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Singular functional differential equations of neutral type in Banach spaces

Said Hadd

Department of Electrical Engineering and Electronics, The University of Liverpool, Brownlow Hill, L69 3GJ, Liverpool, United Kingdom

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Abstract

The well-posedness of a large class of singular partial differential equations of neutral type is discussed. Here the term singularity means that the difference operator of such equations is nonatomic at zero. This fact offers many difficulties in applying the usual methods of perturbation theory and Laplace transform technique and thus makes the study interesting. Our approach is new and it is based on functional analysis of semigroup of operators in an essential way, and allows us to introduce a new concept of solutions for such equations. Finally, we study the well-posedness of a singular reaction–diffusion equation of neutral type in weighted Lebesgue's spaces.

Keywords: Neutral equations; Nonatomic operators; Semigroup; Banach space

1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space and $p \in (1, \infty)$, $r > 0$ be real numbers. Denote

$$\mathscr{X} = X \times L^p\big([-r, 0], X\big) \quad \text{with norm} \quad \left\| \begin{pmatrix} z \\ \varphi \end{pmatrix} \right\| = \|z\| + \|\varphi\|_p,$$

E-mail addresses: s.hadd@liv.ac.uk, saha@fa.uni-tuebingen.de.

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where $(L^p([-r,0],X),\|\cdot\|_p)$ is the Banach space of all p-integrable functions $\varphi:[-r,0]\to X$. Throughout $W^{1,p}([-r,0],X)$ will denote the Sobolev space, the Banach space of all absolutely continuous functions φ such that the derivative φ' is a p-integrable function.

The aim of this paper is to investigate the well-posedness and spectral theory of the evolution equation

$$(\mathscr{C}_{\mathrm{D}})\qquad \begin{cases} \dot{w}(t)=\mathscr{A}_{\mathrm{D}}w(t), & t\geqslant 0,\\ w(0)=\binom{z}{\varphi}\in\mathscr{X}, \end{cases}$$

where $\mathscr{A}_{\mathrm{D}}:\mathscr{D}(\mathscr{A}_{\mathrm{D}})\subset\mathscr{X}\to\mathscr{X}$ is the matrix operator defined by

$$\mathscr{A}_{\mathrm{D}}:=\begin{pmatrix} A & L\\ 0 & \frac{\partial}{\partial\theta}\end{pmatrix},$$
$$\mathscr{D}(\mathscr{A}_{\mathrm{D}}):=\left\{\begin{pmatrix} z\\ \varphi\end{pmatrix}\in\mathscr{D}(A)\times W^{1,p}\big([-r,0],X\big):\ z=D\varphi\right\}. \tag{1.1}$$

Here $A:\mathscr{D}(A)\subset X\to X$ is the generator of a $\mathscr{C}_0$-semigroup $T:=(T(t))_{t\geqslant 0}$ on X, D,L: $W^{1,p}([-r,0],X)\to X$ are linear and bounded. We will focus on operators D that are *nonatomic at zero*, that is, for all $\varepsilon>0$ there exists $\delta\leqslant r$ such that

$$\|D\varphi\|\leqslant\varepsilon\sup_{\theta\in[-r,0]}\big\|\varphi(\theta)\big\|$$

for any $\varphi\in W^{1,p}([-r,0],X)$ with $\operatorname{supp}(\varphi)\subseteq[-\delta,0]$. If $\mu:[-r,0]\to\mathscr{L}(X)$ is a function of bounded variation which is continuous at zero then the following operator

$$D\varphi=\int_{-r}^{0}d\mu(\theta)\,\varphi(\theta),\qquad \varphi\in W^{1,p}\big([-r,0],X\big), \tag{1.2}$$

is nonatomic at zero (see [16, p. 256]). We note that the operator $\mathscr{A}_{\mathrm{D}}$ is closely related to neutral equations with difference operator D and delay operator L (see e.g., [5,28]).

Generally, in neutral equations, the authors consider the difference operator D given by $D\varphi=\varphi(0)-K\varphi$, where K is nonatomic at zero (we say that D is atomic at zero). In this case the Cauchy problem $(\mathscr{C}_{\mathrm{D}})$ have been studied quite intensively in the literature, see e.g., [16, Chapter 9], [18,20,22], which investigate the phase space $C([-r,0],X)$ and [5,9,25] which treat the L^p-setting. In these references the state space X is of finite dimension, so that the operators K and L are represented as in (1.2), due to Riesz's representation theorem [1, p. 248]. This makes the study of neutral equations on $C([-r,0],\mathbb{R}^n)$ and $\mathbb{R}^n\times L^p([-r,0],\mathbb{R}^n)$ similar (see e.g., [17]). The situation is quite different when X is an infinite-dimensional space, where K and L are not necessarily represented as in (1.2). The authors of [14] (see also [2]) show that extra assumptions on K and L (see the conditions (**H**) and (**H**$'$) in Section 2) should be imposed to guarantee the well-posedness of $(\mathscr{C}_{\mathrm{D}})$. These conditions are related to the left shift semigroup on $L^p([-r,0],X)$, and are always satisfied if D and L are given by (1.2), see [15, Theorem 3]. The work [14] is based on the feedback theory of regular linear system [26], [27, Chapter 7], [32] in an essential way and gives a new approach to tackle the well-posedness of $(\mathscr{C}_{\mathrm{D}})$ (see Remark 3.4 for comments on this approach). New results on nonautonomous neutral equations with atomic difference operators in Banach spaces can be found in [12, Section 6]. It is shown in [23,24] that neutral equation with atomic difference operators can be studied as hyperbolic systems (see also [10] for new results on such systems).

But it is of much importance to consider the case of nonatomic operators D and it is natural to expect such a situation in control problems, such as aeroelastic systems (see e.g., [4]) and large-scale or interconnected systems with time lag. To the best of our knowledge, the well-posedness of $(\mathscr{C}_{\mathrm{D}})$ for nonatomic operator D have been only discussed in few papers e.g., [5,6,19,21,29]. These works consider finite dimensional state spaces and $A \equiv 0$, and mainly based on Laplace transform technique. It has been observed that certain neutral equations may be well posed over some choices of state spaces, while being ill-posed over others (see e.g., [5,19,21]). Several concepts of well-posedness for $(\mathscr{C}_{\mathrm{D}})$ were introduced in [6]. In [28] one can find some conditions for the well-posedness of $(\mathscr{C}_{\mathrm{D}})$, the variation of constants formulae for the solutions as well as the relations between alternative well-posedness notions. A typical example of nonatomic operator D is the following singular integral:

$$D\psi = \int_{-r}^{0} |\theta|^{-\alpha} \psi(\theta)\, d\theta, \quad \alpha \in (0,1),\ \psi \in C\big([-r,0],\mathbb{R}^n\big).$$

In this case the neutral equation is well-posed on $\mathbb{R}^n \times L^p([-r,0],\mathbb{R}^n)$ only for $p(1-\alpha) < 1$ (see e.g. [5,21]). This eliminates the Hilbert space $\mathbb{R}^n \times L^2([-r,0],\mathbb{R}^n)$ which is very important for control problem and aeroelastic model. It is shown in [3] and [19,29] that weighted L^2-spaces are the most appropriate state spaces for which well-posedness, dissipativity estimates as well as differentiability of the semigroup solution are guaranteed. This is important in the investigation of numerical analysis for control and identification of some particular models (such as a fluid-structure problem).

By analyzing in profile the aforementioned references it seems that general well-posedness results for $(\mathscr{C}_{\mathrm{D}})$ are yet unknown, and the relations between alternative well-posedness notions are not quite understood. This makes the question of finding general results for the well-posedness of $(\mathscr{C}_{\mathrm{D}})$ with D nonatomic at zero very interesting.

Our interest here is to treat the well-posedness of $(\mathscr{C}_{\mathrm{D}})$ for D nonatomic at zero using a unified approach mainly based on functional analysis of semigroup of operators and indirectly on the closed-loops of well-posed linear systems. In fact, in the previous work [14] we have established that in the case of atomic operators D the matrix operator $\mathscr{A}_{\mathrm{D}}$ coincides with the generator of an appropriate closed-loop system. As we will see in this paper, this is not the case for nonatomic operators D (see Remark 3.4). We will see that the generation property of $\mathscr{A}_{\mathrm{D}}$ follows from a subspace semigroup of a certain closed-loop system. In contrast to the previous aforementioned references, especially, we are interested in working on Banach spaces and with general operators D and L. In addition we characterize $\sigma(\mathscr{A}_{\mathrm{D}})$, the spectrum of $\mathscr{A}_{\mathrm{D}}$.

The organization of this paper is as follows. In Section 2, we present the framework and state our main results, i.e., Theorems 2.3 and 2.4. In this section we also present several results concerning the well-posedness and spectral theory of difference equations as well as of neutral equations with nonatomic difference operators. In Section 3, we prove the main theorems of this paper. Finally, in Section 4 we investigate the well-posedness of a singular reaction–diffusion equation of neutral type in a bounded domain of $\mathbb{R}^n$.

We would like to emphasize that to the best of our knowledge, this paper is the first where the well-posedness of $(\mathscr{C}_{\mathrm{D}})$ on $\mathscr{X}$ is proved for X Banach space and D nonatomic at zero.

2. Framework and main results

Let us introduce some notations and objects we will be working with. First we denote by $S := (S(t))_{t \geqslant 0}$ the left shift semigroup on $L^p([-r,0],X)$ given by

$$\big(S(t)\varphi\big)(\theta) = \begin{cases} 0, & t+\theta \geqslant 0, \\ \varphi(t+\theta), & t+\theta \leqslant 0 \end{cases}$$

for $t \geqslant 0$ and $\theta \in [-r,0]$. Its generator is the operator given by

$$Q\varphi = \frac{\partial \varphi}{\partial \theta}, \quad \mathscr{D}(Q) = \big\{\varphi \in W^{1,p}\big([-r,0],X\big)\colon \varphi(0)=0\big\}.$$

The *Yosida extension* of $P : W^{1,p}([-r,0],X) \to X$ with respect to Q is the linear operator

$$P_{\mathrm{Y}}\varphi := \lim_{\lambda \to +\infty} P\lambda R(\lambda, Q)\varphi, \quad \mathscr{D}(P_{\mathrm{Y}}) := \big\{\varphi \in L^p\big([-r,0],X\big)\colon \text{this limit exists in } X\big\}. \tag{2.1}$$

Denote

$$e_\lambda : X \to L^p\big([-r,0],X\big), \quad (e_\lambda x)(\theta) := e^{\lambda\theta}x, \quad \lambda \in \mathbb{C}. \tag{2.2}$$

Moreover, we define

$$\phi(t)u = \chi_{\{\theta \in [-r,0]\colon t+\theta \geqslant 0\}}(\cdot)u(t+\cdot) \tag{2.3}$$

for $t \geqslant 0$ and $u \in L^p(\mathbb{R}_+,X)$. We set

$$W^{1,p}_{0,\mathrm{loc}}(\mathbb{R}_+,X) := \big\{u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}_+,X)\colon u(0)=0\big\}.$$

It is a dense subspace of $L^p_{\mathrm{loc}}(\mathbb{R}_+,X)$. We note that $(t \mapsto \phi(t)u) \in C(\mathbb{R}_+, W^{1,p}([-r,0],X))$ for any $u \in W^{1,p}_{0,\mathrm{loc}}(\mathbb{R}_+,X)$, by Lemma 2.11.

Definition 2.1. We say that a linear and bounded operator $P : W^{1,p}([-r,0],X) \to X$ satisfies the condition (**H**) if the following hold:

$$\int_0^\alpha \big\|PS(t)\varphi\big\|^p\,dt \leqslant \gamma_1^p \|\varphi\|_p^p \tag{H1}$$

for all $\varphi \in \mathscr{D}(Q)$, constants $\alpha > 0$ and $\gamma_1 := \gamma_1(\alpha) > 0$, the estimate

$$\int_0^\beta \big\|P\phi(t)u\big\|^p\,dt \leqslant \gamma_2^p \|u\|_p^p \tag{H2}$$

holds for any $u \in W^{1,p}_{0,\mathrm{loc}}(\mathbb{R}_+, X)$, constants $\beta > 0$ and $\gamma_2 := \gamma_2(\beta) > 0$, and

$$\mathrm{Range}(e_\lambda) \subset \mathscr{D}(P_{\mathrm{Y}}) \quad \text{for some (hence all) } \lambda \in \rho(Q). \tag{H3}$$

Moreover, we need the following definition.

Definition 2.2. We say that $P : W^{1,p}([-r,0], X) \to X$ satisfies ($\mathbf{H}'$) if P satisfies ($\mathbf{H}$) and the constant $\gamma_2 > 0$ in (H2) satisfies $\gamma_2(\beta) \to 0$ as $\beta \to 0$.

Observe that (H2) implies that the operator $\mathbb{F}^P_\infty : W^{1,p}_{0,\mathrm{loc}}(\mathbb{R}_+, X) \to L^p_{\mathrm{loc}}(\mathbb{R}_+, X)$ defined by $(\mathbb{F}^P_\infty u)(t) := P\phi(t)u$, $t \geqslant 0$, can be extended to a linear and bounded operator on $L^p_{\mathrm{loc}}(\mathbb{R}_+, X)$. Moreover, if $P : W^{1,p}([-r,0], X) \to X$ satisfies ($\mathbf{H}'$) then

$$\left(\mathrm{I}_X - \mathbb{F}^P_\infty\right)^{-1} \text{ exists in } L^p_{\mathrm{loc}}(\mathbb{R}_+, X). \tag{H4}$$

This observation allows us to use linear systems feedback theory to prove the main theorems. On the other hand, one can see that if P satisfies ($\mathbf{H}$) then

$$W^{1,p}\big([-r,0], X\big) \subset \mathscr{D}(P_{\mathrm{Y}}) \quad \text{and} \quad (P_{\mathrm{Y}})_{|W^{1,p}([-r,0],X)} = P. \tag{2.4}$$

As examples of P satisfying the condition ($\mathbf{H}'$), we can include the case when P is given by (1.2) (see [15, Theorem 3]). Thus, by Riesz's representation theorem the condition ($\mathbf{H}'$) is always satisfied if X has a finite dimension.

For $u : [-r, \infty) \to X$ such that $u(s) = \varphi(s)$ for a.e. $s \in [-r, 0]$ we set

$$u(t + \cdot) := S(t)\varphi + \phi(t)u, \quad t \geqslant 0. \tag{2.5}$$

If P satisfies the condition ($\mathbf{H}$) then

$$u(t + \cdot) \in \mathscr{D}(P_{\mathrm{Y}}) \quad \text{a.e. } t \geqslant 0, \tag{2.6}$$

and the function

$$t \in [0, +\infty) \mapsto P_{\mathrm{Y}} u(t + \cdot) \in X \tag{2.7}$$

is locally p-integrable (see [15, Section 4]).

The following two theorems are the main results of this paper.

Theorem 2.3. *Let the operators D and L satisfy the conditions* ($\mathbf{H}'$) *and* ($\mathbf{H}$), *respectively. Then $\mathscr{A}_{\mathrm{D}}$ generates a strongly continuous semigroup $\mathscr{T}_{\mathrm{D}} := (\mathscr{T}_{\mathrm{D}}(t))_{t \geqslant 0}$ on $\mathscr{X}$. Moreover, for any $\binom{z}{\varphi} \in \mathscr{X}$ the unique solution of the Cauchy problem ($\mathscr{C}_{\mathrm{D}}$) satisfies*

$$\mathscr{T}_{\mathrm{D}}(t) \begin{pmatrix} z \\ \varphi \end{pmatrix} = \begin{pmatrix} z(t) \\ u(t + \cdot) \end{pmatrix} \tag{2.8}$$

for $t \geqslant 0$, where $u : [-r, +\infty) \to X$ is such that $u(s) = \varphi(s)$ for a.e. $s \in [-r, 0]$, and the function $z(\cdot) : [0, \infty) \to X$ satisfies the following two equations:

$$z(t) = T(t)z + \int_0^t T(t-\tau)L_{\mathrm{Y}}u(\tau+\cdot)\,d\tau,$$
$$z(t) = D_{\mathrm{Y}}u(t+\cdot) \tag{2.9}$$

for $t \geqslant 0$, where L_{Y} and D_{Y} are the Yosida extensions of L and D defined by (2.1).

Proof. See Section 3 for a proof. □

Let us now consider the nonhomogeneous Cauchy problem

$$(\mathscr{C}_{\mathrm{D}})_F \quad \begin{cases} \dot{w}(t) = \mathscr{A}_{\mathrm{D}}w(t) + F(t), & t \geqslant 0, \\ w(0) = \binom{z}{\varphi} \in \mathscr{X}, \end{cases}$$

where $F = \binom{f}{0}$ with $f \in L^p_{\mathrm{loc}}([0,\infty), X)$ is a forcing term.

Theorem 2.4. *Let the operators D and L satisfy the conditions* ($\mathbf{H}'$) *and* ($\mathbf{H}$)*, respectively. Then for any $\binom{z}{\varphi} \in \mathscr{X}$ and $f \in L^p_{\mathrm{loc}}([0,\infty), X)$ the problem $(\mathscr{C}_{\mathrm{D}})_F$ has a unique solution given by*

$$w(t) = \begin{pmatrix} z(t) \\ u(t+\cdot) \end{pmatrix} \tag{2.10}$$

for $t \geqslant 0$, where $u : [-r, +\infty) \to X$ is such that $u(s) = \varphi(s)$ for a.e. $s \in [-r, 0]$, and the function $z(\cdot) : [0,\infty) \to X$ satisfies the following two equations:

$$z(t) = T(t)z + \int_0^t T(t-\tau)\big[L_{\mathrm{Y}}u(\tau+\cdot) + f(\tau)\big]\,d\tau,$$
$$z(t) = D_{\mathrm{Y}}u(t+\cdot) \tag{2.11}$$

for $t \geqslant 0$, where L_{Y} and D_{Y} are the Yosida extensions of L and D defined by (2.1).

Proof. See Section 3 for a proof. □

In order to characterize the spectrum of $\mathscr{A}_{\mathrm{D}}$ and compute $R(\lambda, \mathscr{A}_{\mathrm{D}})$, the resolvent operator of $\mathscr{A}_{\mathrm{D}}$, for any $\lambda \in \rho(\mathscr{A}_{\mathrm{D}})$ we consider the linear operator

$$Q_{\mathrm{D}}\varphi = \frac{\partial \varphi}{\partial \theta}, \quad \mathscr{D}(Q_{\mathrm{D}}) = \big\{\varphi \in W^{1,p}\big([-r,0], X\big):\ D\varphi = 0\big\}. \tag{2.12}$$

The operator Q_{D} is related to difference equations (see e.g. [21]). The following result shows general conditions for which Q_{D} generates a $\mathscr{C}_0$-semigroup on $L^p([-r,0], X)$.

Proposition 2.5. *Assume that D is nonatomic at zero and satisfies the condition* ($\mathbf{H}'$). *Then Q_{D} generates a $\mathscr{C}_0$-semigroup $S_{\mathrm{D}} := (S_{\mathrm{D}}(t))_{t \geqslant 0}$ on $L^p([-r,0], X)$. If $u \in L^p_{\mathrm{loc}}([-r,+\infty), X)$ is such that $u(s) = \varphi(s)$ for a.e. $s \in [-r, 0]$, then*

$$u(t+\cdot\,;\varphi) = S_{\mathrm{D}}(t)\varphi \quad \textit{and} \quad D_{\mathrm{Y}}u(t+\cdot) = 0, \quad t \geqslant 0, \tag{2.13}$$

where we set $u(t+\cdot) := u(t+\cdot\,;\varphi)$*, where* D_Y *is the Yosida extension of* D *defined by* (2.1). *On the other hand,* $\Theta := \{\lambda \in \mathbb{C}\colon\ De_\lambda \in \mathscr{L}(X) \text{ is invertible}\} \subset \rho(Q_{\mathrm{D}})$ *and*

$$R(\lambda, Q_{\mathrm{D}}) = \big(I - e_\lambda (De_\lambda)^{-1} D\big) R(\lambda, Q) \quad \text{for } \lambda \in \Theta. \tag{2.14}$$

Proof. We want to apply Theorem 2.3. We consider $\mathscr{A}_{\mathrm{D}}$ with $A \equiv 0$ and $L \equiv 0$, so that A and L satisfy the conditions of Theorem 2.3, which means that the semigroup $\mathscr{T}_{\mathrm{D}}$ exists. Moreover, by (2.10) and the first equation in (2.9) this semigroup satisfies

$$\mathscr{T}_{\mathrm{D}}(t)\begin{pmatrix} z \\ \varphi \end{pmatrix} = \begin{pmatrix} z \\ u(t+\cdot\,;\varphi) \end{pmatrix} \tag{2.15}$$

for any $\binom{z}{\varphi} \in \mathscr{X}$. This shows that the first and the second components of $\mathscr{T}_{\mathrm{D}}(t)\binom{z}{\varphi}$ are independent. We then set $S_{\mathrm{D}}(t)\varphi := u(t+\cdot\,;\varphi)$ for $t \geqslant 0$ and $u : [-r,\infty) \to X$ with $u(s) = \varphi(s)$ for $-r \leqslant s \leqslant 0$, so that $(S_{\mathrm{D}}(0)\varphi)(s) = u(s) = \varphi(s)$. The exponential boundedness and the strong continuity of S_{D} follow by those of $\mathscr{T}_{\mathrm{D}}$ using (2.15). Moreover, by (2.15) we obtain

$$\begin{pmatrix} z \\ S_{\mathrm{D}}(t+\tau)\varphi \end{pmatrix} = \begin{pmatrix} z \\ u(t+\cdot\,;u(\tau+\cdot\,;\varphi)) \end{pmatrix} = \begin{pmatrix} z \\ S_{\mathrm{D}}(t)u(\tau+\cdot\,;\varphi) \end{pmatrix} = \begin{pmatrix} z \\ S_{\mathrm{D}}(t)S_{\mathrm{D}}(\tau)\varphi \end{pmatrix}$$

for any $t, \tau \geqslant 0$ and $\binom{z}{\varphi} \in \mathscr{X}$. Thus S_{D} is a $\mathscr{C}_0$-semigroup on $W^{1,p}([-r,0],X)$. Now for $\varphi \in L^p([-r,0],X)$ and $t > 0$ we have

$$\left\| \frac{S_{\mathrm{D}}(t)\varphi - \varphi}{t} - \frac{\partial\varphi}{\partial\theta} \right\|_p = \left\| \frac{\mathscr{T}_{\mathrm{D}}(t)\binom{0}{\varphi} - \binom{0}{\varphi}}{t} - \begin{pmatrix} 0 \\ \partial\varphi/\partial\theta \end{pmatrix} \right\|. \tag{2.16}$$

Observe that $\varphi \in \mathscr{D}(Q_{\mathrm{D}})$ if and only if $\binom{0}{\varphi} \in \mathscr{D}(\mathscr{A}_{\mathrm{D}})$. Thus, by (2.16) one can see that Q_{D} is the generator of S_{D}.

Let $\lambda \in \mathbb{C}$ such that De_λ is bijective. Let $\psi \in L^p([-r,0],X)$ and put $\varphi = e_\lambda (De_\lambda)^{-1} \cdot DR(\lambda,Q)\psi - R(\lambda,Q)\psi$. Thus $\varphi \in \mathscr{D}(Q_{\mathrm{D}})$ and $(\lambda - Q_{\mathrm{D}})\varphi = \psi$. Our aim follows by the closedness of Q_{D}. To show (2.14) let $\lambda \in \rho(Q_{\mathrm{D}})$ and $u \in L^p_{\mathrm{loc}}([-r,+\infty),X)$ such that $u(\theta) = \varphi(\theta)$ for a.e. $\theta \in [-r,0]$. By combining (2.5) and (2.13) we have

$$S_{\mathrm{D}}(t)\varphi = S(t)\varphi + \phi(t)u, \quad t \geqslant 0. \tag{2.17}$$

By tacking Laplace transform in both sides of (2.17) one obtains

$$R(\lambda, Q_{\mathrm{D}})\varphi = R(\lambda, Q)\varphi + e_\lambda \hat{u}(\lambda),$$

where $\hat{u}$ denotes the Laplace transform of u. Moreover, using the definition of the domain of Q_{D} we obtain that $\hat{u}(\lambda) = -(De_\lambda)^{-1} DR(\lambda,Q)\varphi$ for $\lambda \in \Theta$. Thus (2.14) follows. □

Remark 2.6. If we assume in Proposition 2.5 that $X = \mathbb{C}^n$ then we have $\lambda \in \rho(Q_{\mathrm{D}})$ if and only if the $n \times n$ matrix De_λ is invertible. From the proof of Proposition 2.5 it suffices to show that De_λ is injective whenever $\lambda \in \rho(Q_{\mathrm{D}})$. In fact, let $\lambda \in \rho(Q_{\mathrm{D}})$ and let $z \in X$ such that $De_\lambda z = 0$. Then $e_\lambda z \in \ker(\lambda - Q_{\mathrm{D}}) = \{0\}$, so that $z = 0$.

Theorem 2.7. *Let the operators D and L satisfy the conditions* ($\mathbf{H}'$) *and* ($\mathbf{H}$)*, respectively. Then for $\lambda \in \Theta := \{\lambda \in \mathbb{C}\colon De_\lambda \in \mathscr{L}(X)$ is invertible$\}$, we have $\lambda \in \rho(\mathscr{A}_{\mathrm{D}})$ if and only if $\lambda \in \rho(A + Le_\lambda(De_\lambda)^{-1})$. Furthermore,*

$$R(\lambda, \mathscr{A}_{\mathrm{D}}) = \begin{pmatrix} R(\lambda, A + Le_\lambda(De_\lambda)^{-1}) & R(\lambda, A + Le_\lambda(De_\lambda)^{-1})LR(\lambda, Q_{\mathrm{D}}) \\ e_\lambda(De_\lambda)^{-1}R(\lambda, A + Le_\lambda(De_\lambda)^{-1}) & [I + e_\lambda(De_\lambda)^{-1}R(\lambda, A + Le_\lambda(De_\lambda)^{-1})L]R(\lambda, Q_{\mathrm{D}}) \end{pmatrix}$$

for any $\lambda \in \Theta$.

Proof. Let $\lambda \in \Theta$ and define a linear and operator on $\mathscr{X}$ by

$$\mathscr{J}_\lambda = \begin{pmatrix} I & 0 \\ -e_\lambda(De_\lambda)^{-1} & I \end{pmatrix}.$$

Then $\mathscr{J}_\lambda$ is invertible. Observe that for $\binom{z}{\varphi} \in \mathscr{D}(\mathscr{A}_{\mathrm{D}})$ we have $\varphi - e_\lambda(De_\lambda)^{-1}z \in \mathscr{D}(Q_{\mathrm{D}})$ and that $(\lambda - Q_{\mathrm{D}})(\varphi - e_\lambda(De_\lambda)^{-1}z) = (\lambda - \partial/\partial\theta)\varphi$. Thus

$$\mathscr{J}_\lambda\big[\mathscr{D}(A_{\mathrm{D}})\big] \subset \mathscr{D}(A) \times \mathscr{D}(Q_{\mathrm{D}}) \quad \text{and} \quad \lambda - \mathscr{A}_{\mathrm{D}} = \begin{pmatrix} \lambda - A - Le_\lambda(De_\lambda)^{-1} & -L \\ 0 & \lambda - Q_{\mathrm{D}} \end{pmatrix} \mathscr{J}_\lambda.$$

This ends the proof. □

In the rest of this section we introduce new definition of the solution of the nonhomogeneous neutral equation

$$(\mathscr{N}_f) \qquad \begin{cases} \frac{d}{dt}Du(t+\cdot) = ADu(t+\cdot) + Lu(t+\cdot) + f(t), & t \geqslant 0, \\ \lim_{t\to 0} Du(t+\cdot) = z, \quad u(\theta) = \varphi(\theta), & \text{a.e. } -r \leqslant \theta \leqslant 0. \end{cases}$$

Here the operators A, D, L are as above. Moreover, we assume that D is nonatomic at zero and the initial history function $\varphi \in L^p([-r, 0], X)$. We denote by $(\mathscr{N}_0)$ the equation $(\mathscr{N}_f)$ with $f = 0$.

Definition 2.8. A classical solution of the equation $(\mathscr{N}_f)$ is a function $u : [-r, \infty) \to X$ satisfying

(i) $u(\cdot) \in W^{1,p}_{\mathrm{loc}}([-r, \infty), X)$ and $(t \mapsto Du(t+\cdot)) \in W^{1,p}_{\mathrm{loc}}([0, \infty), X)$,
(ii) $Du(t+\cdot) \in \mathscr{D}(A)$ for all $t \geqslant 0$,
(iii) $u(\theta) = \varphi(\theta)$ on $[-r, 0]$ and $K\varphi = z$, and
(iv) u satisfies $(\mathscr{N}_f)$.

A weaker solution of $(\mathscr{N}_f)$ is defined as follows.

Definition 2.9. A generalized solution of the initial value problem $(\mathscr{N}_f)$ is a pair $(z(\cdot), u(\cdot))$ with $z(\cdot) : [0, \infty) \to X$ and $u(\cdot) : [-r, \infty) \to X$ are such that

(i) $z(\cdot)$ is continuous, $u(t+\cdot) \in \mathscr{D}(D_{\mathrm{Y}}) \cap \mathscr{D}(L_{\mathrm{Y}})$ for a.e. $t \geqslant 0$,
(ii) $u(\theta) = \varphi(\theta)$ for a.e. $\theta \in [-r, 0]$, and

(iii) for any $z \in X$ and $t \geqslant 0$,

$$z(t) = T(t)z + \int_0^t T(t-\tau)\big[L_{\mathrm{Y}} u(\tau + \cdot) + f(\tau)\big]\,d\tau,$$
$$z(t) = D_{\mathrm{Y}} u(t + \cdot),$$

where L_{Y} and D_{Y} are the Yosida extension of L and D defined by (2.1).

Proposition 2.10. *Let the operators D and L satisfy the conditions* $(\mathbf{H}')$ *and* $(\mathbf{H})$*, respectively. Then the neutral equation* $(\mathscr{N}_f)$ *has a unique generalized solution.*

Proof. It is an immediate consequence of Theorem 2.4. □

To show the relationship between the generalized solutions and classical solutions of $(\mathscr{N}_f)$ we need the following technical result.

Lemma 2.11. *Let $p \in (1, \infty)$. Define a function $\zeta(t, \cdot) = u(t + \cdot)$ for $t \geqslant 0$ and $u : [-r, \infty) \to X$. The following are equivalent*:

(i) $u \in W^{1,p}_{\mathrm{loc}}([-r, \infty), X)$.
(ii) $\zeta \in C^1(\mathbb{R}_+, L^p([-r, 0], X))$.
(iii) $\zeta \in C(\mathbb{R}_+, W^{1,p}([-r, 0], X))$.

Proof. The proof is similar to that given in [1, Lemma 4.1, p. 256], where $X = \mathbb{R}^\ell$, $\ell \geqslant 1$, was considered. □

Proposition 2.12. *Let the operators D and L satisfy the conditions* $(\mathbf{H}')$ *and* $(\mathbf{H})$*, respectively, and let $(z(\cdot), u(\cdot))$ be the generalized solution of $(\mathscr{N}_f)$ with initial condition $z \in X$ and $\varphi \in L^p([-r, 0], X)$. If $u \in W^{2,p}([-r, \infty), X)$, $\binom{z}{\varphi} \in \mathscr{D}(\mathscr{A}_{\mathrm{D}})$ and $f \in W^{1,1}(\mathbb{R}_+, X)$ then $u : [-r, \infty) \to X$ is a classical solution of $(\mathscr{N}_f)$.*

Proof. Set $g(t, \cdot) = u(t + \cdot)$ and $h(t, \cdot) = \frac{d}{dt} u(t + \cdot)$. Since $\dot{u} \in W^{1,p}([-r, \infty), X)$ then by Lemma 2.11 we have $h \in C(\mathbb{R}_+, W^{1,p}([-r, 0], X))$. From this we deduce that $g \in C^1(\mathbb{R}_+, W^{1,p}([-r, 0], X))$ and the function $\mathbb{R}+ \ni \tau \mapsto Lu(\tau + \cdot)$ (which has sense by (2.4)) is in $C^1(\mathbb{R}_+, X)$. We put $\psi(\tau) := Lu(\tau + \cdot) + f(\tau)$ for $\tau \geqslant 0$. Then by (2.4) we get

$$\int_0^t T(t-\tau)\big[L_{\mathrm{Y}} u(\tau + \cdot) + f(\tau)\big]\,d\tau = \int_0^t T(t-\tau)\psi(\tau)\,d\tau \quad \text{and} \quad z(t) = Du(t + \cdot)$$

for $t \geqslant 0$. Thus our aim now follows from [7, Chapter 7, Corollary 7.6]. □

Definition 2.13. The neutral equation $(\mathscr{N}_f)$ is well-posed in the strong sense if $(\mathscr{N}_0)$ has a unique classical solution, and this solution depends continuously on the initial data.

Proposition 2.14. *Let the operators D and L satisfy the conditions* ($\mathbf{H}'$) *and* ($\mathbf{H}$), *respectively. If the initial conditions of* $(\mathscr{N}_f)$ *satisfy* $\binom{z}{\varphi} \in \mathscr{D}(\mathscr{A}_{\mathrm{D}})$ *then* $(\mathscr{N}_f)$ *is well-posed in the strong sense.*

Proof. It is a consequence of Theorem 2.3 and the definition of well-posed Cauchy problems [7, Chapter 2, Section 6]. □

3. Proof of the main results

The object of this section is to prove Theorems 2.3 and 2.4. To that purpose we need some notation and notions.

Let $\mathscr{X}^{\oplus}$ be a Banach space (in fact we shall use this notation in some proofs below) and let $\mathfrak{T} = (\mathfrak{T}(t))_{t\geqslant 0}$ be a $\mathscr{C}_0$-semigroup on $\mathscr{X}^{\oplus}$ with generator $(\mathfrak{A}, \mathscr{D}(\mathfrak{A}))$. The completion of $\mathscr{X}^{\oplus}$ with respect to the norm $\|x\|_{-1} = \|R(\lambda, \mathfrak{A})x\|$ for some $\lambda \in \rho(\mathfrak{A})$, is called the extrapolation space associated with $\mathscr{X}^{\oplus}$ and A (or $\mathfrak{T}$). We denote this space by $\mathscr{X}^{\oplus}_{-1}$. Note that the norms $\|\cdot\|_{-1}$ are independent of the choice of λ. The extension of $\mathfrak{T}$ on $\mathscr{X}^{\oplus}_{-1}$ is a $\mathscr{C}_0$-semigroup which we denote by $(\mathfrak{T}_{-1}(t))_{t\geqslant 0}$, and whose generator we denote by $\mathfrak{A}_{-1}$. For more details and references on extrapolation theory we refer, e.g., to [7, Chapter II].

Let us consider another Banach space U and a family of bounded linear operators $\Phi(t): L^p([0,t], U) \to \mathscr{X}^{\oplus}$, $t \geqslant 0$, which satisfy the equation

$$\Phi(t+s)u = \Phi(t)(u|_{[s,s+t]}) + \mathfrak{T}(t)\Phi(s)(u|_{[0,s]}) \tag{3.1}$$

for $u \in L^p([0, s+t], U)$ and $t, s \geqslant 0$. It is shown in [31] that there exists a linear bounded operator $B : U \to \mathscr{X}^{\oplus}_{-1}$ such that

$$\Phi(t)u = \int_0^t \mathfrak{T}_{-1}(t-\sigma)Bu(\sigma)\,d\sigma \tag{3.2}$$

for any $t \geqslant 0$ and $u \in L^p([0,t], U)$.

If $C : \mathscr{D}(\mathfrak{A}) \to U$ is a linear and bounded operator then its Yosida extension with respect to $\mathfrak{A}$ is defined as in (2.1), and will be denoted by C_{Y} (see also [30] for details on this operator).

We say that $C : \mathscr{D}(\mathfrak{A}) \to U$ satisfies the condition (**W**) if the following hold:

- For any $z \in \mathscr{D}(\mathfrak{A})$,

$$\int_0^\alpha \left\|C\mathfrak{T}(t)z\right\|^p dt \leqslant \vartheta_1^p \, \|z\|_p^p \tag{W1}$$

  for constants $\alpha > 0$ and $\vartheta_1 := \vartheta_1(\alpha) > 0$.
- For a.e. $t \geqslant 0$ and $u \in L^p_{\mathrm{loc}}(\mathbb{R}_+, U)$,

$$\Phi(t)u \in \mathscr{D}(C_{\mathrm{Y}}) \quad \text{and} \quad t \mapsto C_{\mathrm{Y}}\Phi(t)u \in U \text{ is measurable.} \tag{W2}$$

- For any $u \in L^p([0,t],U)$,

$$\int_0^\beta \left\| C_{\mathrm{Y}} \Phi(t) u \right\|^p dt \leqslant \vartheta_2^p \|u\|_p^p \tag{W3}$$

for constants $\beta > 0$ and $\vartheta_2 := \vartheta_2(\beta) > 0$.
- For some (hence all) $\lambda \in \rho(\mathfrak{A})$, and all $v \in U$,

$$\operatorname{Range}\big(R(\lambda, \mathfrak{A}_{-1})B\big) \subset \mathscr{D}(C_{\mathrm{Y}}) \quad \text{and} \quad \lim_{\mu \to +\infty} C_{\mathrm{Y}} R(\mu, \mathfrak{A}_{-1}) B v = 0. \tag{W4}$$

- The operator $\mathbb{F}_\infty : L^p_{\mathrm{loc}}(\mathbb{R}_+, U) \to L^p_{\mathrm{loc}}(\mathbb{R}_+, U)$ defined by $\mathbb{F}_\infty u = C_{\mathrm{Y}} \Phi(\cdot) u$ satisfies

$$(\mathrm{I}_U - \mathbb{F}_\infty)^{-1} \text{ exists in } L^p_{\mathrm{loc}}(\mathbb{R}_+, U). \tag{W5}$$

The following perturbation theorem is due to Weiss [32] in Hilbert spaces and Staffans [27, Chapter 7].

Theorem 3.1. *Assume that $C : \mathscr{D}(\mathfrak{A}) \to U$ satisfies* (**W**). *Then the operator*

$$\mathscr{R} = \mathfrak{A}_{-1} + B C_{\mathrm{Y}}, \quad \mathscr{D}(\mathscr{R}) := \left\{ Z \in \mathscr{D}(C_{\mathrm{Y}}) \colon \ (\mathfrak{A}_{-1} + B C_{\mathrm{Y}}) Z \in \mathscr{X}^{\oplus} \right\}$$

(the sum is defined in $\mathscr{X}^{\oplus}_{-1}$) *generates the unique $\mathscr{C}_0$-semigroup $\mathscr{U} = (\mathscr{U}(t))_{t \geqslant 0}$ on X satisfying $\mathscr{U}(\sigma) Z \in \mathscr{D}(C_{\mathrm{Y}})$ for almost every $\sigma \geqslant 0$ and*

$$\mathscr{U}(t) Z = \mathfrak{T}(t) Z + \int_0^t \mathfrak{T}_{-1}(t-\tau) B C_{\mathrm{Y}} \mathscr{U}(\sigma) Z \, d\tau$$

for $Z \in \mathscr{X}^{\oplus}$, $t \geqslant 0$, where C_Y is the Yosida extension of C with respect to $\mathfrak{A}$ as in (2.1).

The following remark summarizes the concept of feedback theory of linear systems and will be used later.

Remark 3.2. Let C satisfies (W1)–(W4). Then we say that $(\mathfrak{A}, B, C)$ generates a regular systems (Σ) with state space $\mathscr{X}^{\oplus}$, control space U and output space U (see [32] for more general definitions). The system Σ is completely determined by the following differential system:

$$(\Sigma) \quad \begin{cases} \dot{w}(t) = \mathfrak{A}_{-1} w(t) + B \varpi(t), & t \geqslant 0, \\ y(t) = C w(t), & t \geqslant 0, \end{cases}$$

where $\varpi : [0, \infty) \to U$ is a locally p-integrable function. Moreover, $w(t) \in \mathscr{D}(C_{\mathrm{Y}})$ almost everywere $t \geqslant 0$, and the function

$$[0, \infty) \ni\mapsto C_{\mathrm{Y}} w(t) \in U$$

is a locally p-integrable and it is an extension of the function y (see [27,32]). We then write $y(t) = C_{\mathrm{Y}} w(t)$ for almost every $t \geqslant 0$. The functions w, ϖ and y are respectively called the state trajectory, the input function and output function of (Σ). The state trajectory of (Σ) is given by

$$w(t) = \mathfrak{T}(t) w(0) + \Phi(t)\varpi, \quad t \geqslant 0. \tag{3.3}$$

We now assume that C satisfies all the conditions in (**W**). Then we say that the system (Σ) has I_U as an admissible feedback operator (see [27, Chapter 7], [32] for more general definitions). If we choose the feedback law $\varpi(t) = y(t) = C_{\mathrm{Y}} w(t)$ for a.e. $t \geqslant 0$, then (3.2) and Theorem 3.1 say that $w(t) = \mathscr{U}(t) w(0)$ for $t \geqslant 0$. Then the $\mathscr{C}_0$-semigroup $\mathscr{U}$ is obtained from a feedback property of a regular system. Let us now consider the feedback law $\varpi = y + \varpi_{\mathrm{c}}$, where ϖ_{c} is another input (consign) function. As shown in [32] and [27, Chapter 7] there exists a regular system $(\Sigma)^I$ with state space $\mathscr{X}^{\oplus}$, control space U and output space U, generated by the triple $(\mathscr{R}, B, C_{\mathrm{Y}})$. This system $(\Sigma)^I$ is called the closed loop of (Σ) and has the same state trajectory w and output function y as of (Σ). In addition ϖ_{c} is the control function of $(\Sigma)^I$ and

$$w(t) = \mathscr{U}(t) w(0) + \int_0^t \mathscr{U}_{-1}(t-\tau) B \varpi(\tau)\, d\tau, \quad t \geqslant 0.$$

In the rest of this section we prove the main results of the paper. Let $\mathscr{A}_{\mathrm{D}}$ be the operator defined by (1.1). Moreover define the linear operator

$$\mathscr{A}_{\delta_0} = \begin{pmatrix} A & L \\ 0 & \frac{\partial\varphi}{\partial\theta} \end{pmatrix}, \quad \mathscr{D}(\mathscr{A}_{\delta_0}) = \left\{ \begin{pmatrix} z \\ \varphi \end{pmatrix} \in \mathscr{D}(A) \times W^{1,p}\big([-r,0], X\big) \colon \varphi(0) = z \right\}.$$

We note that (see e.g. [11]) if L satisfies (H1) in Section 2 then the operator $\mathscr{A}_{\delta_0}$ generates a $\mathscr{C}_0$-semigroup $\mathscr{T}_{\delta_0}$ on $\mathscr{X}$. Moreover,

$$\mathscr{T}_{\delta_0}(t) \begin{pmatrix} x \\ \psi \end{pmatrix} = \begin{pmatrix} x(t) \\ x(t+\cdot) \end{pmatrix} = \begin{pmatrix} T(t) z + \int_0^t T(t-\tau) L_{\mathrm{Y}} x(\tau+\cdot)\, d\tau \\ x(t+\cdot) \end{pmatrix} \tag{3.4}$$

for $t \geqslant 0$ and $\binom{x}{\psi} \in \mathscr{X}$, where $x : [-r, \infty) \to X$ is such that $x(s) = \psi(s)$ for $s \in [-r, 0]$.

Define

$$\mathscr{X}^{\oplus} = \mathscr{X} \times \mathscr{X}.$$

We now introduce the linear operator $\mathscr{A}^{\oplus} : \mathscr{D}(\mathscr{A}^{\oplus}) \to \mathscr{X}^{\oplus}$,

$$\mathscr{A}^{\oplus} = \begin{pmatrix} \mathscr{A}_{\mathrm{D}} & 0 \\ 0 & \mathscr{A}_{\delta_0} \end{pmatrix}, \quad \mathscr{D}(\mathscr{A}^{\oplus}) = \mathscr{D}(\mathscr{A}_{\mathrm{D}}) \times \mathscr{D}(\mathscr{A}_{\delta_0}).$$

We first prove the following technical result.

Lemma 3.3. *Assume that the operators D and L satisfy the conditions* (**H**$'$) *and* (**H**)*, respectively. Then the operator*

$$\begin{aligned}
&\mathscr{G} = \mathscr{A}^{\otimes}, \\
&\mathscr{D}(\mathscr{G}) = \big\{(z, \varphi, x, \psi) \in \mathscr{D}(A) \times W^{1,p}\big([-r,0], X\big) \times \mathscr{D}(A) \times W^{1,p}\big([-r,0], X\big): \\
&\qquad (\varphi - \psi)(0) = z - D\varphi,\ \psi(0) = x\big\}
\end{aligned} \tag{3.5}$$

generates a $\mathscr{C}_0$-semigroup $\mathscr{T}^{\oplus} = (\mathscr{T}^{\oplus}(t))_{t \geqslant 0}$ on $\mathscr{X}^{\oplus}$.

Proof. We shall use Theorem 3.1. For this we first define $\mathfrak{A} : \mathscr{D}(\mathfrak{A}) \subset \mathscr{X}^{\oplus} \to \mathscr{X}^{\oplus}$ by

$$\mathfrak{A} = \begin{pmatrix} \mathscr{A} & 0 \\ 0 & \mathscr{A} \end{pmatrix}, \quad \mathscr{D}(\mathfrak{A}) = \mathscr{D}(\mathscr{A}) \times \mathscr{D}(\mathscr{A}),$$

where $\mathscr{A} : \mathscr{D}(\mathscr{A}) \subset \mathscr{X} \to \mathscr{X}$ is given by

$$\mathscr{A} = \begin{pmatrix} A & L \\ 0 & \frac{\partial}{\partial\theta} \end{pmatrix}, \quad \mathscr{D}(\mathscr{A}) = \mathscr{D}(A) \times \mathscr{D}(Q).$$

As shown in [13], the operator $\mathscr{A}$ generates a $\mathscr{C}_0$-semigroup $\mathscr{W} = (\mathscr{W}(t))_{t \geqslant 0}$ on $\mathscr{X}$ given by

$$\mathscr{W}(t) = \begin{pmatrix} T(t) & R(t) \\ 0 & S(t) \end{pmatrix}, \quad t \geqslant 0. \tag{3.6}$$

Here

$$R(t) : L^p\big([-r,0], X\big) \to X, \quad R(t)\varphi = \int_0^t T(t-\tau) L_{\mathrm{Y}} S(\tau)\varphi \, d\tau.$$

Thus $\mathfrak{A}$ generates a diagonal $\mathscr{C}_0$-semigroup $\mathfrak{T} = (\mathfrak{T}(t))_{t \geqslant 0}$ on $\mathscr{X}^{\oplus}$. Let us define

$$\Gamma(t)u = \begin{pmatrix} \int_0^t T(t-\tau) L_{\mathrm{Y}} \phi(\tau) u \, d\tau \\ \phi(t)u \end{pmatrix} \tag{3.7}$$

for $t \geqslant 0$ and $u \in L^p_{\mathrm{loc}}(\mathbb{R}_+, X)$, where $\phi(t)$ are given by (2.3). We now consider $U := X \times X$ and define the family

$$\Phi(t) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \Gamma(t)u \\ \Gamma(t)v \end{pmatrix} \tag{3.8}$$

for $t \geqslant 0$ and $\binom{u}{v} \in L^p_{\mathrm{loc}}(\mathbb{R}_+, U)$. Using [13] one can see that $\Phi(t)$ satisfy (3.1) with respect to the semigroup $\mathfrak{T}$. By taking Laplace transform in (3.8) one can see that the operator $B \in \mathscr{L}(U, \mathscr{X}^{\oplus}_{-1})$ representing $\Phi(t)$ satisfies

$$R(\lambda, \mathfrak{A}_{-1}) B \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} R(\lambda, A) L e_\lambda u_0 \\ e_\lambda u_0 \\ R(\lambda, A) L e_\lambda v_0 \\ e_\lambda v_0 \end{pmatrix} := \mathscr{B}_\lambda, \quad \lambda \in \rho(A). \tag{3.9}$$

We introduce the operator $C : \mathscr{D}(\mathfrak{A}) \to U$ defined by

$$C := \begin{pmatrix} I & -D & -I & 0 \\ 0 & 0 & -I & 0 \end{pmatrix}.$$

Let us now prove that C satisfies the condition (**W**) with respect to $\mathfrak{A}$ and $(\Phi(t))_{t\geqslant 0}$. Using (3.6) and the fact that D satisfy (H1) one can see by a short computation that C satisfies (W1) with respect to the semigroup $\mathfrak{T}$. If we denote by C_{Y} the Yosida extension of C with respect to $\mathfrak{A}$ then

$$X \times \big[\mathscr{D}(D_{\mathrm{Y}}) \cap \mathscr{D}(L_{\mathrm{Y}})\big] \times X \times \big[\mathscr{D}(D_{\mathrm{Y}}) \cap \mathscr{D}(L_{\mathrm{Y}})\big] \subset \mathscr{D}(C_{\mathrm{Y}}),$$
$$C_{\mathrm{Y}} = \begin{pmatrix} I & -D_{\mathrm{Y}} & I & 0 \\ 0 & 0 & I & 0 \end{pmatrix} \quad \text{on } X \times \big[\mathscr{D}(D_{\mathrm{Y}}) \cap \mathscr{D}(L_{\mathrm{Y}})\big] \times X \times \big[\mathscr{D}(D_{\mathrm{Y}}) \cap \mathscr{D}(L_{\mathrm{Y}})\big], \tag{3.10}$$

where L_{Y} and D_{Y} are the Yosida extensions of L and D defined by (2.1) (see also [13] for somehow similar result.) On the other hand, using the fact that $\phi(t)u \in [\mathscr{D}(D_{\mathrm{Y}}) \cap \mathscr{D}(L_{\mathrm{Y}})]$, (3.7), (3.8) and (3.10) one can see that $\Phi(t)$ and C satisfy (W2). Moreover, by using (H2) and Holder's inequality, we obtain

$$\begin{aligned}\int_0^\beta \left\| C_{\mathrm{Y}}\Phi(t)\begin{pmatrix} u \\ v\end{pmatrix}\right\|^p dt &\leqslant c_p\gamma_2(\beta)^p\|u\|_p^p + c_p\int_0^\beta\left\|\int_0^t T(t-\tau)L_{\mathrm{Y}}\phi(t)(u+v)\,d\tau\right\|^p dt \\ &\quad + \int_0^\beta\left\|\int_0^t T(t-\tau)L_{\mathrm{Y}}\phi(t)v\,d\tau\right\|^p dt \\ &\leqslant c_p\big(\gamma_2^p(\beta)+\kappa(\beta)\big)\big(\|u\|_p^p+\|v\|_p^p\big)\end{aligned} \tag{3.11}$$

for any $u, v \in L^p_{\mathrm{loc}}(\mathbb{R}_+, U)$, where $\gamma_2(\beta) \to 0$ is given in (H2) which is associated to D, and $\kappa(\beta) \to 0$ as $\beta \to \infty$. Thus (W3) holds. By combining (H3), (3.9) and (3.10) one easily show that (W4) holds as well. By choosing a sufficiently small $\beta > 0$ in (3.11) the condition (W5) is satisfied. Thus thanks to Theorem 3.1 the following operator

$$\mathscr{R} = \mathfrak{A}_{-1} + BC_{\mathrm{Y}}, \quad \mathscr{D}(\mathscr{R}) := \big\{Z \in \mathscr{D}(C_{\mathrm{Y}}):\ (\mathfrak{A}_{-1} + BC_{\mathrm{Y}})Z \in \mathscr{X}^{\oplus}\big\}, \tag{3.12}$$

generates a unique $\mathscr{C}_0$-semigroup $\mathscr{U} = (\mathscr{U}(t))_{t\geqslant 0}$ on $\mathscr{X}^{\oplus}$.

Let $\mathscr{G}$ be the operator defined by (3.5). Next we prove that $\mathscr{G} = \mathscr{R}$.

First, due to (2.4) and (3.10) we have

$$X \times W^{1,p}\big([-r,0], X\big) \times X \times W^{1,p}\big([-r,0], X\big) \subset \mathscr{D}(C_{\mathrm{Y}}).$$

We show that $\mathscr{D}(\mathscr{R}) \subset \mathscr{D}(\mathscr{G})$ and $\mathscr{R} = \mathscr{G}$ on $\mathscr{D}(\mathscr{R})$. For this, let $(z, \varphi, x, \psi) \in \mathscr{D}(\mathscr{R})$ and $\lambda \in \rho(A)$. From (3.9) and (3.12) we have

$$\begin{aligned}\mathscr{R}(z,\varphi,x,\psi) &= \mathfrak{A}_{-1}\big((z,\varphi,x,\psi) - \mathscr{B}_\lambda C_{\mathrm{Y}}(z,\varphi,x,\psi)\big) \\ &\quad + \lambda\mathscr{B}_\lambda C_{\mathrm{Y}}(z,\varphi,x,\psi) \in \mathscr{X}^{\oplus}.\end{aligned} \tag{3.13}$$

In particular we have $(z,\varphi,x,\psi) - \mathscr{B}_\lambda C_{\mathrm{Y}}(z,\varphi,x,\psi) \in \mathscr{D}(\mathfrak{A})$ and $C_{\mathrm{Y}}(z,\varphi,x,\psi) = C(z,\varphi,x,\psi)$. Now using the expression of $\mathscr{B}_\lambda$ in (3.9) we have

$$(z,\varphi,x,\psi) - \mathscr{B}_\lambda C_{\mathrm{Y}}(z,\varphi,x,\psi) = \begin{pmatrix} z - R(\lambda,A)Le_\lambda(z - D\varphi + x) \\ \varphi - e_\lambda(z - D\varphi + x) \\ x - R(\lambda,A)Le_\lambda x \\ \psi - e_\lambda x \end{pmatrix} \in \mathscr{D}(\mathfrak{A}), \quad (3.14)$$

which implies that $z, x \in \mathscr{D}(A)$, $\varphi,\psi \in W^{1,p}([-r,0],X)$, and $\psi(0) = x$ and $(\varphi - \psi)(0) = z - D\varphi$. In particular, $(z,\varphi,x,\psi) \in \mathscr{D}(\mathscr{G})$. Combining (3.13) together with (3.9) and (3.14), and using the expressions of $\mathfrak{A}$ and C one obtains that $\mathscr{R}(z,\varphi,x,\psi) = \mathscr{G}(z,\varphi,x,\psi)$. The converse can be obtained in a similar way. This ends the proof. □

Proof of Theorem 2.3. We shall use the concept of feedback theory summarized in Remark 3.2. We have seen in the proof of Lemma 3.3 that the triple $(\mathfrak{A}, B, C)$ generates a regular system (Σ) with I_U as admissible feedback operator. Let $w(t) := (z(t), h(t;\cdot), x(t), g(t;\cdot))$, $t \geqslant 0$, be the state trajectory of (Σ). Now let $(z,\varphi,x,\psi) \in \mathscr{X}^{\oplus}$ such that $z(0) = x$, $h(0;\cdot) = \varphi$, $x(0) = x$ and $g(0;\cdot) = \psi$. Then

$$\begin{aligned} w(t;u,v) &= \mathfrak{T}(t)(z,\varphi,x,\psi) + \Phi(t)\begin{pmatrix} u \\ v \end{pmatrix} \\ &= \begin{pmatrix} T(t)z + \int_0^t T(t-s)L_{\mathrm{Y}}u(\tau+\cdot)\,d\tau \\ u(t+\cdot) \\ T(t)x + \int_0^t T(t-s)L_{\mathrm{Y}}v(\tau+\cdot)\,d\tau \\ v(t+\cdot) \end{pmatrix} \end{aligned} \quad (3.15)$$

for $t \geqslant 0$ and $u, v \in L^p_{\mathrm{loc}}([-r,\infty), X)$ and $u(\theta) = \varphi(\theta)$ and $v(\theta) = \psi(\theta)$ for a.e. $\theta \in [-r,0]$. Moreover, if we denote by $y(\cdot) : [0,\infty) \to U = X \times X$ the output function of (Σ) then

$$y(t) = C_{\mathrm{Y}} w(t;u,v) = \begin{pmatrix} z(t) - D_{\mathrm{Y}} h(t;\cdot) + x(t) \\ x(t) \end{pmatrix} \quad (3.16)$$

for almost every $t \geqslant 0$, due to Remark 3.2 and (3.10), where $h(t;\cdot) = u(t+\cdot)$. If we consider the feedback law $\binom{u}{v} = y$ (which has a sense as I_U is an admissible feedback operator for Σ) then the semigroup $\mathscr{U}(t)$ obtained in Lemma 3.3 can be written as

$$\mathscr{U}(t;u,v)(z,\varphi,x,\psi) = \big(z(t), u(t+\cdot), x(t), v(t+\cdot)\big), \quad t \geqslant 0.$$

Let us consider the case when $u = v$ and consider the following closed subspace:

$$\mathscr{E} = \big\{(z,\varphi,x,\psi) \in \mathscr{X}^{\oplus} \colon \varphi = \psi\big\}.$$

Then for all $(z,\varphi,x,\psi) \in \mathscr{E}$ we have

$$\mathscr{U}(t;u,u)(z,\varphi,x,\psi) = \big(z(t), u(t+\cdot), x(t), u(t+\cdot)\big), \quad t \geqslant 0.$$

Thus $\mathscr{U}(t;u,u)\mathscr{E} \subset \mathscr{E}$ for any $t \geqslant 0$. The feedback law $\binom{u}{u} = y$ shows that $x(t) = u(t)$ and $z(t) = D_{\mathrm{Y}}u(t+\cdot)$ for a.e. $t \geqslant 0$. Then, by (3.4) we have

$$\mathscr{U}(t;u,u)(z,\varphi,x,\varphi) = \begin{pmatrix} D_{\mathrm{Y}}u(t+\cdot) \\ u(t+\cdot) \\ \mathscr{T}_{\delta_0}(t)\binom{x}{\varphi} \end{pmatrix}, \quad t \geqslant 0.$$

Since $\mathscr{E}$ is $(\mathscr{U}(t;u,u))_{t\geqslant 0}$-invariant closed subspace of $\mathscr{X}^{\oplus}$, then by [7, p. 61] the restriction of $\mathscr{U}(t;u,u)$ on $\mathscr{E}$ define a strongly continuous semigroup $(\mathscr{U}_{|}(t;u,u))_{t\geqslant 0}$ on $\mathscr{E}$ with generator $(\mathscr{G}_{|}, \mathscr{D}(\mathscr{G}_{|}))$

$$\mathscr{G}_{|} = \mathscr{G}, \quad \mathscr{D}(\mathscr{G}_{|}) = \mathscr{D}(\mathscr{G}) \cap \mathscr{E} = \mathscr{D}\big(\mathscr{A}^{\oplus}\big) \cap \mathscr{E}. \tag{3.17}$$

Put

$$\mathscr{T}_{\mathrm{D}}(t)\begin{pmatrix} z \\ \varphi \end{pmatrix} := \begin{pmatrix} z(t;z,\varphi\cdot) \\ u(t+\cdot;\varphi) \end{pmatrix} = \begin{pmatrix} D_{\mathrm{Y}}u(t+\cdot) \\ u(t+\cdot;\varphi) \end{pmatrix}, \quad t \geqslant 0.$$

Then

$$\mathscr{U}_{|}(t;u,u)(z,\varphi,x,\varphi) = \begin{pmatrix} \mathscr{T}_{\mathrm{D}}(t)\binom{z}{\varphi} \\ \mathscr{T}_{\delta_0}(t)\binom{x}{\varphi} \end{pmatrix} \tag{3.18}$$

for any $(z,\varphi,x,\varphi) \in \mathscr{E}$ and $t \geqslant 0$. In particular

$$\left\| \mathscr{T}_{\mathrm{D}}(t)\begin{pmatrix} z \\ \varphi \end{pmatrix} \right\| \leqslant \left\| \mathscr{U}_{|}(t;u,u)(z,\varphi,0,\varphi) \right\| \leqslant Me^{\omega t}\left\| \begin{pmatrix} z \\ \varphi \end{pmatrix} \right\|$$

for any $\binom{z}{\varphi} \in \mathscr{X}$, $t \geqslant 0$, and constants $\omega \in \mathbb{R}$ and $M \geqslant 1$. Since $\mathscr{T}_{\delta_0}(t)$ is a strongly continuous semigroup on $\mathscr{X}$, then by using (3.18) one can see that $(\mathscr{T}_{\mathrm{D}}(t))_{t\geqslant 0}$ is a $\mathscr{C}_0$-semigroup on $\mathscr{X}$. Let us now compute the generator of this semigroup. For this we consider $(z,\varphi,x,\varphi) \in \mathscr{D}(\mathscr{G}_{|})$. Then by (3.17) we have $\binom{z}{\varphi} \in \mathscr{D}(\mathscr{A}_{\mathrm{D}})$ and $\binom{x}{\varphi} \in \mathscr{D}(\mathscr{A}_{\delta_0})$. Now using (3.18) we get

$$\begin{aligned} &\left\| \frac{1}{t}\left[\mathscr{T}_{\mathrm{D}}(t)\begin{pmatrix} z \\ \varphi \end{pmatrix} - \begin{pmatrix} z \\ \varphi \end{pmatrix} \right] - \mathscr{A}_{\mathrm{D}}\begin{pmatrix} z \\ \varphi \end{pmatrix} \right\| \\ &\quad = \left\| \frac{1}{t}\big[\mathscr{U}_{|}(t;u,u)(z,\varphi,x,\varphi) - (z,\varphi,x,\varphi)\big] - \mathscr{G}_{|}(z,\varphi,x,\varphi) \right\| \\ &\qquad - \left\| \frac{1}{t}\left[\mathscr{T}_{\delta_0}(t)\begin{pmatrix} z \\ \varphi \end{pmatrix} - \begin{pmatrix} x \\ \varphi \end{pmatrix} \right] - \mathscr{A}_{\delta_0}\begin{pmatrix} x \\ \varphi \end{pmatrix} \right\|. \end{aligned}$$

This shows that $\mathscr{A}_{\mathrm{D}}$ is the generator of $\mathscr{T}_{\mathrm{D}}$. Finally, the rest of the proof follows from (3.15). □

Proof of Theorem 2.4. Theorem 2.3 implies that the nonhomogeneous Cauchy problem $(\mathscr{C}_{\mathrm{D}})_F$ has a unique mild solution given by

$$w(t) = \mathscr{T}_{\mathrm{D}}(t)\begin{pmatrix} z \\ \varphi \end{pmatrix} + \int_0^t \mathscr{T}_{\mathrm{D}}(t-\tau)\begin{pmatrix} f(\tau) \\ 0 \end{pmatrix} d\tau \tag{3.19}$$

for any $t \geqslant 0$ and $\binom{z}{\varphi} \in \mathscr{X}$. Let us now show an explicit expression for w in terms of L and f. For this let the operators $\mathfrak{A}$, B and C be as in the proof of Lemma 3.3. Using [13] and [14] one can see that

$$B = \begin{pmatrix} 0 & 0 \\ \mathbb{B} & 0 \\ 0 & 0 \\ 0 & \mathbb{B} \end{pmatrix}$$

where $\mathbb{B} \in \mathscr{L}(X, (L^p([-r, 0], X))_{-1})$ is such that $(\lambda - Q_{-1})e_\lambda = \mathbb{B}$ for any $\lambda \in \mathbb{C}$. We now set $\mathbb{U} := X \times X \times X$ and introduce a new operator $\mathscr{B} : \mathbb{U} \to \mathscr{X}^{\oplus}_{-1}$,

$$\mathfrak{B} = \begin{pmatrix} 0 & 0 & I_X \\ \mathbb{B} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \mathbb{B} & 0 \end{pmatrix}.$$

Define

$$\tilde{\Phi}(t)\varpi = \int_0^t \mathfrak{T}_{-1}(t-\tau)\mathfrak{B}\varpi(\tau)\, d\tau$$

for $t \geqslant 0$ and $\varpi \in L^p_{\mathrm{loc}}([0, \infty), \mathbb{U})$. Now it is not difficult to show that $\tilde{\Phi}$ satisfies (3.1). Moreover, if $\varpi = (u, v, f)$ then by (3.8) we have

$$\begin{aligned} \tilde{\Phi}(t)\varpi &= \Phi(t)\begin{pmatrix} u \\ v \end{pmatrix} + \int_0^t \mathfrak{T}(t-\tau)\big(f(\tau), 0, 0, 0\big)\, d\tau \\ &= \begin{pmatrix} \int_0^t T(t-\tau)[L_{\mathrm{Y}}\phi(\tau)u + f(\tau)]\, d\tau \\ \phi(t)u \\ \int_0^t T(t-\tau)L_{\mathrm{Y}}\phi(\tau)v\, d\tau \\ \phi(t)v \end{pmatrix}, \quad t \geqslant 0. \end{aligned} \tag{3.20}$$

On the other hand, we define a linear operator $\mathfrak{C} : \mathscr{D}(\mathfrak{A}) \to \mathbb{U}$ by

$$\mathfrak{C} := \begin{pmatrix} I & -D & -I & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Similarly to the proof of Lemma 3.3 it can be verified that $\mathfrak{C}$ satisfies the condition (**W**) with respect to $\mathfrak{A}$ and $\tilde{\Phi}$. Then by Theorem 3.1 the following operator

$$\mathfrak{R} = \mathfrak{A}_{-1} + \mathfrak{B}\mathfrak{C}_{\mathrm{Y}}, \quad \mathscr{D}(\mathfrak{R}) := \big\{Z \in \mathscr{D}(C_{\mathrm{Y}}):\ (\mathfrak{A}_{-1} + \mathfrak{B}\mathfrak{C}_{\mathrm{Y}})Z \in \mathscr{X}^{\oplus}\big\},$$

generates a unique $\mathscr{C}_0$-semigroup $\mathfrak{U} = (\mathfrak{U}(t))_{t \geqslant 0}$ on $\mathscr{X}^{\oplus}$, where $\mathfrak{C}_{\mathrm{Y}}$ is the Yosida extension of $\mathfrak{C}$ with respect to $\mathfrak{A}$. Observe that $\mathscr{D}(\mathfrak{C}_{\mathrm{Y}}) = \mathscr{D}(C_{\mathrm{Y}})$ and $\mathfrak{C}_{\mathrm{Y}} = [C_{\mathrm{Y}}\ 0]$. Thus $\mathfrak{B}\mathfrak{C}_{\mathrm{Y}} = BC_{\mathrm{Y}}$ and by the proof of Lemma 3.3 we have $\mathfrak{R} = \mathscr{G}$ and $\mathfrak{U} = \mathcal{U}$. Now using the background of Remark 3.2

we have $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ generates a regular system $(\tilde{\Sigma})$ with $I_{\mathbb{U}}$ as admissible feedback operator. Moreover, the closed-loop system $(\tilde{\Sigma})^I$ associated to $(\tilde{\Sigma})$ is generated by the triple $(\mathscr{G}, \mathfrak{B}, \mathfrak{C}_{\mathrm{Y}})$. Let us consider the feedback law $\omega = \tilde{y} + \omega_{\mathrm{c}}$ with $\omega = (u, u, f)$ the control function of $(\tilde{\Sigma})$ and $\tilde{y} = \binom{y}{0}$ the output function of $(\tilde{\Sigma})$, where y is given by (3.16), and ω_{c} is another new control function. Now Remark 3.2 shows that the state trajectory of $(\tilde{\Sigma})^I$ is given by

$$\tilde{\omega}(t; u, u, f) = \mathscr{U}_{|}(t; u, u)(z, \varphi, x, \varphi) + \tilde{\Phi}(t)\omega_{\mathrm{c}} \tag{3.21}$$

for $(z, \varphi, x, \varphi) \in \mathscr{E}$ and $t \geqslant 0$. Now choose $\omega_{\mathrm{c}} = (0, 0, f)$ and use the fact that $\mathfrak{B}\omega_{\mathrm{c}} = (f, 0, 0, 0)$ and (3.21) one can see that

$$\tilde{\omega}(t; u, u, f) = \mathscr{U}_{|}(t; u, u)(z, \varphi, x, \varphi) + \int_0^t \mathscr{U}_{|}(t - \tau; u, u)\big(f(\tau), 0, 0, 0\big)\, d\tau. \tag{3.22}$$

Moreover, by using (3.18) and (3.19) one obtains that

$$\tilde{\omega}(t; u, u, f) = \begin{pmatrix} w(t) \\ \mathscr{T}_{\delta_0}(t)\binom{x}{\varphi} \end{pmatrix}, \quad t \geqslant 0. \tag{3.23}$$

Since $\tilde{\omega}(\cdot\,; u, u, f)$ is also the state trajectory of $(\tilde{\Sigma})$ corresponding to the input $\omega = (u, u, f)$ then by using (3.20) we obtain

$$\tilde{\omega}(t; u, u, f) = \mathfrak{T}(z, \varphi, x, \varphi) + \tilde{\Phi}(t)\omega = \big(z(t), u(t + \cdot), x(t), u(t + \cdot)\big), \tag{3.24}$$

where

$$z(t) = T(t)z + \int_0^t T(t - \tau)\big[L_{\mathrm{Y}} u(\tau + \cdot) + f(\tau)\big]\, d\tau, \quad t \geqslant 0,$$

and

$$x(t) = T(t)x + \int_0^t T(t - \tau) L_{\mathrm{Y}} u(\tau + \cdot)\, d\tau, \quad t \geqslant 0.$$

Now using the feedback law $\varpi = \binom{y}{0} + \varpi_{\mathrm{c}}$ and (3.16) one can see that

$$x(t) = u(t) \quad \text{and} \quad z(t) = D_{\mathrm{Y}} u(t + \cdot)$$

for almost every $t \geqslant 0$. Now by combining (3.23), (3.24) and (3.4) we have

$$w(t) = \begin{pmatrix} z(t) \\ u(t + \cdot) \end{pmatrix}, \quad t \geqslant 0.$$

This ends the proof. $\square$

Remark 3.4. Here we discuss the difference between well-posedness of neutral equations in the case of atomic and nonatomic difference operators.

Assume that D is atomic, that is $D\varphi = \varphi(0) - K\varphi$ and $K : W^{1,p}([-r,0],X) \to X$ and consider the neutral equation $(\mathscr{N}_0)$ defined above. This equation can be written as

$$\begin{aligned} \dot{z}(t) &= Az(t) + Lu(t+\cdot), \\ u(t) &= z(t) + Ku(t+\cdot). \end{aligned} \tag{3.25}$$

It is shown in [13] that under the condition (**H**) on L the solution $z(\cdot) : [0,\infty) \to X$ of the first equation in (3.25) satisfies

$$w(t) = \mathscr{W}(t)\begin{pmatrix} z \\ \varphi \end{pmatrix} + \Gamma(t)u = \begin{pmatrix} z(t) \\ u(t+\cdot) \end{pmatrix} \tag{3.26}$$

for any $\binom{z}{\varphi} \in \mathscr{X}$, $t \geqslant 0$, and $u \in L^p_{\mathrm{loc}}([-r,\infty),X)$ with $u(\theta) = \varphi(\theta)$ for a.e. $\theta \in [-r,0]$, where $\mathscr{W}$ and Γ are defined above. On the other hand, the second equation in (3.25) can be written as

$$u(t) = [I\ \ K]w(t), \quad t \geqslant 0. \tag{3.27}$$

Thus the solution of $(\mathscr{N}_0)$ can be easily obtained if we close the open loop (3.26) by the input (3.27). To that purpose it suffices to assume that K satisfies $(\mathbf{H}')$ and check that the operator $C := [I\ \ K] : \mathscr{D}(\mathscr{A}) \to X$ satisfies (**W**) with respect to $\mathscr{A}$ and $\Gamma(t)$, and then use Theorem 3.1. It turns that the particular form of the atomic operator D helps to use the feedback theory in a simple way.

Now if D is a nonatomic operator then one can not obtain a relation like (3.27). Thus one can not proceed using a direct feedback theory to solve $(\mathscr{N}_0)$. To overcome this problem we have used an augmented equation by introducing a $\mathscr{C}_0$-semigroup $(\mathscr{U}(t))_{t\geqslant 0}$ on $\mathscr{X}^{\oplus}$ (see Lemma 3.3). In the proof of Theorems 2.3 and 2.4 we have seen that the restriction of the semigroup $\mathscr{U}$ on an appropriate closed subspace helps to solve $(\mathscr{N}_f)$.

4. Well-posedness in weighted spaces

In this section we study the well-posedness of a singular neutral equation in weighted spaces. Here we propose a new approach different from that used in [21] and [29].

We consider the singular neutral reaction–diffusion equation

$$\begin{aligned} &\frac{d}{dt}\left(\int_{-r}^{0} c|\theta|^{-\frac{1}{2}} u(t+\theta,x)\,d\theta\right) \\ &\quad = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}\left(\int_{-r}^{0} c|\theta|^{-\frac{1}{2}} u(t+\theta,x_k)\,d\theta\right) \\ &\qquad + a\int_{-r}^{0} u(t+\theta,x)\,d\xi(\theta) + f(t,x), \quad x \in \Omega,\ t \geqslant 0, \end{aligned}$$

$$\int_{-r}^{0} c|\theta|^{-\frac{1}{2}} u(t+\theta,x)\,d\theta = 0, \qquad x \in \partial\Omega,\ t \geqslant 0,$$

$$x(\theta,x) = \varphi(\theta,x), \qquad \text{a.e. } (\theta,x) \in [-r,0]\times\Omega, \tag{4.1}$$

where $c, a > 0$ are some constants, $x = (x_1, \dots, x_n)$, $\Omega \subset \mathbb{R}^n$ is a bounded open set with boundary $\partial\Omega$ and $\xi : [-1,0] \to [0,1]$ is a function of bounded variation (one can consider ξ as the Cantor function, see [8, Example I.8.15], which is singular with total variation 1).

To adopt the previous abstract result we first define

$$A\psi := \Delta\psi, \quad \mathscr{D}(A) = \left\{ \psi \in H_0^1(\Omega)\colon \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} \psi(x_1,\dots,x_n) \in L^2(\Omega) \right\}.$$

Thus A is the Dirichlet-Laplace, and then generates a $\mathscr{C}_0$-semigroup $(T(t))_{t\geqslant 0}$ on $X := L^2(\Omega)$. On the other hand, we set $g(\theta) = c|\theta|^{-1/2}$ and $\eta := a\xi\ I : [-r,0] \to \mathscr{L}(X)$ which is a function of bounded variation. We now define

$$D\varphi = \int_{-r}^{0} \varphi(\theta) g(\theta)\,d\theta \quad \text{and} \quad L\varphi = \int_{-r}^{0} d\eta(\theta)\,\varphi(\theta) \tag{4.2}$$

for $\varphi \in C([-r,0],X)$. Note that D is bounded on $C([-r,0],X)$; however it is unbounded operator on $L^2([-r,0],X)$. One easily shows that D is nonatomic at zero.

To discuss the well-posedness of (4.1) we need some weighted L^2-spaces. Let $L_g^2([-r,0],X)$ denotes the Lebesgue space $L^2([-r,0],X)$ weighted by g, that is,

$$\|\varphi\|_{2,g} := \left(\int_{-r}^{0} \|\varphi(\theta)\|^2 g(\theta)\,d\theta \right)^{1/2} < \infty.$$

It is to be noted that $L_g^2([-r,0],X)$ endowed with the norm $\|\cdot\|_{2,g}$ is a Banach space, which is densely and continuously embedded in $L^2([-r,0],X)$ with $\|\varphi\|_2 \leqslant cr^{1/2}\|\varphi\|_{2,g}$ for $\varphi \in L_g^2([-r,0],X)$. Further, as in [28, Observation 2.1], $W_g^{1,2}([-r,0],X)$ is continuously injected in $W^{1,2}([-r,0],X)$, where $W_g^{1,2}([-r,0],X)$ is the Sobolev space associated with $L_g^2([-r,0],X)$ defined by

$$W_g^{1,2}\big([-r,0],X\big) = \left\{ \varphi \in W^{1,2}\big([-r,0],X\big)\colon \frac{\partial\varphi}{\partial\theta} \in L_g^2\big([-r,0],X\big) \right\}.$$

It is not difficult to show that $L_g^2([-r,0],X)$ is $(S(t))_{t\geqslant 0}$-invariant subspace. So that the restriction of $S(t)$ to $L_g^2([-r,0],X)$, denoted by $\check{S}(t)$, is a strongly continuous semigroup on $L_g^2([-r,0],X)$ generated by the following operator:

$$\mathscr{D}(\check{Q}) := \left\{ \varphi \in W^{1,2}\big([-r,0],X\big)\colon \frac{\partial\varphi}{\partial\theta} \in L_g^2\big([-r,0],X\big) \text{ and } \varphi(0) = 0 \right\},$$

$$\check{Q}\varphi = \frac{\partial\varphi}{\partial\theta}.$$

We note that $e_\lambda \in L^2_g([-r,0],X)$ for all $\lambda \in \mathbb{C}$.

Let us now define the space

$$\mathscr{X}_g := X \times L^2_g\big([-r,0],X\big) \quad \text{with the norm} \left\| \begin{pmatrix} z \\ \varphi \end{pmatrix} \right\| = \|z\| + \|\varphi\|_{2,g}.$$

We introduce the operator

$$\check{\mathscr{A}}_{\mathrm{D}} := \begin{pmatrix} A & L \\ 0 & \frac{\partial}{\partial\theta} \end{pmatrix},$$

$$\mathscr{D}(\check{\mathscr{A}}_{\mathrm{D}}) := \left\{ \begin{pmatrix} z \\ \varphi \end{pmatrix} \in \mathscr{D}(A) \times W^{1,p}\big([-r,0],X\big) \colon \varphi \in L^2_g\big([-r,0],X\big) \text{ and } z = D\varphi \right\}.$$

Theorem 4.1. *The operator $\check{\mathscr{A}}_{\mathrm{D}}$ generates a $\mathscr{C}_0$-semigroup $\check{T}_{\mathrm{D}} := (\check{T}_{\mathrm{D}}(t))_{t\geqslant 0}$ on $\mathscr{X}_g$. Moreover the singular neutral Eq.* (4.1) *is well-posed in strong sense.*

Proof. It suffices to verify that the operators L and D satisfy the condition (**H**) and (**H**)$'$ with respect to $\check{S}$. Let us first show (H1). Let $\varphi \in \mathscr{D}(\check{Q})$ and $\alpha \in (0, r/2)$. By Hölder's inequality we obtain

$$\begin{aligned}
\int_0^\alpha \big\|DS(t)\varphi\big\|_X^2\,dt &\leqslant \int_0^\alpha \left(\int_{-r}^{-t} g(s)\big\|\varphi(t+s)\big\|\,ds \right)^2 dt \\
&\leqslant \int_0^\alpha \left(\int_{t-r}^{0} \sqrt{g(s)}\big(\sqrt{g(s)}\big\|\varphi(s)\big\|\big)\,ds \right)^2 dt \\
&\leqslant \int_0^\alpha \int_{t-r}^{0} g(s)\,ds \int_{t-r}^{0} g(s)\big\|\varphi(s)\big\|^2\,ds\,dt \\
&\leqslant \alpha\|g\|_1\|\varphi\|_{2,g}^2.
\end{aligned}$$

This shows that (H1) is verified for D with $\gamma_1(\alpha) = \sqrt{\alpha\|g\|_1}$. The fact that L satisfies (H1) follows from [15, Theorem 3] and $\|\varphi\|_2 \leqslant r^{1/2}\|\varphi\|_{2,g}^2$.

By [15, Theorem 3] L satisfies (H2). Let us show that D satisfies (H2) as well. For this let $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}_+, X)$ with $u(0) = 0$ and let $\beta > 0$. A similar argument as above shows that

$$\int_0^\beta \big\|D\phi(t)u\big\|_X^2\,dt \leqslant \int_0^\tau \left(\int_{-t}^{0} g(s)\big\|u(t+s)\big\|\,ds \right)^2 dt$$

$$\begin{aligned}
&\leqslant \gamma \int_0^\beta \int_{-t}^0 g(s)\big\|u(t+s)\big\|^2\,ds\,dt\\
&= \gamma \int_0^\beta \int_0^t (t-\sigma)^{-\frac{1}{2}}\big\|u(\sigma)\big\|^2\,d\sigma\,dt\\
&= \gamma \int_0^\beta \int_\sigma^\beta (t-\sigma)^{-\frac{1}{2}}\,dt\,\big\|u(\sigma)\big\|^2\,d\sigma\\
&= 2\gamma \int_0^\beta (\beta-\sigma)^{\frac{1}{2}}\big\|u(\sigma)\big\|^2\,d\sigma\\
&= 2\gamma\beta^{\frac{1}{2}} \int_0^\beta \big\|u(\sigma)\big\|^2\,d\sigma.
\end{aligned} \tag{4.3}$$

This shows that D satisfies (H2). Observe that $\phi(t)u \in L_g^2([-r,0],X)$ for any $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+,X)$ and $t \geqslant 0$. In fact for $t > r$ we have,

$$\int_{-r}^0 g(s)\big\|\big(\phi(t)u\big)(s)\big\|^2\,ds = \int_{-r}^0 g(s)\big\|u(t+s)\big\|^2\,ds \leqslant \frac{\|u\|_2^2}{\sqrt{t-r}}.$$

Now for $t \in (0,r)$ we have

$$\int_{-r}^0 g(s)\big\|\big(\phi(t)u\big)(s)\big\|^2\,ds \leqslant \big\|\phi(r)u\big\|^2\|g\|_1 \leqslant \|g\|_1\|u\|_2^2.$$

The fact that L satisfies (H3) follows from [15, Theorem 3 and Corollary 1]. Since $R(\lambda,\check{Q})$ is the restriction of $R(\lambda,Q)$ on $L_g^2([-r,0],X)$ for $\lambda \in \mathbb{C}$ then for $z \in X$ and $\mu > \operatorname{Re}\lambda$,

$$\big\|D\mu R(\mu,\check{Q})e_\lambda z\big\| \leqslant \frac{\mu}{\mu-\operatorname{Re}\lambda}\int_{-r}^0 g(\theta)e^{\mu\theta}\,d\theta\,\|z\|.$$

Using Lebesgue's dominated convergence theorem one shows that $D\mu R(\mu,\check{Q})e_\lambda z$ goes to zero as $\mu \to +\infty$. Now if we set $\mathbb{F}_\infty^D u = D\phi(\cdot)u$ for $u \in W^{1,2}_{0,\mathrm{loc}}(\mathbb{R}+,X)$ by choosing a small β in (4.3) one can see that D satisfies (H4). Now by proceeding exactly as in the proofs of Theorems 2.3 and 2.4 (see Section 3) our aim follows. □

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