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Control Systems on Lie Groups*

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1. INTRODUCTION

In this article we study the controllability properties of systems which are described by an evolution equation in a Lie group G of the form:

$$\frac{dx}{dt}(t) = X_0(x(t)) + \sum_{i=1}^{m} u_i(t)\, X_i(x(t)) \qquad (*)$$

where $X_0, \ldots, X_m$ are right-invariant vector fields on G. Systems described by (*) we term right-invariant. This study is based on the results of [11], and is related to the work of Brockett [1]. As remarked by Brockett, there are many important applications in engineering and in physics which are not treated by classical control theory because of the assumption that the state space is a vector space. In particular, when controlling the orientation of a rigid body relative to some fixed set of axes, the state space is the tangent bundle of $SO(3)$ (the group of 3×3 real orthogonal matrices M such that $\det M = 1$). The evolution equation describing this system is of the form given by (*) [1]. Instead of restricting our study to groups of matrices, we consider systems described in an abstract Lie group G. This generalization in no essential way affects the nature of the problem.

From the theoretical point of view a study of systems of the form (*) appears natural for several reasons. For instance, the algebraic criteria developed in [11] can be used to obtain global results by exploiting the algebraic structure of the state space and the sets attainable from the identity. In this regard, the analogy with the controllability of linear systems is striking.

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In this article we shall look for necessary and sufficient conditions for a right-invariant system to be controllable. A necessary condition is that the system have the "accessibility property" [11]. We show that this condition is also sufficient if G is connected and if either (a) the system is homogeneous (i.e., $X_0 = 0$) or (b) G is compact. When neither (a) nor (b) hold, accessibility (plus the connectedness of G) is not sufficient for controllability. In this case we give some sufficient conditions, and a necessary condition, and we single out a particular situation in which a necessary and sufficient condition can be obtained.

An obvious necessary condition for controllability is that the set $\mathbf{A}(e)$ of points reachable from the identity of G be a subgroup of G. Thus, the controllability problem reduces to the following:

(a) When is $\mathbf{A}(e)$ a subgroup?, and
(b) If $\mathbf{A}(e)$ is a subgroup, when is $\mathbf{A}(e) = G$?

Question (b) is much easier to answer than question (a). In Theorem 4.6 we show that if $\mathbf{A}(e)$ is a subgroup; then, necessarily, this subgroup is the connected Lie subgroup $\mathbf{S}$ of G whose Lie algebra is the subalgebra $\mathbf{L}$ generated by $X_0, \ldots, X_m$. From this it follows that the system will be controllable if and only if (i) $\mathbf{A}(e)$ is a subgroup, (ii) G is connected, and (iii) $\mathbf{L}$ is the Lie algebra of G. This shows that the crucial question is that of determining when $\mathbf{A}(e)$ is a subgroup.

This question is (partially) answered in Sections 5 and 6.

The organization of the article is as follows: In Section 2 we introduce notation and basic concepts; in addition, we quote a result about Lie groups which will be used later. In Section 3 we single out the relevant Lie algebras induced by a right invariant system. In Section 4 we derive the basic properties of attainable sets. In Section 5 we study the homogeneous case, and in Section 6 we study the general case. In Section 7 we interpret our results in terms of controllability. Finally, Section 8 contains examples.

2. Preliminaries

We shall assume that the reader is familiar with the basic facts about Lie groups [2, 4, or 5].

Throughout this paper, G will denote a Lie group, and $L(G)$ will denote the Lie algebra of G. We shall think of $L(G)$ as the set of vector fields on G that are invariant under *right* translations. It is known that every $X \in L(G)$ is analytic, and that $L(G)$ is a Lie algebra with the obvious vector operations, and with the Lie product defined by $[X, Y] = XY - YX$.

The exponential map from $L(G)$ into G is denoted by exp. Recall that $\exp(0) = e$ (the identity of G), and that, for each $X \in L(G)$, the curve $t \to \exp(tX)$ is an integral curve of X.

We recall that there is a one-to-one correspondence between Lie subalgebras of $L(G)$ and connected Lie subgroups of G. If H is a connected Lie subgroup of G, the Lie algebra $L(H)$ is naturally identified with a subalgebra of $L(G)$. We shall also denote this subalgebra by $L(H)$.

Let $X_0, \ldots, X_m$ be elements of $L(G)$. We shall consider the following control system defined on G:

$$\dot{x}(t) = X_0(x(t)) + \sum_{i=1}^{m} u_i(t) X_i(x(t)) \tag{1}$$

where $u = (u_1, \ldots, u_m)$ belongs to the class of admissible controls U. Throughout the article we shall assume that U is one of the classes U_u, U_r or U_b, defined as follows:

(i) U_u is the class of all locally bounded and measurable functions defined on the interval $[0, \infty)$ and having values in $\mathbf{R}^m$.

(ii) U_r is the subset of U_u consisting of all elements which take values in the cube $\{x \in \mathbf{R}^m : |x_i| \leqslant 1, i = 1, \ldots, m\}$.

(iii) U_b is the class of all piecewise constant functions defined on $[0, \infty)$ with values in $\mathbf{R}^m$ such that the components of its elements only take values 1 and -1.

We will refer to U_u, U_r and U_b as the class of *unrestricted*, *restricted* and "*bang-bang*" controls, respectively.

If $\mathbf{X} = (X_0, \ldots, X_m)$ is an $m + 1$-tuple of elements of $L(G)$, and if U is a class of admissible controls, then the system described by Eq. (1) will be termed *right-invariant*. For notational convenience, we will denote such a system by $(\mathbf{X}, U)$. We will also adopt the convention that if in a particular statement U is not specified explicitly, we will mean that such a statement is true for any class of admissible controls (i.e., U_u, U_r or U_b).

We have the following basic fact:

LEMMA 2.1. *Let* $(\mathbf{X}, U)$ *be a right-invariant system on* G, *and let* $u \in U$. *Then for every* $g \in G$, *there exists a unique solution*[1] x *of* (1), *defined for* $0 \leqslant t < \infty$, *such that* $x(0) = g$.

Proof. Uniqueness and local existence follow from the standard results on ordinary differential equation. Moreover, we know from these results

[1] A solution of (1) is an absolutely continuous G-valued function of the real variable t, with the property that (1) is satisfied for almost every t.

that there is a *maximal* interval $[0, T)(T > 0)$ on which there exists a solution x of (1) with $x(0) = g$. We show that $T = \infty$. Assume $T < \infty$. Let $y(t)$ be a solution of (1) defined for $T - \delta < t < T + \delta$, where $\delta > 0$, and such that $y(T) = e$. Let $g' = y(T - \frac{1}{2}\delta)$, $g'' = x(T - \frac{1}{2}\delta)$. Let $z(t)$ be defined by

$$z(t) = x(t) \qquad \text{for} \quad 0 \leqslant t \leqslant T - \tfrac{1}{2}\delta,$$
$$z(t) = y(t)\, g'^{-1} g'' \qquad \text{for} \quad T - \tfrac{1}{2}\delta < t \leqslant T + \delta.$$

Then $z(t)$ is a solution of (1) which satisfies $z(0) = g$ and is defined for $0 \leqslant t < T + \delta$. This contradicts the maximality of the interval $[0, T)$. Therefore $T = \infty$ and our proof is complete.

If $u \in U$ and $g \in G$ we will denote the solution x of (1) which satisfies $x(0) = g$ by $\pi(g, u, \cdot)$; i.e., $x(t) = \pi(g, u, t)$ for all $t \in [0, \infty)$. If, for some $t \geqslant 0$, $\pi(g, u, t) = g'$, we say that the control *u steers g into g' in t units of time.* If there exists $u \in U$ which steers g into g' in t units of time, we say that g' is *attainable* (or *reachable*) *from g at time t.* The set of all $g' \in G$ which are attainable from g at time t will be denoted by $A(g, t)$. We shall also use the notations

$$\mathbf{A}(g, T) = \bigcup_{0 \leqslant t \leqslant T} A(g, t)$$

$$\mathbf{A}(g) = \bigcup_{0 \leqslant t \leqslant \infty} A(g, t).$$

We shall refer to $\mathbf{A}(g)$ as the *set attainable from g.*

From the right invariance of our control systems it follows trivially that $A(g, T) = A(e, T)g$, $\mathbf{A}(g, T) = \mathbf{A}(e, T)g$, and $\mathbf{A}(g) = \mathbf{A}(e)\, g$.[2] Therefore, without loss of generality, *we can limit ourselves to the study of the sets attainable from the identity.*

We finish this section by quoting a result about Lie groups whose proof can be found in [12] (cf. also [5, pp. 275]).

THEOREM 2.2. *Let G be a Lie group, and let H be a path-connected subgroup of G. Then H is a Lie subgroup of G.*

3. THE ASSOCIATED LIE SUBALGEBRAS

To every right-invariant control system $(\mathbf{X}, U)$ on a Lie group G, we shall associate the following three Lie subalgebras of $L(G)$:

[2] If A is a subset of G, and $g \in G$, we use Ag to denote the set of all products ag, where $a \in A$.

(1) The subalgebra $\mathbf{L}$ generated by $X_0, ..., X_m$,

(2) The ideal of $\mathbf{L}$ generated by $X_1, ..., X_m$. This ideal will be denoted by $\mathbf{L}_0$.

(3) The subalgebra L of $L(G)$ generated by $X_1, ..., X_m$.

We denote the corresponding connected Lie subgroups by $\mathbf{S}$, $\mathbf{S}_0$ and S. We have

LEMMA 3.1. (i) $L \subset \mathbf{L}_0 \subset \mathbf{L}$ *and* $S \subset \mathbf{S}_0 \subset \mathbf{S}$,

(ii) $\mathbf{L}_0$ *is a subspace of* $\mathbf{L}$ *of codimension zero or one*,

(iii) $\mathbf{S}_0$ *is a normal subgroup of* $\mathbf{S}$.

Proof. (i) and (ii) are trivial. (iii) follows from the fact that a connected Lie subgroup H of a connected Lie group K is a normal subgroup of K if and only if $L(H)$ is an ideal of $L(K)$ (cf. [2, p. 124]).

We shall use the notation $\mathbf{S}_0^t$ for the coset of $\mathbf{S}$ modulo $\mathbf{S}_0$ which contains $\exp(tX_0)$.

4. ELEMENTARY PROPERTIES OF THE ATTAINABLE SETS

If $(\mathbf{X}, U)$ is a right-invariant control system on G, then the vector fields $X_0, ..., X_m$ belong to the Lie algebra of $\mathbf{S}$. Therefore, we can consider $(\mathbf{X}, U)$ as a right-invariant control system on $\mathbf{S}$, and Lemma 2.1 will be valid if G is replaced by $\mathbf{S}$. This gives

LEMMA 4.1. *If* $(\mathbf{X}, U)$ *is a right-invariant system on* G, *then* $\mathbf{A}(e)$ *is contained in* $\mathbf{S}$.

The following lemma states a similar result for the sets $A(e, t)$.

LEMMA 4.2. *If* $(\mathbf{X}, U)$ *is a right-invariant system on* G, *then for each* $t > 0$ $A(e, t)$ *is contained in* S_0^t.

It would be easy to prove this lemma directly, but since this result is included in that of Lemma 6.1 we omit the proof.

We next derive some elementary topological properties of the attainable sets. If $T \geqslant 0$ we will denote the set of all restrictions of elements of U to $[0, T]$ by $U(T)$.

LEMMA 4.3. *Let* $(\mathbf{X}, U)$ *be a right-invariant control system on* G. *The mapping* $(u, t) \to \pi(g, u, t)$ *from* $U(T) \times [0, T]$ *into* G *is continuous for each* g *and each* $T \geqslant 0$, *if* $U(T)$ *is given the topology of weak convergence.*

The proof of this result appears in [10], and therefore we will omit it.[3] From this we obtain:

LEMMA 4.4. *Let* $(\mathbf{X}, U)$ *be a right-invariant control system on* G.

(i) *The sets* $\mathbf{A}(e, T)$, $\mathbf{A}(e)$, $A(e, T)$ *are path-connected, for each* $T \geqslant 0$.

(ii) *If* $U = U_r$ *then* $\mathbf{A}(e, T)$ *and* $A(e, T)$ *are compact.*

Proof. (i) will be an immediate consequence of the fact that $U(T)$ is path-connected and of Lemma 4.3. The path-connectedness of $U(T)$ is trivial in the unrestricted and in the restricted case. In the "bang-bang" case, let u and v belong to $U(T)$. For each t such that $0 \leqslant t \leqslant T$, let w_t be defined by

$$w_t(\tau) = v(\tau) \qquad \text{if} \quad 0 \leqslant \tau \leqslant t,$$

$$w_t(\tau) = u(\tau) \qquad \text{if} \quad t < \tau \leqslant T.$$

Then $w_t \in U(T)$. Moreover, $w_0 = u$ and $w_T = v$. Since $t \to w_t$ is a continuous path in $U(T)$, it follows that $U(T)$ is indeed path-connected.

To prove (ii) we remark that, if U is the class of restricted controls, then $U(T)$ is compact in the weak topology. The proof is now complete.

In regard to the algebraic properties of the attainable sets we have the following:

LEMMA 4.5. *Let* $(\mathbf{X}, U)$ *be a right-invariant control system on* G. *Then the set* $\mathbf{A}(e)$ *is a semi-group.*

Proof. Let g and g' belong to $\mathbf{A}(e)$. Let $g = \pi(e, u, t)$, $g' = \pi(e, u', t')$. Let the control v be defined by

$$v(\tau) = u(\tau) \qquad \text{for} \quad 0 \leqslant \tau \leqslant t,$$

$$v(\tau) = u'(\tau - t) \qquad \text{for} \quad \tau > t.$$

Then $\pi(e, v, t + t') = g'g$, and therefore, $g'g \in \mathbf{A}(e)$. The proof is then complete.

We cannot assert, in general, that $\mathbf{A}(e)$ is a group. However, the following theorem tells us that, if $\mathbf{A}(e)$ is a group, then it *must* be the group $\mathbf{S}$.

THEOREM 4.6. *Let* $(\mathbf{X}, U)$ *be a right-invariant control system on* G. *If* $\mathbf{A}(e)$ *is a subgroup of* G, *then* $\mathbf{A}(e) = \mathbf{S}$.

[3] The result is proved in [10] for groups of matrices, but the proof is valid for arbitrary Lie groups. Alternatively, one could use Ado's Theorem [4] to go from the result of [10] to a "local" version of Lemma 4.3, and then deduce the general result.

Proof. We know that $\mathbf{A}(e)$ is path-connected. If $\mathbf{A}(e)$ is a subgroup, it follows from Theorem 2.2 that it is a Lie subgroup of G. Let Λ be its Lie algebra. Then $\Lambda \subset \mathbf{L}$, because $\mathbf{A}(e) \subset \mathbf{S}$ (Lemma 4.1). On the other hand, let $\mathbf{a} = (a_1, ..., a_m)$ be an m-tuple such that each a_i is ± 1. Let u be the constant control $u \equiv (a_1, ..., a_m)$. Then $u \in U$ and, therefore, the curve $t \to \pi(e, u, t)$ $(0 \leqslant t < \infty)$ is contained in $\mathbf{A}(e)$. In other words, if we let

$$X(\mathbf{a}) = X_0 + \sum_{i=1}^{m} a_i X_i ,$$

it follows that $\exp(tX(\mathbf{a}))$ belongs to $\mathbf{A}(e)$ for all $t \geqslant 0$. Since $\mathbf{A}(e)$ is a subgroup, this will be true for all real t. Therefore [4, p. 94], we can conclude that $X(\mathbf{a})$ belongs to Λ. Since the elements $X(\mathbf{a})$ form a system of generators of $\mathbf{L}$, we conclude that $\mathbf{L} \subset \Lambda$ and, therefore, $\mathbf{L} = \Lambda$ and $\mathbf{A}(e) = \mathbf{S}$.

5. The Homogeneous Case

A right-invariant control system $(\mathbf{X}, U)$ is *homogeneous* if $X_0 = 0$. As an introduction to the general case, we consider these systems first.

The result stated in the next theorem appeared first in a study by R. W. Brockett [1].

Theorem 5.1. *Let $(\mathbf{X}, U)$ be a homogeneous right-invariant control system on G. Then the set attainable from the identity is the subgroup $\mathbf{S}$. Moreover, if U is unrestricted then, for each $T > 0$, $A(e, T) = \mathbf{A}(e) = \mathbf{S}$.*

Proof. To prove the first statement it is sufficient, in view of Theorem 4.6, to show that $\mathbf{A}(e)$ is a subgroup. We know that $\mathbf{A}(e)$ is a semigroup. It remains to be shown that, if $g \in \mathbf{A}(e)$, then $g^{-1} \in \mathbf{A}(e)$. Let $\pi(e, u, t) = g$, where $u \in U$, $t \geqslant 0$. Let

$$\begin{aligned} v(s) &= -u(t - s) \qquad \text{for} \quad 0 \leqslant s \leqslant t, \\ v(s) &= u(s) \qquad\qquad \text{for} \quad s > t. \end{aligned}$$

Obviously, $v \in U$. Let

$$f(s) = \pi(e, u, t - s).$$

Then

$$\dot{f}(s) = \sum_{i=1}^{m} v_i(s)\, X_i(f(s)).$$

Therefore, f is a solution of the evolution equation corresponding to the control v. By the right-invariance we must have $f(s) = \pi(e, v, s)h$, where

$h = f(0) = g$. But $f(t) = \pi(e, u, 0) = e$. Therefore, $\pi(e, v, t) = g^{-1}$, and we have shown that $g^{-1} \in \mathbf{A}(e)$.

To prove the second statement, assume that U is unrestricted. Let $g = \pi(e, u, t)$ for some $u \in U$ and $t > 0$. Let $s > 0$, and define a control v by

$$v(\tau) = (t/s)\, u(\tau t/s) \qquad \text{for} \quad 0 \leqslant \tau < \infty.$$

An easy computation shows that $\pi(e, v, s) = g$. We have therefore shown that $A(e, t) \subset A(e, s)$. Similarly, $A(e, s) \subset A(e, t)$. Thus $A(e, s) = A(e, t)$ for all t, s such that $0 < t$, $0 < s$. Our proof is then complete.

Remark. The previous theorem implies that, for a homogeneous system

(a) The attainable set $\mathbf{A}(e)$ is a subgroup of G.

(b) The set $\mathbf{A}(e)$ is the same for the three classes of controls, so that, in particular, every $g \in G$ that can be reached from the identity by means of an arbitrary control, can also be reached by means of a "bang-bang" control (possibly at a later time).

(c) If $U = U_u$, then every $g \in G$ that can be reached from the identity can in fact be reached in an arbitrarily short time.

We shall see later that neither (a), nor (b), nor (c) need be true in the non-homogeneous case.

6. The General Case

Our subsequent study will be based on the following lemma:

Lemma 6.1. *Let* $(\mathbf{X}, U)$ *be a right-invariant control system on* G. *Then for each* $T > 0$,

(i) $\mathbf{A}(e, T)$ *is contained in* $\mathbf{S}$, *and the interior of* $\mathbf{A}(e, T)$ *is dense (in the topology of* $\mathbf{S}$*) in* $\mathbf{A}(e, T)$.

(ii) $A(e, T)$ *is contained in* S_0^T, *and the interior of* $A(e, T)$ *is dense (in the topology of* S_0^T*) in* $A(e, T)$.

Proof. We shall use the results of [11]. Our system is of the form considered in the remark following Example 5.2 of [11], with $M = G$, and with G acting on G by left translations. In the notations of [11], we have $\Omega = \mathbf{R}^m$, or $\Omega = \mathbf{C}^m$ (the unit cube in $\mathbf{R}^m$), or $\Omega = \mathbf{V}^m$ (the set of vertices of $\mathbf{C}^m$) in the unrestricted, restricted and "bang-bang" cases, respectively. In each of the three cases, the assumptions of [11] are satisfied, and an easy computation shows that $\mathscr{I}(D) = \mathbf{L}$ and that $\mathscr{I}_0(D) = \mathbf{L}_0$. Since $\mathbf{S}$ is the integral manifold of $\mathbf{L}$ through e [2, p. 108], our first statement follows from [11].

Similarly, it is easy to verify that $\mathbf{S}_0^T$ is precisely the submanifold $I_0^T(D, e)$ of [11], and the second part of our lemma follows.

In particular, it follows from Lemma 6.1 that *the interior of* $\mathbf{A}(e)$ *relative to* $\mathbf{S}$ *is nonempty.*

We shall also need the following:

LEMMA 6.2. *Let H be a connected Lie group, and let $L_1,..., L_n$ be elements of $L(H)$ that generate $L(H)$. Then every $h \in H$ is a finite product of elements of the form* $\exp(tL_i)$, *where t is real and $i = 1,..., n$.*

Proof. The set H' of all finite products of elements of the form $\exp(tL_i)$ is obviously a path-connected subgroup of H. Therefore, H' is a connected Lie subgroup of H (cf. Theorem 2.2). Obviously, H' contains the one-parameter subgroups generated by $L_1,..., L_n$. Therefore, [4, p. 94] $L_1,..., L_n$ belong to $L(H')$. Then, $H' = H$, and our proof is complete.

LEMMA 6.3. *Let* $(\mathbf{X}, U)$ *be a right-invariant control system on G. If the set attainable from the identity is dense in* $\mathbf{S}$, *then it is equal to* $\mathbf{S}$.

Proof. Let $g \in \mathbf{A}(e)$ belong to the interior of $\mathbf{A}(e)$ relative to $\mathbf{S}$ (cf. Lemma 6.1). Let $V \subset \mathbf{A}(e)$ be relatively open in $\mathbf{S}$ and such that $g \in V$. Let $W = \{h^{-1}: h \in V\}$. Then W is a nonempty relatively open subset of $\mathbf{S}$. Our assumption implies that W contains an element h of $\mathbf{A}(e)$; then the set Vh [cf. footnote 2] is relatively open in $\mathbf{S}$, and is contained in $\mathbf{A}(e)$. Moreover, Vh contains the identity. Therefore, the semigroup $\mathbf{A}(e)$ contains a neighbourhood of the identity in $\mathbf{S}$. Since $\mathbf{S}$ is connected, we have that $\mathbf{A}(e) = \mathbf{S}$, and our proof is complete.

LEMMA 6.4. *Let* $(\mathbf{X}, U)$ *be a right-invariant control system on G with* $U = U_u$. *Then* $\bar{S} \subset \overline{\mathbf{A}(e)}$ *(the closure is taken relative to* $\mathbf{S}$*).*

Proof. By Lemma 6.2, every element of S is a product of elements of the form $\exp(tX_i)$ $(-\infty < t < \infty,\ i = 1,..., m)$. We show that $\exp(tX_i)$ belongs to $\overline{\mathbf{A}(e)}$ for every real t and for every $i = 1,..., m$. Since $\overline{\mathbf{A}(e)}$ is a semigroup, this will imply that $S \subset \overline{\mathbf{A}(e)}$, and the desired conclusion will follow immediately.

Let t be a real number, and let $1 \leqslant i \leqslant m$. Let u_n be the constant control $(0,..., 0, n, 0,..., 0)$ where n appears in the i-th position. Then $u_n \in U$ for each $n > 0$. We have

$$\begin{aligned}\pi(e, u_n, t/n) &= \exp(X_0 + nX_i)(t/n)\\ &= \exp((t/n)\, X_0 + tX_i).\end{aligned}$$

Letting $n \to \infty$, we conclude that $\exp(tX_i) \in \overline{\mathbf{A}(e)}$, and our proof is complete.

Remark. If U is not unrestricted, then $\bar{S}$ need not be contained in $\overline{\mathbf{A}(e)}$ (cf. Example 8.4).

We can now prove:

THEOREM 6.5. *Let* $(\mathbf{X}, U)$ *be a right-invariant control system on* G. *Assume that the subgroup* $\mathbf{S}$ *is compact. Then*

(i) $\mathbf{A}(e) = \mathbf{S}$.
(ii) *There exists* $T > 0$ *such that* $\mathbf{A}(e, T) = \mathbf{A}(e)$.

Proof. Let H be the closure of $\mathbf{A}(e)$ relative to $\mathbf{S}$. Then H is a semigroup. We show that H is a group. Let $h \in H$. Then, for every positive integer n, $h^n \in H$. The sequence $\{h^n\}_{n=1,...}$ has a convergent subsequence $\{h^{n(k)}\}_{k=1,...}$, and we can assume that $n(k) < n(k+1)$ for all k. Now, as $k \to \infty$, $h^{-1} = \lim h^{n(k+1)-n(k)-1} = \lim h_k$. Since $n(k+1) - n(k) - 1$ is nonnegative, it follows that h_k belongs to H for each k. Since H is closed, $h^{-1} \in H$. Therefore, H is a group. Since $\mathbf{A}(e) \subset H$ and $\mathbf{A}(e)$ has a nonempty interior relative to $\mathbf{S}$, the same is true for H. Since H is a group and $\mathbf{S}$ is connected, we conclude that $H = \mathbf{S}$. Therefore, $\mathbf{A}(e)$ is dense in $\mathbf{S}$, and (i) follows from Lemma 6.3.

To prove (ii) we let $W(t)$ denote, for each $t > 0$, the interior, relative to $\mathbf{S}$, of $\mathbf{A}(e, t)$. It is easy to see that the union of all the sets $W(t)$ is $\mathbf{S}$ (if $g \in \mathbf{S}$, let $g \in \mathbf{A}(e, T)$, let h be interior to $\mathbf{A}(e, T')$, and let $h^{-1} \in \mathbf{A}(e, T'')$; then, $g \in W(T + T' + T'')$.

Since the sets $W(t)$ are increasing, it follows that $W(t) = \mathbf{S}$ for sufficiently large T, and our proof is complete.

Remark. Theorem 6.5 shows that, if $\mathbf{S}$ is compact, then conditions (a) and (b) of the remark following Theorem 5.1 are satisfied. However, in this case *condition* (c) *need not be satisfied.* Even if U is unrestricted, it may not be possible to reach every element of $\mathbf{S}$ in an arbitrarily small time (cf. Example 8.1).

If $\mathbf{S}$ is not compact, then $\mathbf{A}(e)$ need not be equal to $\mathbf{S}$. The following theorem gives a sufficient condition under which $\mathbf{A}(e) = \mathbf{S}$; we do not know if this condition is also necessary.

THEOREM 6.6. *Let* $(\mathbf{X}, U)$ *be a right-invariant control system on* G *with* $U = U_u$. *If there exists a constant control* u *and a sequence of positive numbers* $\{t_n\}$ *with* $t_n \geq \epsilon > 0$ *for some* ϵ, *with the property that* $\lim \pi(e, u, t_n)$ *exists and belongs to* $\bar{S}$ (*the closure is relative to* $\mathbf{S}$), *then* $\mathbf{A}(e) = \mathbf{S}$.

Proof. Let u and $\{t_n\}$ satisfy the conditions of the theorem, and let $\lim \pi(e, u, t_n) = x$. If $X = X_0 + \sum_{i=1}^{m} u_i X_i$ then, since u is constant, $\pi(e, u, t) = \exp(tX)$. We first show that $\exp(tX) \in \overline{\mathbf{A}(e)}$ for every real number t. If $\{t_n\}$ is bounded, then there exists a positive number T such that $\exp(TX) \in \bar{S}$. Let t be any real number, and let n be a natural number with $nT + t > 0$. Since $\bar{S}$ is a group we have that $\exp(-TX) \in \bar{S}$, and hence $\exp(-TnX) \in \bar{S}$. By Lemma 6.4, it follows that $\exp(-TnX) \in \bar{\mathbf{A}}(e)$. Since, obviously $\exp((nT + t)\, X) \in \mathbf{A}(e)$, we have that

$$\exp(tX) = \exp(-TnX) \cdot \exp((+\,Tn + t)\, X),$$

and hence, $\exp(tX) \in \overline{\mathbf{A}(e)}$. If $\{t_n\}$ is unbounded, let $\{t_{n_k}\}$ be a subsequence of $\{t_n\}$ with $t_{n_{k+1}} - t_{n_k} > k$, and let $\tau_k = t_{n_{k+1}} - t_{n_k}$. We have that $\tau_k \to \infty$ and $\exp \tau_k X \to e$ as $k \to \infty$. Thus for any real number t, $\exp(tX) = \lim_{k\to\infty} \exp((t + \tau_k)\, X)$.

If k is sufficiently large, then $t + \tau_k$ is positive. Therefore $\exp((t + \tau_k)\, X)$ belongs to $\mathbf{A}(e)$ for k large. It follows that $\exp(tX) \in \bar{\mathbf{A}}(e)$.

By Lemma 6.4, $\exp(tX_i)$ belongs to $\bar{\mathbf{A}}(e)$ for every real t and every $i = 1,..., m$. Since $\bar{\mathbf{A}}(e)$ is a semigroup, it follows that every product of elements of the form $\exp(tY)$ (t real, $Y \in \{X, X_1 ,..., X_m\}$) belongs to $\mathbf{A}(e)$. Clearly, the elements $X, X_1 ,..., X_m$ generate $\mathbf{L}$. By Lemma 6.2, $\bar{\mathbf{A}}(e) = \mathbf{S}$, and by Lemma 6.1, $\mathbf{A}(e) = S$. This completes the proof.

The following corollary is immediate:

COROLLARY 6.7. *Let* $(\mathbf{X}, U)$ *be a right-invariant control system on* G *with* $U = U_u$. *If there exists a constant control* u *such that* $t \to \pi(e, u, t)$ *is periodic, then* $\mathbf{A}(e) = \mathbf{S}$.

The following lemma gives a necessary condition for $\mathbf{A}(e)$ to be equal to $\mathbf{S}$; however, this condition is not sufficient (see Example 8.3).

LEMMA 6.8. *Let* $(\mathbf{X}, U)$ *be a right-invariant control system on* G, *and let* $\mathbf{A}(e) = \mathbf{S}$. *Then, there exists a nonzero number* T *such that* $\exp(X_0 T) \in \mathbf{S}_0$.

Proof. Our assumption implies that $\exp(-X_0)$ belongs to $A(e, t)$ for some $t \geqslant 0$. Therefore, by Lemma 4.2, $\exp(-X_0) = \exp(tX_0)g$ where $g \in \mathbf{S}_0$. To complete the proof, take $T = -1 - t$.

There is one important case when Theorem 6.6 and Lemma 6.8 yield a necessary and sufficient condition for $\mathbf{A}(e) = \mathbf{S}$, namely, when $S = \mathbf{S}_0$. This will happen if and only if $L = \mathbf{L}_0$. It is easy to check that this equality holds if and only if all the brackets $[X_0, X_i]$ belong to L ($i = 1,..., m$).

THEOREM 6.9. *Let* $(\mathbf{X}, U)$ *be a right-invariant control system on* G *with* $U = U_u$. *If* $L = \mathbf{L}_0$, *then a necessary and sufficient condition for* $\mathbf{A}(e)$ *to be equal to* $\mathbf{S}$ *is that there exist a number* T, $T \neq 0$, *such that* $\exp(TX_0)$ *belongs to* S.

Remark. The condition $L = \mathbf{L}_0$ holds, in particular, when $[X_0, X_i] = 0$ ($i = 1,..., m$), i.e., when $\exp(tX_0)$ commutes with the elements of S.

7. Controllability

Let $(\mathbf{X}, U)$ be a right invariant control system on G, and let $g \in G$. We say that $(\mathbf{X}, U)$ is *controllable from* g if $\mathbf{A}(g) = G$. We say that $(\mathbf{X}, U)$ is *controllable* if it is controllable from every $g \in G$.

THEOREM 7.1. *A necessary condition for* $(\mathbf{X}, U)$ *to be controllable is that* G *be connected and that* $\mathbf{L} = L(G)$. *If* G *is compact, or if the system is homogeneous, the condition is also sufficient.*

Proof. The condition of the theorem holds if and only if $G = \mathbf{S}$. By Lemma 4.1, the condition is necessary. The second part of the statement follows from Theorems 5.1 and 6.5 (and from the obvious fact that, if $\mathbf{A}(e) = G$, then $\mathbf{A}(g) = G$ for every g).

In the compact case, we can prove stronger controllability properties.

THEOREM 7.2. *Let* G *be compact, and let* $(\mathbf{X}, U)$ *be controllable. Then there exists* $T > 0$ *such that, for every* $g \in G$, $g' \in G$, *there is a control that steers* g *into* g' *in less than* T *units of time. If* G *is semisimple, then there exists* $T > 0$ *such that, for every* $g \in G$, $g' \in G$, *there is a control that steers* g *into* g' *in exactly* T *units of time.*

Proof. The first statement follows from Theorem 6.5(ii). To prove the second statement, we observe that, if G is semisimple, then $(\mathbf{X}, U)$ has the "strong accessibility property," i.e., the set $A(e, t)$ has a nonempty interior for every $t > 0$ (for a proof of this, see [11]). From this fact the conclusion follows as in the proof of Theorem 6.5(ii).

Finally, Theorem 6.9 can also be interpreted as a controllability result.

THEOREM 7.3. *Assume that the necessary conditions of Theorem* 7.1 *hold, and that* (i) $U = U_u$, *and* (ii) $L = \mathbf{L}_0$ *(or, equivalently,* L *is an ideal of* $\mathbf{L}$*). Then* $(\mathbf{X}, U)$ *is controllable if and only if* $\exp(TX_0)$ *belongs to* S $(=\mathbf{S}_0)$ *for some* $T \neq 0$.

COROLLARY 7.4. *If* G *is connected,* $\mathbf{L} = L(G)$, $U = U_u$ *and* X_0 *belongs to the Lie algebra generated by* $X_1, \ldots, X_m$, *then* $(\mathbf{X}, U)$ *is controllable.*

8. Examples

In most of the following examples, we shall work with groups of matrices. Our groups will be Lie subgroups of $GL(n, \mathbf{R})$, the group of all $n \times n$ nonsingular real matrices. Recall that $GL(n, \mathbf{R})$ is an open subset of $M(n, \mathbf{R})$

(the set of all $n \times n$ real matrices). Since $M(n, \mathbf{R})$ is a vector space, we can identify the tangent space to $GL(n, \mathbf{R})$ at each point with $M(n, \mathbf{R})$. With this identification, a right-invariant vector field corresponds to a mapping $X \to AX$ from $GL(n, \mathbf{R})$ into $M(n, \mathbf{R})$, where A is a fixed matrix. If $X_0, \ldots, X_m$ are right-invariant vector fields, given by $X \to A_i X$ ($i = 0, \ldots, m$), then the evolution equation becomes

$$\dot{X}(t) = \left(A_0 + \sum_{i=1}^{m} u_i(t)\, A_i\right) X(t).$$

EXAMPLE 8.1. Let $G = SO(3)$, the set of all 3×3 real orthogonal matrices with positive determinant. The algebra $L(G)$ is the set of all 3×3 antisymmetric matrices. A basis for $L(G)$ is given by the matrices

$$K_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad K_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

and

$$K_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

It is easy to check that $[K_1, K_2] = K_3$, $[K_2, K_3] = K_1$ and $[K_3, K_1] = K_2$. Thus $L(G)$ is isomorphic to three-dimensional real space, with the Lie bracket corresponding to the vector product. Using this correspondence, it is obvious that, if A and B are any two linearly independent elements of $L(G)$, then $\{A, B, [A, B]\}$ is a basis for $L(G)$.

Let A and B be any linearly independent 3×3 anti-symmetric matrices, and let our right-invariant control system on $SO(3)$ be described by

$$\dot{X}(t) = (A + uB)\, X(t)$$

where u belongs to any class of admissible controls. Since $SO(3)$ is compact and connected, Theorem 6.5 applies, and our system is controllable. Moreover, there is a $T > 0$ such that, given any two elements P, Q of $SO(3)$ there is a "bang-bang" control u that steers P into Q in less than T units of time. In this connection, it is interesting to observe that, in general, there may not exist arbitrarily small numbers T with the above property, *even if the control u is completely unrestricted.* Take, for instance, $A = K_1$ and $B = K_2$. If u is an arbitrary control, and if $X(t)$ is the solution of the evolution equation corresponding to u with initial condition $X(0) = I$, write $X = (x_{ij})_{i,j=1,2,3}$. Then we have

$$\dot{x}_{12} = x_{22} + ux_{32}$$

and

$$\dot{x}_{32} = -ux_{12}.$$

Multiplying the first equation by x_{12}, the second equation by x_{32} and adding, we get

$$\tfrac{1}{2}\, d/dt\, (x_{12}^2 + x_{32}^2) = x_{22}x_{12}.$$

Since $x_{12}^2 + x_{32}^2$ vanishes at $t = 0$, we have

$$(x_{12}^2 + x_{32}^2)(t) = 2 \int_0^t x_{22}(\tau)\, x_{12}(\tau)\, d\tau.$$

But $x_{22}(\tau)$ and $x_{12}(\tau)$ are entries of orthogonal matrices. Hence they are bounded in absolute value by 1. Therefore, we conclude that

$$(x_{12}^2 + x_{32}^2)(t) \leqslant 2t.$$

This shows that *a matrix* (a_{ij}) *for which* $a_{12}^2 + a_{32}^2 = 1$ *cannot be reached from the identity in less than* $\frac{1}{2}$ *units of time.*

EXAMPLE 8.2. The considerations of the previous example can be generalized to $G = SO(n)$. In this case the Lie algebra of G is the set of all $n \times n$ anti-symmetric matrices.

Let $A = (a_{ij})$ and $B = (b_{ij})$ be matrices defined as follows: $a_{i,i+1} = 1$ for $i = 1,..., n-2$, $a_{i,i-1} = -1$ for $i = 2,..., n-1$, $a_{ij} = 0$ otherwise, and let $b_{n-1,n} = 1$, $b_{n,n-1} = -1$, $b_{ij} = 0$ otherwise. It is easy to show that the smallest subalgebra that contains A and B is exactly $L(G)$. Thus, even though $SO(n)$ is $\frac{1}{2}n(n-1)$-dimensional, the system $\dot{X} = (A + uB)X$, in which only *one* control is involved, is controllable. Moreover, as before, we can limit u to be "bang-bang." An easy argument shows that this fact, which has been shown to be true for the particular matrices A and B defined above, is in fact true for "almost all" pairs $(A, B) \in L(G) \times L(G)$. Precisely, the set of pairs (A, B) such that A and B generate $L(G)$ is open and dense in $L(G) \times L(G)$.

Remark. If G is an arbitrary connected Lie group such that $L(G)$ is generated by two elements, then Theorem 5.1 enables us to conclude, in a way similar to that of the previous examples that the homogeneous system on G of the form $\dot{X}(t) = (uA + vB)\, X(t)$ is controllable for "almost all" pairs $(A, B) \in L(G) \times L(G)$. This result holds even if we restrict u and v to be "bang-bang."

The previous statement holds, in particular, when $G = SL(n, \mathbf{R})$, the set of all $n \times n$ real matrices whose determinant is 1, or when $G = G_+L(n, \mathbf{R})$, the set of all $n \times n$ real matrices whose determinant is greater than 0.

EXAMPLE 8.3. We show that, if A, B generate $L(G)$, and if G is connected and not compact, then the system $\dot{X} = (A + uB)X$ need not be controllable, even if $L(G)$ is a simple Lie algebra. In particular, this will show that the necessary condition of Lemma 6.8 is not sufficient. Take $G = SL(2, \mathbf{R})$. Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is clear that A and B generate $L(G)$ and that $L(G)$ is simple. Let u be an arbitrary control, and let $X(t)$ be the solution of the evolution equation corresponding to u, with initial condition $X(0) = I$. Let $X = (x_{ij})_{i,j=1,2}$. Then $\dot{x}_{11} = x_{11} + ux_{21}$ and $\dot{x}_{21} = ux_{11} - x_{21}$. Multiplying the first equation by x_{11}, the second one by x_{21} and subtracting, we get

$$\frac{1}{2} \frac{d}{dt} (x_{11}^2 - x_{21}^2) = x_{11}^2 + x_{21}^2 .$$

Thus the function $x_{11}^2(t) - x_{21}^2(t)$ is nondecreasing for every trajectory of our system. Since its value for $t = 0$ is 1, it follows that every element of $SL(2, \mathbf{R})$ that can be reached from the identity in positive time satisfies the inequality $x_{11}^2 \geqslant x_{21}^2 + 1$. Hence, the system is not controllable. In the notations of Section 3, it is clear that $\mathbf{L} = L(G)$. Thus, we have shown that $\mathbf{A}(e)$ is not a group. However, $\mathbf{L}_0 = L(G)$ (because $L(G)$ is simple), and hence $\mathbf{S}_0 = G$. Therefore, $\exp(tA)$ belongs to $\mathbf{S}_0$ for all $t \geqslant 0$. This shows that the condition of Lemma 6.8 is satisfied.

EXAMPLE 8.4. In this example we show that Lemma 6.4 and Theorem 6.6 need not be valid if U is not assumed to be unrestricted. Let $G = \mathbf{R} \times S^1$, the product of the real line and the unit circle. Let X_0 be the generator of the one parameter group $t \to (t, e^{2\pi i t})$, and let X_1 be the generator of the one parameter group $t \to (t, 1)$. Let $U = U_r$, or $U = U_b$. Then $\mathbf{A}(e) = [0, \infty) \times S^1$, which is not a group. But if $u = 0$, then $\pi(e, u, 1) = \exp X_0 = (1, 1)$ which belongs to S. Thus, Theorem 6.6 does not hold. As for Lemma 6.4 it is clear that S and $\mathbf{A}(e)$ are closed, but $S \not\subset \mathbf{A}(e)$.

EXAMPLE 8.5. In view of Theorem 6.6 it might seem that a necessary condition for a right-invariant system to be controllable is that $\exp tX_0$ "gets arbitrarily close" to S for some nonzero values of t. This example shows that such a statement is false.

Let $G = SL(2, \mathbf{R})$, and let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Consider the system $X = (A + Bu)X$ where u belongs to the class of unbounded controls.

Let u be the constant control $u = 1$. Then the trajectory $t \to \pi(I, u, t)$ is the curve $t \to e^{t(A+B)}$, which is periodic with period 2π. By Corollary 6.7, the system is controllable.

We have that

$$e^{tA} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \qquad \text{and} \qquad e^{\tau B} = \begin{pmatrix} 1 - \tau & \tau \\ -\tau & 1 + \tau \end{pmatrix}.$$

Now it is obvious that e^{tA} stays away from S for all positive values of t.

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