Model conversions of uncertain linear systems using interval multipoint Pade approximation

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Many dynamic systems and industrial control processes can be represented by a multirate sampled-data uncertain system, which consists of a continuous-time uncertain subsystem and a multirate discrete-time uncertain subsystem. The uncertainties in these systems arise from unmodeled dynamics, parameter variations, sensor noises, actuator constraints, etc. As is the common practice the sampled-data uncertain system needs to be converted to a purely continuous-time or discrete-time uncertain model, so that the well-established analysis and design methods in the continuous-time or discrete-time domain can be directly applied to the equivalent model. This paper presents a new interval multipoint Pade approximation method for converting a continuous-time (discrete-time) uncertain linear system to an equivalent discrete-time (continuous-time) uncertain model. The system matrices characterizing the state-space descriptions of the original uncertain systems are represented by interval matrices. Using the approximate uncertain models obtained based on interval analysis and multipoint Pade approximation the dynamic states of the resulting models have been shown to be able to closely match those of the original uncertain systems for a relatively longer sampling period. © 1997 by Elsevier Science Inc.

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1. Introduction

Dynamic systems are often formulated using continuous-time and/or discrete-time equations with uncertain parameters. The uncertainties of these systems arise from unmodeled dynamics, parameter variations, sensor noises, actuator constraints, load disturbances, nonlinear effects, etc. These uncertainties generally do not follow any of the known probability distributions and are most often quantified in terms of bounds. Hence, uncertain systems are usually represented by continuous-time and/or discrete-time uncertain models with interval parameters. In order to carry out digital simulation and digital design for these continuous-time uncertain systems it is often desired to find an equivalent discrete-time uncertain model from the original continuous-time uncertain model. On the other hand, identification of these continuous-time uncertain systems using the discrete-time input-output data observed in the systems may also result in discrete uncertain models. In this case it is required to find the original continuous-time uncertain systems from the identified discrete-time uncertain models. These procedures are called model conversions.

The model conversion of a nominal continuous-time (discrete-time) linear system to an equivalent nominal discrete-time (continuous-time) linear model has been studied by Shieh et al. and elsewhere in the literature. However, methods for model conversions of uncertain linear state-space models have not been sufficiently explored. Recently, Ezzine and Johnson used a perturbation method to convert a continuous-time uncertain linear system to an equivalent discrete-time uncertain model, but they did not solve the converse of the problem. Also, Shieh et al. developed a bilinear and inverse-bilinear approximation method, the Pade and inverse-Pade method, and the geometric series approximation method to convert a continuous-time (discrete-time) uncertain system to an equivalent discrete-time (continuous-time) uncertain model. In these approaches the model conversions are carried out by neglecting the high-order terms in the sampling period and perturbation parameter matrices in the state-space settings. The uncertain models so obtained usually retain explicitly the structured uncertainties of the original uncertain systems, which contain a sufficiently small sampling period and some small parametric uncertainties. All the approximate models studied in Refs.
5, 6 and 7 approximate the original uncertain models well, but only around the "origin." Therefore the error in the approximated models introduced by using these methods increases as the norm of the system matrix increases.\textsuperscript{8, 9}

In this paper we present a new method for the model conversion of unstructured uncertain linear state-space models using the multipoint Pade approximation method together with interval arithmetic and interval analysis. The principle and algorithms of the nominal multipoint Pade approximation method\textsuperscript{10, 11} are introduced in Section 2 and Appendix A, respectively. Model conversions of unstructured uncertain linear models using the interval multipoint Pade approximation method are discussed in Section 3. A numerical example is given in Section 4, and conclusions are summarized in Section 5. In Appendix B, moreover, interval arithmetic and interval analysis\textsuperscript{12-14} are briefly reviewed for the reader's convenience.

2. Preliminaries

2.1 Multipoint Pade approximation

Consider the Taylor series expansion\textsuperscript{8-10} of a given function

\[ F(s) = \sum_{j=0}^{\infty} f_{i,j}(s-s_{i})^{j}, \quad i = 1, 2, \ldots, r \]

where \( s_{i}, i = 1, 2, \ldots, r \), are the distinct interpolating points, or expanding points, in the complex plane. If a rational function

\[ R_{LM}(s) = \frac{P_{L}(s)}{Q_{M}(s)} = \frac{p_{0} + p_{1}s + \cdots + p_{L}s^{L}}{q_{0} + q_{1}s + \cdots + q_{M}s^{M}}, \quad L \leq M \]

exists and has the property

\[ R_{LM}(s) = \sum_{j=0}^{n_{i}-1} \hat{f}_{i,j}(s-s_{i})^{j} + O[(s-s_{i})^{n}], \quad i = 1, 2, \ldots, r \]

where

\[ L + M + 1 \triangleq N = \sum_{i=1}^{r} n_{i} \]

\( n_{i} \) is the order number of the Taylor series expanded at the \( i \)th interpolating point \( s_{i} \), and \( O[(s-s_{i})^{n}] \) represents a power series in which the order of \( O[(s-s_{i})^{n}] \) is not less than the \( n_{i} \)th order, then the rational function \( R_{LM}(s) \) is called the \( r \)-point Pade approximant of the function \( F(s) \). It is noticed that when all \( s_{i} \) equal zeros (i.e., \( r = 1, s_{i} = 0 \)), then the approximate function \( R_{LM}(s) \) is simply the ordinary (single-point) Pade approximant. More details about the method and equations for finding the multipoint Pade approximant are included in Appendix A.

From the definition of the multipoint Pade approximation it is clear that the multipoint Pade approximant equals the original function \( F(s) \) exactly at the interpolating points \( s_{i}, i = 1, 2, \ldots, r \). Hence the multipoint Pade approximant can approximate the original function \( F(s) \) over a wider range than the ordinary Pade approximant without losing accuracy.

2.2 Model conversions of nominal systems via multipoint Pade approximation

Discrete-time model conversion. Consider a continuous-time linear system

\[ \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_{0} \]

where \( x(t) \in \mathbb{R}^{n \times 1} \) is the state vector, \( u(t) \in \mathbb{R}^{m \times 1} \) is the input vector, \( A \in \mathbb{R}^{n \times n} \) is the system matrix, and \( B \in \mathbb{R}^{n \times m} \) is the input matrix.

The discrete-time system model corresponding to equation (5) for a piecewise-constant input \( u_{d}(t) \) is

\[ x_{d}(kT + T) = Gx_{d}(kT) + Hu_{d}(kT), \quad x_{d}(0) = x_{0} \]

where

\[ G = e^{AT} \]

\[ H = \int_{0}^{1} e^{AT}Bd\tau = (G - I_{n})A^{-1}B \]

\[ u_{d}(kT) = u_{c}(t), \quad kT < t < (k+1)T \]

where \( G \in \mathbb{R}^{n \times n} \), \( H \in \mathbb{R}^{n \times m} \), and \( T \) is the sampling period. It is desired to find the discrete-time system matrices \((G, H)\) from the continuous time system matrices \((A, B)\) using the multipoint Pade approximation method. According to the principle of multipoint Pade approximation described in Section 2.1 the exponential function \( e^{s} \) can be expressed by a rational function (denoted by \( R_{s}(s) \)), which has the form of equation (2) such that \( R_{s}(s) \) has the property

\[ R_{s}(s) = \sum_{j=0}^{n_{i}-1} \frac{e^{s_{i}t}}{j!}(s-s_{i})^{j} + O[(s-s_{i})^{n}], \quad i = 1, 2, \ldots, r \]

where the expanding points \( s_{i} \) are selected near the eigenvalues of \( A \). Then the discrete-time system matrix \( G \) is approximated by

\[ G = e^{AT} \approx R_{s}(AT) \]

\[ - \left[ q_{0}I_{n} + q_{1}AT + \cdots + q_{M}(AT)^{M} \right]^{-1} \times \left[ p_{0}I_{n} + p_{1}AT + \cdots + p_{L}(AT)^{L} \right] \]

\[ = \left[ I_{n} + b_{1}AT + \cdots + b_{M}(AT)^{M} \right]^{-1} \times \left[ I_{n} + a_{1}AT + \cdots + a_{L}(AT)^{L} \right] \]
where the sampling period $T$ has to be suitably chosen such that the matrix inversion exists, $p_i = p_{i-1}/p_0$, for $i = 1, \ldots, L$, and $q_i = q_{i-1}/q_0$, for $i = 1, \ldots, M$. It is worthy to note that if one of the interpolating points is zero, $p_0$ must be equal to $q_0$ (i.e., $p_m = 1$).

### Continuous-time model conversion

If a given linear system model is described by equation (6a) and the desired system model to be found is described by equation (5), then the system matrices $(A, B)$ need to be solved from the system matrices $(G, H)$. They have the following relations:

$$A = \frac{1}{T} \ln(G)$$

and

$$B = A(G - I_p)^{-1} H$$

Similarly the logarithmic function $\ln(s)$ can be expressed by the rational function $R_D(s)$, which has the form of equation (2), such that

$$R_D(s) = \ln(s) + \sum_{j=1}^{\frac{n-1}{2}} \frac{(-1)^{j+1}(s_j)^{-j}}{j!} (s - s_j)^j + O((s - s_j)^{n})$$

Then the desired matrix $A$ is approximated by

$$A = \frac{1}{T} \ln(G) \approx \frac{1}{T} R_D(G)$$

$$= \frac{1}{T} \left[ q_0 I_n + q_1 G + \cdots + q_M (G)^M \right]^{-1}$$

$$\times \left[ p_0 I_n + p_1 G + \cdots + p_M (G)^L \right],$$

$L \leq M$ (11)

The multipoint Pade approximation method gives a solution much more accurate and with a much wider range for system matrices $(A, G)$ than that obtained by using the ordinary Pade approximation method. When all the eigenvalues of the argument matrix are available and used as interpolating points in the approximant the multipoint Pade method gives the exact solution.

### 3. Model conversions of uncertain systems via multipoint Pade approximation

#### 3.1 Discrete-time model conversion

An unstructured continuous-time uncertain linear system model can be expressed in the following interval form:

$$\dot{x}_c(t) = A(t)x_c(t) + B(t)u_c(t) \quad \quad x_c(0) = x_{c0}$$

$$y_c(t) = Cx_c(t)$$

where the interval system matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{p \times m}$ contain the degenerate interval matrices (i.e., real matrices) $A_r \in \mathbb{R}^{n \times n}$ and $B_r \in \mathbb{R}^{p \times m}$, respectively, such that $A_r \subseteq A$, $A_r = \frac{1}{2} (A + A^T)$, and $B_r \subseteq B$, $B_r = \frac{1}{2} (B + B^T)$. Then the desired matrix $A$ is approximated by $A = (G^T)^{-1} B^T$.

$$x_d(kT + T) = G^T x_d(kT) + H^T u_d(kT)$$

$$y_d(kT) = Cx_d(kT)$$

where

$$G^T = e^{A'T}$$

$$H^T = \int_0^T e^{A'T} B(t) \, dt = (G^T - I_n)(A^T)^{-1} B^T$$

$$u_d(kT) = u_c(t), \quad \text{for } kT \leq t < (k+1)T$$

$T$ is the sampling period, and the interval matrices, $G \in \mathbb{R}^{n \times n}$ and $H \in \mathbb{R}^{p \times m}$, contain the degenerate (real) matrices, $G_r \in \mathbb{R}^{n \times n}$ and $H_r \in \mathbb{R}^{p \times m}$, such that

$$G_r = e^{A'T} \in \mathbb{R}^{n \times n}$$

$$H_r = (G_r - I_n) A_r^{-1} B_r \in \mathbb{R}^{p \times m} = (G^T - I_n)(A^T)^{-1} B^T$$

Here it is very important to remark that the expression $(G - I_n)(A)^{-1}$ in equation (14b) is well defined even if $A$ is singular, since $G = e^{A'T} I + A T + \cdots$ so that $(G - I_n)(A)^{-1}$ is only a convenient notation. We will use this notation frequently throughout the paper, which should not cause confusion.

It is desired to find the discrete-time interval system matrices $(G', H')$ in equation (13) from the continuous-time interval system matrices $(A', B')$ given in equation (12), using the multipoint Pade approximation method with its interval algorithm.

Following the method in Ref. 6 the approximate discrete-time interval matrices $(G', H')$ shown in equation (13) can be obtained by using the interval ordinary (single-point) Pade approximation method as follows:

$$G' = e^{A'T} = D_n^{-1}(A'T)N_k(A'T) = G_p$$

where

$$N_k(A'T) = I_n + \sum_{i=1}^{k} \frac{k(k-1) \cdots (k-i+1)}{2k(2k-1) \cdots (2k-i+1)i!}$$

$$\times (A'T)^i$$

$$k \leq M$$ (15b)
and
\[ D_k(A^T) = N_k(-A^T) \]  

The sampling period \( T \) must be suitably selected to ensure that the matrix \( D_k(A^T) \) is invertible. When \( k = 2 \) the second-order interval ordinary Pade approximate model is given by
\[ G' \approx D_2^{-1}(A^T)N_2(A^T) \]

The alternative representation of the expression (16a) in (16b) gives a better numerical result (less conservative result) for the evaluation of \( G_p \). The associated second-order interval ordinary Pade approximate input matrix is as follows:
\[ H_p = \left[ I_n - \frac{1}{2} A'T + \frac{1}{12} (A'T)^2 \right] B'T \]

Without losing generality we will concentrate on the analysis and comparison of the second-order ordinary Pade approximate model and the second-order multipoint Pade approximate model in this paper. In this section we first propose the interval multipoint Pade approximation method for finding the approximants of the discrete-time uncertain system matrices \( G' \) and \( H' \) given in equation (13). Note that the evaluation of the interval approximants of \( G' \) and \( H' \), directly using the interval multipoint Pade approximation method, is practically impossible due to the nature of interval arithmetic and the inherent conservativeness of interval arithmetic operations. For example, \( A'-A = 0 \), \( (A')^{-1}A' \neq I \), and \( (A')^2 + A' \geq A'(A'^2 + I_n) \). In general, interval analysis is carried out using real (i.e., degenerate interval) analysis to find the desired real result. Then the inclusive theorem (i.e., Theorem 1 in Appendix B) is applied to the result sought by replacing the real variables and real arithmetic operations with interval variables and interval arithmetic operations, respectively. The interval result thus obtained is guaranteed to be inclusion monotonic.\(^{12,14} \)

From equation (8) the matrix-valued function \( e^{A'T} \in S^{n \times n} \) with the degenerate interval matrix \( A' \in \mathbb{R}^{n \times n} \) can be approximated by the second-order multipoint Pade approximation, which was discussed in Section 2.2, as follows:
\[ G_r = e^{A'T} \cong \left[ p_0 I_n + p_1 A'T + p_2 (A'T)^2 \right]^{-1} \times \left[ I_n + a_1 A'T + a_2 (A'T)^2 \right] \triangleq G_{mp} \]

where \( (p_0, p_1, p_2) \) and \( (q_0, q_1, q_2) \) are the results of the Pade approximation, \( p_m = p_0/q_0, a_1 = p_1/p_0, a_2 = p_2/p_0, b_1 = q_1/q_0, \) and \( b_2 = q_2/q_0 \). A less conservative result is obtained in equation (18b) as compared to equation (18a), if the interval arithmetic is applied to the evaluation of \( G_{mpr} \). When one of the interpolating points is zero, \( p_m \) in equation (18) equals one, and the second-order multipoint Pade approximant of \( G_r \) becomes
\[ G_r = e^{A'T} \cong \left[ I_n + b_1 A'T + b_2 (A'T)^2 \right]^{-1} \times \left[ I_n + a_1 A'T + a_2 (A'T)^2 \right] \triangleq \tilde{G}_{mpr} \]

The degenerate interval input matrices associated with equations (18) and (19) are, respectively,
\[ H_r = (G_r - I_n) A'^{-1} B_r \cong (G_{mpr} - I_n) A'^{-1} B_r \triangleq \tilde{H}_{mpr} \]

Without losing generality we consider the case of \( p_m = 1 \) in the following sections of the paper. In fact, \( p_m \) is equal to one only if one of the expanding points is chosen to be zero, but this can be determined by the user. Using interval arithmetic the multipoint Pade approximants of the discrete-time uncertain linear system model can be presented as
\[ x_d(kT + T) = x_d(kT) = G_{mp}^l x_d(kT) + H_{mp}^l u_d(kT), \]
\[ y_d(kT) = Cx_d(kT) \equiv Cx_d(kT) \]

where
\[ G_{mp}^l = \left[ I_n + b_1 A'T + b_2 (A'T)^2 \right]^{-1} \times \left[ I_n + a_1 A'T + a_2 (A'T)^2 \right] \triangleq \tilde{G}_{mp}^l \]

and

\[ H_{mp} = \left[ I_n + b_1 A'T + b_2 (A'T)^2 \right]^{-1} \times \left( (a_1 - b_1) I_n + (a_2 - b_2) A'T \right) B'T \]  

\[ \text{(21e)} \]

### 3.2 Continuous-time model conversion

Consider a given discrete-time uncertain linear system

\[ x_d(kT + T) = G' x_d(kT) + H' u_d(kT), \quad k = 1, 2, \ldots \]  

\[ \text{(22a)} \]

\[ y_d(kT) = C x_d(kT) \]  

\[ \text{(22b)} \]

where \( G' \in \mathbb{R}^{n \times n} \) and \( H' \in \mathbb{R}^{n \times m} \) are interval matrices containing the respective degenerate interval matrices \( G, H \in \mathbb{R}^{n \times n} \) with \( \det(G) \neq 0 \) and \( G, H \in \mathbb{R}^{n \times m} \) with \( H \in \mathbb{H} \). It is desired to find an equivalent continuous-time uncertain linear system model, shown below in equation (23), from the discrete-time uncertain system matrices \( (G', H') \) given in equation (22) in the form

\[ \dot{x}_c(t) = A' x_c(t) + B' u_c(t) \]  

\[ \text{(23a)} \]

\[ y_c(t) = C x_c(t) \]  

\[ \text{(23b)} \]

where \( A' \in \mathbb{R}^{n \times n} \) and \( B' \in \mathbb{R}^{n \times m} \) are interval system matrices containing respective degenerate interval matrices \( A, B \in \mathbb{R}^{n \times n} \) and \( A, B \in \mathbb{R}^{n \times m} \), such that

\[ A_r = \frac{1}{T} \ln(G_r) \equiv \frac{1}{T} \ln(G') \in \mathbb{R}^{n \times n} \]  

\[ \text{(24a)} \]

\[ B_r = A_r (G_r - I_n)^{-1} H_r \in B' \]  

\[ = A'(G' - I_n)^{-1} H' \in \mathbb{R}^{n \times m} \]  

\[ \text{(24b)} \]

The ordinary Pade approximants of \( A' \) and \( B' \) can be determined following the methods in Refs. 2 and 6 as

\[ A' \approx \frac{2}{T} R' \left[ I_n - \frac{1}{3} (R')^3 \right]^{-1} \triangleq A'_r \]  

\[ \text{(25a)} \]

\[ B' \equiv A'_r (G' - I_n)^{-1} H' \triangleq B'_r \]  

\[ \text{(25b)} \]

where

\[ R' = (G' - I_n)(G' + I_n)^{-1} \]  

\[ \text{(25c)} \]

As mentioned in Section 2.2 the multipoint Pade approximant of the matrix-valued function \( \ln(G_r) \) can be found to be in the following form:

\[ \ln(G_r) = [q_0 I_n + q_1 G_r + q_2 G_r^2]^{-1} \times [p_0 I_n + p_1 G_r + p_2 G_r^2] \]  

\[ - \beta [I_n + d_1 G_r + d_2 G_r^2]^{-1} [I_n + c_1 G_r + c_2 G_r^2] \]  

\[ - \beta [I_n + d_1 G_r + d_2 G_r^2]^{-1} [I_n + c_1 G_r + c_2 G_r^2] \]  

\[ \times [(c_1 - d_1) I_n + (c_2 - d_2) G_r] \]  

\[ \text{(26a)} \]

\[ \text{(26b)} \]

\[ \text{where} \quad \beta = \frac{p_0}{q_0}, \quad c_1 = \frac{p_1}{p_0}, \quad c_2 = \frac{p_2}{p_0}, \quad d_1 = \frac{q_1}{q_0}, \quad \text{and} \quad d_2 = \frac{q_2}{q_0}. \]

Here, equation (26b) gives a less conservative result if the interval arithmetic is applied. Hence the approximants of the degenerate interval matrices \( A, B \) can be expressed, respectively, as

\[ A_r = \frac{1}{T} \ln(G_r) \equiv \frac{\beta}{T} \left[ I_n + d_1 G_r + d_2 G_r^2 \right]^{-1} \times \left[ I_n + c_1 G_r + c_2 G_r^2 \right] \triangleq A'_{m_r} \]  

\[ \text{(27a)} \]

\[ = \frac{\beta}{T} \left[ I_n + d_1 G_r + d_2 G_r^2 \right]^{-1} \times \left[ I_n + c_1 G_r + c_2 G_r^2 \right] \triangleq B'_{m_r} \]  

\[ \text{(27b)} \]

and

\[ B_r = A_r (G_r - I_n)^{-1} H_r \equiv \frac{\beta}{T} \left[ I_n + d_1 G_r + d_2 G_r^2 \right]^{-1} \times \left[ I_n + c_1 G_r + c_2 G_r^2 \right] \]  

\[ \times (G_r - I_n)^{-1} H_r \triangleq B'_{m_r} \]  

\[ \text{(27c)} \]

\[ = \frac{\beta}{T} \left[ I_n + d_1 G_r + d_2 G_r^2 \right]^{-1} \times \left[ I_n + c_1 G_r + c_2 G_r^2 \right] \]  

\[ \times (G_r - I_n)^{-1} H_r \triangleq B'_{m_r} \]  

\[ \text{(27d)} \]

Then the approximants of the discrete-time uncertain linear system matrices can be presented in the following interval form:

\[ A' = \frac{1}{T} \ln(G') \equiv \frac{\beta}{T} \left[ I_n + d_1 G' + d_2 (G')^2 \right]^{-1} \times \left[ I_n + c_1 G' + c_2 (G')^2 \right] \triangleq A'_{m} \]  

\[ \text{(28a)} \]

\[ \geq \frac{\beta}{T} \left[ I_n + d_1 G' + d_2 (G')^2 \right]^{-1} \times \left[ I_n + c_1 G' + c_2 (G')^2 \right] \]  

\[ \times (G' - I_n)^{-1} H' \triangleq B'_{m} \]  

\[ \text{(28b)} \]

\[ B' = A' (G' - I_n)^{-1} H' \]  

\[ \equiv \frac{\beta}{T} \left[ I_n + d_1 G' + d_2 (G')^2 \right]^{-1} \times \left[ I_n + c_1 G' + c_2 (G')^2 \right] \]  

\[ \times (G' - I_n)^{-1} H' \triangleq B'_{m} \]  

\[ \text{(29a)} \]

\[ \geq \frac{\beta}{T} \left[ I_n + d_1 G' + d_2 (G')^2 \right]^{-1} \times \left[ I_n + c_1 G' + c_2 (G')^2 \right] \]  

\[ \times (G' - I_n)^{-1} H' \triangleq B'_{m} \]  

\[ \text{(29b)} \]
The ordinary Pade approximate system model in equations (16) and (17) requires that the sampling period satisfy \( T < \frac{2}{\| A' \|} \), and the mode in equation (25) requires that the norm of the bilinear transform matrix satisfy \( \| R' \| < 1. \) The multipoint interval Pade approximate system models have no such requirements because with a different sampling period \( T \) and system matrix \( A'(G') \), a different multipoint Pade approximant can be found such that the degenerate (nominal) interval approximant matches the degenerate (nominal) system model exactly (or much closely) instead of always using a unique approximant to fit different system matrices as in the ordinary Pade approximation method. But it is noticed that both the ordinary Pade method and the multipoint Pade method cannot guarantee the stability of the approximated models.

4. Numerical example

Consider an asymptotically stable linear \( R - L - C \) circuit described by the uncertain state-space model in equation (12a). The nominal and perturbation system matrices of the model are given by

\[
\begin{align*}
A_0 &= \begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix} \\
R_0 &= \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\
\Delta A &= \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0 \end{bmatrix} \\
\Delta B &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{align*}
\] (30)

The initial state vector \( x_0(0) = 0_{2 \times 1} \), and the input vector \( u_0(t) \) is a unit-step function. The corresponding interval form of the uncertain system model is

\[
\begin{align*}
A' &= \begin{bmatrix} -2.1 & -1.9 \\ 0.9 & 1.1 \end{bmatrix} \\
B' &= \begin{bmatrix} 2,2 \\ 2,2 \end{bmatrix}
\end{align*}
\] (31)

4.1 Model conversion of continuous-time uncertain linear systems

Two uncertain model conversion methods, the interval ordinary Pade method and the interval multipoint Pade method, are used to find the approximate models of \( (G', H') \) and the solution of \( x_k(kT) \), \( k = 1, 2, \ldots \), from the continuous-time uncertain system model given by equations (12a) and (31).

Using the implicit Euler integration method the Euler approximate discrete-time interval model was obtained in Ref. 13 as follows:

\[
(I_2 - A'T)x_k(kT + T) = \hat{x}_k(kT) + B'Tu_k(kT)
\] (32a)

with \( \hat{x}_k(kT) = \hat{x}_k(kT) \) or \( \bar{x}_k(kT) \), where \( \hat{x}_k(kT) \) of \( \hat{x}_k(kT) \) represents the vertex corresponding to the upper (lower) endpoint of the current interval solution \( x_k(kT) \) in equation (32a). Solving the interval matrix equation in (32a) recursively it was reported in Ref. 13 that the approximate steady-state solution \( x_k(kT) \) at \( t = t_p = 6 \) sec is

\[
x_k(t_p) = \begin{bmatrix} 0.952381, 1.052632 \\ 0.952381, 1.052632 \end{bmatrix}
\] for \( T = 0.1 s \). (32b)

Based on the interval ordinary Pade approximants shown in equations (16) and (17) the approximate system matrices \( \left( G_p', H_p' \right) \) and states \( \bar{x}(t) \), \( t = t_p = 6 \) sec and \( T = 0.1 s \), can be computed as

\[
G_p' = \begin{bmatrix} 0.089993, 0.827566 \\ 0.068839, 0.870060 \end{bmatrix}
\]
\[
H_p' = \begin{bmatrix} 0.180395, 0.182147 \\ 0.180372, 0.182176 \end{bmatrix}
\] (33a)

and

\[
x_{dp}(t_p) = \begin{bmatrix} 0.949363, 1.056323 \\ 0.948308, 1.057496 \end{bmatrix}
\] (33b)

The corresponding approximate nominal system matrices can be calculated as

\[
G_{0p} = \frac{1}{2} \left( G_p' + G_p' \right) = \begin{bmatrix} 0.818688, 0.000000 \\ 0.077864, 0.40821 \end{bmatrix}
\]
\[
H_{0p} = \frac{1}{2} \left( H_p' + H_p' \right) = \begin{bmatrix} 0.181268, 0.181270 \end{bmatrix}
\] (33c)

The multipoint Pade approximant of the system matrix \( G_{mp}' \) shown in equation (21d) can be determined first by the method described in Appendix A. The five interpolating points are chosen as \( 0, 0.1T \times \text{trace}(A_0)/n, T \times s_1, 1 \times \text{trace}(A_0)/n, \) and \( T \times s_2, \) respectively, where \( A_0 \) is the nominal system matrix given in equation (30), \( T \) is the sampling period, and \( s_1 = -2 \) and \( s_2 = -3 \) are the eigenvalues of \( A_0 \).

For \( T = 0.1s \) the coefficients in equations (18) and (21d) are \( p_m = 1, a_1 = 0.487092, a_2 = 0.077149, b_1 = -0.512907, \) and \( b_2 = 0.090084, \) and the multipoint Pade approximate system matrices \( G_{mp}', H_{mp}' \) and states \( \bar{x}_{dm}(t) \) at \( t = t_p = 6, \) sec are

\[
G_{mp}' = \begin{bmatrix} 0.810253, 0.827225 \\ 0.069201, 0.866940 \end{bmatrix}
\]
\[
H_{mp}' = \begin{bmatrix} 0.180340, 0.182195 \\ 0.180315, 0.182226 \end{bmatrix}
\] (34a)

and

\[
x_{dm}(t_p) = \begin{bmatrix} 0.950420, 1.054512 \\ 0.950420, 1.054512 \end{bmatrix}
\] (34b)
The corresponding approximate nominal system matrices are computed as

\[
G_{0m_p} = \frac{1}{2} \begin{bmatrix} G^L_{mp} + G^U_{mp} \end{bmatrix} = \begin{bmatrix} 0.818739 & 0 \\ 0.077948 & 0.740818 \end{bmatrix}
\]

\[
H_{0m_p} = \frac{1}{2} \begin{bmatrix} H^L_{mp} + H^U_{mp} \end{bmatrix} = \begin{bmatrix} 0.181268 \\ 0.181270 \end{bmatrix}
\]

(34c)

The exact solutions of the nominal system matrices are calculated as

\[
G_0 = e^{A_0 T} = \begin{bmatrix} 0.818731 \\ 0.740818 \end{bmatrix}
\]

\[
H_0 = (G_0 - I_n) A_0^{-1} B_0 = \begin{bmatrix} 0.181270 \\ 0.181270 \end{bmatrix}
\]

(35)

For \( T = 0.3s \) and \( t = t_p = 6 \) set, the solutions computed by the ordinary Pade method are

\[
G^p = \begin{bmatrix} 0.519845, 0.577332 \\ 0.112403, 0.170800 \end{bmatrix}
\]

\[
H^p = \begin{bmatrix} 0.444914, 0.457291 \\ 0.444506, 0.457788 \end{bmatrix}
\]

(36a)

and

\[
x_{dp}(t_p) = \begin{bmatrix} 0.926510, 1.081883 \\ 0.925096, 1.083429 \end{bmatrix}
\]

(36b)

The corresponding approximate nominal system matrices are

\[
G_{0p} = \begin{bmatrix} 0.548881 & 0 \\ 0.142466 & 0.406570 \end{bmatrix}
\]

\[
H_{0p} = \begin{bmatrix} 0.451165 \\ 0.451220 \end{bmatrix}
\]

(37c)

The exact solution of the nominal system matrices are calculated as

\[
G_0 = e^{A_0 T} = \begin{bmatrix} 0.548812 \\ 0.142242 \end{bmatrix}
\]

\[
H_0 = (G_0 - I_n) A_0^{-1} B_0 = \begin{bmatrix} 0.451188 \\ 0.451188 \end{bmatrix}
\]

(38)

It is observed that each entry of the discrete-time nominal system matrices in equations (34c) and (37c) approximated by the interval multipoint Pade method matches more closely with each associated entry of the exact nominal system matrices shown in equations (35) and (38), respectively, than the corresponding discrete-time nominal system matrices given in equations (33c) and (36c), which were approximated by the interval ordinary Pade method.

The unit-step responses, \( x_{dp}(t) \) and \( x_{dp}(t) \), of the uncertain linear system described in equations (12a) and (31), with \( T = 0.1s \) and \( T = 0.3s \), are plotted in Figures 1 and 2, respectively. From the response curves in Figures 1 and 2, it is observed that the approximate solutions

![Figure 1. The unit-step responses of states with \( T = 0.1s \).](image-url)
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\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{The unit step responses of states with \( T = 0.3 \text{ sec.} \)}
\end{figure}

The second \( T = 0.3 \text{ sec.} \). Therefore the interval multipoint Pade approximation method, as expected, gives a more accurate and a less conservative interval solution than the interval ordinary Pade approximation method.

4.2 Model conversion of discrete-time uncertain linear systems

For the continuous-time model conversion the discrete-time uncertain system model is given in the form of equation (22), and it is desired to find the corresponding continuous-time uncertain model described in equation (23).

For \( T = 0.1 \text{ s} \) the discrete-time uncertain system matrices are given in equation (34a). The continuous-time uncertain system matrices computed by using the interval multipoint Pade approximation method are

\begin{align}
A'_m &= \begin{bmatrix}
-2.104069 & -1.896682 \\
0.883258 & 1.118475 \\
0.000000 & 0.000000
\end{bmatrix}
\end{align}

\begin{align}
B'_m &= \begin{bmatrix}
-2.999866 \\
-2.999866 \\
1.993110
\end{bmatrix}
\end{align}

The corresponding nominal system matrices (denoted by \( A_{0m} \) and \( B_{0m} \)) and the perturbation system matrices (denoted by \( \Delta A_m \) and \( \Delta B_m \)) are computed as

\begin{align}
A_{0m} &= \begin{bmatrix}
-2.013217 & 0 \\
0.971384 & -2.982067
\end{bmatrix}
\end{align}

\begin{align}
B_{0m} &= \begin{bmatrix}
2.019294 \\
2.005955
\end{bmatrix}
\end{align}

and

\begin{align}
\Delta A_m &= \begin{bmatrix}
0.101286 & 0.000000 \\
0.117674 & 0
\end{bmatrix}
\end{align}

\begin{align}
\Delta B_m &= \begin{bmatrix}
0.210506 \\
0.326764
\end{bmatrix}
\end{align}

For \( T = 0.3 \text{ s} \) the discrete-time uncertain system matrices are given in equation (37a). The continuous-time system matrices computed by using the ordinary Pade method are

\begin{align}
A'_f &= \begin{bmatrix}
-2.114503 & -1.911931 \\
0.854636 & 1.088133 \\
-2.983067 & -2.983067
\end{bmatrix}
\end{align}

\begin{align}
B'_f &= \begin{bmatrix}
1.808788 & 2.229800 \\
1.619191 & 2.3527301
\end{bmatrix}
\end{align}

The corresponding nominal system matrices (denoted by \( A_{0f} \) and \( B_{0f} \)) and the perturbation system matrices (denoted by \( \Delta A_f \) and \( \Delta B_f \)) are evaluated as

\begin{align}
A_{0f} &= \begin{bmatrix}
-1.999715 & 0.000000 \\
0.994964 & -2.990070
\end{bmatrix}
\end{align}

\begin{align}
B_{0f} &= \begin{bmatrix}
2.007774 \\
1.985236
\end{bmatrix}
\end{align}
For $T = 0.3s$ the coefficients in equations (28) and (29) are $\beta = -2.14297$, $c_{T} = -0.762757$, $c_{2} = -0.214703$, $d_{T} = 1.458163$, and $d_{2} = 0.181178$, and the continuous-time uncertain system matrices (denoted by $A_m'$ and $B_m'$) computed by using the interval multipoint Pade method are

$$
A_m' = \begin{pmatrix} -2.14297 & -1.932584 \\ 0.759049 & 1.044808 \end{pmatrix},
B_m' = \begin{pmatrix} 1.821236 & 2.268964 \\ 1.669135 & 2.382025 \end{pmatrix}
$$

The corresponding nominal system matrices (denoted by $A_{0m}$ and $B_{0m}$) and the perturbation system matrices (denoted by $\Delta A_m$ and $\Delta B_m$) are calculated as

$$
A_{0m} = \begin{pmatrix} -2.14297 & 0 \\ 0 & -2.935576 \end{pmatrix},
B_{0m} = \begin{pmatrix} 2.045099 \\ 2.025580 \end{pmatrix}
$$

and

$$
\Delta A_m = \begin{pmatrix} 0.104807 & 0.000000 \\ 0.142879 & 0 \end{pmatrix},
\Delta B_m = \begin{pmatrix} 0.223865 \\ 0.356445 \end{pmatrix}
$$

Comparing the results in equation 39 with the results in equation (40) and the results in equation (41) with those in equation (42) it is observed that the approximate solutions obtained by the multipoint Pade approximation method are more accurate and less conservative than those determined by the ordinary Pade approximation method.

5. Conclusions

The degenerate (real) multipoint Pade approximation method has been extended to the interval multipoint Pade approximation method and applied to carry out the model conversions of continuous-time and discrete-time uncertain models.

For a relatively larger perturbed system matrix and/or a sampling period in the model conversions the proposed interval multipoint Pade approximation method gives more accurate and less conservative interval approximate models and solutions than those obtained by the ordinary interval Pade approximation method. An example is given to demonstrate the effectiveness of the proposed method.

**Appendix A: Algorithms for multipoint Pade approximation**

According to the definition of the multipoint Pade approximation given in Section 2 the coefficients $p_0 \ldots p_L$ and $q_0 \ldots q_M$ in equation (2) can be found by comparing the coefficients of the Taylor series of $F(s)$ and the Taylor series of $R_{LM}(s)$, which can be obtained by long division. This method for finding the coefficients of the multipoint Pade approximant is called the "direct method," which needs to solve $L$ linear equations and $M$ simultaneous equations. Therefore the inverse of an $M \times M$ Toeplitz matrix is required.

Another effective algorithm for finding the coefficients of a multipoint Pade approximant is the multipoint continued-fraction expansion method, which is given in Ref. 11. In this algorithm it is assumed that equations (1) and (2) exist and that equation (2) satisfies the property equation (3). Then the equivalent multipoint continued-fraction expansion of $R_{LM}(s)$ can be expressed as

$$
R_{LM}(s) = \frac{1}{h_1 + \frac{1}{h_2 + \frac{1}{\ddots + \frac{1}{h_{N-1} + \frac{s-a_{N-1}}{h_N(s-a_N)R(s)}}}}}
$$

where

$$
a_{(n+1)} = a_{(n+2)} = \ldots = a_{(n+1)} = s_i
$$

and $R(s)$ is the continued fraction expansion at the interpolating point $s = \infty$, namely,

$$
R(s) = \frac{1}{z_{N+1} + \frac{1}{z_{N+2} + \frac{1}{\ddots + \frac{1}{z_{K-1} + \frac{1}{z_K}}}}}
$$

where

$$
K = \begin{cases} 2M, & L < M \\
2M + 1, & L = M
\end{cases}
$$

and

$$
z_i = \begin{cases} h_i & i \text{ is even} \\
h_i s & i \text{ is odd}
\end{cases}
$$
and \( h_i, i = 1, 2, \ldots, k, \) are called partial quotients of the multipoint continued fraction expansion. After the desired quotients are found the corresponding multipoint Pade approximant, as shown in equation (2), can be determined via the inversion operator of equation (43).

A recursive algorithm for finding the partial quotients in equation (43) is reported in Refs. 10 and 11 and is described below. First we need to build a Routh array for each expanding point, including the expanding point \( s = s_n. \) A total of \((r + 1)\) Routh arrays need to be built. The Routh array for the \( i \)th interpolating point is listed in Table 1. Then the desired partial quotients can be found from the following equation:

\[
\frac{h_j}{f_{j+1,1}} = f_{j,k} = \begin{cases} 
\frac{(f_{j,k-1} + h_{j-2}f_{j-1,k} - f_{j-2,k})/(a_{j-2} - s_j),} {i = 2, 3, \ldots, r; j = 3, 4, \ldots, n_i; k = 1, 2, \ldots, n_i} \\
\frac{f_{j-2,k+1} + h_{j-2}f_{j-1,k+1},} {i = 1, 2, \ldots, r; j = \bar{n}_i + 3, \bar{n}_i + 4, \ldots, \bar{n}_i + 1;} \\
a_{j-2}f_{j,k-1} - h_{j-2}f_{j-1,k-1} + f_{j-2,k},} 
& \quad \text{if } i = r + 1; j = 3, 5, 7, \ldots, \bar{n}_i + 2; k = 1, 2, \ldots, n_i \\
a_{j-2}f_{j,k-1} - h_{j-2}f_{j-1,k} + f_{j-2,k} & \quad \text{if } i = r + 1; j = 4, 6, 8, \ldots, \bar{n}_i + 2; k = 1, 2, \ldots, n_i 
\end{cases}
\]

The rest of the elements in Table 1 are determined by the following recursive formulas:

\[
f_{1,k} = \begin{cases} 
1, & k = 1, \ldots, n_i \\
0, & k = 1, 2, \ldots, n_i
\end{cases}
\]

\[
f_{2,k} = \begin{cases} 
\hat{f}_{i,k-1}, & k = 1, 2, \ldots, n_i; i = 1, 2, \ldots, r \\
0, & k = 1, 2, \ldots, n_i; i = r + 1
\end{cases}
\]

Appendix B: Interval analysis preliminaries

An interval number \([a, b]\) is defined to be the set of \( x \in \mathbb{R} \) such that \( a \leq x \leq b. \) The arithmetic operations on intervals are defined as follows:

\[
\begin{align*}
[a, b] + [c, d] &= [a + c, b + d]; \\
[a, b] \times [c, d] &= [\min(ac, ad, bc, bd), \\
&\quad \max(ac, ad, bc, bd)]; \\
[a, b] - [c, d] &= [a - d, b - c]; \\
[a, b] \div [c, d] &= [a, b] \times \left[ \frac{1}{d}, \frac{1}{c} \right] \quad \text{if } 0 \notin [c, d].
\end{align*}
\]

Note that the interval addition and multiplication are associative and commutative, and that the distributive law is subdistributive, i.e.,

\[
[a, b] \times ([c, d] + [e, f]) \subseteq [a, b] \times [c, d] + [a, b] \times [e, f].
\]
Interval multipoint Padé approximations: F. Feng et al.

Table 1. The i-th Routh array of multipoint continued-fraction expansion

<table>
<thead>
<tr>
<th>f_{1,1}</th>
<th>f_{1,2}</th>
<th>f_{1,3}</th>
<th>...</th>
<th>f_{1,n-1}</th>
<th>f_{1,n}</th>
</tr>
</thead>
<tbody>
<tr>
<td>f_{2,1}</td>
<td>f_{2,2}</td>
<td>f_{2,3}</td>
<td>...</td>
<td>f_{2,n-1}</td>
<td>f_{2,n}</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

h_{n+1} = \frac{f_{n+1,1}}{f_{n+2,1}} \cdot h_{n+1,1}

h_{m+1} = \frac{f_{m+1,1}}{f_{m+2,1}} \cdot h_{m+1,1}

h_{n+1} = \frac{f_{n+1,1}}{f_{n+2,1}} \cdot h_{n+1,1}

The interval matrix addition and subtraction operations are associative and commutative. However the interval matrix multiplication is, in general, not associative, and is only subdistributive.

Theorem 1

If \( F(x_1, x_2, \ldots, x_n) \) is a rational expression or an irrational monotonic real function in the interval variables \( x_1, x_2, \ldots, x_n \), i.e., a finite combination of \( x_1, x_2, \ldots, x_n \) and a finite set of constant intervals with interval arithmetic operations, then \( x'_1 \subseteq x_1, \ldots, x'_n \subseteq x_n \) implies \( F(x'_1, x'_2, \ldots, x'_n) \subseteq F(x_1, x_2, \ldots, x_n) \) for every set of interval numbers \( x_1, x_2, \ldots, x_n \) for which the interval arithmetic operations in \( F \) are defined.

In this paper we use the Hansen’s method\(^{12,14} \) to estimate the inverse of an interval matrix as shown below.

Let an interval matrix \( A' = \{a'_{ij}\} \) be a degenerate interval matrix with each entry \( a'_{ij} = (a_{ij} + \tilde{a}_{ij})/2 \). Thus the inversion of the constant real matrix \( A_0 = \{a_{ij}\} \) can be found by any matrix inversion algorithm. Also, let \( E' \in \mathcal{IR}^{n \times n} \) be an interval error matrix given by

If \( \|E'\| < 1 \), the desired \( (A')^{-1} \) satisfies

\[
(A')^{-1} \subseteq A_0^{-1} \left[ S_0' + P_0' \right] \subseteq \mathcal{IR}^{n \times n}
\]

where

\[
S_0' = I_n + E'(I_n + E'(I_n + E'(I_n + E'(\ldots))))
\]

has \( m \) additions, and \( P_0' = \{p_{0,i,j}'\} \) is an \( n \times n \) interval matrix with identical elements, each of which is the interval

\[
p_{0,i,j}' = \left[ \frac{-\|E'\|^{m+1}}{1 - \|E'\|}, \frac{\|E'\|^{m+1}}{1 - \|E'\|} \right] (\text{all } i,j)
\]

The Hansen’s method gives a less conservative estimate of \( (A')^{-1} \) than any direct matrix inversion method with interval matrix operations. To reduce the inherent conservativeness of interval arithmetic operations, various improvements such as the nested form, centered form, etc. are proposed in Refs. 12–14.

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References


Interval multipoint Pade approximations: F. Feng et al.


