



Some Pachpatte type inequalities on time scales[☆]

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ABSTRACT

In this paper, using the comparison theorem, we investigate some Pachpatte type integral inequalities on time scales, which provide explicit bounds on unknown functions. Our results extend some known dynamic inequalities on time scales, unify and extend some continuous inequalities and their corresponding discrete analogues. Some applications of the main results are given in the end of this paper.

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1. Introduction

The theory of time scales, which has recently received a lot of attention, was initiated by Hilger [1] in his Ph.D. thesis in 1988 in order to contain both difference and differential calculus in a consistent way. Since then many authors have expounded on various aspects of the theory of dynamic equations on time scales. For example, we refer the reader to the papers [2–7], the monographs [8,9] and the references cited therein. At the same time, a few papers [10–14] have studied the theory of integral inequalities on time scales.

In this paper, we study some Pachpatte type inequalities on time scales, which extend some known dynamic inequalities on time scales, unify and extend some continuous inequalities and their corresponding discrete analogues. This paper is organized as follows: In the next section we present some basic definitions and preliminary results with respect to the calculus on time scales, which can also be found in [8,9]. In Section 3 we deal with our Pachpatte type inequalities on time scales. In Section 4 we give some applications of our main results.

2. Some preliminaries on time scales

In what follows, \mathbb{R} denotes the set of real numbers, \mathbb{Z} denotes the set of integers.

A *time scale* \mathbb{T} is an arbitrary nonempty closed subset of \mathbb{R} . The *forward jump operator* σ on \mathbb{T} is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \in \mathbb{T} \quad \text{for all } t \in \mathbb{T}.$$

In this definition we put $\inf \emptyset = \sup \mathbb{T}$, where \emptyset is the empty set. If $\sigma(t) > t$, then we say that t is *right-scattered*. If $\sigma(t) = t$ and $t < \sup \mathbb{T}$, then we say that t is *right-dense*. The *graininess* $\mu : \mathbb{T} \rightarrow [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$. The set \mathbb{T}^{κ} is derived from \mathbb{T} as follows: If \mathbb{T} has a left-scattered maximum m , then $\mathbb{T}^{\kappa} = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$.

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We say that $f : \mathbb{T} \rightarrow \mathbb{R}$ is *rd-continuous* provided f is continuous at each right-dense point of \mathbb{T} and has a finite left-sided limit at each left-dense point of \mathbb{T} . As usual, the set of rd-continuous functions is denoted by C_{rd} . We say that $p : \mathbb{T} \rightarrow \mathbb{R}$ is *regressive* provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$. We denote by \mathcal{R} the set of all regressive and rd-continuous functions. We define the set of all positively regressive functions by $\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}$. Obviously, if $p \in C_{rd}$ and $p(t) \geq 0$ for $t \in \mathbb{T}$, then $p \in \mathcal{R}^+$.

Theorem 2.1. *If $p \in \mathcal{R}$ and fix $t_0 \in \mathbb{T}$, then the exponential function $e_p(\cdot, t_0)$ is for the unique solution of the initial value problem*

$$x^\Delta = p(t)x, \quad x(t_0) = 1 \text{ on } \mathbb{T}.$$

Theorem 2.2. *Let $t_0 \in \mathbb{T}^\kappa$ and $w : \mathbb{T} \times \mathbb{T}^\kappa \rightarrow \mathbb{R}$ be continuous at (t, t) , where $t \geq t_0$, $t \in \mathbb{T}^\kappa$ with $t > t_0$. Assume that $w^\Delta(t, \cdot)$ is rd-continuous on $[t_0, \sigma(t)]$. If for any $\varepsilon > 0$, there exists a neighborhood U of t , independent of $\tau \in [t_0, \sigma(t)]$, such that*

$$|w(\sigma(t), \tau) - w(s, \tau) - w^\Delta(t, \tau)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s| \text{ for all } s \in U,$$

where w^Δ denotes the derivative of w with respect to the first variable, then

$$g(t) := \int_{t_0}^t w(t, \tau) \Delta \tau$$

implies

$$g^\Delta(t) = \int_{t_0}^t w^\Delta(t, \tau) \Delta \tau + w(\sigma(t), t).$$

The following theorem is a foundational result in dynamic inequalities.

Theorem 2.3 (Comparison Theorem). *Suppose $u, b \in C_{rd}$, $a \in \mathcal{R}^+$. Then*

$$u^\Delta(t) \leq a(t)u(t) + b(t), \quad t \geq t_0, t \in \mathbb{T}^\kappa$$

implies

$$u(t) \leq u(t_0)e_a(t, t_0) + \int_{t_0}^t e_a(t, \sigma(\tau))b(\tau) \Delta \tau, \quad t \geq t_0, t \in \mathbb{T}^\kappa.$$

3. Main results

In this section, we deal with Pachpatte type inequalities on time scales. For convenience, we always assume that $t \geq t_0$, $t \in \mathbb{T}^\kappa$.

Theorem 3.1. *Assume that $u, f, p \in C_{rd}$, $u(t), f(t)$ and $p(t)$ are nonnegative, and u_0 is a nonnegative constant. If $w(t, s)$ is as defined in Theorem 2.2 such that $w(t, s) \geq 0$ and $w^\Delta(t, s) \geq 0$ for $t, s \in \mathbb{T}$ with $s \leq t$, then*

$$u(t) \leq u_0 + \int_{t_0}^t [f(\tau)u(\tau) + p(\tau)] \Delta \tau + \int_{t_0}^t f(\tau) \left(\int_{t_0}^\tau w(\tau, s)u(s) \Delta s \right) \Delta \tau, \quad t \in \mathbb{T}^\kappa \tag{E1}$$

implies

$$u(t) \leq u_0 + \int_{t_0}^t \left\{ p(\tau) + f(\tau) \left[u_0 e_{f+A}(\tau, t_0) + \int_{t_0}^\tau e_{f+A}(\tau, \sigma(s))p(s) \Delta s \right] \right\} \Delta \tau, \quad t \in \mathbb{T}^\kappa, \tag{I1}$$

where

$$A(t) = w(\sigma(t), t) + \int_{t_0}^t w^\Delta(t, s) \Delta s. \tag{3.1}$$

Proof. Define a function $z(t)$ by the right hand of (E1). Then $z(t_0) = u_0$, $u(t) \leq z(t)$, and

$$\begin{aligned} z^\Delta(t) &= f(t)u(t) + p(t) + f(t) \int_{t_0}^t w(\tau, s)u(s) \Delta s \\ &\leq p(t) + f(t) \left[z(t) + \int_{t_0}^t w(\tau, s)z(s) \Delta s \right], \quad t \in \mathbb{T}^\kappa. \end{aligned} \tag{3.2}$$

Let

$$v(t) = z(t) + \int_{t_0}^t w(\tau, s)z(s)\Delta s. \tag{3.3}$$

Obviously, $v(t_0) = z(t_0) = u_0$, $z(t) \leq v(t)$, and $z^\Delta(t) \leq p(t) + f(t)v(t)$. Using Theorem 2.2, we have

$$\begin{aligned} v^\Delta(t) &= z^\Delta(t) + w(\sigma(t), t)z(t) + \int_{t_0}^t w^\Delta(t, s)z(s)\Delta s \\ &\leq p(t) + \left(f(t) + w(\sigma(t), t) + \int_{t_0}^t w^\Delta(t, s)\Delta s \right) v(t) \\ &= p(t) + (f(t) + A(t))v(t), \quad t \in \mathbb{T}^\kappa, \end{aligned}$$

where $A(t)$ is as defined in (3.1). It is easy to see that $(f + A) \in \mathcal{R}^+$. Therefore, using Theorem 2.3, from the above inequality, we have

$$v(t) \leq u_0 e_{f+A}(t, t_0) + \int_{t_0}^t e_{f+A}(t, \sigma(s))p(s)\Delta s, \quad t \in \mathbb{T}^\kappa. \tag{3.4}$$

Combining (3.2)–(3.4), we obtain

$$z^\Delta(t) \leq p(t) + f(t) \left[u_0 e_{f+A}(t, t_0) + \int_{t_0}^t e_{f+A}(t, \sigma(s))p(s)\Delta s \right], \quad t \in \mathbb{T}^\kappa. \tag{3.5}$$

Setting $t = \tau$ in (3.5), integrating it from t_0 to t , and noting $z(t_0) = u_0$ and $u(t) \leq z(t)$, we easily obtain the desired inequality (I1). The proof is complete. \square

Remark 3.1. If $p(t) = 0$ and $w(t, s) = w(s)$ in Theorem 3.1, then the inequality given in (I1) reduces to the inequality in [13, Theorem 1].

Remark 3.2. Let $p(t) = 0$ in Theorem 3.1. If $\mathbb{T} = \mathbb{R}$, then we can obtain Theorem 2.1(a_1) in [15]. If $\mathbb{T} = \mathbb{Z}$, then we easily obtain Theorem 2.3 (c_1) in [15].

Remark 3.3. Let $w(t, s) = w(s)$ in Theorem 3.1. If $\mathbb{T} = \mathbb{R}$, then the inequality established in Theorem 3.1 reduces to the inequality established by Pachpatte in [16, Theorem 1.7.2 (i)]. If $\mathbb{T} = \mathbb{Z}$, then from Theorem 3.1, we easily obtain Theorem 1.8.7 in [17].

Theorem 3.2. Assume that $u, f, p \in C_{rd}$, $u(t), f(t)$ and $p(t)$ are nonnegative, and u_0 is a nonnegative constant. If $w(t, s)$ is as defined in Theorem 2.2 such that $w(t, s) \geq 0$ and $w^\Delta(t, s) \geq 0$ for $t, s \in \mathbb{T}$ with $s \leq t$, then

$$u(t) \leq u_0 + \int_{t_0}^t f(\tau)u(\tau)\Delta\tau + \int_{t_0}^t f(\tau) \left(\int_{t_0}^\tau [w(\tau, s)u(s) + p(s)]\Delta s \right) \Delta\tau, \quad t \in \mathbb{T}^\kappa \tag{E2}$$

implies

$$u(t) \leq u_0 + \int_{t_0}^t f(\tau) \left[u_0 e_{f+A}(\tau, t_0) + \int_{t_0}^\tau e_{f+A}(\tau, \sigma(s))p(s)\Delta s \right] \Delta\tau, \quad t \in \mathbb{T}^\kappa, \tag{I2}$$

where $A(t)$ is as defined in (3.1).

Proof. Define a function $z(t)$ by the right hand of (E2). Then $z(t_0) = u_0$, $u(t) \leq z(t)$, and

$$\begin{aligned} z^\Delta(t) &= f(t)u(t) + f(t) \int_{t_0}^t [w(\tau, s)u(s) + p(s)]\Delta s \\ &\leq f(t) \left[z(t) + \int_{t_0}^t [w(\tau, s)z(s) + p(s)]\Delta s \right], \quad t \in \mathbb{T}^\kappa. \end{aligned} \tag{3.6}$$

Let

$$m(t) = z(t) + \int_{t_0}^t [w(\tau, s)z(s) + p(s)]\Delta s, \quad t \in \mathbb{T}^\kappa. \tag{3.7}$$

It is easy to see that $m(t_0) = z(t_0) = u_0, z(t) \leq m(t)$, and

$$\begin{aligned} m^\Delta(t) &= z^\Delta(t) + w(\sigma(t), t)z(t) + \int_{t_0}^t w^\Delta(t, s)z(s)\Delta s + p(t) \\ &\leq \left(f(t) + w(\sigma(t), t) + \int_{t_0}^t w^\Delta(t, s)\Delta s \right) m(t) + p(t) \\ &= (f(t) + A(t))m(t) + p(t), \quad t \in \mathbb{T}^\kappa, \end{aligned} \tag{3.8}$$

where $A(t)$ is as defined in (3.1). Using Theorem 2.3, from (3.8), we obtain

$$m(t) \leq u_0 e_{f+A}(t, t_0) + \int_{t_0}^t e_{f+A}(t, \sigma(s))p(s)\Delta s, \quad t \in \mathbb{T}^\kappa. \tag{3.9}$$

Therefore,

$$z^\Delta(t) \leq f(t) \left\{ u_0 e_{f+A}(t, t_0) + \int_{t_0}^t e_{f+A}(t, \sigma(s))p(s)\Delta s \right\}, \quad t \in \mathbb{T}^\kappa. \tag{3.10}$$

Setting $t = \tau$ in (3.10), integrating it from t_0 to t , and noting $z(t_0) = u_0$ and $u(t) \leq z(t)$, we easily obtain the desired inequality (12). The proof of Theorem 3.2 is completed. \square

Remark 3.4. By taking $w(t, s) = w(s)$, from Theorem 3.2, we easily obtain Theorem 4.8 (ii) in [14].

Remark 3.5. Letting $w(t, s) = w(s)$ in Theorem 3.2, we can obtain the inequality established in [16, Theorem 1.7.2 (ii)] if $\mathbb{T} = \mathbb{R}$, and the inequality established in [17, Theorem 1.4.6 (ii)] if $\mathbb{T} = \mathbb{Z}$.

Theorem 3.3. Assume that $u, f, g \in C_{rd}, u(t), f(t)$ and $g(t)$ are nonnegative, and u_0 is a nonnegative constant. If $w(t, s)$ is as defined in Theorem 2.2 such that $w(t, s) \geq 0$ and $w^\Delta(t, s) \geq 0$ for $t, s \in \mathbb{T}$ with $s \leq t$, then

$$u(t) \leq u_0 + \int_{t_0}^t f(\tau)u(\tau)\Delta\tau + \int_{t_0}^t g(\tau) \left(u(\tau) + \int_{t_0}^\tau w(\tau, s)u(s)\Delta s \right) \Delta\tau, \quad t \in \mathbb{T}^\kappa \tag{E3}$$

implies

$$u(t) \leq u_0 \left[e_f(t, t_0) + \int_{t_0}^t e_f(t, \sigma(\tau))g(\tau)e_{f+g+A}(\tau, t_0)\Delta\tau \right], \quad t \in \mathbb{T}^\kappa, \tag{13}$$

where $A(t)$ is as defined in (3.1).

Proof. Define a function $z(t)$ by the right hand of (E3). Then $z(t_0) = u_0, u(t) \leq z(t)$, and

$$\begin{aligned} z^\Delta(t) &= f(t)u(t) + g(t) \left(u(t) + \int_{t_0}^t w(t, s)u(s)\Delta s \right) \\ &\leq f(t)z(t) + g(t) \left(z(t) + \int_{t_0}^t w(t, s)z(s)\Delta s \right), \quad t \in \mathbb{T}^\kappa. \end{aligned} \tag{3.11}$$

Let

$$v(t) = z(t) + \int_{t_0}^t w(t, s)z(s)\Delta s, \quad t \in \mathbb{T}^\kappa. \tag{3.12}$$

Then $v(t_0) = z(t_0) = u_0, z(t) \leq v(t)$, and

$$\begin{aligned} v^\Delta(t) &= z^\Delta(t) + w(\sigma(t), t)z(t) + \int_{t_0}^t w^\Delta(t, s)z(s)\Delta s \\ &\leq (f(t) + g(t) + A(t))v(t), \quad t \in \mathbb{T}^\kappa, \end{aligned} \tag{3.13}$$

where $A(t)$ is as defined in (3.1). It is easy to see that $(f + g + A) \in \mathcal{R}^+$. Therefore, by using Theorem 2.3, from (3.13), we easily have

$$v(t) \leq u_0 e_{f+g+A}(t, t_0), \quad t \in \mathbb{T}^\kappa. \tag{3.14}$$

Combining (3.11), (3.12) and (3.14), we obtain

$$z^\Delta(t) \leq f(t)z(t) + u_0 g(t) e_{f+g+A}(t, t_0), \quad t \in \mathbb{T}^\kappa,$$

which implies

$$z(t) \leq u_0 \left[e_f(t, t_0) + \int_{t_0}^t e_f(t, \sigma(\tau))g(\tau)e_{f+g+A}(\tau, t_0)\Delta\tau \right], \quad t \in \mathbb{T}^\kappa. \tag{3.15}$$

It is obvious that the desired inequality (I3) follows from $u(t) \leq z(t)$ and (3.15). The proof of Theorem 3.3 is complete. \square

Remark 3.6. Let $w(t, s) = w(s)$ in Theorem 3.3. Then we observe that Theorem 1.7.2 (iii) in [16] is a peculiar case of Theorem 3.3 if $\mathbb{T} = \mathbb{R}$, and Theorem 1.4.6 (iii) in [17] is also a peculiar case of Theorem 3.3 if $\mathbb{T} = \mathbb{Z}$.

Theorem 3.4. Assume that $u, a, f, p \in C_{rd}$, $u(t), a(t), f(t)$ and $p(t)$ are nonnegative. If $a(t)$ and $p(t)$ are nondecreasing, and $w(t, s)$ is as defined in Theorem 2.2 such that $w(t, s) \geq 0$ and $w^\Delta(t, s) \geq 0$ for $t, s \in \mathbb{T}$ with $s \leq t$, then

$$u(t) \leq a(t) + p(t) \left[\int_{t_0}^t f(\tau)u(\tau)\Delta\tau + \int_{t_0}^t f(\tau)p(\tau) \left(\int_{t_0}^\tau w(\tau, s)u(s)\Delta s \right) \Delta\tau \right], \quad t \in \mathbb{T}^\kappa \tag{E4}$$

implies

$$u(t) \leq a(t) \left\{ 1 + p(t) \left[\int_{t_0}^t f(\tau) \left(1 + p(\tau) \int_{t_0}^\tau e_{p(f+A)}(\tau, \sigma(s))(f(s) + A(s))\Delta s \right) \Delta\tau \right] \right\}, \quad t \in \mathbb{T}^\kappa, \tag{I4}$$

where $A(t)$ is as defined in (3.1).

Proof. Define

$$v(t) = \int_{t_0}^t f(\tau)u(\tau)\Delta\tau + \int_{t_0}^t f(\tau)p(\tau) \left(\int_{t_0}^\tau w(\tau, s)u(s)\Delta s \right) \Delta\tau, \quad t \in \mathbb{T}^\kappa. \tag{3.16}$$

Then $v(t_0) = 0, u(t) \leq a(t) + p(t)v(t)$, and

$$\begin{aligned} v^\Delta(t) &= f(t)u(t) + f(t)p(t) \int_{t_0}^t w(t, s)u(s)\Delta s \\ &\leq f(t) \left\{ a(t) + p(t) \left[v(t) + \int_{t_0}^t w(t, s)(a(s) + p(s)v(s))\Delta s \right] \right\}, \quad t \in \mathbb{T}^\kappa. \end{aligned} \tag{3.17}$$

Letting

$$y(t) = v(t) + \int_{t_0}^t w(t, s)(a(s) + p(s)v(s))\Delta s, \tag{3.18}$$

and noting $a(t)$ and $p(t)$ are nondecreasing, we easily see that $y(t_0) = v(t_0) = 0, v(t) \leq y(t), v^\Delta(t) \leq f(t)[a(t) + p(t)y(t)]$, and

$$\begin{aligned} y^\Delta(t) &= v^\Delta(t) + w(\sigma(t), t)(a(t) + p(t)v(t)) + \int_{t_0}^t w^\Delta(t, s)(a(s) + p(s)v(s))\Delta s \\ &\leq a(t) \left(f(t) + w(\sigma(t), t) + \int_{t_0}^t w^\Delta(t, s)\Delta s \right) + p(t) \left(f(t) + w(\sigma(t), t) + \int_{t_0}^t w^\Delta(t, s)\Delta s \right) y(t) \\ &= a(t)(f(t) + A(t)) + p(t)(f(t) + A(t))y(t), \quad t \in \mathbb{T}^\kappa \end{aligned} \tag{3.19}$$

where $A(t)$ is as defined in (3.1). It is easy to see that $p(f + A) \in \mathcal{R}^+$. Therefore, using Theorem 2.3, from (3.19), we obtain

$$\begin{aligned} y(t) &\leq \int_{t_0}^t e_{p(f+A)}(t, \sigma(s))a(s)(f(s) + A(s))\Delta s \\ &\leq a(t) \int_{t_0}^t e_{p(f+A)}(t, \sigma(s))(f(s) + A(s))\Delta s, \quad t \in \mathbb{T}^\kappa. \end{aligned} \tag{3.20}$$

Therefore,

$$v^\Delta(t) \leq a(t)f(t) \left\{ 1 + p(t) \int_{t_0}^t e_{p(f+A)}(t, \sigma(s))(f(s) + A(s))\Delta s \right\}, \quad t \in \mathbb{T}^\kappa. \tag{3.21}$$

Setting $t = \tau$ in (3.21), integrating it from t_0 to t , and noting $v(t_0) = 0, u(t) \leq a(t) + p(t)v(t)$, and $a(t)$ and $p(t)$ are nondecreasing, we easily obtain the desired inequality (I4). This completes the proof. \square

Remark 3.7. Letting $w(t, s) = w(s)$, $p(t) = 1$ in Theorem 3.4, we immediately obtain Theorem 2(a) in [13], Theorem 1.7.4 in [16] if $\mathbb{T} = \mathbb{R}$, and Theorem 1.4.2 in [17] if $\mathbb{T} = \mathbb{Z}$.

Theorem 3.5. Assume that $u, f, g \in C_{rd}$, $u(t), f(t)$ and $g(t)$ are nonnegative, and u_0 is a nonnegative constant. If $w(t, s)$ is as defined in Theorem 2.2 such that $w(t, s) \geq 0$ and $w^\Delta(t, s) \geq 0$ for $t, s \in \mathbb{T}$ with $s \leq t$, then

$$\begin{aligned} u(t) &\leq u_0 + \int_{t_0}^t f(\tau)u(\tau)\Delta\tau + \int_{t_0}^t f(\tau) \left(\int_{t_0}^\tau g(s)u(s)\Delta s \right) \Delta\tau \\ &\quad + \int_{t_0}^t f(\tau) \left[\int_{t_0}^\tau g(s) \left(\int_{t_0}^s w(s, \xi)u(\xi)\Delta\xi \right) \Delta s \right] \Delta\tau, \quad t \in \mathbb{T}^\kappa, \end{aligned} \quad (E5)$$

implies

$$u(t) \leq u_0 \left\{ 1 + \int_{t_0}^t f(\tau) \left[e_f(\tau, t_0) + \int_{t_0}^\tau e_f(\tau, \sigma(s))g(s)e_{f+g+A}(s, t_0)\Delta s \right] \Delta\tau \right\}, \quad t \in \mathbb{T}^\kappa, \quad (I5)$$

where $A(t)$ is as defined in (3.1).

Proof. Define a function $z(t)$ by the right hand of (E5). Then $z(t_0) = u_0$, $u(t) \leq z(t)$, and

$$\begin{aligned} z^\Delta(t) &= f(t)u(t) + f(t) \int_{t_0}^t g(s)u(s)\Delta s + f(t) \int_{t_0}^t g(s) \left(\int_{t_0}^s w(s, \xi)u(\xi)\Delta\xi \right) \Delta s \\ &\leq f(t) \left[z(t) + \int_{t_0}^t g(s)z(s)\Delta s + \int_{t_0}^t g(s) \left(\int_{t_0}^s w(s, \xi)z(\xi)\Delta\xi \right) \Delta s \right], \quad t \in \mathbb{T}^\kappa. \end{aligned} \quad (3.22)$$

Let

$$v(t) = z(t) + \int_{t_0}^t g(s)z(s)\Delta s + \int_{t_0}^t g(s) \left(\int_{t_0}^s w(s, \xi)z(\xi)\Delta\xi \right) \Delta s, \quad t \in \mathbb{T}^\kappa. \quad (3.23)$$

Obviously, $v(t_0) = z(t_0) = u_0$, $z(t) \leq v(t)$, and

$$v^\Delta(t) \leq f(t)v(t) + g(t) \left[v(t) + \int_{t_0}^t w(s, \xi)v(\xi)\Delta\xi \right], \quad t \in \mathbb{T}^\kappa. \quad (3.24)$$

Setting

$$m(t) = v(t) + \int_{t_0}^t w(s, \xi)v(\xi)\Delta\xi, \quad t \in \mathbb{T}^\kappa, \quad (3.25)$$

we easily see that $m(t_0) = v(t_0) = u_0$, $v(t) \leq m(t)$, and

$$\begin{aligned} m^\Delta(t) &= v^\Delta(t) + w(\sigma(t), t)v(t) + \int_{t_0}^t w^\Delta(t, \xi)v(\xi)\Delta\xi \\ &\leq (f(t) + g(t) + A(t))m(t), \quad t \in \mathbb{T}^\kappa, \end{aligned} \quad (3.26)$$

where $A(t)$ is as defined in (3.1). Using Theorem 2.3, from (3.26), we have

$$m(t) \leq u_0 e_{f+g+A}(t, t_0), \quad t \in \mathbb{T}^\kappa. \quad (3.27)$$

Then from (3.24), (3.25) and (3.27) we obtain

$$v^\Delta(t) \leq f(t)v(t) + u_0 g(t) e_{f+g+A}(t, t_0), \quad t \in \mathbb{T}^\kappa, \quad (3.28)$$

which implies the estimate for $v(t)$ such that

$$v(t) \leq u_0 \left[e_f(t, t_0) + \int_{t_0}^t e_f(t, \sigma(s))g(s)e_{f+g+A}(s, t_0)\Delta s \right], \quad t \in \mathbb{T}^\kappa. \quad (3.29)$$

Combining (3.22), (3.23) and (3.29), we obtain

$$z^\Delta(t) \leq u_0 f(t) \left[e_f(t, t_0) + \int_{t_0}^t e_f(t, \sigma(s))g(s)e_{f+g+A}(s, t_0)\Delta s \right], \quad t \in \mathbb{T}^\kappa. \quad (3.30)$$

Setting $t = \tau$ in (3.30), integrating it from t_0 to t , and noting $z(t_0) = u_0$ and $u(t) \leq z(t)$, we easily obtain the desired inequality (I5). This completes the proof. \square

Remark 3.8. Let $w(t, s) = w(s)$ in Theorem 3.5. If $\mathbb{T} = \mathbb{R}$, then the inequality established in Theorem 3.5 reduces to the inequality established by Pachpatte in [16, Theorem 1.7.3 (i)]. If $\mathbb{T} = \mathbb{Z}$, then from Theorem 3.5, we easily obtain Theorem 1.4.7 (iv) in [17].

Remark 3.9. Using our main results, we can obtain many dynamic inequalities for some peculiar time scales. Due to limited space, their statements are omitted here.

4. Some applications

In this section, we present some applications of Theorem 3.1 to investigate certain properties of solutions of the following dynamic equation

$$u^\Delta(t) = F\left(t, u(t), \int_{t_0}^t H(t, s, u(s))\Delta s\right), \quad u(t_0) = C, \quad t \in \mathbb{T}^\kappa, \tag{4.1}$$

where C is a constant, $F : \mathbb{T}^\kappa \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, and $H : \mathbb{T}^\kappa \times \mathbb{T}^\kappa \times \mathbb{R} \rightarrow \mathbb{R}$ is also a continuous function.

Theorem 4.1. Assume that

$$\begin{cases} |F(t, u, v)| \leq f(t)(|u| + |v|), \\ |H(t, s, u(s))| \leq w(t, s)|u(s)|, \quad t, s \in \mathbb{T}^\kappa. \end{cases} \tag{4.2}$$

If $u(t)$ is a solution of Eq. (4.1), then

$$|u(t)| \leq |C| \left[1 + \int_{t_0}^t f(\tau) e_{f+A}(\tau, t_0) \Delta \tau \right], \quad t \in \mathbb{T}^\kappa, \tag{4.3}$$

where $f \in C_{rd}$, $f(t) \geq 0$, $w(t, s)$ is as defined in Theorem 2.2 such that $w(t, s) \geq 0$ and $w^\Delta(t, s) \geq 0$ for $t, s \in \mathbb{T}$ with $s \leq t$, and A is as defined in (3.1).

Proof. Clearly, the solution $u(t)$ of Eq. (4.1) satisfies the following equivalent equation

$$u(t) = C + \int_{t_0}^t F\left(\tau, u(\tau), \int_{t_0}^\tau H(\tau, s, u(s))\Delta s\right) \Delta \tau, \quad t \in \mathbb{T}^\kappa. \tag{4.4}$$

It follows from (4.4) and (4.2) that

$$\begin{aligned} |u(t)| &\leq |C| + \int_{t_0}^t \left| F\left(\tau, u(\tau), \int_{t_0}^\tau H(\tau, s, u(s))\Delta s\right) \right| \Delta \tau \\ &\leq |C| + \int_{t_0}^t f(\tau) \left(|u(\tau)| + \int_{t_0}^\tau |H(\tau, s, u(s))| \Delta s \right) \Delta \tau \\ &\leq |C| + \int_{t_0}^t f(\tau) \left(|u(\tau)| + \int_{t_0}^\tau w(\tau, s) |u(s)| \Delta s \right) \Delta \tau, \quad t \in \mathbb{T}^\kappa. \end{aligned} \tag{4.5}$$

Using Theorem 3.1, the desired inequality (4.3) is obtained from (4.5). The proof of Theorem 4.1 is complete. \square

Theorem 4.2. Assume that

$$\begin{cases} |F(t, u_1, u_2) - F(t, v_1, v_2)| \leq f(t)(|u_1 - v_1| + |u_2 - v_2|), \\ |H(t, s, u_1) - H(t, s, u_2)| \leq w(t, s)|u_1 - u_2|, \quad t, s \in \mathbb{T}^\kappa, \end{cases} \tag{4.6}$$

where f and w are as defined in Theorem 4.1. Then the Eq. (4.1) has at most one solution.

Proof. Let $u_1(t)$ and $u_2(t)$ be two solutions of Eq. (4.1). Then we have

$$\begin{aligned} u_1(t) - u_2(t) &= \int_{t_0}^t \left[F\left(\tau, u_1(\tau), \int_{t_0}^\tau H(\tau, s, u_1(s))\Delta s\right) \right. \\ &\quad \left. - F\left(\tau, u_2(\tau), \int_{t_0}^\tau H(\tau, s, u_2(s))\Delta s\right) \right] \Delta \tau, \quad t \in \mathbb{T}^\kappa. \end{aligned} \tag{4.7}$$

It follows from (4.6) and (4.7) that

$$\begin{aligned} |u_1(t) - u_2(t)| &\leq \int_{t_0}^t f(\tau) \left[|u_1(\tau) - u_2(\tau)| + \int_{t_0}^{\tau} |H(\tau, s, u_1(s)) - H(\tau, s, u_2(s))| \Delta s \right] \Delta \tau \\ &\leq \int_{t_0}^t f(\tau) \left[|u_1(\tau) - u_2(\tau)| + \int_{t_0}^{\tau} w(\tau, s) |u_1(s) - u_2(s)| \Delta s \right] \Delta \tau, \quad t \in \mathbb{T}^{\kappa}. \end{aligned} \quad (4.8)$$

By Theorem 3.1, we have $|u_1(t) - u_2(t)| \equiv 0, t \in \mathbb{T}^{\kappa}$. Therefore, $u_1(t) = u_2(t)$, i.e., the Eq. (4.1) has at most one solution. This completes the proof of Theorem 4.2. \square

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