The Finite Element Method in Anisotropic Sobolev Spaces

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Abstract—This paper deals with various aspects of the theory and implementation of finite element methods for elliptic boundary value problems whose variational formulation is posed on anisotropic Sobolev spaces. The theory is applied to the Onsager pancake equation which arises in the study of high speed gas centrifuges. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

We consider elliptic boundary value problems of the form

\[ Au = f, \quad \text{in } \Omega, \]
\[ Bj u = gj, \quad \text{on } \Gamma, \]

where \( A \) is the partial differential operator defined by

\[ Au = (-1)^m D_x^m (a(x) D_x^m u) + (-1)^n D_y^n (b(y) D_y^n u), \]

\( B_j \) represents a differential operator on the boundary, \( \Gamma \), of the domain, \( \Omega \subset \mathbb{R}^2 \), and \( f, g_j \) are given. Here \( D_x^m u \) and \( D_y^n u \) denote \( \frac{\partial^m u}{\partial x^m} \) and \( \frac{\partial^n u}{\partial y^n} \), respectively, and \( m, n \geq 1 \) are integers. For simplicity, \( \Omega \) will represent the rectangular domain in \( \mathbb{R}^2 \) given by \( \Omega = (0, 1) \times (0, 1) \). We assume that the coefficients \( a(x) \) and \( b(y) \) are sufficiently smooth and that

\[ 0 < a \leq a(x) \leq \bar{a}, \quad \text{for } 0 \leq x \leq 1, \quad \text{and} \quad 0 < b \leq b(y) \leq \bar{b}, \quad \text{for } 0 \leq y \leq 1. \]

No use of results which are peculiar to two dimensions will be made and the methods used will apply to product domains in higher dimensions.

There are many examples of boundary value problems of this type when \( m = n \). For example, the Dirichlet problem for Poisson's equation is of the form (1.1),(1.2) with \( m = n = 1 \) and \( a = b = 1 \) in (1.3) and \( u = g \) on \( \Gamma \).
The goal of this paper is to study the various aspects of the theory and implementation of finite element methods for the elliptic boundary value problem given by (1.1), (1.2) with \( m \neq n \) in (1.3). A review of some basic results on anisotropic Sobolev spaces is presented in Section 2. In Section 3, a variational problem for (1.1), (1.2) is studied. In Section 4, some basic convergence results for the Galerkin approximation are given and examples of conforming finite element spaces are constructed. In the last section, the results obtained in the previous sections are illustrated by applying the theory to the particular example of the Onsager pancake equation.

2. ANISOTROPIC SOBOLEV SPACES

In this section, we present some basic results on anisotropic Sobolev spaces and indicate that isotropic spaces are actually a special case of anisotropic spaces. Let \( H^m(\Omega) \) for nonnegative integer \( m \) denote the Sobolev space defined by

\[
H^m(\Omega) = \{ w \in L^2(\Omega) | D_i^j w \in L^2(\Omega), i, j \geq 0, i + j \leq m \},
\]

where \( i, j \) are positive integers. Clearly, \( H^0(\Omega) = L^2(\Omega) \). \( H^m(\Omega) \) is a Hilbert space under the inner product

\[
(w, v)_m = \sum_{i+j \leq m} (D_i^j w, D_i^j v),
\]

where \( (\cdot, \cdot) \) represents the usual inner product on \( L^2(\Omega) \). A norm on \( H^m(\Omega) \) is

\[
\|w\|_m = \left( \sum_{i+j \leq m} \|D_i^j w\|_0^2 \right)^{1/2},
\]

where \( \| \cdot \|_0 \) denotes the usual norm on \( L^2(\Omega) \). These spaces are said to be isotropic in the sense that they require equal orders of differentiability in each direction. We also define, in the usual manner, the trace spaces \( H^s(\Gamma) \) of functions defined on the boundary. The reader is referred to [1] for a detailed discussion of these spaces.

Anisotropic Sobolev spaces were introduced in the papers of Nikol’skii [2,3]. We define the anisotropic Sobolev space \( H^{m,n}(\Omega) \) for nonnegative integers \( m, n \), as

\[
H^{m,n}(\Omega) = \{ w \in L^2(\Omega) | D^m_x w, D^n_y w \in L^2(\Omega) \}.
\]

This is a Hilbert space with inner product

\[
(w, v)_{m,n} = (w, v) + (D^m_x w, D^m_x v) + (D^n_y w, D^n_y v).
\]

The following notation is employed for a seminorm and norm associated with these spaces:

\[
|w|_{m,n} = \left( \|D^m_x w\|_0^2 + \|D^n_y w\|_0^2 \right)^{1/2}
\]

and

\[
\|w\|_{m,n} = \left( \|w\|_0^2 + |w|_{m,n}^2 \right)^{1/2}.
\]

Isotropic spaces are actually a special case of the anisotropic ones with \( m = n \). This is not immediately obvious since the norms for isotropic spaces contain terms involving mixed and intermediate derivatives, whereas the only terms that appear explicitly in the anisotropic norms are unmixed derivatives of the highest order and of the function itself. However, in [4], the \( L^2 \)-norms of the intermediate and mixed derivatives of a function in \( H^{m,n}(\Omega) \) are shown to be related to the \( L^2 \)-norms of the function and its highest derivatives. In particular, we have the inequalities given in the following lemma.
LEMMA 2.1. For fixed nonnegative integers $m, n$, there exist positive constants $C_1, C_2, C_3$ such that for every positive $\epsilon$ and for all $w$ in $H^{m,n}(\Omega)$,

$$
\|D_i^i w\|_0 \leq C_1 \left( \epsilon^{i} \|w\|_0 + \epsilon^{m-i} \|D_x^m w\|_0 \right), \quad 0 < i < m, 
$$

(2.8)

$$
\|D_j^j w\|_0 \leq C_2 \left( \epsilon^{j} \|w\|_0 + \epsilon^{n-j} \|D_y^n w\|_0 \right), \quad 0 < j < n, 
$$

(2.9)

and

$$
\|D_i^i w D_j^j w\|_0 \leq C_3 \|w\|_{m,n}, \quad i, j \geq 0, \quad \frac{i}{m} + \frac{j}{n} \leq 1. 
$$

(2.10)

Thus, the isotropic norm $\| \cdot \|_m$ can be shown to be equivalent to the anisotropic norm $\| \cdot \|_{m,n}$ by using (2.8)-(2.10) with $\epsilon = 1$ and the definitions of the norms (2.3) and (2.7).

3. WEAK FORMULATION

In this section, we obtain a weak formulation for the elliptic boundary value problem (1.1),(1.2) and address the existence, uniqueness, and regularity of the solution. We will consider the problem where only homogeneous boundary conditions are specified for both the essential and natural conditions. Once this problem has been analyzed, we will indicate how the results can be extended for problems which involve inhomogeneous natural boundary conditions. The case of inhomogeneous essential boundary conditions can be treated in the usual manner; see [5,6] for details.

Before writing the variational formulation of (1.1),(1.2), we must specify the type of boundary operators which we will allow. We define the trace operators, $\tau_i^{(x)}, \tau_j^{(y)}$ for $w \in C^\infty(\hat{\Omega})$, by

$$
\tau_i^{(x)} w \equiv D_x^{i-1} w \big|_{x=0,1}, \quad \text{for } i = 1, \ldots, m, 
$$

(3.1)

and

$$
\tau_j^{(y)} w \equiv D_y^{j-1} w \big|_{y=0,1}, \quad \text{for } j = 1, \ldots, n. 
$$

(3.2)

In [4], imbedding theorems are proved that guarantee that these trace operators are bounded. In particular, we have the following result.

THEOREM 3.1. For fixed positive integers $m, n$, there exist positive constants $C_{i,1}, i = 1, \ldots, m,$ and $C_{j,2}, j = 1, \ldots, n,$ such that for all $w \in C^\infty(\hat{\Omega})$,

$$
\left\| \tau_i^{(x)} w \right\|_{\alpha,n} \leq C_{i,1} \|w\|_{m,n}, \quad \text{for } \alpha_i = 1 - \frac{i}{m} + \frac{1}{2m}, \quad i = 1, \ldots, m, 
$$

(3.3)

and

$$
\left\| \tau_j^{(y)} w \right\|_{\beta,j} \leq C_{j,2} \|w\|_{m,n}, \quad \text{for } \beta_j = 1 - \frac{j}{n} + \frac{1}{2n}, \quad \text{for } j = 1, \ldots, n. 
$$

(3.4)

Using these results and the fact that $C^\infty(\hat{\Omega})$ is dense in $H^{m,n}(\Omega)$, we are able to extend the trace operators; i.e.,

$$
\tau_i^{(x)} : H^{m,n}(\Omega) \to H^{\alpha,n}((0,1)), \quad \text{for } x = 0, 1, 
$$

and

$$
\tau_j^{(y)} : H^{m,n}(\Omega) \to H^{\beta,j}((0,1)), \quad \text{for } y = 0, 1. 
$$

The homogeneous essential boundary conditions, i.e., those which involve the trace operators defined by (3.1),(3.2), describe a closed subspace $H_x^{m,n}(\Omega)$ of $H^{m,n}(\Omega)$. This follows from the continuity of the trace operators.
We also allow the possibility of certain supplementary boundary operators which arise due to integration by parts of the operator $A$ in (1.3). We define the following operators for $w \in C_c(\Omega)$:

$$
\sigma_i^{(x)} w \equiv D_x^{m-i} a(x) D_x^n w \big|_{x=0,1}, \quad \text{for } i = 1, \ldots, m,
$$

and

$$
\sigma_j^{(y)} w \equiv D_y^{n-j} b(y) D_y^m w \big|_{y=0,1}, \quad \text{for } j = 1, \ldots, n.
$$

Although these supplementary operators cannot be extended to $H^{m,n}(\Omega)$ as the trace operators were, we will define the following space as a convenient setting in which they can be extended. For any nonnegative integer $m$, $n$ define

$$
H^{m,n}(\Omega; A) \equiv \{ w \in H^{m,n}(\Omega) : Aw \in L^2(\Omega) \},
$$

which is a Hilbert space with the graph norm

$$
\| w \|_{m,n; A}^2 = \| w \|_{m,n}^2 + \| Aw \|_0^2.
$$

In order to prove a result analogous to Theorem 3.1 for the operators defined by (3.5),(3.6), we need the following inverse imbedding lemma [4].

**Lemma 3.2.** For fixed positive integers $m$, $n$, and for any $g_i \in C_0^\infty([0,1] \times \{0,1\})$, $i = 1, \ldots, m$, there exists a constant $C_g > 0$ and $v \in H^{m,n}(\Omega)$ such that

$$
\tau_i^{(x)} v = g_i, \quad \text{for } i = 1, \ldots, m,
$$

$$
\tau_j^{(y)} v = 0, \quad \text{for } j = 1, \ldots, n,
$$

and

$$
\| v \|_{m,n} \leq C_g \left( \sum_{i=1}^m \| g_i \|_{\alpha_i} \right), \quad \text{for } \alpha_i = 1 - \frac{i}{m} + \frac{1}{2m}.
$$

An analogous result holds for any $g_j \in C_0^\infty((0,1) \times \{0,1\})$, $j = 1, \ldots, n$. We now state and prove the result analogous to Theorem 3.1 for the operators defined by (3.5) and (3.6).

**Theorem 3.3.** For fixed positive integers $m$, $n$, there exist positive constants $C_{i,1}$, $i = 1, \ldots, m$, and $C_{j,2}$, $j = 1, \ldots, n$, such that for all $w \in C_c(\Omega)$,

$$
\left\| \sigma_i^{(x)} w \right\|_{\alpha_i} \leq C_{i,1} \left\| w \right\|_{m,n;A}, \quad \text{for } \alpha_i = 1 - \frac{i}{m} + \frac{1}{2m}, \quad i = 1, \ldots, m,
$$

$$
\left\| \sigma_j^{(y)} w \right\|_{\beta_j} \leq C_{j,2} \left\| w \right\|_{m,n;A}, \quad \text{for } \beta_j = 1 - \frac{j}{n} + \frac{1}{2n}, \quad j = 1, \ldots, n,
$$

where the norm on the left-hand side denotes the standard operator norm on the appropriate dual space. Thus, the supplementary operators may be extended by continuity, i.e.,

$$
\sigma_i^{(x)} : H^{m,n}(\Omega; A) \to \left( H_0^{m,n}(\{0,1\} \times \{0,1\}) \right)',
$$

and

$$
\sigma_j^{(y)} : H^{m,n}(\Omega; A) \to \left( H_0^{n,m}(\{0,1\} \times \{0,1\}) \right)',
$$

where $W'$ denotes the dual of the Hilbert space $W$.

**Proof.** We first note that integration by parts gives

$$
(a(x)D_x^{m-i} w, D_x^{m} v) + (b(y)D_y^{n-j} w, D_y^{n} v) - (Aw, v)
$$

$$
= \sum_{i=1}^m (-1)^{m-i} \int_0^1 D_x^{m-i} a D_x^n w D_x^{i-1} v \big|_{x=0,1} dy
$$

$$
+ \sum_{j=1}^n (-1)^{n-j} \int_0^1 D_y^{n-j} b D_y^m w D_y^{j-1} v \big|_{y=0,1} dx.
$$

(3.15)
Thus, an estimate for the absolute value of the combined sum of the boundary integrals is obtained by bounding the left-hand side of (3.15); i.e.,

\[
\frac{1}{m} \sum_{i=1}^{m} (-1)^{m-i} \int_0^1 D_x^{m-i} a(x) D_x^{m-i} v \bigg|_{x=0}^{x=1} \, dy + \sum_{j=1}^{n} (-1)^{n-j} \int_0^1 D_y^{n-j} b D_y^{n-j} w D_y^{n-j-1} v \bigg|_{y=0}^{y=1} \, dx \leq C_1 \| w \|_{m,n;\mathcal{A}} \| v \|_{m,n},
\]

(3.16)

for some positive constant $C_1$. We now use this estimate along with Lemma 3.2 to prove (3.11). We take $g_i$, $i = 1, \ldots, m$ in Lemma 3.2 to be zero except at a single index, $i = k$, and on one edge, say $x = 0$. For nonzero $g$ if $g_k \big|_{x=0} = g$, then Lemma 3.2 guarantees that we can find a $v \in H^{m,n}(\Omega)$ satisfying

\[
\tau_k(x) v \big|_{x=0} = g, \quad \tau_k(x) v \big|_{x=1} = 0,
\]

and where all the other traces of $v$ vanish. We have

\[
\left| \int_0^1 \sigma_k^{(x)} w \bigg|_{x=0} \, dy \right| \leq C_1 \| w \|_{m,n;\mathcal{A}} \| v \|_{m,n} \leq C \| w \|_{m,n;\mathcal{A}} \| g \|_{\alpha_{k,n}}.
\]

(3.17)

To obtain the first inequality in (3.17), we have used (3.16) and the fact that all traces except $\tau_k(x) v \big|_{x=0}$ vanish. The last inequality is obtained by using (3.10). To complete the proof, we divide through by $\| g \|_{\alpha_{k,n}}$ and take the supremum over all such $g$ to give

\[
\| \sigma_k^{(x)} w \|_{\alpha_{k,n}} \leq C \| w \|_{m,n;\mathcal{A}}.
\]

The other estimates are similar. Since $C^\infty(\overline{\Omega})$ is dense in $H^{m,n}(\Omega; \mathcal{A})$, the proof is complete.

We now specify the particular problem with homogeneous boundary data which we will consider; i.e.,

\[
\begin{align*}
Au &= f, & \text{in } \Omega, \\
B_j u &= 0, & \text{on } \Gamma,
\end{align*}
\]

(3.18) (3.19)

where $A$ is defined by (1.3) and $B$ represents differential boundary operators as specified by (3.1), (3.2), (3.5), and (3.6). In particular, on each edge $x = 0$ and $x = 1$, there will be $m$ conditions and at each edge $y = 0$ and $y = 1$, there will be $n$ conditions. For simplicity, we consider the case in which only one boundary operator appears in each condition. Specifically, we assume that for $j = 1, \ldots, n$, the boundary conditions include

\[
\text{either } \sigma_j^{(y)} u \big|_{y=0} = 0 \quad \text{or} \quad \tau_j^{(y)} u \big|_{y=0} = 0,
\]

(3.20)

and

\[
\text{either } \sigma_j^{(y)} u \big|_{y=1} = 0 \quad \text{or} \quad \tau_j^{(y)} u \big|_{y=1} = 0.
\]

(3.21)

For $i = 1, \ldots, m$, the boundary conditions include

\[
\text{either } \sigma_i^{(x)} u \big|_{x=0} = 0 \quad \text{or} \quad \tau_i^{(x)} u \big|_{x=0} = 0,
\]

(3.22)

and

\[
\text{either } \sigma_i^{(x)} u \big|_{x=1} = 0 \quad \text{or} \quad \tau_i^{(x)} u \big|_{x=1} = 0.
\]

(3.23)
hold, where we require that homogeneous essential boundary conditions are imposed at one of the boundaries; i.e.,

\[ \text{either } \tau_i^{(x)} u \big|_{x=0} = 0 \quad \text{or} \quad \tau_i^{(x)} u \big|_{x=1} = 0, \quad \text{for } i = 1, \ldots, m. \]  

(3.24)

This assumption will be used to apply a Poincaré-type inequality in the proof of coercivity of our particular bilinear form. Alternatively, we could assume a similar condition for \( \tau_j^{(y)} u, \ j = 1, \ldots, n \) at \( y = 0 \) or \( y = 1 \).

We consider the following weak formulation of (3.18),(3.19): given \( f \in L^2(\Omega) \), seek \( u \in H^{m,n}_* (\Omega) \) satisfying

\[ L(u,v) = (f,v), \quad \forall \ v \in H^{m,n}_*(\Omega), \]  

(3.25)

where

\[ L(w,v) = (a(x)D^m_w w, D^m_v v) + (b(y)D^n_y w, D^n_y v), \quad \forall \ w, v \in H^{m,n}_*(\Omega). \]  

(3.26)

Recall that the subscript on the space \( H^{m,n}_*(\Omega) \) indicates the imposition of the homogeneous essential boundary conditions. The following theorem demonstrates that there is a unique solution to the variational problem (3.25).

**THEOREM 3.4.** Let \( f \in L^2(\Omega) \) and let the right-hand side of (3.25) represent a bounded linear functional on \( H^{m,n}_*(\Omega) \). Then there exists a unique solution \( u \in H^{m,n}_*(\Omega) \) satisfying (3.25). Moreover, we have the estimate

\[ \|u\|_{m,n} \leq C \|f\|_0, \]  

(3.27)

for some positive constant \( C \).

**PROOF.** We must demonstrate that \( L(\cdot,\cdot) \) defined by (3.26) is bounded and coercive on \( H^{m,n}_*(\Omega) \times H^{m,n}_*(\Omega) \) so that the Lax-Milgram theorem can be applied. To show the boundedness, we apply Hölder's inequality to both terms in (3.26) to obtain

\[ |(aD^m_x w, D^m_v v)| \leq \tilde{a} \|w\|_{m,0} \|v\|_{m,0} \]  

and

\[ |(bD^n_y w, D^n_y v)| \leq \tilde{b} \|w\|_{0,n} \|v\|_{0,n}, \]  

where we have used (1.4). These combine to give

\[ |L(w,v)| \leq 4 (\tilde{a} + \tilde{b}) \|w\|_{m,n} \|v\|_{m,n}. \]  

(3.28)

To show that \( L(\cdot,\cdot) \) is coercive on \( H^{m,n}_*(\Omega) \), we need the Poincaré-type inequality

\[ \|w\|_0 \leq C \|D_x w\|_0, \]  

(3.29)

where \( w \in H^{1,0}(\Omega) \) satisfying either \( w = 0 \) at \( x = 0 \) or at \( x = 1 \). We have that

\[ L(w,w) \geq \min\{a,b\} \left( \|D^m_x w\|^2_0 + \|D^n_y w\|^2_0 \right). \]

With assumption (3.24), we can use induction on (3.29) to obtain

\[ \|w\|_0 \leq C^m \|D^m_x w\|_0, \quad \forall \ w \in H^{m,0}(\Omega). \]  

(3.30)

Using the fact that (3.30) holds for all \( w \in H^{m,n}_*(\Omega) \), we obtain

\[ L(w,w) \geq \min\{a,b\} \left[ \left( \frac{1}{2C^{2m}} \right) \|w\|_0^2 + \|D^n_y w\|^2_0 + \frac{1}{2} \|D^n_y w\|^2_0 \right] \geq \min\{a,b\} \min \left\{ \frac{1}{2}, \frac{1}{2C^{2m}} \right\} \|w\|_{m,n}^2. \]
We have now verified the hypotheses of the Lax-Milgram theorem and so are guaranteed uniqueness to the solution of the variational problem and continuous dependence of the solution on the data. In particular, we have the estimate

\[ \|u\|_{m,n} \leq C \sup_{v \in H^m,n(\Omega)} \frac{|(f,v)|}{\|v\|_{m,n}} \leq C \|f\|_0, \quad \text{for } v \neq 0, \tag{3.31} \]

which concludes the proof.

The unique solution \( u \in H^{m,n}(\Omega) \) to the variational problem (3.25) guaranteed by Theorem 3.4 is also a weak or generalized solution to (3.18),(3.19). This is seen by taking \( v \in C^\infty(\bar{\Omega}) \) in (3.25), comparing (3.25) with (3.15) and applying the boundary conditions.

A regularity estimate is proved by standard means; specifically, we have the following theorem.

**THEOREM 3.5.** Let \( f \in L^2(\Omega) \). Then there exists \( u \in H^{2m,2n}_*(\Omega) \) satisfying the homogeneous problem (3.18),(3.19).

This result can be used to show that the inhomogeneous problem (3.1),(3.2) has a solution \( u \in H^{2m,2n}(\Omega) \) by standard arguments. In particular, \( w \in H^{2m,2n}(\Omega) \) is chosen so that \( Bw = g \) on the boundary of \( \Omega \) and problem (1.1),(1.2) for \( u - w \) is analyzed where the boundary conditions for \( u - w \) are homogeneous.

## 4. THE FINITE ELEMENT METHOD IN ANISOTROPIC SOBOLEV SPACES

In this section, we obtain optimal error estimates for the Galerkin approximation. As an example, we show how to construct conforming finite element spaces which will have optimal accuracy. Consistent with the restriction to rectangular domains, we consider the tensor product of two one-dimensional spline spaces.

Let \( S_h \) be a parameterized family of conforming finite element spaces; i.e., \( S_h \subset H^{m,n}_*(\Omega) \). Then standard finite element arguments give the following result.

**THEOREM 4.1.** There exists a unique solution \( u^h \in S_h \) to the discrete variational problem

\[ L(u^h, v^h) = (f, v^h), \quad \forall v^h \in S_h. \tag{4.1} \]

Moreover, there exists a positive constant \( C, \) independent of \( u, h \), such that

\[ \|u - u^h\|_{m,n} \leq C \inf_{w^h \in S_h} \|u - w^h\|_{m,n}. \]

If we assume that there exists a Hilbert space \( V \) which is continuously imbedded in \( H^{m,n}_*(\Omega) \) such that the approximation property

\[ \inf_{w^h \in S_h} \|w - w^h\|_{m,n} \leq C_h \|w\|_V, \quad \forall w \in V \tag{4.2} \]

holds, then if the solution \( u \) of (3.18),(3.19) is also in \( V \), we have

\[ \|u - u^h\|_{m,n} \leq C_h \|u\|_V. \tag{4.3} \]

The technique devised by Aubin-Nitsche can be used to demonstrate that the rate of convergence in the \( L^2 \)-norm is one order of \( h \) higher than the estimate in the \( W^{m,n}_* \)-norm.
THEOREM 4.2. If the solution $u$ to the continuous variational problem (3.25) is in $V$ and (4.2) holds, then

$$\|u - u^h\|_0 \leq Ch^2 \|u\|_V.$$  \hspace{1cm} (4.4)

PROOF. Let $v$ be the solution of the adjoint problem

$$L(w, v) = (w, u - u^h), \quad \forall w \in H^{m,n}_*(\Omega).$$

Choosing $w = u - u^h \in H^{m,n}_*(\Omega)$, we obtain

$$\|u - u^h\|_0^2 = L(u - u^h, v).$$

Combining this result with the orthogonality condition

$$L(u - u^h, v^h) = 0, \quad \forall v^h \in S_h,$$

we have

$$\|u - u^h\|_0^2 \leq L(u - u^h, v - v^h), \quad \forall v^h \in S_h.$$ 

Now using the boundedness of $L(\cdot, \cdot)$ and taking the infimum over all $v^h \in S_h$, we have

$$\|u - u^h\|_0^2 \leq C \|u - u^h\|_{m,n} \inf_{v^h \in S_h} \|v - v^h\|_{m,n}.$$

Using (4.3) and Theorem 3.5, we obtain

$$\|u - u^h\|_{m,n} \leq Ch^2 \|u\|_V \|u - u^h\|_0.$$  \hspace{1cm} (4.5)

The goal of the remainder of this section is to show the existence of a finite element space $S_h \in H^{m,n}_*(\Omega)$ which satisfies (4.1) with $V$ appropriately given. First, we construct finite element spaces for the model problem by using a tensor product construction. Let $S_h^{(x)}$, $S_h^{(y)}$ be one-dimensional spline spaces such that $S_h^{(x)} \subseteq H^r(0, 1)$ and $S_h^{(y)} \subseteq H^s(0, 1)$, where the subscript again indicates the imposition of the homogeneous essential boundary conditions at $x = 0, 1$ and $y = 0, 1$. The tensor product of these spaces is given by

$$S_h \equiv S_h^{(x)} \otimes S_h^{(y)} = \left\{ \sum_{i,j} \phi_i(x)\psi_j(y) \mid \phi_i \in S_h^{(x)}, \psi_j \in S_h^{(y)} \right\},$$  \hspace{1cm} (4.6)

and is contained in $H^{m,n}_*(\Omega)$.

In order to estimate the error of a function in $H^{m,n}_*(\Omega)$ and its best approximation in our tensor product space (4.5), we must first estimate the error in a function in $H^{m,n}_*(\Omega)$ and its $S_h$-interpolant. To this end, we define the one-dimensional interpolation operators

$$I_h^{(x)} : C^\infty[0, 1] \to S_h^{(x)} \subseteq H^r(0, 1),$$  \hspace{1cm} (4.7)

which satisfy the properties of the type

$$\|w - I_h^{(x)}w\|_r \leq C_{i,r}h^r|w|_{i+r}, \quad \forall w \in C^\infty[0, 1],$$  \hspace{1cm} (4.8)

and

$$\|w - I_h^{(y)}w\|_s \leq C_{j,s}h^s|w|_{j+s}, \quad \forall w \in C^\infty[0, 1],$$  \hspace{1cm} (4.9)

for $r, s \geq 0$. The natural extensions to $\tilde{\Omega}$ of these interpolation operators are given by

$$I_h^{(x)} : C^\infty(\tilde{\Omega}) \to S_h^{(x)}$$

such that $(I_h^{(x)}w)(x, y) = (I_h^{(x)}w(\cdot, y))(x)$ \hspace{1cm} (4.10)

and

$$I_h^{(y)} : C^\infty(\tilde{\Omega}) \to S_h^{(y)}$$

such that $(I_h^{(y)}w)(x, y) = (I_h^{(y)}w(x, \cdot))(y).$ \hspace{1cm} (4.11)

The next proposition demonstrates that these extensions satisfy estimates analogous to (4.8) and (4.9).
PROPOSITION 4.3. The interpolation operators defined by equations (4.10) and (4.11) satisfy the estimates

$$\|w - I_h^{(x)} w\|_{i,0} \leq C_1 h^r |w|_{i+r,0}, \quad \forall w \in C^\infty_*(\Omega),$$

and

$$\|w - I_h^{(y)} w\|_{0,j} \leq C_2 h^s |w|_{0,j+s}, \quad \forall w \in C^\infty_*(\Omega),$$

respectively.

PROOF. For fixed $y$ in $[0,1]$ and $w \in C^\infty_*(\Omega)$, we have that $w(\cdot, y) \in C^\infty_*(0,1]$. Using this fact and equation (4.8), we obtain

$$\|w - I_h^{(x)} w\|_{i,0}^2 \leq C_1 \int_0^1 \|w(\cdot, y) - I_h^{(x)} w(\cdot, y)\|_{i}^2 dy$$

$$\leq (C_2 h^r)^2 \int_0^1 |w(\cdot, y)|_{i+r}^2 dy$$

$$= (C_2 h^r)^2 w_{i+r,0}^2.\]$$

The estimate given by equation (4.13) is obtained in a similar manner.

We define the $S_h$-interpolant of $w \in C^\infty_*(\Omega)$ as $I_h w \equiv I_h^{(x)} I_h^{(y)} w$ where for fixed $(x, y) \in \Omega$, we define $(I_h^{(x)} I_h^{(y)} w)(x, y) = (I_h^{(x)} (I_h^{(y)} w(\cdot, y)))(x)$ and $(I_h^{(x)} I_h^{(y)} w)(x, y) = (I_h^{(y)} (I_h^{(x)} w(x, \cdot)))(y)$.

Furthermore, we have assumed the commutativity properties

$$I_h^{(z)} D_y w = D_y I_h^{(z)} w, \quad I_h^{(y)} D_x w = D_x I_h^{(y)} w, \quad \text{and} \quad I_h^{(x)} I_h^{(y)} w = I_h^{(y)} I_h^{(x)} w. \quad (4.14)$$

We are now ready to give an estimate for the error in a function in $H^{m,n}_*(\Omega)$ and its interpolation operator on $S_h$.

THEOREM 4.4. If the interpolation operator $I_h^{(x)}$ satisfies (4.12) for $r = 1$ when $i = 0$ or $i = m$ and $I_h^{(y)}$ satisfies (4.13) for $s = 1$ when $j = 0$ or $j = n$, then the interpolation operator $I_h$ satisfies

$$\|w - I_h w\|_{m,n} \leq Ch \left(|D_x^m w|_{1,1} + |D_y^m w|_{1,1} \right), \quad \forall w \in C^\infty_*(\Omega). \quad (4.15)$$

PROOF. In order to prove equation (4.15), we estimate each of the terms on the right-hand side of the inequality to obtain

$$\|w - I_h w\|_{m,n} \leq \|w - I_h w\|_{m,0} + \|w - I_h w\|_{0,n}. \quad (4.16)$$

By adding and subtracting the appropriate quantities, we can bound the first term in this inequality by

$$\|w - I_h w\|_{m,0} \leq \|w - I_h^{(x)} w\|_{m,0} + \|w - I_h^{(y)} w\|_{m,0}$$

$$+ \left\| \left( w - I_h^{(y)} w \right) - I_h^{(x)} \left( w - I_h^{(y)} w \right) \right\|_{m,0}. \quad (4.17)$$

The first term on the right-hand side of (4.17) is estimated by

$$\|w - I_h^{(x)} w\|_{m,0} \leq C h |w|_{m+1,0} = C h |D_x^m w|_{1,0}, \quad (4.18)$$

where we have used (4.15) with $r = 1$ and $i = m$. The second term on the right-hand side of (4.17) can be estimated in the following manner:

$$\|w - I_h^{(y)} w\|_{m,0} = \left[ 2 \left\| w - I_h^{(y)} w \right\|_0^2 + \left\| D_x^m \left( w - I_h^{(y)} w \right) \right\|_0^2 \right]^{1/2}$$

$$\leq \left\| w - I_h^{(y)} w \right\|_0 + \left\| D_x^m \left( w - I_h^{(y)} w \right) \right\|_0$$

$$\leq C h |D_y^m w|_{0,1} + C h |D_x^m w|_{0,1}, \quad (4.19)$$
where we have obtained the last line by using the two inequalities
\[ \| w - I_h^{(y)} w \|_0 \leq \| w - I_h^{(y)} w \|_{0,n} \leq C h |w|_{0,n+1} = C h |D_h^n w|_{0,1} \]
and
\[ \| D_x^m (w - I_h^{(y)} w) \|_0 = \| D_x^m w - I_h^{(y)} D_x^m w \|_0 \leq C h |D_x^m w|_{0,1} , \]
which we obtained by setting \( s = 1, j = n \), and \( s = 1, j = 0 \), respectively, in equation (4.13). It remains to estimate the last term on the right-hand side of (4.19). We do this by applying (4.12) with \( r = 0 \) and \( i = m \) to obtain
\[ \left\| \left( w - I_h^{(y)} w \right) - I_h^{(x)} \left( w - I_h^{(y)} w \right) \right\|_{m,0} \leq C \left\| w - I_h^{(y)} w \right\|_m \leq C \left\| w - I_h^{(y)} w \right\|_{m,0} , \]
which can in turn be estimated by using (4.19). Combining these results, we obtain
\[ \| w - I_h w \|_{m,0} \leq C h \left( |D_x^m w|_{1,0} + |D_x^m w|_{0,1} + |D_y^m w|_{1,0} + |D_y^m w|_{0,1} \right) \]
\[ \leq C h \left( |D_x^m w|_{1,1} + |D_y^m w|_{1,1} \right) . \]
In a similar manner, one can obtain the estimate
\[ \| w - I_h w \|_{0,n} \leq C h \left( |D_x^m w|_{1,1} + |D_y^m w|_{1,1} \right) . \]

We have shown the existence of a finite element space \( S_h \) defined by (4.5) such that \( S_h \subset H_0^{m,n}(\Omega) \) and such that it satisfies an interpolation condition of the form (4.14).

5. ONSAGER PANCAKE EQUATION

The Onsager pancake approximation [7] arises in analyzing gas centrifuges which are used to separate isotopes such as uranium. It models axially symmetric flow of a viscous compressible fluid rotating almost rigidly at a nearly constant radius. The model involves the single partial differential equation
\[ A u = (e^x (e^x u_{xx})_{xx})_{xx} + b u_{yy} = f(x, y) \]
along with the set of boundary conditions
\[ u_x = u_{xx} = 0, \quad \text{at } x = 0, \quad \text{(5.2)} \]
\[ u = u_x = 0, \quad \text{at } x = 1, \quad \text{(5.3)} \]
\[ (e^x (e^x u_{xx})_{xx})_x = g(y), \quad \text{at } x = 0, \quad \text{(5.4)} \]
\[ (e^x u_{xx})_x = 0, \quad \text{at } x = 1, \quad \text{(5.5)} \]
\[ -b u_y = d \left( e^{x/2} u_x \right)_x + \gamma_0(x), \quad \text{at } y = 0, \quad \text{(5.6)} \]
and
\[ b u_y = d \left( e^{x/2} u_x \right)_x + \gamma_1(x), \quad \text{at } y = 1. \quad \text{(5.7)} \]

In the equation and boundary conditions, \( b \) and \( d \) are positive constants and \( \gamma_0 \) and \( \gamma_1 \) are given. We note that the variable coefficient in the operator given by (5.1) plays a slightly different role than does the one in our model problem (1.3). However, in the Onsager pancake approximation, the different role of the variable coefficient only results in the addition of some lower-order terms which will cause no difficulties. Another difference occurs in the boundary conditions at \( y = 0, 1 \) where there appear both normal and tangential derivative terms. For simplicity, this case was
not treated in the general setting, but it will require only the addition of a boundary integral in our weak formulation.

The variational formulation of equations (5.1)–(5.7) that we consider is the following: seek \( u \in H^{3,1}_*(\Omega) \) satisfying

\[
L(u, v) = - \int_\Omega f v \, d\Omega - \int_0^1 g(y)v|_{y=0} \, dy \\
- \int_0^1 (\gamma_1 v|_{y=1} + \gamma_0 v|_{y=0}) \, dx, \quad \forall v \in H^{3,1}_*(\Omega),
\]

where \( L(w, v) \) is the bilinear form defined by

\[
L(w, v) = \int_\Omega (e^x w_{xx})_x (e^x v_{xx})_x \, dx + \int_\Omega b w v_y \, d\Omega \\
+ \int_0^1 de^{x/2} w_x (v_x|_{y=0} + v_x|_{y=1}) \, dx, \quad \forall w, v \in H^{3,1}_*(\Omega).
\]

(5.8)

(5.9)

Here \( H^{3,1}_*(\Omega) \) is the constrained space given by

\[
H^{3,1}_*(\Omega) = \{ w \in H^{3,1}(\Omega) : w_x = w_{xx} = 0 \text{ at } x = 0, w = w_x = 0 \text{ at } x = 1 \}.
\]

Verification that the solution of (5.1)–(5.7) is also a solution to (5.8) is done in the usual manner by integrating by parts; in this case the integration by parts is performed three times in \( x \) and once in \( y \).

In the next result, we apply the Lax-Milgram theorem [8] to prove that the solution to (5.8) is unique.

**Theorem 5.1.** There exists a unique \( u \in H^{3,1}_*(\Omega) \) such that equation (5.8) is satisfied, provided the right-hand side of (5.8) denotes a bounded linear functional on \( H^{3,1}_*(\Omega) \).

**Proof.** In order to apply the Lax-Milgram theorem, we must show that \( L(w, v) \) is bounded and coercive on \( H^{3,1}_*(\Omega) \times H^{3,1}_*(\Omega) \). To demonstrate the boundedness of \( L(w, v) \), we only need to treat the boundary integral since the remaining integrals can be treated as in the proof of Theorem 3.4 for the general bilinear form (3.26). From the imbedding result (3.3) for the trace operators, we have that since \( w \in H^{3,1}(\Omega) \), \( w_x|_{y=0}, w_x|_{y=1} \in H^{1/2}(\Gamma) \). Thus, \( L(\cdot, \cdot) \) is bounded on \( H^{3,1}_*(\Omega) \times H^{3,1}_*(\Omega) \). For coercivity, we must show that there exists a constant \( C > 0 \) such that 

\[
L(w, w) \geq C \| w \|^2_{3,1} \text{ for all } w \in H^{3,1}_*(\Omega),
\]

where

\[
L(w, w) = \int_\Omega ((e^x w_{xx})_x)^2 \, d\Omega + \int_\Omega b w^2 \, d\Omega + \int_0^1 de^{x/2} [w_x^2|_{y=0} + w_x^2|_{y=1}] \, dx.
\]

(5.10)

We first note that since the boundary integral is positive definite, it is sufficient to prove that the first two terms on the right-hand side of (5.10) are bounded below by \( C \| w \|^2_{3,1} \). This is shown in a similar manner to the proof in Theorem 3.4; however, the exponential term must be handled. First note that for \( w \in H^{2,1}_*(\Omega) \), \( e^x w_{xx} = 0 \) at \( x = 0 \); thus, we can apply Poincaré-type inequality (3.29) to obtain

\[
\| w_{xx} \|_0 \leq \| e^x w_{xx} \|_0 \leq C_1 \| (e^x w_{xx})_x \|_0.
\]

(5.11)

The triangle inequality and (5.11) then give that

\[
\| w_{xxx} \|_0 \leq \| e^x w_{xxx} \|_0 \leq \| (e^x w_{xx})_x \|_0 + \| e^x w_{xx} \|_0 \\
\leq (1 + C_1) \| (e^x w_{xx})_x \|_0.
\]
The combination of these results and the fact that \( \|w\|_0 \leq C \|w_{xx}\|_0 \) gives

\[
L(w, w) \geq C_2 \left( \|w_{xx}\|_0^2 + \|w_y\|_0^2 \right) \geq C_3 \|w\|_{\alpha, 1}^2, \quad \forall w \in H^{3,1}(\Omega).
\]

For our example, we consider two conforming finite element spaces. The first is the tensor product of cubic B-splines in \( x \) and piecewise linear functions in \( y \). Each one-dimensional spline space admits a Lagrange interpolant; thus, each interpolation operator commutes with each other and with the derivatives as we assumed in (4.14). The following approximation properties of these spaces are known if we assume a uniform spacing; see [9]. We have

\[
\inf_{w_h \in S_h} \|w - w_h\|_1 \leq C h^s |w|_{i+s},
\]

where for cubic B-splines, we have that \( 0 < i + s \leq 4 \) for \( i = 0, 1, 2, 3 \) and for piecewise linear functions \( 0 < i + s \leq 2 \) for \( i = 0, 1 \). Thus, Theorem 4.4 guarantees that

\[
\|w - I_h w\|_{3,1} \leq C h \left( |D_x^2 w|_{1,1} + |D_y w|_{1,1} \right).
\]

Theorems (4.1) and (4.2) provide the error estimates

\[
\|u - u_h\|_{3,1} \leq C h \left( |D_x^3 u|_{1,1} + |D_y u|_{1,1} \right)
\]

and

\[
\|u - u_h\|_0 \leq C h^2 \left( |D_x^3 u|_{1,1} + |D_y u|_{1,1} \right),
\]

for this choice of approximating spaces. For the second choice of an approximating space, we choose cubic B-splines in both the \( x \)- and \( y \)-directions. However, the rate of convergence for this choice should be the same as our first pair. This is due to the fact that when cubic B-splines are also chosen in \( y \), the term \( hD^2_y u \) is still the governing factor.

Computations were performed for two different examples. The first is a problem which is artificial in nature but which has a closed form solution given by \( u = x^3(x - 1)^2(x - 1.3) \cos y \). In the second example, we took \( f = 0 \) in (5.1) and \( g = 1 \) in (5.4). This is similar to problems of physical interest and corresponds to the case of a countercurrent driven by a linear temperature profile along the outside wall. No closed form solution is available for this problem. For both problems, the predicted rates were obtained asymptotically. For example, in the first problem using cubic B-splines in the \( x \)-direction and linears in the \( y \)-direction, the relative \( L^2 \)-error for \( h = 1/10 \) was 0.1378 and for \( h = 1/15 \) was 0.0617 which gives a rate of 1.98, while using cubic B-splines in both directions gives relative \( L^2 \)-errors of 0.0838 and 0.0373 for \( h = 1/10 \) and \( h = 1/15 \), respectively, giving a rate of 2.00.

REFERENCES