}$ to v_1. Then by an argument similar to that for the previous subcase (α) we obtain $g_4(P) \geq 3$.

So we can assume that both $C(v_2; p_1, p_4)$ and $C(v_4; p_4, p_1)$ just contain the point q by symmetry.

If $C(v_2; p_4, v_4) \cong \emptyset$, then $p_1v_2v_4p_4$ is an empty convex quadrilateral and the remaining points of P are contained in $C(v_1; p_1, p_4)$, that is, the convex hull of the remaining points does not contain any point of $\{p_1, v_2, v_4, p_4\}$, so there are three empty convex quadrilaterals by Lemma 4. Thus we can assume that both $C(v_2; p_4, v_4)$ and $C(v_4; p_1, v_2)$ are not empty. Let $t_1 = A(v_2; p_4, v_4)$, $t_2 = A(v_4; p_1, v_2)$. Consider the following two subcases.

(1) $t_1 = t_2$; let $t_1 = t_2 = t$. Without loss of generality, we assume that t is on the same side of v_1q as p_4.

(a) $C(v_4; t, v_2)$ is not empty. Let $l_1 = A(v_4; t, v_2)$.

(i) $l_1 \in C(v_1; p_1, q)$.

(A) $C(v_4; l_1, v_2)$ is not empty. Let $l_2 = A(v_4; l_1, v_2)$.

(I) $l_2 \in C(v_1; p_1, l_1)$. Then $v_1p_1l_2l_1$ and v_4p_4qt are two empty quadrilaterals disjoint from the other five points.

(II) $l_2 \in C(v_1; l_1, q)$. Then $v_1l_1l_2q$ and $v_4p_4p_1t$ are two empty quadrilaterals disjoint from the other five points.

(III) $l_2 \in C(v_1; q, p_4)$. If l_2 is on the opposite side of p_4t to v_4, then $v_1l_2tp_4$ and $p_1ql_1A(l_1; p_1, v_2)$ are two empty quadrilaterals disjoint from the other five points. If l_2 is on the same side of p_4t as v_4, then $v_1p_1l_1q$ and $v_4p_4tl_2$ are two empty quadrilaterals disjoint from the other five points.

(B) $C(v_4; l_1, v_2) \cong \emptyset$. By a discussion similar to that for the previous subcase (A), we can obtain $g_4(P) \geq 3$.

(ii) $l_1 \in C(v_1; q, t)$.

(A) $C(q; v_2, l_1) \cong \emptyset$. Then the empty convex quadrilateral $v_2p_1ql_1$ is separated from the other nine points; $g_4(P) \geq 3$ holds.

(B) $C(q; v_2, l_1)$ is not empty.

(I) $C(v_2; t, l_1)$ is not empty. Let $l_3 = A(v_2; t, l_1)$.

(α)$l_3 \in C(v_1; p_1, q)$.

If l_1 is on the opposite side of p_4t to v_4, then $v_2l_3qp_1$ and $v_1l_1tp_4$ are two empty convex quadrilaterals disjoint from the other five points. If l_1 is on the same side of p_4t as v_4, then $v_1p_1l_3q$ and $v_4p_4tl_1$ are two empty convex quadrilaterals disjoint from the other five points.

(β) $l_3 \in C(v_1; q, l_1)$.

- $C(q; v_2, l_3) \cong \emptyset$. Then the empty quadrilateral $v_2p_1ql_3$ is separated from the other nine points.
- $C(q; v_2, l_3)$ is not empty. Then $v_2p_1l_3A(l_3; v_2, p_2)$ and v_1qtp_4 are two empty quadrilaterals separated from the other five points.

(II) $C(v_2; t, l_1) \cong \emptyset$. Then $v_2p_1l_1A(l_1; v_2, p_2)$ and v_1qtp_4 are two empty quadrilaterals separated from the other five points.

(iii) $l_1 \in C(v_1; t, p_4)$. Then $v_2 p_1 q A(q; v_2, p_2)$ and $v_1 t l_1 p_4$ are two empty quadrilaterals separated from the other five points.

(b) $C(v_4; t, v_2) \cong \emptyset$. Let $l_4 = A(v_4; v_2, p_2)$; by a discussion similar to that for subcase (a) we reach the result $g_4(P) \geq 3$.

(2) $t_1 \neq t_2$.

(a) t_1 and t_2 are on the same side of $v_1 q$; without loss of generality, let t_1 and t_2 be on the same side of $v_1 q$ as p_1. If $C(q; v_4, t_2) \cong \emptyset$, then $v_4 p_4 q t_2$ is an empty convex quadrilateral separated from the other nine points; otherwise $v_4 p_4 t_2 A(t_2; v_4, p_3)$ and $v_1 p_1 t_1 q$ are two empty quadrilaterals separated from the other five points.

(b) t_1 and t_2 are on the opposite side of $v_1 q$.

(i) $C(p_1; v_2, t_1)$ is not empty. Then $v_1 q t_2 p_4$ and $t_1 p_1 v_2 A(t_1; v_2, p_2)$ are two empty quadrilaterals separated from the other five points.

(ii) Both $C(p_1; v_2, t_1)$ and $C(p_4; v_4, t_2)$ are empty by symmetry.

If $C(t_1; p_4, t_2)$ is not empty, let $s = A(t_1; p_4, t_2)$. If s is on the opposite side of $v_1 q$ to p_1, then $v_2 p_1 p_4 t_1$ and $v_1 q s A(s; p_4, t_2)$ are two empty quadrilaterals separated from the other five points. Otherwise $v_1 p_1 t_1 s$ and $q p_4 v_4 t_2$ are two empty quadrilaterals separated from the other five points. So we may suppose $C(t_1; p_4, t_2) \cong \emptyset$.

By the same discussion we suppose that $C(t_1; t_2, v_4) \cong \emptyset$ and $C(v_4; t_1, v_2) \cong \emptyset$. Then $t_1 t_2 v_4 v_2$ is an empty convex quadrilateral and the convex hull of the remaining points does not contain any point of $\{t_1, t_2, v_2, v_4\}$, so we may find three empty convex quadrilaterals by Lemma 4.

Case 3: $|V(P)| = 3$. There exists some T_i, say T_1, containing at most two points of P by (∗∗). If T_1 contains at most one point, by the reasoning similar to that for Cases 1 and 2 we obtain $g_4(P) \geq 3$. Then we just consider the case of T_1 containing exactly two points. Let $p = A(v_2; p_1, p_3)$, $q = A(v_3; p_3, p_1)$. If p is not in T_1, then $v_1 p_1 p A(p; v_1, p_3)$ is an empty convex quadrilateral separated from the other nine points; thus we can assume that both $p, q \in T_1$ by symmetry.

A finite set of points in the plane is called in convex position if it forms the set of vertices of a convex polygon. If $p = q$ or $\{p, q, p_1, p_3\}$ are not in convex position, the conclusion $g_4(P) \geq 3$ is obvious. So we only need to consider the case of $p \neq q$ and $\{p, q, p_1, p_3\}$ are in convex position. Consider the following two subcases.

(1) $C(p_1; p_3, v_3)$ is not empty. Let $q_1 = A(p_1; p_3, v_3)$. Consider three possible positions of q_1.

(a) $q_1 \in C(v_1; p_1, p)$. Then $v_1 p_1 q_1 p$ and $q p_3 v_3 A(q; v_3, p_2)$ are two empty quadrilaterals separated from the other five points.

(b) $q_1 \in C(v_1; p, q)$.

(i) $C(p_3; p_1, q_1)$ is not empty. Let $q_2 = A(p_3; p_1, q_1)$.

(A) $q_2 \in C(v_1; p_1, p)$. Then $v_1 p_1 q_2 p$ and $p_3 q q_1 A(q_1; p_3, v_3)$ are two empty quadrilaterals separated from the other five points.

(B) $q_2 \in C(v_1; p, q_1)$.

(I) $C(p_3; q_2, q_1)$ is not empty. Let $q_3 = A(p_3; q_2, q_1)$.

(α) $q_3 \in C(v_1; p_1, p)$. Then $v_1 p_1 q_3 p$ and $p_3 q q_2 A(q_2; p_3, q_1)$ are two empty quadrilaterals separated from the other five points.

(β) $q_3 \in C(v_1; p, q_2)$. Then $v_1 p q_3 q_2$ and $p_1 q_1 p_3 q$ are two empty quadrilaterals separated from the other five points.

(γ) $q_3 \in C(v_1; q_2, q_1)$. Then $v_1 q_2 q_3 q$ and $p_1 q_1 p_3 p$ are two empty quadrilaterals separated from the other five points.

(II) $C(p_3; q_2, q_1) \cong \emptyset$. Then $v_1 q_2 q_1 q$ and $v_2 p_1 p A(p; v_2, p_2)$ are two empty quadrilaterals separated from the other five points.

(ii) $C(p_3; p_1, q_1) \cong \emptyset$.

(A) $C(p_3; q_1, v_2)$ is not empty. Let $q_4 = A(p_3; q_1, v_2)$.

(I) $q_4 \in C(v_1; p_1, p)$. Then $v_1 p_1 q_4 p$ and $p_3 q q_1 A(q_1; p_3, v_3)$ are two empty quadrilaterals separated from the other five points.

(II) $q_4 \in C(v_1; p, q_1)$.

(α) $C(p; v_2, q_4) \cong \emptyset$. Then $v_2 p_1 p q_4$ is an empty convex quadrilateral separated from the other nine points.

(β) $C(p; v_2, q_4)$ is not empty.

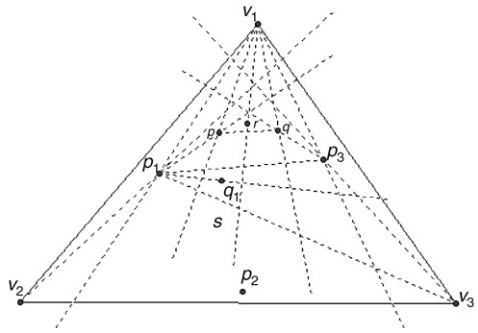

Fig. 5.

- $C(q_4; p_1, v_2) \cong \emptyset$. Then $v_1 pq_1 q$ and $v_2 p_1 q_4 A(q_4; v_2, p_2)$ are two empty quadrilaterals separated from the other five points.
- $C(q_4; p_1, v_2)$ is not empty. Let $r = A(q_4; p_1, v_2)$.

If $r \in C(v_1; p_1, p)$, then $v_1 p_1 rp$ and $p_3 qq_1 A(q_1; p_3, v_3)$ are two empty quadrilaterals separated from the other five points. If $r \in C(v_1; p, q_4)$, then $v_1 pq_4 r$ and $p_1 q_1 p_3 q$ are two empty quadrilaterals separated from the other five points.

(III) $q_4 \in C(v_1; q_1, q)$. Then $v_1 q_1 q_4 q$ and $v_2 p_1 pA(p; v_2, p_2)$ are two empty quadrilaterals separated from the other five points.

(IV) $q_4 \in C(v_1; q, p_3)$. Then $v_1 qq_4 p_3$ and $p_1 pq_1 A(q_1; p_1, v_2)$ are two empty quadrilaterals separated from the other five points.

(B) $C(p_3; q_1, v_2) \cong \emptyset$. Let $q_5 = A(p_3; v_2, p_2)$. Like in the previous subcase (A) we get $g_4(P) \geq 3$.

(c) $q_1 \in C(v_1; q, p_3)$. Then $v_1 qq_1 p_3$ and $v_2 p_1 pA(p; v_2, p_2)$ are two empty quadrilaterals separated from the other five points.

(2) Both $C(p_1; p_3, v_3)$ and $C(p_3; p_1, v_2)$ are empty by symmetry. Let $q_6 = A(p_1; v_3, p_2)$; we get $g_4(P) \geq 3$ by an argument similar to that for subcase (1).

For all cases mentioned above we have $g_4(P) \geq 3$. On other hand, for any P with $|P| = 13$ obviously $g_4(P) \leq 3$. Thus $g_4(P) = 3$ for any P with $|P| = 13$. By the definition, we obtain the result $G_4(13) = 3$. □

Proof of Lemma 6. Let $|P| = 17$. If $|V(P)| \geq 4$, we are done by the same argument as for Lemma 5. Next we discuss the remaining case of $|V(P)| = 3$.

There exists some T_i, say T_1, containing at most three points of P by (∗∗). If T_1 contains at most two points, by exactly the same argument as for Lemma 5 we reach the conclusion that $g_4(P) \geq 4$. Then we only need to consider the case of T_1 containing exactly three points. Here p and q are defined in the same way as Case 3 of Lemma 5. Since if $p = q$, we can easily find an empty quadrilateral to be separated, we suppose that $p \neq q$.

Let r be the remaining point in T_1. If q is on the opposite side of pp_3 to v_1, then $p_3 v_1 qA(q; v_1, p)$ is an empty convex quadrilateral separated from the other 13 points. Thus we may assume that q is on the same side of pp_3 as v_1 and p is on the same side of $p_1 q$ as v_1 by symmetry. If $r \in C(v_1; p_1, p)$, then $v_1 prp_1$ and $v_3 p_3 qA(q; v_3, p_2)$ are two empty quadrilaterals separated from the other nine points. Thus we suppose that r is not in $C(v_1; p_1, p)$; then r is also not in $C(v_1; q, p_3)$ by symmetry, and $r \in C(v_1; p, q)$. If r is on the opposite side of pq to v_1, then $v_1 pqr$ is an empty convex quadrilateral separated from the other 13 points, so we can assume that r is on the same side of pq as v_1. If $r \in C(p_1; p, v_1)$ or $r \in C(p_3; q, v_1)$, then $v_1 p_1 pr$ or $v_1 rqp_3$ is an empty convex quadrilateral separated from the other 13 points. Now consider the case where r is in the interior of the triangular domain determined by $p_1 p$, $p_3 q$ and $\overline{pq}$. There are two subcases to discuss in the following.

(1) $C(p_1; p_3, v_3)$ is not empty. Let $q_1 = A(p_1; p_3, v_3)$; then q_1 has four possible positions (see Fig. 5).

(a) $q_1 \in C(v_1; p_1, p)$. Then $v_1 p_1 q_1 p$ and $p_3 qrA(r; p_3, v_3)$ are two empty quadrilaterals separated from the other nine points.

(b) $q_1 \in C(v_1; p, r)$. Then $v_1 pq_1 r$ and $v_3 p_3 qA(q; v_3, p_2)$ are two empty quadrilaterals separated from the other nine points.

(c) $q_1 \in C(v_1; r, q)$.

(i) $C(p_3; p_1, q_1) \cong \emptyset$. Then $v_1 r q_1 q$ and $v_2 p_1 p A(p; v_2, p_2)$ are two empty quadrilaterals separated from the other nine points.

(ii) $C(p_3; p_1, q_1)$ is not empty. Let $q_2 = A(q_1; p_1, v_2)$.

(A) $q_2 \in C(v_1; p_1, p)$. Then $v_1 p_1 q_2 p$ and $r q p_3 q_1$ are two empty quadrilaterals separated from the other nine points.

(B) $q_2 \in C(v_1; p, r)$. Then $v_1 p q_2 r$ and $p_1 q_1 p_3 q$ are two empty quadrilaterals separated from the other nine points.

(C) $q_2 \in C(v_1; r, q_1)$. Then $v_1 r q_2 q_1$ and $p_1 p q p_3$ are two empty quadrilaterals separated from the other nine points.

(d) $q_1 \in C(v_1; q, p_3)$. Then $v_1 q q_1 p_3$ and $p_1 p r A(r; p_1, v_2)$ are two empty quadrilaterals separated from the other nine points.

(2) $C(p_1; p_3, v_3) \cong \emptyset$. Also $C(p_3; p_1, v_2) \cong \emptyset$ by symmetry. Let $s = A(p_1; v_3, p_2)$. We obtain the result $g_4(P) \geq 4$ by an argument similar to that for subcase (1).

By the argument above, we obtain the result $g_4(P) \geq 4$. Thus $g_4(P) = 4$ holds by the fact that $g_4(P) \leq 4$ when $|P| = 17$. So we get the conclusion $G_4(17) = 4$. □

Proof of Theorem 7. Let P be a set of n points in the plane, no three collinear. Any 38 points can be divided into nine empty convex quadrilaterals by Lemma 6 and Lemma 9. Take a line l not parallel to any line determined by any two points of P. We move l in a direction orthogonal to itself until exactly 38 points are on one side of l. Continue in the same way so that we obtain $\lceil \frac{n}{38} \rceil$ disjoint convex regions such that each region contains 38 points of P except probably the last region R which contains m points of P, $1 \leq m \leq 38$. If $4k + 1 \leq m \leq 4k + 4$ for $k = 0, 1, 2, 3$, then there exist at least k empty convex quadrilaterals. If $17 \leq m \leq 21$, we can find at least four empty convex quadrilaterals by Lemma 6. If $4k + 2 \leq m \leq 4k + 5$ for $k = 5, 6, 7, 8$, there exist at least k empty convex quadrilaterals. If $m = 38$, we can find at least nine empty convex quadrilaterals. Therefore, we can obtain at least $\lfloor \frac{9n}{38} \rfloor$ empty convex quadrilaterals. □

Proof of Theorem 8. Notice that the inequality to be proved is equivalent to the following inequality:

$$G_4(21 \times 2^{k-1} - 4) \geq 5 \times 2^{k-1} - 1 \ (k \geq 1).$$

Let $\alpha(k) = 21 \times 2^{k-1} - 4$, $\beta(k) = 5 \times 2^{k-1} - 1$. Using Lemma 9, the inequality can be proved easily by induction on k. This completes the proof for $n = 21 \times 2^{k-1} - 4$. □

3. Remark

We expect $G_4(4k + 1) = k$ to also hold for $k \geq 5$. Therefore we conjecture that $G_4(n) = \lfloor \frac{n-1}{4} \rfloor$.

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