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SHORT COMMUNICATION

An improved quadratic program for unweighted Euclidean *1-center* location problem

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Abstract In this paper, an improved quadratic programing formulation for the solution of unweighted Euclidean *1-center* location problem is presented. The original quadratic program is proposed by Nair and Chandrasekaran in 1971. Besides, they proposed a geometric approach for problem solving. Then, they concluded that the geometric approach is more efficient than the quadratic program. This conclusion is true only when all decision variables are treated as nonnegative variables. To improve the quadratic program, one of those variables should be an unrestricted variable as it is presented here. Numerically we proved that the improved quadratic program leads to the optimal solution of the problem in parts of second regardless of the size of the problem. Moreover, constrained version of the problem is solved optimally via the improved quadratic program in parts of second.

1. Introduction

1-Center Euclidean location problem is introduced originally by Sylvester (1857). The problem involves enclosing m known points in the plane within a circle of minimum radius. Contrary to what a person might think, the problem cannot be solved by vision (or at least no one has yet been able to do so) (Francis et al., 1992).

This problem is also known as the circle covering problem, minimum spanning circle, smallest enclosing circle (or disk) and single facility minimax location problem. In the circle covering problem as shown in Fig. 1, it is wished to locate a new facility with respect to m demand points so as to minimize the maximum Euclidean distance from the new facility to the demand points. Thus, the objective is to minimize the function $g(x, y)$ defined by:

$$g(x,y) = \max\{[(x-a_i)2 + (y-b_i)^2]^{1/2} : 1 \le i \le m\} \tag{1}$$

where (x, y) are the new facility coordinates; (a_i, b_i) are the demand points (P_i) coordinates $i = 1, \ldots, m$.

A problem equivalent to minimizing $g(x, y)$ is to minimize the maximum Euclidean distance (Z) as follows (Farahani and Hekmatfar, 2009):

$$\begin{aligned} &\text{Min} && Z \\ &\text{Subject to :} && [(x-a_i)^2 + (y-b_i)^2]^{1/2} \le Z \quad i = 1, \ldots, m \end{aligned} \tag{2}$$

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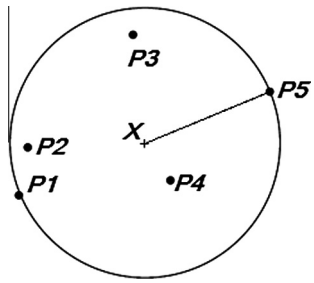

Figure 1 Circle covering problem example.

The circle covering problem may be of interest in locating a radio transmitter, a radio receiver, radar station, hospital for emergency cases, fire station and police office. Also, the problem of stationing a helicopter so as to minimize the maximum time for it to respond to an emergency at any one of m sites is closely related to this problem.

2. Review of literature

Many solution approaches are suggested in the literature to solve the circle covering problem as shown in Table 1.

Nair and Chandrasekaran (1971) proposed geometric approach and quadratic program to solve the problem. Elizinga and Hearn (1972) proposed geometric method for the solution of the problem. Unweighted Euclidean and rectilinear distances are considered. The complexity of the algorithm is $O(n^2)$, where n is the number of demand points. Drezner and Wesolowsky (1980) presented a fast iterative method for locating one center on the plane. Weighted and unweighted distances are considered. The general lp-norm ($p \geqslant 1$) is used as distance measure. A 3000 demand point problem in Euclidean distance is solved in parts of second. Chandrasekaran (1982) presented a polynomial algorithm to solve the weighted Euclidean *1-center* problem. The algorithm is proposed to minimize the ratio of convex quadratic and an affine function offer a polynomial set. The complexity of the algorithm is polynomial in the dimension of space. Megiddo (1983) presented a linear time algorithm to solve the problem. Datta (1996) proposed an algorithm based on the concept of self-organizing neural networks to solve the problem. The worst-case complexity of the proposed algorithm is $O(\log n)$ where n is the number of demand points. Ohsawa and Imai (1997) presented a procedure to construct contour lines and compute the area of the region where the objective function value is equal or less than a constant value for the problem based on the farthest point Voronoi diagram. Matsutomi and Ishii (1998) proposed a solution procedure to solve the problem when A-distance is considered. The procedure is an extending of the geometric approach proposed by Elizinga and Hearn (1972) to A-distance case. Then they applied the procedure to the location of an ambulance service station in an area. Das et al. (1999) proposed an algorithm to solve the problem when the demand points are spread over a hemisphere. The algorithm is based on geometry and having a time complexity $O(n^2)$ where n is the number of demand points. Li et al. (2002) considered two fuzzy versions of the circle covering problem when the locations of points are not precise but fuzzy. Polynomial algorithms are proposed for both versions. Brimberg and Wesolowsky (2002) formulated the problem on the continuous plane where the demand points and service center may be represented by areas instead of points. The distance function measures the shortest distance between any point on the service center and any point in the demand area. Also, a general methodology for optimization was developed and can lead to efficient solution methods. Roy et al. (2009) proposed a heuristic algorithm to solve the circle covering problem, where the center is constrained to lie on a query line segment. The time complexity of the algorithm is $O(\log^2 n)$ where, n is the number of demand points.

3. Quadratic programing model

Nair and Chandrasekaran (1971) proposed a quadratic programing formulation for the solution of the problem. The problem is converted into an equivalent quadratic programing problem as follows:

Table 1 Literature summary.

Authors	Year	Distance	Method/Formula
Nair and Chandrasekaran	1971	Unweighted Euclidean	Geometrical approach Quadratic program
Elzinga and Hearn	1972	Unweighted Euclidean and Rectilinear	Geometrical approach
Drezner and Wesolowsky	1980	Weighted and unweighted lp-norm	Heuristic approach
Chandrasekaran	1982	Weighted Euclidean	Polynomial algorithm
Megiddo	1983	Unweighted Euclidean, weighted rectilinear and weighted tree network	Heuristic approach
Datta	1996	Unweighted Euclidean	Heuristic approach based on Neural network
Ohsawa and Imai	1997	Unweighted Euclidean	Geometric approach to construct contour lines
Matsutomi and Ishii	1998	A-distance	Geometric approach
Das et al.	1999	Unweighted Euclidean on a hemisphere	Heuristic approach
Li et al.	2002	Unweighted Euclidean	Heuristic approach
Brimberg and Wesolowsky	2002	Unweighted Euclidean with area facility	General methodology
Roy et al.	2009	Unweighted Euclidean with constraints	Heuristic approach

Let the coordinates of the points P_i be denoted by (a_i, b_i), $i = 1, \ldots, m$ and those of the point P by (x, y) (Li et al., 2002):

Define $d(x, y) = \max d(P, P_i), \quad i = 1, \ldots, m.$

Then $d(x, y) \geqslant d(P, P_i), \quad i = 1, \ldots, m$ or equivalently,

$$d^2(x, y) \geqslant (x - a_i)^2 + (y - b_i)^2, \quad i = 1, \ldots, m. \tag{3}$$

By defining new variable $\lambda = d^2 - x^2 - y^2$, the problem is reduced to the following quadratic programing model:

$$\text{Minimize} \quad f(\lambda, x, y) = \lambda + x^2 + y^2 \tag{4}$$

$$\text{Subject to}: \quad 2a_i x + 2b_i y + \lambda \geq a_i^2 + b_i^2, \quad i = 1, \ldots, m \tag{5}$$

Nair and Chandrasekaran (1971) did not attempt to compare their two proposed methods. They stated that, in all practical problems the geometric method will be done manually while the quadratic program will be solved on a computer. When the two methods require such different means a comparison of computational time is not meaningful. However, based on their prior experience with quadratic programs, they strongly believe that the geometric method will be more efficient and it is very simple.

One can notice that this model is not complete, since, bounded constraints i.e., nonnegative constraints, are not defined. Thus, when it is wanted to solve this model, we will assume that all defined variables i.e., x, y, and λ, are nonnegative variables. In this case, Nair and Chandrasekaran's conclusions about the quadratic model will be true. If we assume that first quarter space will be considered, variables x and y will be nonnegative. The third variable λ should be investigated as follows:

The objective function i.e., $f(\lambda, x, y) = \lambda + x^2 + y^2$, is to minimize the maximum Euclidean distance. Hence, the distance d will never be negative. In addition, if $x = y = 0$, then λ must be positive. Oppositely, if the optimal solution of the problem for example is $x = 4$, $y = 5$ and $d = 2$, then if we substitute these values in the equation $\lambda = d^2 - x^2 - y^2$, we will find that $\lambda = (2)^2 - (4)^2 - (5)^2 = 4 - 16 - 25 = -37$ which is a negative value. Thus, variable λ is an unrestricted variable.

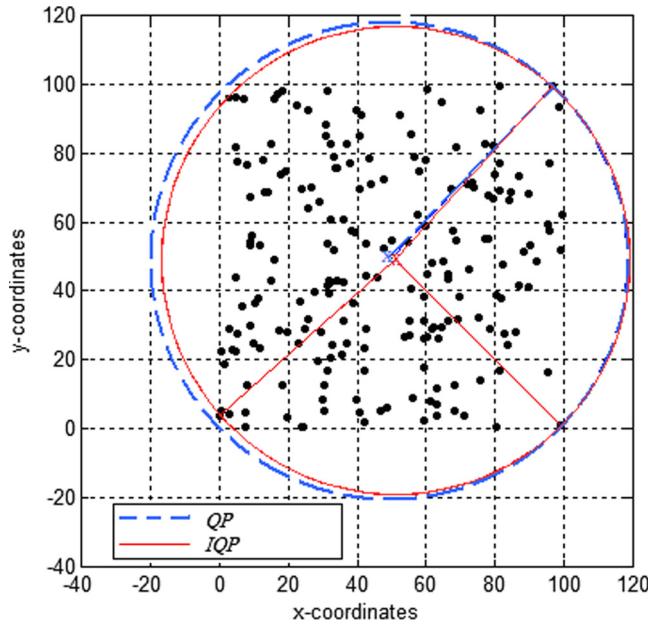

Figure 3 IQP vs. QP solution, $m = 200$.

4. Numerical results

To find out how effective is the improved quadratic program model, i.e., λ is the unrestricted variable, random coordinate data sets for 10 and 200 points, representing the demand points, is generated from a uniform distribution U(0, 1000) and third data set for 500 points is generated from a uniform distribution between U(0, 1000) by first generating x-coordinate values then generating y-coordinate values.

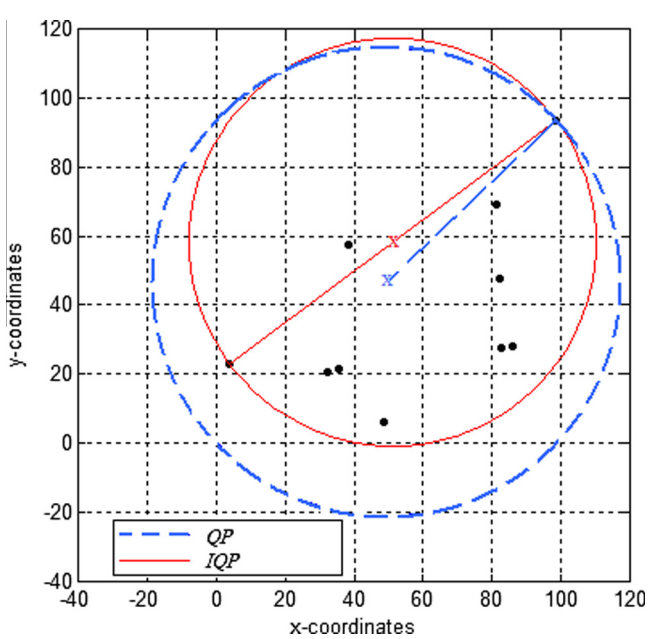

Figure 2 IQP vs. QP solution, $m = 10$.

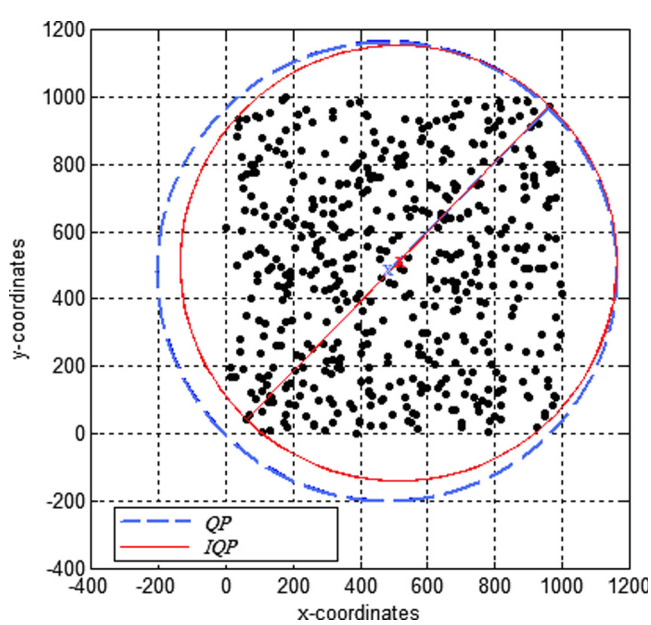

Figure 4 IQP vs. QP solution, $m = 500$.

Table 2 IQP vs. QP results summary.

Number of Demand points (m)	Solution	IQP	QP
10	x-Coordinate	51.33886	49.40251
	y-Coordinate	58.33431	46.83893
	Circle radius	59.1794	68.0771
	Iterations	10	8
200	x-Coordinate	51.17656	49.25605
	y-Coordinate	48.92306	48.87072
	Circle radius	68.0492	69.3866
	Iterations	9	9
500	x-Coordinate	512.5	482.5
	y-Coordinate	506.5	485.5
	Circle radius	648.4724	684.4827
	Iterations	15	11

Table 3 IQP vs. D&W algorithm (Drezner and Wesolowsky algorithm).

Number of demand points (m)	Solution	IQP	D&W algorithm
500	x-Coodinate	483.9797	483.9797
	y-Coordinate	497.8540	497.8540
	Circle radius	684.5700	684.5700
1000	x-Coodinate	498	498
	y-Coordinate	507.5	507.5
	Circle radius	687.6767	687.6767
3000	x-Coodinate	50.5	50.5
	y-Coordinate	50	50
	Circle radius	69.6509	69.6509
5000	x-Coodinate	503.4948	503.4948
	y-Coordinate	506.9365	506.9365
	Circle radius	694.0741	694.0741

Table 4 CPU time in seconds for IQP and nonlinear program.

Number of demand points (m)	IQP	Nonlinear program
500	< 1	43
1000	< 1	172
3000	< 1	1085
5000	< 1	4240

The IQP (improved quadratic program) is compared to the QP (original quadratic program) proposed by Nair and Chandrasekaran to solve the unweighted Euclidean *1-center* location problem. IQP and QP are modeled in LINGO 11 (LINDO systems Inc.) then, the comparisons are run on T7200 2 GHz with 2 GB RAM.

It is known that, in the optimal solution of the circle covering problem, the circle is determined by two or three demand points lying on its circumference (Francis et al., 1992). Regarding this fact, an instance of 10 demand points is solved via both IQP and QP as shown in Fig. 2. Then, an instance of 200 demand points is solved as shown in Fig. 3. Finally, instance of 500 demand points is solved as shown in Fig. 4. It can be seen clearly that IQP leads to the optimal solution of the problem in all cases while, QP does not. The results are summarized in Table 2.

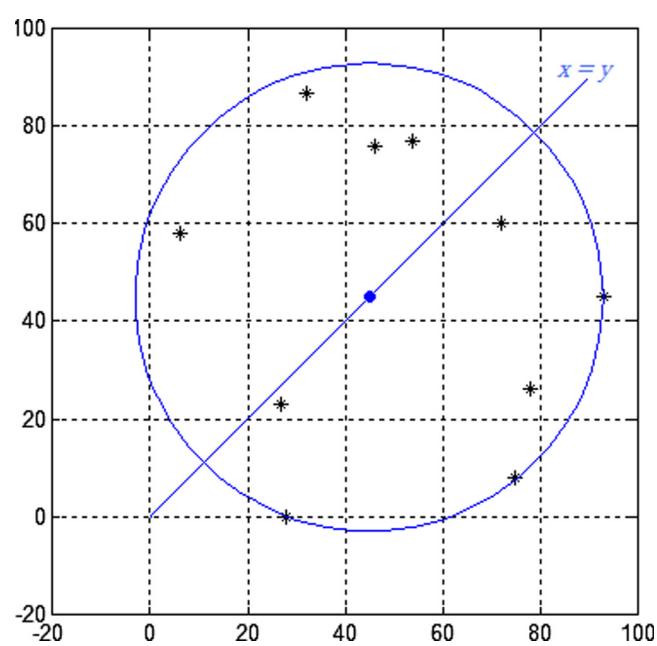

Figure 5 Constrained *1-center*, $m = 10$.

Moreover, the IQP is compared to a fast heuristic algorithm proposed by Drezner and Wesolowsky (1980). D&W algorithm (Drezner and Wesolowsky algorithm) is coded on MATLAB (Mathwork Inc.) then, the comparisons are run. Large size instance problems; i.e., 500, 1000, 3000 and 5000 points; are generated from a uniform distribution U(0, 1000). In all cases, the solution values obtained by both methods were exactly similar as shown in Table 3. However, the solutions are obtained in parts of second via both methods.

Finally, the IQP is compared to the nonlinear program (presented in Section 1) to solve constrained; i.e., the center must or must not lie on a specific point; and unconstrained version of the problem. The novelty of the IQP is due to the solution time and its ability to solve the constrained version of the problem. Thus, optimal solutions of the problems are obtained in parts of second. The nonlinear program leads to the optimal solution in long CPU time; e.g. 5000 demand points are solved optimally in more than 70 min. CPU times for IQP and nonlinear program are shown in Table 4. In Fig. 5 the 10 demand points instance solution when the center is constrained to lie on a line having the equation $x = y$ is illustrated.

5. Conclusions

1-Center location problem is one of the best known location problems. Geometric approaches and a quadratic program are proposed in the literature to solve the problem. The authors who proposed the quadratic program concluded that the geometric approach is more efficient than the quadratic program. This conclusion is true only when all decision variables are treated as nonnegative variables in the quadratic program, which is not right. One of those variables should be an unrestricted variable as it is presented in the improved quadratic program. The comparison between the improved quadratic program and the original one shows that the improved quadratic program leads to the optimal solution while, the original one does not. Moreover, the improved quadratic program is compared to a fast heuristic algorithm proposed by Drezner and Wesolowsky (1980). Numerically we proved that the improved quadratic program leads to the optimal solution of the problem in parts of second regardless of the size of the problem. In addition, constrained version of the problem is solved optimally via the IQP in parts of second.

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